Circular edge singularities for the Laplace equation and the elasticity system in 3-D domains

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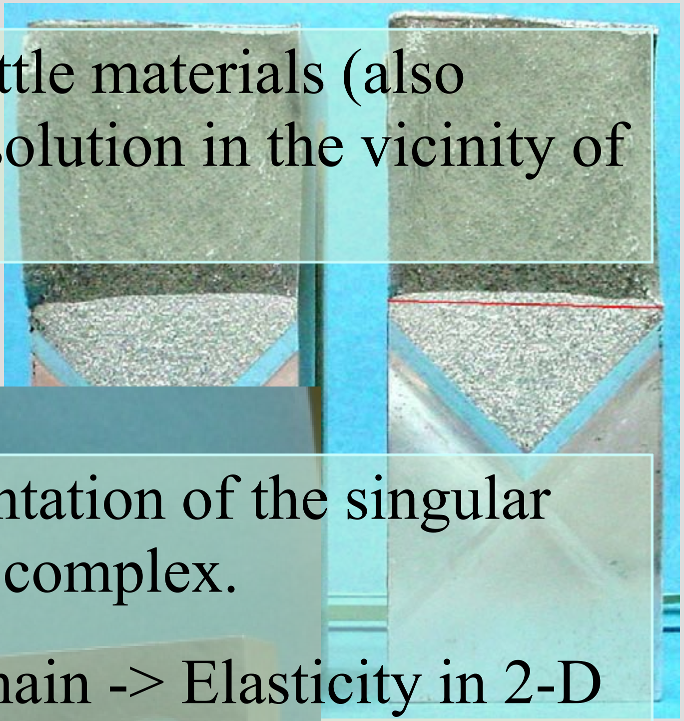
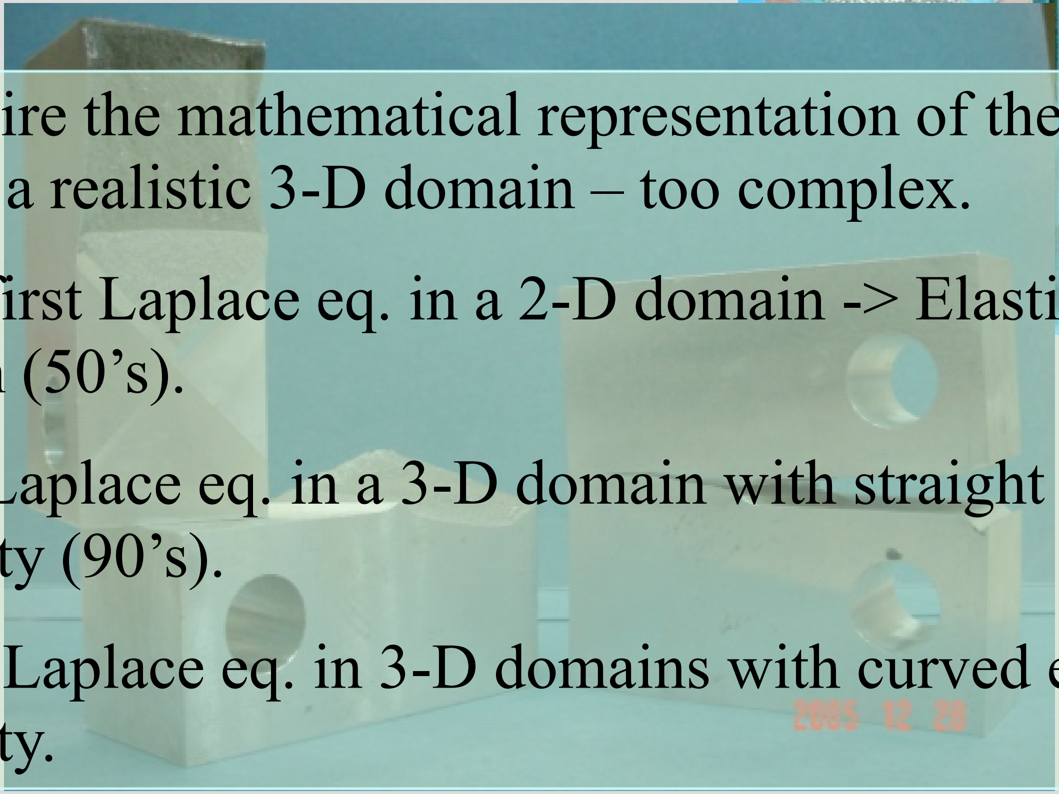
Motivation - Outline



1. Failure initiation and propagation in brittle materials (also metals) can be correlated to the elastic solution in the vicinity of the singular point –Failure laws.



2. These require the mathematical representation of the singular solution in a realistic 3-D domain – too complex.
   a. Study first Laplace eq. in a 2-D domain -> Elasticity in 2-D domain (50's).
   b. Study Laplace eq. in a 3-D domain with straight edges , then elasticity (90's).
   c. Now – Laplace eq. in 3-D domains with curved edges and elasticity.

My experience as an engineer is that Murphy's laws hold:

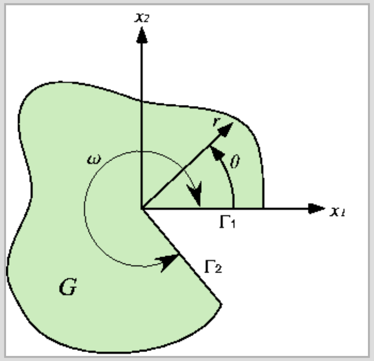
If anything can go wrong, it will.

(Murphy's Law - Book three, by Arthur Bloch, 1987)

Aloha Airlines Accident Flight 243, April 28, 1988, near Maui, Hawaii

A section of the upper fuselage of a Boeing 737-200 was torn away at 24,000 feet due to a crack in the fuselage at an altitude of 24,000 feet, after 89,681 flight cycles. One flight attendant killed, 8 people injured.

2-D Solution:



The solution is of the form:

$$\tau(r,\theta)=\sum_{i=1}^{\infty}A_i\Phi_i(r,\theta)$$

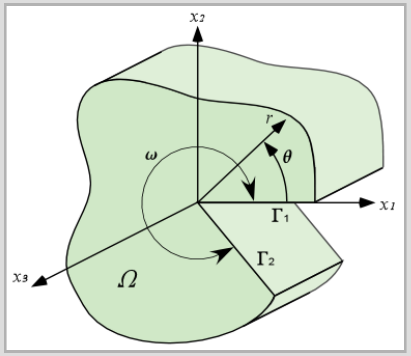
where $\Phi_i(r,\theta)=r^{\alpha_i}\varphi_i(\theta)$

The dual solution is of the form:

$$K(r,\theta)=\sum_{i=1}^{\infty}B_i\Psi_i(r,\theta)$$

where $\Psi_i(r,\theta)=r^{-\alpha_i}\psi_i(\theta)$

3-D Solution:



The solution is of the form:

$$\tau(r,\theta,x_3)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\partial_3^j A_i(x_3)\Phi_{ij}(r,\theta)$$

where $\Phi_{ij}(r,\theta)=r^{\alpha_i+j}\varphi_{ij}(\theta)$

The dual solution is of the form:

$$K(r,\theta,x_3)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\partial_3^j B_i(x_3)\Psi_{ij}(r,\theta)$$

where $\Psi_{ij}(r,\theta)=r^{-\alpha_i+j}\psi_{ij}(\theta)$

$$\begin{aligned}\tau(r,\theta,x_3)&=A_1(x_3)\Phi_{10}(r,\theta)+\partial_3A_1(x_3)\Phi_{11}(r,\theta)+\partial_3^2A_1(x_3)\Phi_{12}(r,\theta)+\cdots\\&+A_2(x_3)\Phi_{20}(r,\theta)+\partial_3A_2(x_3)\Phi_{21}(r,\theta)+\partial_3^2A_2(x_3)\Phi_{22}(r,\theta)+\cdots\\&\vdots\end{aligned}$$

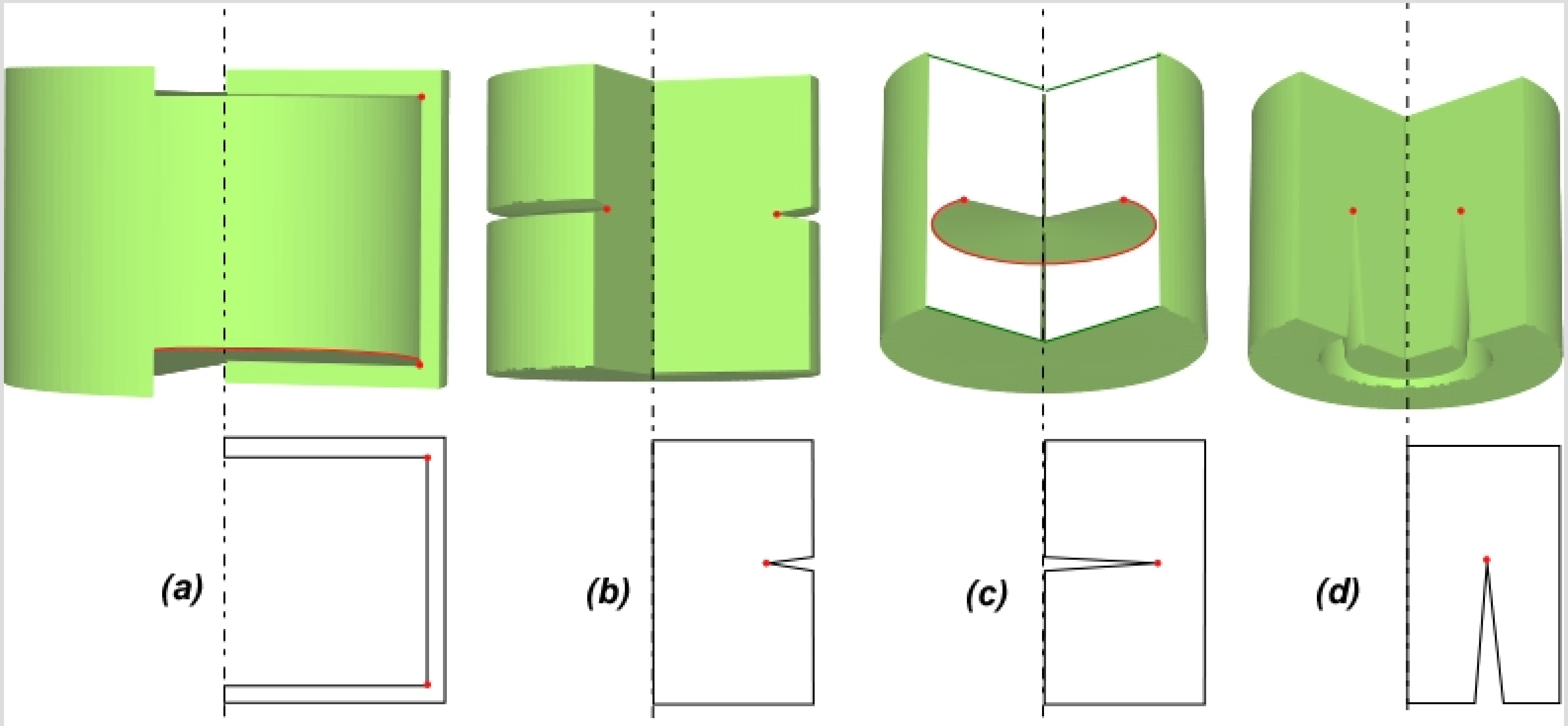
The 3D Solution is of the Form:

$$\vec{u}(r,\theta,x_3) = A_1(x_3)r^{\alpha_1}\vec{\varphi}_0^{(\alpha_1)}(\theta) + \partial_3 A_1(x_3)r^{\alpha_1+1}\vec{\varphi}_1^{(\alpha_1)}(\theta) + \partial_3^2 A_1(x_3)r^{\alpha_1+2}\vec{\varphi}_2^{(\alpha_1)}(\theta) + \cdots$$
$$+ A_2(x_3)r^{\alpha_2}\vec{\varphi}_0^{(\alpha_2)}(\theta) + \partial_3 A_2(x_3)r^{\alpha_2+1}\vec{\varphi}_1^{(\alpha_2)}(\theta) + \partial_3^2 A_2(x_3)r^{\alpha_2+2}\vec{\varphi}_2^{(\alpha_2)}(\theta) + \cdots$$
$$+ A_3(x_3)r^{\alpha_3}\vec{\varphi}_0^{(\alpha_3)}(\theta) + \partial_3 A_3(x_3)r^{\alpha_3+1}\vec{\varphi}_1^{(\alpha_3)}(\theta) + \partial_3^2 A_3(x_3)r^{\alpha_3+2}\vec{\varphi}_2^{(\alpha_3)}(\theta) + \cdots$$
$$\vdots$$

For isotropic materials:

$$\vec{u}(r,\theta,x_3) = A_1(x_3)r^{\alpha_1}\begin{pmatrix} u_0^{(\alpha_1)}(\theta) \\ v_0^{(\alpha_1)}(\theta) \\ 0 \end{pmatrix} + \partial_3 A_1(x_3)r^{\alpha_1+1}\begin{pmatrix} 0 \\ 0 \\ w_1^{(\alpha_1)}(\theta) \end{pmatrix} + \partial_3^2 A_1(x_3)r^{\alpha_1+2}\begin{pmatrix} u_2^{(\alpha_1)}(\theta) \\ v_2^{(\alpha_1)}(\theta) \\ 0 \end{pmatrix} + \cdots$$
$$+ A_2(x_3)r^{\alpha_2}\begin{pmatrix} u_0^{(\alpha_2)}(\theta) \\ v_0^{(\alpha_2)}(\theta) \\ 0 \end{pmatrix} + \partial_3 A_2(x_3)r^{\alpha_2+1}\begin{pmatrix} 0 \\ 0 \\ w_1^{(\alpha_2)}(\theta) \end{pmatrix} + \partial_3^2 A_2(x_3)r^{\alpha_2+2}\begin{pmatrix} u_2^{(\alpha_2)}(\theta) \\ v_2^{(\alpha_2)}(\theta) \\ 0 \end{pmatrix} + \cdots$$
$$+ A_3(x_3)r^{\alpha_3}\begin{pmatrix} 0 \\ 0 \\ w_0^{(\alpha_3)}(\theta) \end{pmatrix} + \partial_3 A_3(x_3)r^{\alpha_3+1}\begin{pmatrix} u_1^{(\alpha_3)}(\theta) \\ v_1^{(\alpha_3)}(\theta) \\ 0 \end{pmatrix} + \partial_3^2 A_3(x_3)r^{\alpha_3+2}\begin{pmatrix} 0 \\ 0 \\ w_2^{(\alpha_3)}(\theta) \end{pmatrix} + \cdots$$
$$\vdots$$

Circular singular edges



Past works on the topic

First three singular terms for the solution of the ***Laplace*** equation in the vicinity of a circular edge with ***homogeneous Dirichlet*** boundary conditions was analyzed from a theoretical viewpoint in:

- von Petersdorff T. & Stephan, E.: Singularities of the solution of the Laplacian in domains with circular edges, *Applicable Analysis* **45**(1-4), 281-294 (1992)

Asymptotic series for the elastic ***axi-symmetric*** case was given in:

- Leung, A. & Su, R.: Eigenfunction expansion for penny-shaped and circumferential cracks, Int. Jour. Fracture, 89, 205-222 (1998)

Systematic computation of the entire series solution up to an arbitrary order for any circular edge based on the methods in:

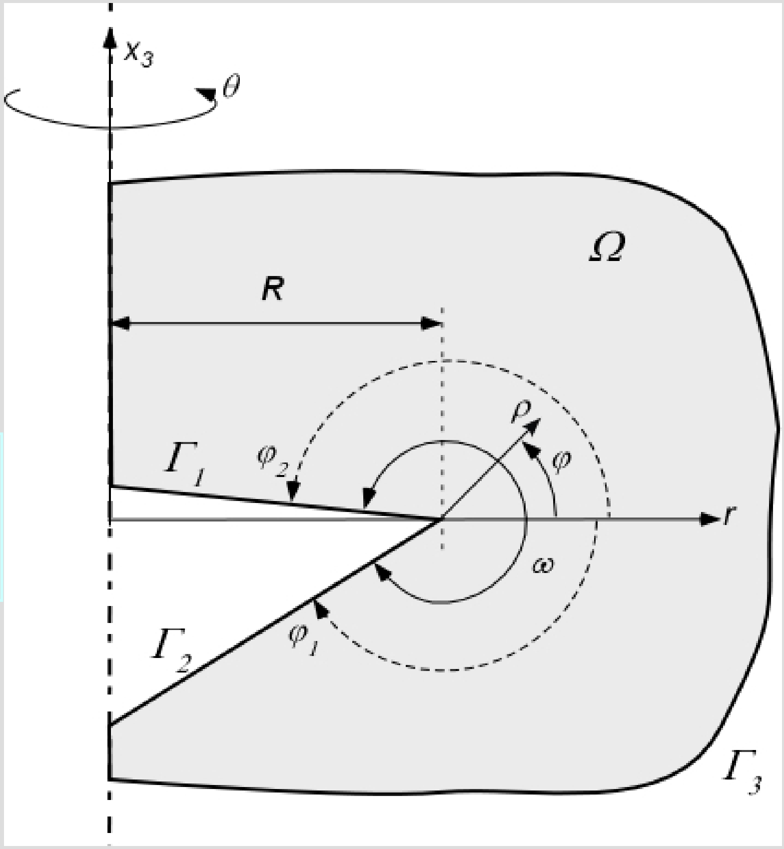
- Costabel, M., Dauge, M., Yosibash, Z.: A quasidual function method for extracting edge stress intensity functions, SIAM Jour. Math. Anal, 35(5), 1177-1202 (2004)

Laplace eq. -Circular singular edges

$$\triangle^{3D}\tau \overset{\text{def}}{=} \left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \partial_{33}\right)\tau = 0$$

We are interested in the solution in the vicinity of $\rho \to 0$.

$$r = \rho\cos\varphi + R, \qquad x_3 = \rho\sin\varphi$$



$$\triangle^{3D}\tau = \left[\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\varphi\varphi} + \frac{1}{r}\left(\cos\varphi\partial_\rho - \frac{1}{\rho}\sin\varphi\partial_\varphi\right) + \frac{1}{r^2}\partial_{\theta\theta}\right]\tau = 0$$

Axi-symmetric solution $\partial_{\theta\theta} = 0$

$$\triangle^{Axi} = \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\varphi\varphi} + \frac{1}{r}\left(\cos\varphi\partial_\rho - \frac{1}{\rho}\sin\varphi\partial_\varphi\right)$$

$$r = \rho\cos\varphi + R$$

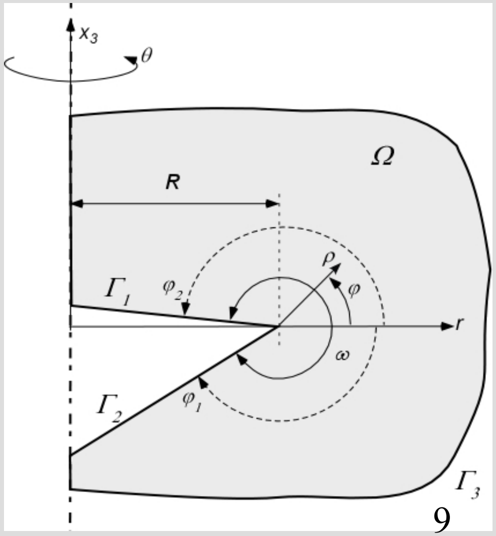
Notice that as r→∞, i.e. R→∞, then: $\triangle^{Axi} \xrightarrow{R\to\infty} \triangle^{2D}$

Consider now $\rho^2 \frac{r}{R} \triangle^{Axi} \tau = 0$:

$$\underbrace{\left[(\rho\partial_\rho)^2 + \partial_{\varphi\varphi}\right]}_{\rho^2\triangle^{2D}} \tau + \frac{\rho}{R}\left[\cos\varphi(\rho\partial_\rho) - \sin\varphi\partial_\varphi + \cos\varphi\left((\rho\partial_\rho)^2 + \partial_{\varphi\varphi}\right)\right]\tau = 0.$$

For ρ/R→0, we may look for an asymptotic expansion similar to 2D, composed of eigen-pairs. For a specific 2D eigen-pair $\alpha, \phi(\varphi)$ we consider:

$$\tau = A\rho^\alpha \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^i \phi_i(\varphi)$$



Axi-symmetric solution $\partial_{\theta\theta} = 0$

$$
\begin{aligned}
A \quad & \left\{\left[\alpha^2\phi_0 + \phi_0''\right]\right. \\
& + \frac{\rho}{R}\left[\left((\alpha+1)^2\phi_1 + \phi_1''\right) + \alpha\cos\varphi\phi_0 - \sin\varphi\phi_0' + \cos\varphi\left(\alpha^2\phi_0 + \phi_0''\right)\right] \\
& + \frac{\rho^2}{R^2}\left[\left((\alpha+2)^2\phi_2 + \phi_2''\right) + (\alpha+1)\cos\varphi\phi_1 - \sin\varphi\phi_1' + \cos\varphi\left((\alpha+1)^2\phi_1 + \phi_1''\right)\right] \\
& + \frac{\rho^3}{R^3}\left[\left((\alpha+3)^2\phi_3 + \phi_3''\right) + (\alpha+2)\cos\varphi\phi_2 - \sin\varphi\phi_2' + \cos\varphi\left((\alpha+2)^2\phi_2 + \phi_2''\right)\right] \\
& \left. + \cdots \right\} = 0
\end{aligned}
$$

$$
\begin{aligned}
\alpha^2\phi_0 + \phi_0'' &= 0, \quad \varphi_1 < \varphi < \varphi_2 \\
(\alpha+1)^2\phi_1 + \phi_1'' &= -\left(\alpha\cos\varphi\phi_0 - \sin\varphi\phi_0'\right), \quad \varphi_1 < \varphi < \varphi_2 \\
(\alpha+i)^2\phi_i + \phi_i'' &= -\left[(\alpha+i)(\alpha+i-1)\cos\varphi\phi_{i-1} - \sin\varphi\phi_{i-1}' + \cos\varphi\phi_{i-1}''\right] \\
& \qquad i \geq 2, \quad \varphi_1 < \varphi < \varphi_2
\end{aligned}
$$

Second order ODE (e-value problem) to determine α, ϕ_0 (primal eigen-function).

A recursive system of ODEs for the shadows: $\phi_1, \phi_2, \cdots$

Axi-symmetric solution $\partial_{\theta\theta} = 0$

The solution has an infinite primal eigen-pairs, $\alpha_k, \phi_{k,0}$, thus is a double sum series:

$$\tau = \sum_k A_k \rho^{\alpha_k} \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^i \phi_{k,i}(\varphi)$$

The ODEs are complemented by the homogeneous BCs:

$$\phi_{k,i}(\varphi = \varphi_1, \varphi_2) = 0 \qquad \text{Dirichlet BCs}$$
$$\phi'_{k,i}(\varphi = \varphi_1, \varphi_2) = 0 \qquad \text{Neumann BCs}$$

Axisymmetric solution $\partial_{\theta\theta} = 0$
Penny-shaped crack with homogeneous Neumann BCs.

Primal Eigen-pairs

k	α_k	$\phi_{k,0}(\varphi)$
0	0	1
1	$\frac{1}{2}$	$\sin\frac{\varphi}{2}$
2	1	$\cos\varphi$
3	$\frac{3}{2}$	$\sin\frac{3\varphi}{2}$
4	2	$\cos 2\varphi$

$$\tau = A_0 + A_1\rho^{\frac{1}{2}}\Big[\sin\frac{\varphi}{2} \quad \left(\frac{1}{12}\sin\frac{\varphi}{2} - \frac{3}{32}\sin\frac{3\varphi}{2}\right)$$

$$+\left(\frac{\rho}{R}\right)^3 \quad + \frac{5}{128}\sin\frac{5\varphi}{2}\Big) + \cdots\Big]$$

$$+ A_2\rho\Big[\cos\varphi - \left(\frac{\rho}{R}\right) \quad \frac{\rho}{R}\Big)^3\left(\frac{9}{128} + \frac{5}{64}\cos 2\varphi\right) + \cdots\Big]$$

$$+ A_3\rho^{\frac{3}{2}}\Big[\sin\frac{3\varphi}{2} - \left(\frac{\rho}{R}\right)\frac{1}{4}\sin\frac{\varphi}{2} - \left(\frac{\rho}{R}\right)^2\frac{1}{32}\left(3\sin\frac{\varphi}{2} - \frac{16}{5}\sin\frac{3\varphi}{2}\right)$$

$$+\left(\frac{\rho}{R}\right)^3\left(-\frac{3}{40}\sin\frac{\varphi}{2} + \frac{5}{128}\sin\frac{3\varphi}{2} - \frac{3}{70}\sin\frac{5\varphi}{2}\right) + \cdots\Big] + \cdots$$

We enforce orthogonality conditions on the shadow terms to make them unique:

$$\int_{\varphi_1=-\pi}^{\varphi_2=\pi} \phi_{k,i}(\varphi)\,\phi_{k+i,0}(\varphi)\,d\varphi = 0, \quad k = 0,1,2,3 \ \text{ and } \ i = 1,2,3.$$

Laplace eq. – General circular singular edges

$$\left(\frac{r}{R}\right)^2 \rho^2 \triangle^{3D} \tau = 0$$

$$\begin{aligned}
&(1+\frac{\rho}{R}\cos\varphi)^2\left[(\rho\partial_\rho)^2+\partial_{\varphi\varphi}\right]\tau \\
&\quad +\frac{\rho}{R}(1+\frac{\rho}{R}\cos\varphi)\Big[\cos\varphi(\rho\partial_\rho)-\sin\varphi\partial_\varphi\Big]\tau \\
&\quad +\left(\frac{\rho}{R}\right)^2\partial_{\theta\theta}\tau = 0.
\end{aligned}$$

$$\tau = \sum_{\ell=0,2,4,\ldots}\sum_{k=0}\partial_\theta^\ell A_k(\theta)\ \rho^{\alpha_k}\left(\frac{\rho}{R}\right)^\ell \sum_{i=0}^{\infty}\left(\frac{\rho}{R}\right)^i \phi_{\ell,k,i}(\varphi)$$

Notice that $\phi_{0,k,i} = \phi_{k,i}$ (associated with the curvature for an axisymmetric case), so these are known for the axi-symmetric analysis.

Laplace eq. – General circular singular edges

Substituting the series expansion one obtains:

$$
\begin{aligned}
0 \quad = \quad & A_k(\theta) \times \{[\alpha_k^2 \phi_{0,k,0} + \phi''_{0,k,0}] \\
& + \left(\frac{\rho}{R}\right) [(\alpha_k+1)^2 \phi_{0,k,1} + \phi''_{0,k,1} + (\alpha_k \phi_{0,k,0} \cos\varphi - \phi'_{0,k,0} \sin\varphi)] \\
& + \left(\frac{\rho}{R}\right)^2 [(\alpha_k+2)^2 \phi_{0,k,2} + \phi''_{0,k,2} + \big((\alpha_k+1)\phi_{0,k,1} \cos\varphi - \phi'_{0,k,1} \sin\varphi\big) \\
& \qquad - \cos\varphi \left(\alpha_k \phi_{0,k,0} \cos\varphi - \phi'_{0,k,0} \sin\varphi\right)] + \cdots \} \\
+ \quad & A''_k(\theta) \times \left\{ \left(\frac{\rho}{R}\right)^2 [(\alpha_k+2)^2 \phi_{2,k,0} + \phi''_{2,k,0} + \phi_{0,k,0}] \right. \\
& + \left(\frac{\rho}{R}\right)^3 [(\alpha_k+3)^2 \phi_{2,k,1} + \phi''_{2,k,1} + \big((\alpha_k+2)\phi_{2,k,0} \cos\varphi - \phi'_{2,k,0} \sin\varphi\big) \\
& \qquad + (\phi_{0,k,1} - 2\cos\varphi \phi_{0,k,0})] \\
& + \left(\frac{\rho}{R}\right)^4 [(\alpha_k+4)^2 \phi_{2,k,2} + \phi''_{2,k,2} + ((\alpha_k+3)\phi_{2,k,1} \cos\varphi - \phi'_{2,k,1} \sin\varphi) \\
& \qquad - \cos\varphi((\alpha_k+2)\phi_{2,k,0} \cos\varphi - \phi'_{2,k,0} \sin\varphi) \\
& \qquad + \left. \left(\phi_{0,k,2} - 2\cos\varphi \phi_{0,k,1} + 3\cos^2\varphi \phi_{0,k,0}\right)] + \cdots \right\} \\
+ \quad & \cdots
\end{aligned}
$$

Laplace eq. – General circular singular edges

The following recursive system of ODEs for the shadows $\phi_{\ell,k,i}$ is obtained:

ℓ - The shadow # due to 3-D

k - The eigen-value #

i - Curvature shadow #

$$\ell = 0$$

Equations for the axi-symmetric case hold.

$$\ell = 2, 4, 6 \cdots, \qquad i \geq 0$$

$$\begin{aligned}(\alpha_k + i + \ell)^2 \phi_{\ell,k,i} + \phi''_{\ell,k,i} = & -(\ell + i + \alpha_k - 1)\left[2(\ell + i + \alpha_k) - 1\right] \cos\varphi \phi_{\ell,k,(i-1)} \\ & + \sin\varphi \phi'_{\ell,k,(i-1)} - 2\cos\varphi \phi''_{\ell,k,(i-1)} \\ & -(\ell + \alpha_k + i - 2)(\ell + \alpha_k + i - 1)\cos^2\varphi \phi_{\ell,k,(i-2)} \\ & + \cos\varphi \sin\varphi \phi'_{\ell,k,(i-2)} - \cos^2\varphi \phi''_{\ell,k,(i-2)} - \phi_{(\ell-2),k,i}\end{aligned}$$

ODEs are complemented by the homogeneous BCs:

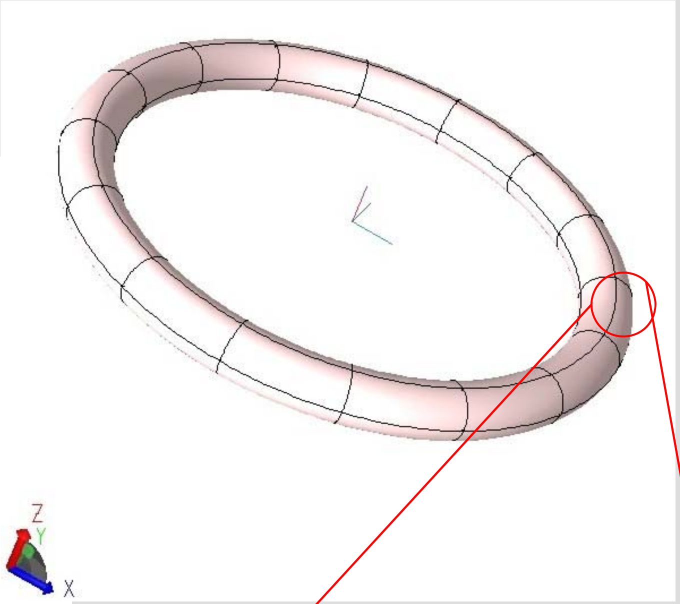
$$\phi_{\ell,k,i}(\varphi = \varphi_1, \varphi_2) = 0 \qquad \text{Dirichlet BCs}$$

$$\phi'_{\ell,k,i}(\varphi = \varphi_1, \varphi_2) = 0 \qquad \text{Neumann BCs}$$

Laplace eq. - Penny-shaped crack with homogeneous Neumann BCs.

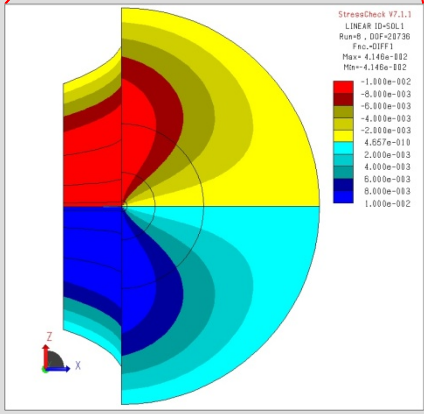
$$
\begin{aligned}
\tau &= A_0(\theta) \\
&+ A_0''(\theta)\left(\frac{\rho}{R}\right)^2\left[-\frac{1}{4}+\left(\frac{\rho}{R}\right)\frac{5}{16}\cos\varphi-\left(\frac{\rho}{R}\right)^2\left(\frac{19}{128}+\frac{11}{64}\cos 2\varphi\right)+\cdots\right]+\cdots \\
&+ A_1(\theta)\rho^{\frac{1}{2}}\left[\sin\frac{\varphi}{2}+\left(\frac{\rho}{R}\right)\frac{1}{4}\sin\frac{\varphi}{2}+\left(\frac{\rho}{R}\right)^2\left(\frac{1}{12}\sin\frac{\varphi}{2}-\frac{3}{32}\sin\frac{3\varphi}{2}\right)+\right. \\
&\qquad\left.+\left(\frac{\rho}{R}\right)^3\left(\frac{1}{16}\sin\frac{\varphi}{2}-\frac{1}{30}\sin\frac{3\varphi}{2}+\frac{5}{128}\sin\frac{5\varphi}{2}\right)+\cdots\right] \\
&+ A_1''(\theta)\rho^{\frac{1}{2}}\left(\frac{\rho}{R}\right)^2\left[-\frac{1}{6}\sin\frac{\varphi}{2}+\left(-\frac{1}{8}\sin\frac{\varphi}{2}+\frac{7}{60}\sin\frac{3\varphi}{2}\right)\left(\frac{\rho}{R}\right)+\cdots\right]+\cdots
\end{aligned}
$$

Laplace eq. - General singular circular Penny-shaped crack with homogeneous Neumann BCs.



Taking $A_1 = 10\cos\theta$, $A_k = 0$, $k \neq 1$, $(\rho/R = 1/10)$ we prescribed on the outer surface:

$$
\begin{aligned}
\tau = \quad & 10\cos\theta\sqrt{\frac{1}{10}}\left[\sin\left(\frac{\varphi}{2}\right) + \frac{1}{4}\sin\frac{\varphi}{2}\left(\frac{1}{10}\right)\right. \\
& + \left(\frac{1}{12}\sin\frac{\varphi}{2} - \frac{3}{32}\sin\frac{3\varphi}{2}\right)\left(\frac{1}{10}\right)^2 + \\
& \left.\left(\frac{1}{16}\sin\frac{\varphi}{2} - \frac{1}{30}\sin\frac{3\varphi}{2} + \frac{5}{128}\sin\frac{5\varphi}{2}\right)\left(\frac{1}{10}\right)^3\right] \\
& -10\cos\theta\sqrt{\frac{1}{10}}\left[-\frac{1}{6}\sin\frac{\varphi}{2}\left(\frac{1}{10}\right)^2\right. \\
& \left. + \left(-\frac{1}{8}\sin\frac{\varphi}{2} + \frac{7}{60}\sin\frac{3\varphi}{2}\right)\left(\frac{1}{10}\right)^3\right]
\end{aligned}
$$



Error plot – order 10^{-3}

The elasticity system in the vicinity of a singular circular edge

Consider the system of 3 PDEs (equilibrium equations) in terms of the 6 stress tensor components, in ρ, φ, θ coordinates:

$$
\begin{aligned}
0 &= \frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\rho\varphi}}{\partial \varphi} + \frac{\sigma_{\rho\rho} - \sigma_{\varphi\varphi}}{\rho} + \frac{1}{r}\left(\frac{\partial \sigma_{\rho\theta}}{\partial \theta} + (\sigma_{\rho\rho} - \sigma_{\theta\theta})\cos\varphi - \sigma_{\rho\varphi}\sin\varphi\right) \\
0 &= \frac{1}{\rho}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \sigma_{\rho\varphi}}{\partial \rho} + \frac{2}{\rho}\sigma_{\rho\varphi} + \frac{1}{r}\left(\frac{\partial \sigma_{\varphi\theta}}{\partial \theta} + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\sin\varphi + \sigma_{\rho\varphi}\cos\varphi\right) \\
0 &= \frac{\partial \sigma_{\rho\theta}}{\partial \rho} + \frac{1}{\rho}\sigma_{\rho\theta} + \frac{1}{\rho}\frac{\partial \sigma_{\varphi\theta}}{\partial \varphi} + \frac{1}{r}\left(\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\sigma_{\rho\theta}\cos\varphi - 2\sigma_{\varphi\theta}\sin\varphi\right),
\end{aligned}
$$

$$r = \rho\cos\varphi + R$$

We use the Hooke constitutive law, and the kinematic connections between strains and displacements to finally obtain the complicated system of PDEs to be solved.

The elasticity system in the vicinity of a singular circular edge

The Navier-Lamè system in terms of $u_\rho, u_\varphi, u_\theta$:

$$
\begin{aligned}
0 \; = \; & (\lambda+2\mu)\left(\frac{1}{\rho}\frac{\partial u_\rho}{\partial \rho}+\frac{\partial^2 u_\rho}{\partial \rho^2}-\frac{1}{\rho^2}u_\rho\right)+\mu\frac{1}{\rho^2}\frac{\partial^2 u_\rho}{\partial \varphi^2}-(\lambda+3\mu)\frac{1}{\rho^2}\frac{\partial u_\varphi}{\partial \varphi}+(\lambda+\mu)\frac{1}{\rho}\frac{\partial^2 u_\varphi}{\partial \rho \partial \varphi} \\
& +\frac{1}{r}\left[(\lambda+2\mu)\cos\varphi\frac{\partial u_\rho}{\partial \rho}-(\lambda+\mu)\sin\varphi\frac{\partial u_\varphi}{\partial \rho}+\mu\frac{\sin\varphi}{\rho}\left(u_\varphi-\frac{\partial u_\rho}{\partial \varphi}\right)+(\lambda+\mu)\frac{\partial^2 u_\theta}{\partial \rho \partial \theta}\right] \\
& +\frac{1}{r^2}\left[(\lambda+2\mu)\cos\varphi(u_\varphi\sin\varphi-u_\rho\cos\varphi)+\mu\frac{\partial^2 u_\rho}{\partial \theta^2}-(\lambda+3\mu)\cos\varphi\frac{\partial u_\theta}{\partial \theta}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 \; = \; & \mu\left(\frac{\partial^2 u_\varphi}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u_\varphi}{\partial \rho}-\frac{1}{\rho^2}u_\varphi\right)+(\lambda+2\mu)\frac{1}{\rho^2}\frac{\partial^2 u_\varphi}{\partial \varphi^2}+(\lambda+3\mu)\frac{1}{\rho^2}\frac{\partial u_\rho}{\partial \varphi}+(\lambda+\mu)\frac{1}{\rho}\frac{\partial^2 u_\rho}{\partial \rho \partial \varphi} \\
& +\frac{1}{r}\left[(\lambda+\mu)\cos\varphi\frac{1}{\rho}\left(\frac{\partial u_\rho}{\partial \varphi}-u_\varphi\right)+\mu\cos\varphi\frac{\partial u_\varphi}{\partial \rho}-(\lambda+2\mu)\sin\varphi\frac{1}{\rho}\left(\frac{\partial u_\varphi}{\partial \varphi}+u_\rho\right)+(\lambda+\mu)\frac{1}{\rho}\frac{\partial^2 u_\theta}{\partial \varphi \partial \theta}\right] \\
& +\frac{1}{r^2}\left[(\lambda+2\mu)\sin\varphi(u_\rho\cos\varphi-u_\varphi\sin\varphi)+\mu\frac{\partial^2 u_\varphi}{\partial \theta^2}+(\lambda+3\mu)\sin\varphi\frac{\partial u_\theta}{\partial \theta}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 \; = \; & \mu\left(\frac{\partial^2 u_\theta}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u_\theta}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u_\theta}{\partial \varphi^2}\right) \\
& +\frac{1}{r}\left[\mu\left(\cos\varphi\frac{\partial u_\theta}{\partial \rho}-\sin\varphi\frac{1}{\rho}\frac{\partial u_\theta}{\partial \varphi}\right)+(\lambda+\mu)\left(\frac{1}{\rho}\left(\frac{\partial u_\rho}{\partial \theta}+\frac{\partial^2 u_\varphi}{\partial \varphi \partial \theta}\right)+\frac{\partial^2 u_\rho}{\partial \rho \partial \theta}\right)\right] \\
& +\frac{1}{r^2}\left[-\mu u_\theta+(\lambda+2\mu)\frac{\partial^2 u_\theta}{\partial \theta^2}+(\lambda+3\mu)\left(\cos\varphi\frac{\partial u_\rho}{\partial \theta}-\sin\varphi\frac{\partial u_\varphi}{\partial \theta}\right)\right]
\end{aligned}
$$

The elasticity system in the vicinity of a singular circular edge

The Navier-Lamé equations are complemented by homogeneous boundary conditions on the faces intersecting at the singular edge:

$$u_\rho = u_\varphi = u_\theta = 0 \qquad \text{on } \Gamma_1 \cup \Gamma_2 \qquad \text{Clamped BCs}$$
$$t_\varphi = t_\rho = t_\theta = 0 \qquad \text{on } \Gamma_1 \cup \Gamma_2 \qquad \text{Traction Free BCs,}$$

Similar to the Laplace equation, we herein consider a series expansion of the form:

$$\boldsymbol{u} = \sum_{\ell=0}\sum_{k=0} \partial_\theta^\ell A_k(\theta)\, \rho^{\alpha_k} \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^{i+\ell} \begin{Bmatrix} \phi^\rho(\varphi) \\ \phi^\varphi(\varphi) \\ \phi^\theta(\varphi) \end{Bmatrix}_{\ell,k,i} = \sum_{\ell=0}\sum_{k=0} \partial_\theta^\ell A_k(\theta)\, \rho^{\alpha_k} \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^{i+\ell} \boldsymbol{\phi}_{\ell,k,i}$$

Substituting the series expansion into the Navier-Lamè system results in a “messy” system of different orders of ρ/R, $\partial^\ell A(\theta)$

The elasticity system in the vicinity of a singular circular edge

$$
\begin{aligned}
[m_0]\boldsymbol{\phi}_{\ell,k,i} = & -(2\cos\varphi[m_0]+[m_{01}])\,\boldsymbol{\phi}_{\ell,k,i-1} - \left(\cos^2\varphi[m_0]+\cos\varphi[m_{01}]+[m_{02}]\right)\boldsymbol{\phi}_{\ell,k,i-2} - [m_{10}]\boldsymbol{\phi}_{\ell-1,k,i} \\
& -(\cos\varphi[m_{10}]+[m_{11}])\,\boldsymbol{\phi}_{\ell-1,k,i-1} - [m_2]\boldsymbol{\phi}_{\ell-2,k,i}, \qquad \ell \ge 0,\ i \ge 0
\end{aligned}
$$

where $\boldsymbol{\phi}$'s with negative indices are set to zero, and

$$
[m_0]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} (\lambda+2\mu)\left(\beta^2-1\right)+\mu\partial_{\varphi\varphi} & ((\lambda+\mu)\beta-(\lambda+3\mu))\,\partial_\varphi & 0 \\ ((\lambda+\mu)\beta+(\lambda+3\mu))\,\partial_\varphi & \mu\left(\beta^2-1\right)+(\lambda+2\mu)\partial_{\varphi\varphi} & 0 \\ 0 & 0 & \mu\left(\beta^2+\partial_{\varphi\varphi}\right)\boldsymbol{\phi}_{l,k,i} \end{pmatrix}\boldsymbol{\phi}_{\ell,k,i}
$$

$$
[m_{01}]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} (\lambda+2\mu)\cos\varphi\beta-\mu\sin\varphi\partial_\varphi & \sin\varphi\,(\mu-(\lambda+\mu)\beta) & 0 \\ -(\lambda+2\mu)\sin\varphi+(\lambda+\mu)\cos\varphi\partial_\varphi & \cos\varphi\,(\mu\,(\beta-1)-\lambda)-(\lambda+2\mu)\sin\varphi\partial_\varphi & 0 \\ 0 & 0 & \mu\,(\beta\cos\varphi-\sin\varphi\partial_\varphi) \end{pmatrix}\boldsymbol{\phi}_{\ell,k,i}
$$

$$
[m_{02}]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} -(\lambda+2\mu)\cos^2\varphi & (\lambda+2\mu)\cos\varphi\sin\varphi & 0 \\ (\lambda+2\mu)\sin\varphi\cos\varphi & -(\lambda+2\mu)\sin^2\varphi & 0 \\ 0 & 0 & -\mu \end{pmatrix}\boldsymbol{\phi}_{\ell,k,i}
$$

$$
[m_{10}]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} 0 & 0 & (\lambda+\mu)\beta \\ 0 & 0 & (\lambda+\mu)\partial_\varphi \\ (\lambda+\mu)\beta & (\lambda+\mu)\partial_\varphi & 0 \end{pmatrix}\boldsymbol{\phi}_{\ell,k,i}
$$

$$
[m_{11}]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} 0 & 0 & -(\lambda+3\mu)\cos\varphi \\ 0 & 0 & (\lambda+3\mu)\sin\varphi \\ (\lambda+3\mu)\cos\varphi & -(\lambda+3\mu)\sin\varphi & 0 \end{pmatrix}\boldsymbol{\phi}_{\ell,k,i}, \qquad [m_2]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda+2\mu) \end{pmatrix}\boldsymbol{\phi}_{\ell,k,i}
$$

$$\beta = (\alpha_k+\ell+i)$$

The elasticity system in the vicinity of a singular circular edge

Clamped or traction free boundary conditions are a bit more complicated:

Clamped:

$$\boldsymbol{\phi}_{\ell,k,i}(\varphi_1) = \boldsymbol{\phi}_{\ell,k,i}(\varphi_2) = \mathbf{0}, \quad \forall \ell, \alpha_k, i$$

Traction Free:

$$[t_0]\boldsymbol{\phi}_{\ell,k,i} = -\left(\cos\varphi[t_0] + [t_{01}]\right)\boldsymbol{\phi}_{\ell,k,i-1} - [t_1]\boldsymbol{\phi}_{\ell-1,k,i} = \mathbf{0}, \qquad \varphi = \varphi_1, \varphi_2$$

$$[t_0]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} 2\mu + \lambda(\beta+1) & (\lambda+2\mu)\partial_\varphi & 0 \\ \mu\partial_\varphi & \mu(\beta-1) & 0 \\ 0 & 0 & \mu\partial_\varphi \end{pmatrix} \boldsymbol{\phi}_{\ell,k,i}$$

$$[t_{01}]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} \lambda\cos\varphi & -\lambda\sin\varphi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu\sin\varphi \end{pmatrix} \boldsymbol{\phi}_{\ell,k,i}, \qquad [t_1]\boldsymbol{\phi}_{\ell,k,i} = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix} \boldsymbol{\phi}_{\ell,k,i}$$

Traction free penny shaped crack
Axisymmetric solution $\partial_{\theta\theta} = 0$

Computing the displacements, one may then evaluate the stresses:

$$
\begin{Bmatrix} \sigma_{\rho\rho} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sigma_{\rho\theta} \\ \sigma_{\rho\varphi} \\ \sigma_{\theta\varphi} \end{Bmatrix}
= \frac{K_I}{\sqrt{2\pi\rho}} \left[\begin{pmatrix} -5\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2} \\ -\frac{4\lambda}{\lambda+\mu}\cos\frac{\varphi}{2} \\ -3\cos\frac{\varphi}{2} - \cos\frac{3\varphi}{2} \\ 0 \\ -\sin\frac{\varphi}{2} - \sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} -\frac{5\lambda+13\mu}{4(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{\lambda+9\mu}{4(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ -\frac{2(2\lambda+\mu)(\lambda+5\mu)}{(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{3\lambda+2\mu}{\lambda+\mu}\cos\frac{3\varphi}{2} \\ -\frac{3(\lambda+9\mu)}{4(\lambda+\mu)}\cos\frac{\varphi}{2} - \frac{\lambda+9\mu}{4(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \\ \frac{\lambda-7\mu}{4(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{\lambda-7\mu}{4(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots \right]
$$

$$
+ \frac{K_{II}}{\sqrt{2\pi\rho}} \left[\begin{pmatrix} -\frac{5}{3}\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2} \\ -\frac{4\lambda}{3(\lambda+\mu)}\sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} - \sin\frac{3\varphi}{2} \\ 0 \\ \frac{1}{3}\left(\cos\frac{\varphi}{2} + 3\cos\frac{3\varphi}{2}\right) \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} -\frac{51\lambda+107\mu}{60(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{\lambda+9\mu}{12(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ \frac{2\left(34\lambda^2+83\lambda\mu+45\mu^2\right)}{15(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{3\lambda+2\mu}{3(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ -\frac{\lambda+9\mu}{12(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{\lambda+9\mu}{12(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \\ -\frac{23\lambda+31\mu}{60(\lambda+\mu)} cos\frac{\varphi}{2} + \frac{-\lambda+7\mu}{12(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots \right]
$$

$$
+ \frac{K_{III}}{\sqrt{2\pi\rho}} \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}\sin\frac{\varphi}{2} \\ 0 \\ \frac{1}{2}\cos\frac{\varphi}{2} \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{7}{8}\sin\frac{\varphi}{2} - \frac{1}{2}\sin\frac{3\varphi}{2} \\ 0 \\ \frac{5}{8}\cos\frac{\varphi}{2} - \frac{1}{2}\cos\frac{3\varphi}{2} \end{pmatrix} + \cdots \right]
$$

Traction free penny shaped crack
Non-Axisymmetric solution

Consider the "mode I" component of stresses:

$$
\begin{Bmatrix} \sigma_{\rho\rho} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sigma_{\rho\theta} \\ \sigma_{\rho\varphi} \\ \sigma_{\theta\varphi} \end{Bmatrix}
= \frac{K_I(\theta)}{\sqrt{2\pi\rho}} \left[\begin{pmatrix} -5\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2} \\ -\frac{4\lambda}{\lambda+\mu}\cos\frac{\varphi}{2} \\ -3\cos\frac{\varphi}{2} - \cos\frac{3\varphi}{2} \\ 0 \\ -\sin\frac{\varphi}{2} - \sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} -\frac{5\lambda+13\mu}{4(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{\lambda+9\mu}{4(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ -\frac{2(2\lambda+\mu)(\lambda+5\mu)}{(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{3\lambda+2\mu}{\lambda+\mu}\cos\frac{3\varphi}{2} \\ -\frac{3(\lambda+9\mu)}{4(\lambda+\mu)}\cos\frac{\varphi}{2} - \frac{\lambda+9\mu}{4(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \\ \frac{\lambda-7\mu}{4(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{\lambda-7\mu}{4(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots \right]
$$

$$
+ \frac{K_I'(\theta)}{\sqrt{2\pi\rho}} \left(\frac{\rho}{R}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{2(\lambda-\mu)}{\lambda+\mu}\cos\frac{\varphi}{2} - \frac{2(\lambda+3\mu)}{\lambda+\mu}\cos\frac{3\varphi}{2} \\ 0 \\ \frac{2(\lambda+3\mu)}{\lambda+\mu}\sin\frac{\varphi}{2} + \frac{2(\lambda+3\mu)}{\lambda+\mu}\sin\frac{3\varphi}{2} \end{pmatrix} + \cdots
$$

$$
+ \frac{K_I''(\theta)}{\sqrt{2\pi\rho}} \left(\frac{\rho}{R}\right)^2 \begin{pmatrix} \frac{-3\lambda+5\mu}{6(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{21\lambda+61\mu}{18(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ \frac{2(3\lambda+2\mu)}{\lambda+\mu}\cos\frac{\varphi}{2} - \frac{4(4\lambda+3\mu)(3\lambda+7\mu)}{9(\lambda+\mu)^2}\cos\frac{3\varphi}{2} \\ \frac{3\lambda-5\mu}{6(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{3\lambda-5\mu}{18(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \\ -\frac{3\lambda+11\mu}{6(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{3\lambda+11\mu}{6(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots
$$

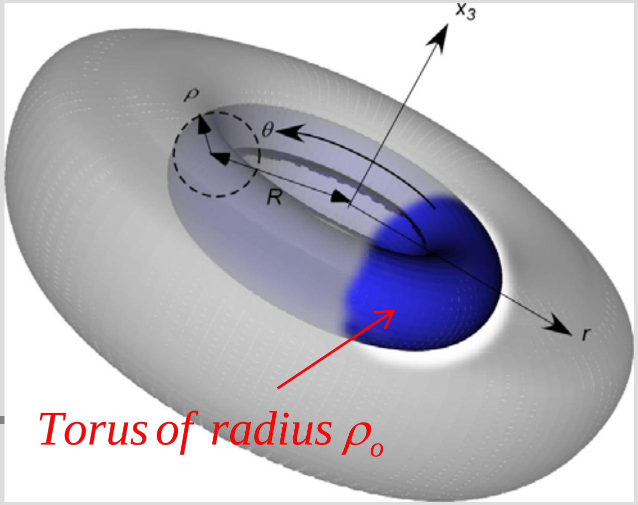
What is the next step?

Next – computation of Edge Flux Intensity Functions (ESIFs):

$$\tau(\rho,\varphi,\theta) = \sum_{\ell=0} \sum_{k=1} \partial_\theta^\ell A_k(\theta) \rho^{\alpha_k} \left(\frac{\rho}{R}\right)^\ell \sum_{i=0} \left(\frac{\rho}{R}\right)^i \phi_{\ell,k,i}(\varphi)$$

Extending the Quasi-Dual Function Method for Extracting ESIFs - $J[\rho_o]$*Integral*
First we consider the Axisymmetric case

Can we extend the boundary integral $J[R](\vec{u},\vec{v})$ [Costabel, Dauge & Yosibash, SIAM J. Math. Anal. (2004)] to circular edges?



$$J[\rho_o](\tau,K) \equiv \int_{\Gamma_R} ([T]\tau\cdot K - \tau\cdot[T]K)d\Gamma$$

$$= \int_{\theta=0}^{2\pi}\int_{\varphi=\varphi_1}^{\varphi_1+\omega} (\partial_\rho \tau\cdot K - \tau\cdot\partial_\rho K)\rho\left(R+\rho\cos\varphi\right)\Big|_{\rho_o} d\theta d\varphi$$

The quasidual extraction function $K_n^{(\alpha_i)}$ is constructed by the duals:

$$K_m^{(\alpha_k)} \square B_k \rho^{-\alpha_k} \sum_{i=0}^{m} \left(\frac{\rho}{R}\right)^i \psi_{k,i}(\varphi)$$

Are there orthonormal relations between primal and dual shadows? What about the Bs?

$$J[\rho_o](\tau, K_m^{(\alpha_k)}) \overset{????}{=} A_k + O(\rho_o{}^{\alpha_1-\alpha_k+m+???})$$

Extending the Quasi-Dual Function Method for Extracting ESIFs - $J[\rho_o]$ Integral
First we consider the Axisymmetric case

What about B_k ?

$$B_k = \left[\int_{\theta=0}^{2\pi} \int_{\varphi=\varphi_1}^{\varphi_1+\omega} (\partial_\rho \Phi_{k,0} \cdot \Psi_{k,0} - \Phi_{k,0} \cdot \partial_\rho \Psi_{k,0}) \rho \left(R + \rho \cos\varphi \right) |_{\rho_o} \, d\theta d\varphi \right]^{-1}$$

$$= \left[2\pi \int_{\varphi=\varphi_1}^{\varphi_1+\omega} 2\alpha_k \, \phi_{k,0} \, \psi_{k,0} \left(R + \rho \cos\varphi \right) |_{\rho_o} \, d\varphi \right]^{-1}$$

For homogeneous Neumann BCs and circular crack we obtain for example:

$$B_1 \equiv B(\alpha_1 = \frac{1}{2}) = \frac{1}{\pi^2 R \left(2 - \rho / R \right)} \overset{\rho/R \to 0}{=} \frac{1}{2\pi^2 R}$$

$$B_3 \equiv B(\alpha_3 = \frac{3}{2}) = \frac{1}{6\pi^2 R}$$

$$B_5 \equiv B(\alpha_5 = \frac{5}{2}) = \frac{1}{10\pi^2 R}$$

QDFM – Axisymmetric, homogeneous Neumann BCs, circular crack

Take:

$$\tau = A_1 \rho^{\frac{1}{2}} \left[\sin\frac{\varphi}{2} + \left(\frac{\rho}{R}\right)\frac{1}{4}\sin\frac{\varphi}{2} + \left(\frac{\rho}{R}\right)^2 \left(\frac{1}{12}\sin\frac{\varphi}{2} - \frac{3}{32}\sin\frac{3\varphi}{2}\right) + \left(\frac{\rho}{R}\right)^3 \left(\frac{1}{16}\sin\frac{\varphi}{2} - \frac{1}{30}\sin\frac{3\varphi}{2} + \frac{5}{128}\sin\frac{5\varphi}{2}\right) + \cdots + O\left(\frac{\rho}{R}\right)^8 \right]$$

We compute A_1 by the QDFM with increasing orders of the dual functions:

$$K_0^{(\alpha_1)} \approx \frac{1}{2\pi^2 R} \rho^{-1/2} \psi_{1,0}(\varphi)$$

$$K_1^{(\alpha_1)} \approx \frac{1}{2\pi^2 R} \rho^{-1/2} \left[\psi_{1,0}(\varphi) + \left(\frac{\rho}{R}\right) \psi_{1,1}(\varphi) \right]$$

$$K_2^{(\alpha_1)} \approx \frac{1}{2\pi^2 R} \rho^{-1/2} \left[\psi_{1,0}(\varphi) + \left(\frac{\rho}{R}\right) \psi_{1,1}(\varphi) + \left(\frac{\rho}{R}\right)^2 \psi_{1,2}(\varphi) \right]$$

$$K_3^{(\alpha_1)} \approx \frac{1}{2\pi^2 R} \rho^{-1/2} \left[\psi_{1,0}(\varphi) + \left(\frac{\rho}{R}\right) \psi_{1,1}(\varphi) + \left(\frac{\rho}{R}\right)^2 \psi_{1,2}(\varphi) + \left(\frac{\rho}{R}\right)^3 \psi_{1,3}(\varphi) \right]$$

QDFM – Axisymmetric, homogeneous Neumann BCs, circular crack

We compute A_1 by the QDFM with increasing orders of the dual functions:

$$J[\rho](\tau_1, K_0^{(\alpha_1)}) = A_1\left[1 + O\left(\frac{\rho}{R}\right)^3\right]$$

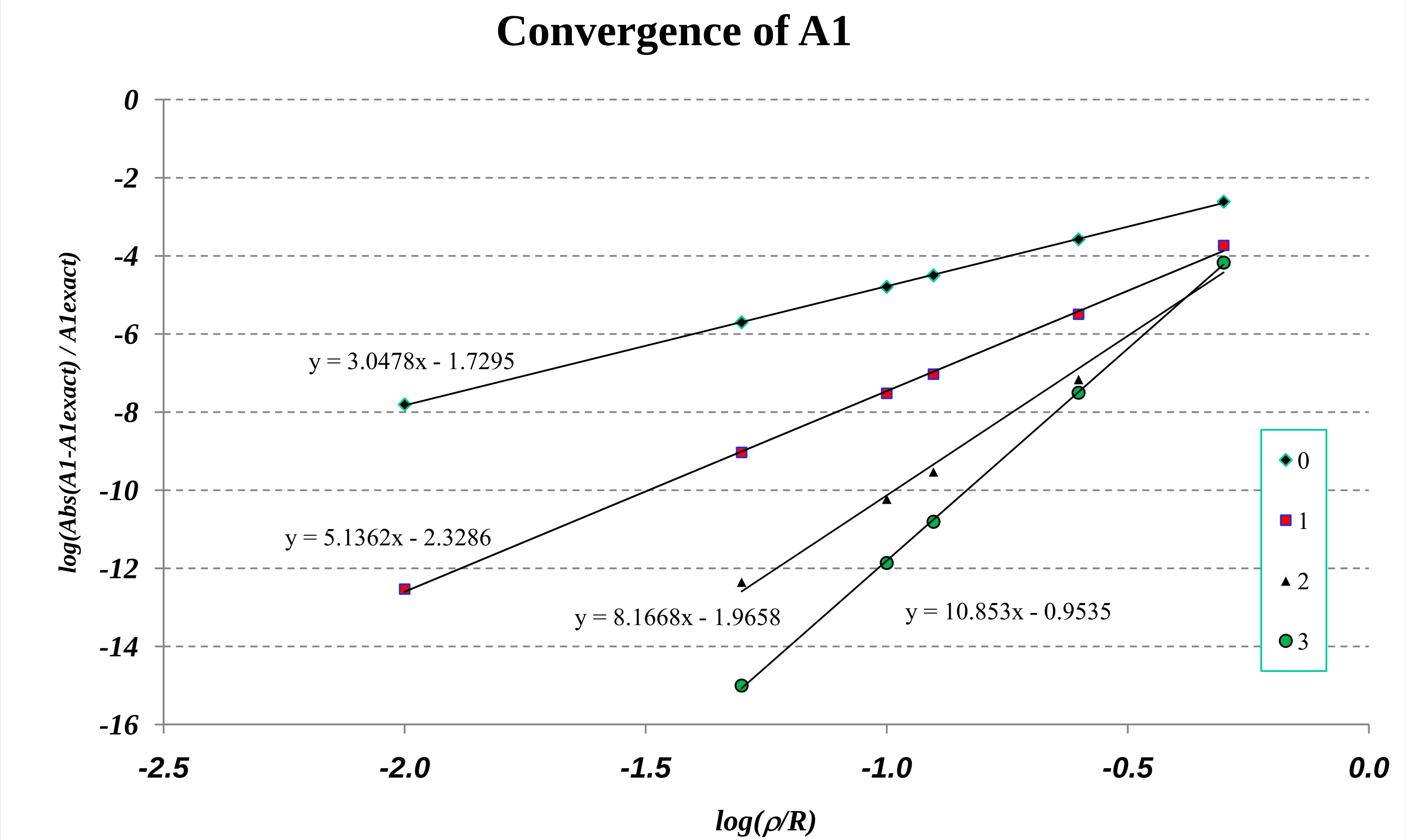
$$J[\rho](\tau_1, K_1^{(\alpha_1)}) = A_1\left[1 + O\left(\frac{\rho}{R}\right)^5\right]$$

$$J[\rho](\tau_1, K_2^{(\alpha_1)}) = A_1\left[1 + O\left(\frac{\rho}{R}\right)^7\right]$$

$$J[\rho](\tau_1, K_3^{(\alpha_1)}) = A_1\left[1 + O\left(\frac{\rho}{R}\right)^9\right]$$

QDFM – Axisymmetric, homogeneous Neumann, penny shaped crack

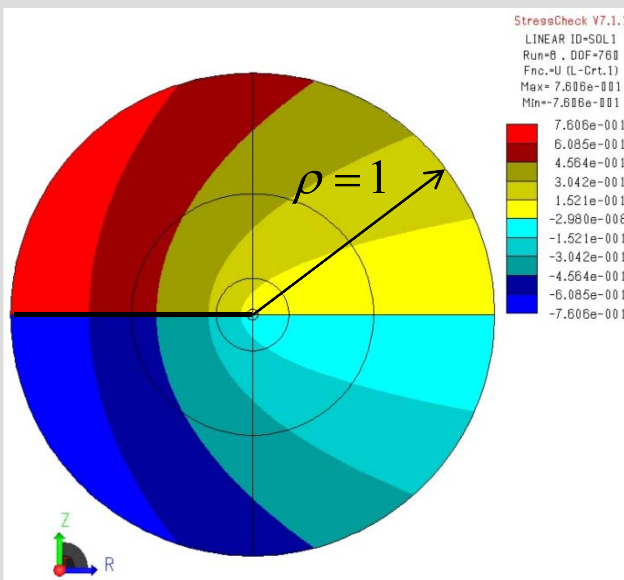
Taking a numerical example to visualize the actual convergence rate:

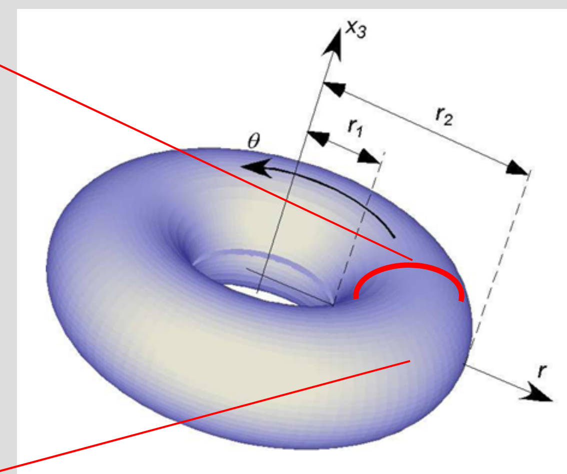


Extraction of EFIFs by post-processing the FE solution Homog. Neumann BCs, circular crack - Axisymmetric

Finite Element approximation τ_{FE} of the exact solution τ :

$$J(\tau, K_m^{(\alpha_i)}) = 2\pi \sum_{k=1}^{nG} \frac{\omega}{2} w_k \left([T]\tau_{FE} \cdot K_m^{(\alpha_i)} - \tau_{FE} \cdot [T] K_m^{(\alpha_i)}\right) \rho \left(R + \rho \cos\varphi(\xi)\right)\Big|_{\xi_k(\varphi)}$$





$R = 2$

Coming back to this picture – the aim is to be able to compute the ESIF for the curved edge…

Summary

- The explicit series expansion of the solutions in the vicinity of a circular edge can be computed analytically or by p-FEMs.
- The quasi-dual function method (QDFM) for extracting EFIFs is being extended to circular edges, in conjunction with p-FE methods.
- Future plans - extend the methods to ESIFs in elasticity.

That's it – Thank you for your attention.