Singular limits in compressible fluids and shape optimization of drag

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Non stationary flow around obstacle

- Equations in dimensionless form
- Isenotropic flow
- Existence and compactness of weak solutions
- Existence of an optimal shape for work minimization
- Numerical results for stationary problem and the drag minimization

Obstacle in bounded domain

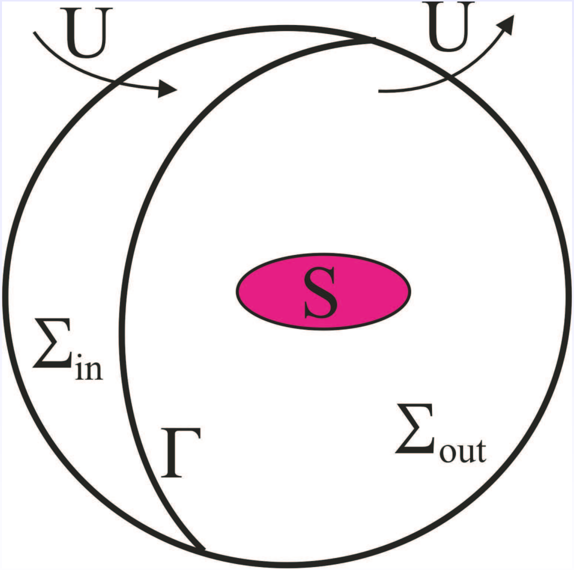


Figure: Flow domain $\Omega = B \setminus \overline{S}$

Stationary boundary value problem

- Proofs in the stationary case for drag minimization are given in the papers already published, with one exception: IECN preprint (2009) is revised for SICON.
- **Global generalized solutions**: Compactness and the existence of optimal shapes for drag minimization (stationary case, SICON 2006).
- **Local approximate solutions**: Uniqueness and the shape differentiability (stationary case, SIMA 2008).
- **Non stationary problems**: Presentation of non published results. Monograph in preparation.

Equations of compressible gases

mass balance

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \ \text{ in } \ \Omega, \tag{1a}$$

balance of momentum

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \varrho \mathbf{f} + \mathbf{h} + \operatorname{div} \mathbb{S} \ \text{ in } \ \Omega, \tag{1b}$$

,

and energy conservation law

$$\partial_t E + \operatorname{div}((E + p)\mathbf{u}) = \operatorname{div}(\mathbb{S}\mathbf{u}) + \operatorname{div}(\kappa \nabla \vartheta) + (\varrho \mathbf{f} + \mathbf{h})\mathbf{u} + \varrho Q. \tag{1c}$$

Given vector fields **f** and **h** denote the densities of external mass and volume forces, the *heat conduction coefficient* κ is a positive constant, a given function Q is the intensity of the external energy flux, the viscous stress tensor $\mathbb{S}$ has the form

$$\mathbb{S}(\mathbf{u}) = \nu_1(\nabla\mathbf{u} + \nabla\mathbf{u}^\top - \frac{2}{3}\operatorname{div}\mathbf{u}\,\mathbb{I}) + \nu_2\operatorname{div}\mathbf{u}\,\mathbb{I}, \tag{2}$$

in which the viscous coefficients $\nu_i, i = 1, 2$ satisfy the inequality $\frac{4}{3}\nu_1 + \nu_2 > 0$, the energy density E is given by

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e,$$

where e is the density of *internal energy*.

The physical properties of a gas are reflected through constitutive equations relating the state variables to the pressure p and the internal energy density e. The common point of view is that p and e can be represented as functions of ϱ and ϑ. Under the assumption that the fluid is at the thermodynamical equilibrium, the functions $p(\varrho, \vartheta)$ and $e(\varrho, \vartheta)$ are not arbitrary but should satisfy the Gibbs equation

$$\frac{1}{\vartheta} de(\varrho, \vartheta) - \frac{p(\varrho, \vartheta)}{\vartheta \varrho^2} d\varrho = ds(\varrho, \vartheta),$$

which means that the left hand side is the exact differential of some function s named *entropy*.

Hydrodynamical Forces and Work

Stress tensor in viscous compressible flow

$$\mathbb{T} = \mathbb{S}(\mathbf{u}) - p\,\mathbb{I}, \tag{3}$$

the force acting from the side of flow at the boundary point $x \in \partial\Omega_t$

$$\mathbf{R}_f = -\mathbb{T}\,\mathbf{n} = (-\mathbb{S}(\mathbf{u}) + p\,\mathbb{I})\,\mathbf{n}. \tag{4}$$

Total work of the hydrodynamical forces over the time period $[0, T]$ is

$$W_\Omega = \int_0^T \int_{\partial\Omega_t} (-\mathbb{S}(\mathbf{u}) + p\,\mathbb{I})\,\mathbf{n} \cdot \mathbf{V}_s\, ds\, dt, \tag{5}$$

$\mathbf{V}_s(x, t)$ is the velocity of boundary points at $\partial\Omega_t$.

Isentropic Flows

The flow is *barotropic* if the pressure depends only on the density. The most important example of such flows are *isentropic flows*. In order to deduce the governing equations for isentropic flows we note that for perfect fluid with $\nu_i = \kappa = 0$, the entropy takes a constant value at each material point. Hence in this case the governing equations have a family of explicit solutions with the entropy $s = \mathrm{const}$. In this case we have

$$p(\varrho) = (\gamma - 1)\exp(s_c)\varrho^\gamma,$$

where a positive constant s_c is a characteristic value of the entropy (without loss of generality we can take $(\gamma - 1)\exp(s_c) = 1$). Exponent γ depends on the physical properties of the gas, $\gamma = 5/3$ for mono-atomic, $\gamma = 7/5$ for diatomic and $\gamma = 4/3$ for polyatomic gases. Assuming that this relation holds for $\nu_i \neq 0, i = 1, 2$, we arrive at the system of *compressible Navier-Stokes equations* for isentropic flows of viscous compressible fluid in the dimensionless form

Governing Equations

$$\mathbb{N}\mathrm{r}\partial_t \varrho \mathbf{u} + \varrho \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\mathbb{M}\mathrm{a}^2} \nabla p = \frac{1}{\mathbb{R}\mathrm{e}} \operatorname{div} \mathbb{S} + \frac{1}{\mathbb{F}\mathrm{r}_m^2} \varrho \mathbf{f} + \frac{1}{\mathbb{F}\mathrm{r}_v^2} \mathbf{h}, \tag{6a}$$

$$\mathbb{N}\mathrm{r}\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \,, \quad \text{in } \ \Omega, \tag{6b}$$

where we denote : the Reynolds number, the Pecle number, the Mach number, the Strouhal number, and the viscosity ratio,

$$\mathbb{R}\mathrm{e} = \frac{\varrho_{\mathrm{c}} \mathrm{u}_{\mathrm{c}} \mathrm{l}_{\mathrm{c}}}{\nu_1}, \quad \mathbb{P}\mathrm{r} = \frac{\mathrm{p}_{\mathrm{c}} \mathrm{l}_{\mathrm{c}} \mathrm{u}_{\mathrm{c}} \kappa_{\mathrm{c}}}{\kappa}, \quad \mathbb{M}\mathrm{a}^2 = \frac{\varrho_{\mathrm{c}} \mathrm{u}_{\mathrm{c}}^2}{\mathrm{p}_{\mathrm{c}}}, \quad \mathbb{N}\mathrm{r} = \frac{\mathrm{l}_{\mathrm{c}}}{\mathrm{T}_{\mathrm{c}} \mathrm{u}_{\mathrm{c}}},$$

$$\lambda = \frac{1}{3} + \frac{\nu_2}{\nu_1},$$

here u_c, ϱ_c, p_c, ϑ_c, are the characteristic values of velocity, density, pressure and temperature, and l_c and T_c the characteristic values of length scale and time intervals.

In addition, f_c, h_c, Q_c are the characteristic values of mass and volume forces, and heat influx. They form dimensionless combinations

$$\mathbb{F}\mathrm{r}_m^2 = \frac{u_c^2}{f_c l_c}, \quad \mathbb{F}\mathrm{r}_v^2 = \frac{\varrho_c u_c^2}{h_c l_c}, \quad \Theta = \frac{\varrho_c Q_c l_c}{p_c u_c}.$$

Dimensionless viscous stress tensor is defined by

$$\mathbb{S} = (\nabla \mathbf{u} + \nabla \mathbf{u}^\top + (\lambda - 1)\,\mathrm{div}\,\mathbf{u}\mathbb{I}), \quad \mathrm{div}\,\mathbb{S} = \Delta \mathbf{u} + \lambda \nabla\,\mathrm{div}\,\mathbf{u} \quad (7)$$

Note that here the characteristic quantities ϱ_c, ϑ_c, and p_c should be compatible with the constitutive law. For instance, with the pressure formula we have, $p_c = R_m\,\varrho_c\,\vartheta_c$. Note that the specific values of the constants γ, λ, and $\mathbb{P}\mathrm{r}$ depend only on physical properties of a fluid. For example, for the air under standard conditions, we have $\gamma = 7/5$, $\lambda = 1/3$, and $\mathbb{P}\mathrm{r} = 7/10$.

Boundary Conditions

Dirichlet-type condition

$$\mathbf{u} = \mathbf{U} \ \text{ on } \ \partial\Omega, \tag{8}$$

Neumann-type condition

$$(\mathbb{S}(\mathbf{u}) - p\,\mathbb{I})\mathbf{n} = \mathbb{S}_n \ \text{ on } \ \partial\Omega, \tag{9}$$

where $\mathbf{n}$ is the outward normal vector to $\partial\Omega$, $\mathbf{U}$ and $\mathbb{S}_n$ are given vector fields. The important particular cases are the *no-slip boundary condition* with $\mathbf{U} = 0$, and zero normal stress condition with $\mathbb{S}_n = 0$. The third physically and mathematically reasonable condition is the *no-stick boundary condition*

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad ((\mathbb{S}(\mathbf{u}) - p\,\mathbb{I})\mathbf{n}) \times \mathbf{n} = 0 \ \text{ on } \ \partial\Omega,$$

which corresponds to the case of frictionless boundary.

Assume that that velocity **u** satisfies the Dirichlet boundary condition, and split the boundary of flow region into three disjoint sets called the inlet $\Sigma_{\rm in}$, the outgoing set $\Sigma_{\rm out}$, and the characteristic set Σ_0, the sets are defined by the relations

$$\Sigma_{\rm in} = \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} < 0\}, \quad \Sigma_{\rm out} = \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} > 0\}, \\ \Sigma_0 = \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} = 0\}. \tag{10}$$

The density distribution should be given on the inlet

$$\varrho = \varrho_\infty \text{ on } \Sigma_{\rm in}. \tag{11}$$

The boundary conditions for the density are not needed in the case of $\Sigma_{\rm in} = \emptyset$. In particular, there are no boundary conditions for the density if the velocity satisfies the no-slip and no-stick conditions whenever $\Sigma_{\rm in} = \Sigma_{\rm out} = \emptyset$.

Moving Frame

If the flow region varies in time, then it is convenient, for technical and practical reasons, to reduce the corresponding boundary problem for fluid dynamics equation to a problem in a fixed domain by a change of independent variables. We describe such a change of variables in the case when Ω_t evolves like an absolutely rigid body. Recall that one-parameter family of the mappings $y \mapsto x(y, t)$ represents a rigid motion in the Euclidian space $\mathbb{R}^d$ if and only if

$$x = \mathbb{U}(t)y + \mathbf{a}(t), \tag{12}$$

where the $\mathbb{U}(t)$ is an arbitrary one-parameter family of orthogonal matrices, i.e., $\mathbb{U}\,\mathbb{U}^\top = \mathbb{I}$ and $\mathbf{a}(t)$ is an arbitrary vector function.

Navier-Stokes Equations in Moving Frame

Let us consider the solid body motion of the fixed domain Ω_0,

$$\Omega_t = \mathbb{U}(T_c t)\,\Omega_0 + \mathbf{a}(T_c t).$$

We reduce the governing equations defined in Ω_t to the fixed domain Ω_0, as a result **the Coriolis and centrifugal forces** will appear in the equations. To this end, an appropriate change of unknown functions is performed: $(\mathbf{u}, \varrho)$ satisfy equations in Ω_t iff

$$\mathbf{v}(y,t) = \mathbb{U}^\top(t)\,\mathbf{u}(x(y,t),t) - \mathbb{N}\mathrm{r}\,\mathbf{W}(y,t), \quad \rho(y,t) = \varrho(x(y,t),t)$$

satisfy the equations

$$\mathbb{N}\mathrm{r}\,\partial_t(\rho\,\mathbf{v}) + \mathrm{div}(\rho\,\mathbf{v}\otimes\mathbf{v}) - \frac{1}{\mathbb{R}\mathrm{e}}\,\mathrm{div}\,\mathbb{S}(\mathbf{v}) + \tag{13a}$$
$$\frac{1}{\mathbb{M}\mathrm{a}^2}\nabla p(\rho) + \mathbb{N}\mathrm{r}\;\mathbb{C}\,\mathbf{v} = \rho\,\mathbf{f} + \mathbf{h} \quad \text{in} \quad \Omega_0,$$

$$\mathbb{N}\mathrm{r}\partial_t(\rho) + \mathrm{div}(\rho\mathbf{v}) = 0 \quad \text{in} \quad \Omega_0, \tag{13b}$$

where the viscous stress tensor

$$\mathbb{S}(\mathbf{v}) = (\nabla \mathbf{v} + \nabla \mathbf{v}^\top + (\lambda - 1)\,\mathrm{div}\,\mathbf{v}\mathbb{I}),$$

the antisymmetric matrix $\mathbb{C} = (C_{ij})_{d\times d}$, the vector fields **f**, **h**,

$$C_{ij} = \frac{\partial W_i}{\partial y_j} - \frac{\partial W_j}{\partial y_i},$$

$$\mathbf{f} = \frac{1}{\mathbb{F}\mathrm{r}_m^2}\mathbb{U}^\top \mathbf{f}(x(y,t),t) - \mathbb{N}\mathrm{r}^2\frac{\partial \mathbf{W}}{\partial t} + \mathbb{N}\mathrm{r}^2\frac{1}{2}\nabla|\mathbf{W}|^2,$$

$$\mathbf{h} = \frac{1}{\mathbb{F}\mathrm{r}_m^2}\mathbb{U}^\top \mathbf{h}(x(y,t),t),$$

and

$$\mathbf{W}(y,t) = \mathbb{U}^\top(t)\,\mathbf{v}(y,t) = \mathbb{U}^\top(t)\dot{\mathbb{U}}(t)y + \mathbb{U}^\top(t)\dot{\mathbf{a}}(t). \qquad (14)$$

Expressions for dimensionless power and work now become

$$J_\Omega = -\mathbb{N}\mathrm{r} \int\limits_{\partial\Omega_0} \big(\nabla \mathbf{v} + (\nabla \mathbf{v})^\top + (\lambda - 1)\,\mathrm{div}\,\mathbf{v}\mathbb{I} - \sigma p(\rho)\,\mathbb{I}\big)\,\mathbf{n} \cdot \mathbf{W}\, ds,$$

$$W_\Omega = \int\limits_0^T J_\Omega dt.$$

We consider the following setting:

$$\Omega := \Omega_0 = B \setminus S,$$

where B is *hold all* domains and S represents a moving body which shape is to be optimized. The work W_Ω becomes our shape functional, written for $\mathbf{u}$ and ϱ,

$$J(S) = -\int_0^T \int_{\partial S} (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top + (\lambda - 1)\,\mathrm{div}\,\mathbf{u}\mathbb{I} - \sigma p(\varrho)\,\mathbb{I})\,\mathbf{n} \cdot \mathbf{U}\, ds dt$$

and we pose $\mathbb{N}\mathrm{r} = \mathbb{R}\mathrm{e} = 1$, $\lambda = \frac{1}{3} + \frac{\nu_2}{\nu_1}$, $\sigma = \frac{\mathbb{R}\mathrm{e}}{\mathbb{M}\mathrm{a}^2}$.

For the pressure, we assume the general constitutive law $p = p(\varrho)$ with the function $p \in C^2[0, \infty)$ which satisfies the following conditions

$$p(0) = 0, \quad p'(\varrho) \geq 0, \quad p''(\varrho) \geq 0,$$

and for all $\varrho > 1$,

$$\begin{aligned} c^{-1}\varrho^{\gamma} \leq p(\varrho) \leq c\varrho^{\gamma}, \quad c^{-1}\varrho^{\gamma-1} \leq p'(\varrho) \leq c\varrho^{\gamma-1}, \\ c^{-1}\varrho^{\gamma-2} \leq p''(\varrho) \leq c\varrho^{\gamma-2} \end{aligned} \tag{15}$$

Notation

$\Omega \in \mathbb{R}^d$, $d = 2, 3$, is a bounded with boundary of class C^3. For given $T > 0$ denote by Q_T the cylinder with lateral surface S_T defined by

$$Q_T = \Omega \times (0, T), \quad S_T = \partial\Omega \times (0, T). \tag{16}$$

Furthermore, assume that given vector fields $\mathbf{U}$, $\mathbf{f}$, $\mathbf{h}$ and a function ϱ_∞ satisfy

$$\varrho_\infty, \mathbf{U} \in C^\infty(Q_T), \quad \mathbf{U} \in C^\infty(Q_T), \quad \mathbf{f}, \mathbf{h} \in C(Q_T). \tag{17}$$

Governing Equations

Thus, we arrive at the following problem
Find velocity and density distributions satisfying the following equations and boundary conditions

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\,\mathbf{u}\otimes\mathbf{u}) + \sigma\nabla p(\varrho) + \mathbb{C}\,\mathbf{u} = \tag{18a}$$
$$\operatorname{div}\mathbb{S}(\mathbf{u}) + \varrho\,\mathbf{f} + \mathbf{h} \quad \text{in} \;\; \Omega\,,$$
$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in} \;\; \Omega, \tag{18b}$$

$$\begin{gathered} \mathbf{u} = \mathbf{U} \;\; \text{on} \;\; \partial\Omega, \qquad \varrho = \varrho_\infty \;\; \text{on} \;\; \Sigma_{\text{in}}\,, \\ \mathbf{u}\Big|_{t=0} = \mathbf{U}, \qquad \varrho\Big|_{t=0} = \varrho_\infty \;\; \text{in} \;\; \Omega, \end{gathered} \tag{18c}$$

where $\mathbb{S}(\mathbf{u})$ is the viscous stress tensor $\mathbb{C}$ is a smooth skew symmetric matrix, $\mathbf{f}, \mathbf{h}$ are given continuous functions. The specific expressions for $\mathbb{C}$, $\mathbf{f}, \mathbf{h}$, are not important for the mathematical theory.

Existence Theory: Main Theorem for $\gamma > d$

There is a weak renormalized solution to Problem (18)

$$\mathbf{u}-\mathbf{U}\in L^2(0,T;W^{1,2}_0(\Omega)),\quad \varrho\in L^\infty(0,T;L^\gamma(\Omega)),$$

which satisfies

$$\int\limits_{Q_T}\varrho\mathbf{u}\cdot\partial_t\zeta\,dxdt+\int\limits_{Q_T}\varrho\mathbf{u}\otimes\mathbf{u}:\nabla\zeta\,dxdt+\int\limits_{\Omega\times\{0\}}\varrho_\infty\mathbf{U}\cdot\zeta\,dx+$$

$$\int\limits_{Q_T}\Big(p(\varrho)\mathbb{I}-\mathbb{S}(\mathbf{u})\Big):\nabla\zeta\,dxdt+\int\limits_{Q_T}\Big(\varrho\mathbf{f}+\mathbf{h}+\mathbb{C}\mathbf{u}\big)\cdot\zeta\,dxdt=0.$$

and

$$\int\limits_{Q_T}\varphi(\varrho)(\partial_t\psi+\nabla\psi\cdot\mathbf{u})\,dxdt-\int\limits_{Q_T}\big(\varphi'(\varrho)\varrho-\varphi(\varrho)\big)\,\mathrm{div}\,\mathbf{u}\psi\,dxdt$$

$$+\int\limits_{\Omega}\varphi(\varrho_\infty)(\cdot,0)\psi(\cdot,0)\,dx-\int\limits_{S_T}\varphi\varrho_\infty\mathbf{U}\cdot\mathbf{n}dsdt=0$$

The first integral identity is satisfied for all $\zeta \in C^1(Q_T)$, $Q_T = \Omega \times (0, T)$,

$$\zeta(x, T) = 0 \ \text{ in } \ \Omega, \quad \zeta = 0, \ \text{ on } \ S_T = \partial\Omega \times (0, T).$$

The second identity holds true for any functions φ and ψ satisfying the conditions

$$\psi \in C^1(Q_T), \quad \psi(\cdot, T) = 0,$$

$$\psi(x, t) = 0 \ \text{ on } \ S_T \setminus \Sigma_{\text{in}}.$$

$$\sup_{s \in \mathbb{R}^+} |\varphi''(s)| \le c,$$

Solution Scheme

The basic element of these scheme is the standard parabolic regularization of the governing equations proposed by P.L. Lions and E.Feireisel

$$\partial_t(\varrho\,\mathbf{u}) + \operatorname{div}\Big((\varrho\,\mathbf{u} - \varepsilon\nabla\varrho)\otimes\mathbf{u}\Big) + \nabla p + \mathbb{C}\,\mathbf{u} =$$
$$= \operatorname{div}\mathbb{S}(\mathbf{u}) + \varrho\,\mathbf{f} + \mathbf{h},$$
$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = \varepsilon\Delta\varrho.$$

These equations can be considered as a mathematical model of compressible flows with a mass diffusion and are of independent interest.

Numerical example in two spatial dimensions

Recall that in our framework the hydro-dynamical force acting on the body S is defined by the formula,

$$\mathbf{J}(S) = -\int_{\partial S} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T + (\lambda - 1)\mathrm{div}\mathbf{u} I - \frac{R}{\delta} p I)\mathbf{n}\, dS.$$

In a frame attached to the moving body the drag is the component of **J** parallel to $\mathbf{U}_\infty$,

$$J_D(S) = \mathbf{U}_\infty \cdot \mathbf{J}(S), \tag{19}$$

and the lift is the component of **J** in the direction orthogonal to $\mathbf{U}_\infty$.

State equations

For the fixed data, the drag can be regarded as a functional depending on the shape of the obstacle S.
The system of compressible Navier-Stokes equations transforms to

$$\begin{aligned} \Delta \mathbf{u} - \nabla q &= R\varrho \nabla \mathbf{u} \cdot \mathbf{u} \\ \operatorname{div}(\mathbf{u}) &= \sigma_0 p(\varrho) - \frac{1}{\lambda} q \\ \mathbf{u}^T \nabla \varrho + \sigma_0 p(\varrho)\varrho - \frac{1}{\lambda} q \varrho &= 0 \end{aligned} \tag{20}$$

where $\sigma_0 = R/(\lambda\delta)$, $\sigma = \sigma_0 p_0 \gamma$.
Here, the effective viscous pressure is used

$$q = \frac{R}{\delta} p(\varrho) - \lambda \operatorname{div}(\varrho \mathbf{u}) \tag{21}$$

Approximate solutions

In addition, if we introduce a smooth function η defined in Ω and satisfying boundary conditions $\eta = 1$ on ∂S, $\eta = 0$ on Σ, then the expression for drag takes on the form

$$J_D(S) = -\mathbf{U}_\infty \cdot \int_\Omega \big(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \mathrm{div}\,(\mathbf{u}) I \\ - qI - R\varrho \mathbf{u} \otimes \mathbf{u}\big) \nabla \eta \, dx. \tag{22}$$

Assuming $\lambda \gg 1$ and $R \ll 1$ (weakly compressible flow) we may approximate solution to (20) by means of the small perturbation with respect to the solution of the Stokes problem

$$\begin{gathered} \Delta \mathbf{u}_0 - \nabla q_0 = 0 \\ \mathrm{div}(\mathbf{u}_0) = 0 \\ \mathbf{u}_0 = \mathbf{U} \ \text{ on } \ \Sigma, \quad \mathbf{u}_0 = 0 \ \text{ on } \ \partial S, \ \ M(q_0) = 0 \end{gathered} \tag{23}$$

where $M(\cdot)$ denotes mean value on Ω.

We assume these perturbations in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \qquad \varrho = \varrho_0 + \phi \tag{24}$$

$$q = q_0 + \lambda\sigma_0 p_0 + \pi + m \tag{25}$$

where $\mathbf{v}$, ϕ, π are unknown functions and m the unknown constant. Taking into account (23) the system of equations for $\mathbf{v}$, ϕ, π is

$$\begin{aligned} \Delta\mathbf{v} - \nabla\pi &= R\varrho\nabla\mathbf{u}\cdot\mathbf{u} \\ \mathrm{div}(\mathbf{v}) &= \sigma_0 p(\varrho) - \frac{1}{\lambda}q \\ \mathbf{u}^T\nabla\phi + \sigma\phi &= \frac{1}{\lambda}\varrho(q_0 + \pi) \\ &\quad - \sigma_0\varrho(p - p_0) + \sigma_0 p_0 p'(\varrho_0)(\varrho - \varrho_0) \end{aligned} \tag{26}$$

with boundary conditions

$$\mathbf{v} = 0 \ \text{ on } \ \partial\Omega, \quad \phi = 0 \ \text{ on } \ \Sigma_{in}, \quad M(\pi) = 0$$

and the condition $M(\mathrm{div}(\mathbf{v})) = 0$, which translates to

$$m = \frac{\sigma_0}{|\Omega|}\int_\Omega \big[p(\varrho) - p(\varrho_0)\big]\,dx. \tag{27}$$

It is convenient to introduce an additional equation

$$\begin{aligned} -\,\mathrm{div}(\mathbf{u}\zeta) + \sigma\zeta = \sigma \ \text{ in } \ \Omega \\ \zeta = 0 \ \text{ on } \ \Sigma_{out} \end{aligned} \tag{28}$$

and express m

$$m = \varkappa \int\limits_\Omega \big(\varrho_0^{-1}\Psi_1[\vartheta]\zeta - \mathfrak{g}\Psi[\vartheta]\big)\,dx, \tag{29}$$

$$\varkappa = \Big(\int\limits_\Omega \mathfrak{g}(1 - \zeta - \varrho_0^{-1}\zeta\varphi)\,dx\Big)^{-1}. \tag{30}$$

Material derivatives

In order to introduce the perturbation of the obstacle we introduce the transformation of the domain Ω by means of the mapping

$$\mathcal{T}(\mathbf{x}) = \mathbf{x} + \varepsilon \mathbf{T}(\mathbf{x}) \tag{31}$$

where $\mathbf{T}(\mathbf{x}) = 0$ on Σ and $\mathbf{T}_{|\partial S}$ describes the movement of the boundary of S. We assumed $\mathbf{T} = [t_1, t_2]^T$ in the particular form, where t_i satisfy equations

$$\begin{gathered} \Delta t_i = 0 \ \text{ in } \ \Omega, \quad t_i = 0 \ \text{ on } \ \Sigma \\ t_i = h_i \ \text{ on } \ \partial S, \ \ i = 1, 2. \end{gathered} \tag{32}$$

Here $h_i(\mathbf{x})$ represent the shift of the point $\mathbf{x}$ on ∂S. In the sequel we denote the solutions of the same equations (26),(27),(28) in the transformed domain $\Omega_\varepsilon = \mathcal{T}(\Omega)$ by $\mathbf{v}(\varepsilon)$, $\phi(\varepsilon)$, $\pi(\varepsilon)$, $m(\varepsilon)$, $\zeta(\varepsilon)$. By means of the inverse transformation $\mathcal{T}^{-1}$ all these functions may be shifted again to the unperturbed domain Ω, together with defining equations.

Material derivatives

Therefore we may consider them as functions defined on Ω and formally compute derivatives

$$\begin{aligned}
\mathbf{w} &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathbf{v} - \mathbf{v}(\varepsilon)] \\
\omega &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\pi - \pi(\varepsilon)] \\
\xi &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\zeta - \zeta(\varepsilon)] \qquad (33) \\
\psi &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\phi - \phi(\varepsilon)] \\
n &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[m - m(\varepsilon)]
\end{aligned}$$

Material derivatives

Let us denote by $\mathcal{P}$ the set of solutions in the unperturbed domain, $\mathcal{P} = [\mathbf{v}, \phi, \pi, m, \zeta]$.

$$\begin{aligned} \Delta \mathbf{w} - \nabla \omega &= \mathbf{F}_1(\mathcal{P}, \mathbf{w}, \psi, D) \\ \operatorname{div}(\mathbf{w}) &= F_2(\mathcal{P}, \psi, n, \omega, D) \\ \mathbf{u}^T \nabla \psi + \sigma \psi &= F_3(\mathcal{P}, \psi, n, \omega, D) \\ -\operatorname{div}(\mathbf{u}\xi) + \sigma \xi &= F_4(\mathcal{P}, \omega, D) \end{aligned} \tag{34}$$

with boundary conditions

$$\mathbf{w} = 0 \;\text{ on }\; \partial\Omega, \quad \psi = 0 \;\text{ on }\; \Sigma_{in}, \quad \xi = 0 \;\text{ on }\; \Sigma_{out}$$

as well $M(\omega) = 0$ and

$$n = \int_\Omega F_5(\psi, \omega, \xi, D)\, dx.$$

The matrix D characterizing the transformation is given by

$$D = \mathrm{div}(\mathbf{T})I - \nabla \mathbf{T}.$$

The functions $\mathbf{F}_1, F_2, F_3, F_4, F_5$ are complicated expressions in terms its arguments. For illustration we show only $\mathbf{F}_1$:

$$\begin{aligned}
\mathbf{F}_1(\mathcal{P}, \mathbf{w}, \psi, D) &= R^2\big(\phi\mathbf{u}\nabla\mathbf{u} + \varrho\mathbf{w}\nabla\mathbf{u} + \varrho\mathbf{u}\nabla\mathbf{w}\big) \\
&+ R\mathbf{u}\nabla(D\mathbf{u}) + RD^T(\mathbf{u}\nabla\mathbf{u}) \\
&+ \mathrm{div}\Big[(D + D^T)\nabla\mathbf{u} - \frac{1}{2}Tr(D)\nabla\mathbf{u}\Big] \\
&- D\Delta\mathbf{u} - \Delta(D\mathbf{u})
\end{aligned}$$

Shape derivative of drag

The expression for the shape derivative of the drag takes on the form

$$\frac{d}{d\varepsilon} J_D(S_\varepsilon)_{|\varepsilon=0} = L_1 + L_2 + L_3 + L_4 + L_5 \tag{35}$$

where

Shape derivative of drag

$$L_1 = \int_\Omega \mathrm{div}(\mathbf{T})(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \mathrm{div}(\mathbf{u}) I)\nabla\eta \cdot \mathbf{U}_\infty \, dx$$

$$L_2 = -\int_\Omega \big(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \mathrm{div}(\mathbf{u}) I - qI \\ - R\varrho \mathbf{u} \otimes \mathbf{u}\big) D^T \nabla\eta \cdot \mathbf{U}_\infty \, dx$$

$$L_3 = -\int_\Omega \big(D^T \nabla \mathbf{u} + \nabla \mathbf{u}^T D + \nabla(D\mathbf{u}) \\ + \nabla(D\mathbf{u})^T\big) \nabla\eta \cdot \mathbf{U}_\infty \, dx$$

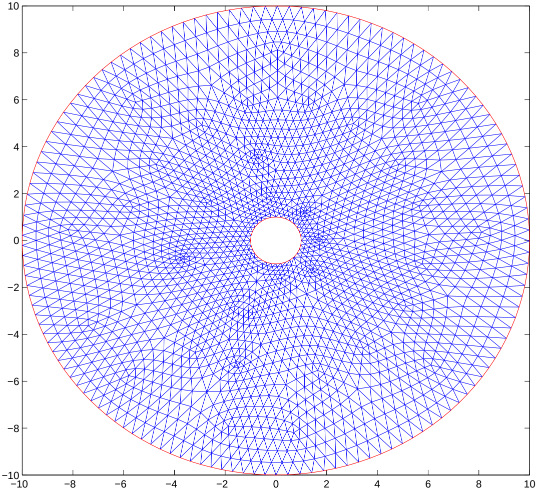
$$L_4 = \int_\Omega \mathbf{w} \cdot \big(\Delta\eta \mathbf{U}_\infty + R\varrho(\mathbf{u} \cdot \nabla\eta)\mathbf{U}_\infty \\ + R\varrho(\mathbf{u} \cdot \mathbf{U}_\infty)\nabla\eta\big) \, dx$$

$$L_5 = \int_\Omega \big[\omega \nabla\eta \cdot \mathbf{U}_\infty + \psi(\mathbf{u} \cdot \nabla\eta)(\mathbf{u} \cdot \mathbf{U}_\infty)\big] \, dx$$

Material derivatives

It can be shown that under reasonable regularity assumptions $\mathbf{v} \in [C^1(\Omega)]^3$ and $\pi,\ \phi,\ \zeta\ \in\ C(\Omega)$. However, the convergence of limits in (31) takes place in very weak spaces, see above. The preliminary numerical computations were performed in $\mathbb{R}^2$. The domain B constituted a ball $B = B(0, R)$ and the initial obstacle was $S = B(0, r)$ with $R/r = 10$. The domain $\Omega = B \setminus S$ was triangulated (see Fig.1) For solving the Stokes Problem (23) the flow velocity $\mathbf{u}_0$ was approximated by piecewise P_1 (first order polynomial) functions on triangles, while for q_0 piecewise P_0 (constant) functions were used. For regularization of the pressure q_0 the penalty term containing interelement jumps was applied. The same elements were used for approximating $\mathbf{v}, \pi$. The functions $\phi,\ \zeta$ were approximated by P_1 elements.

Computational domain



Numerical solution

However, the system (26) is nonlinear. Therefore it was solved iteratively, using Ishikawa [2] fixed point procedure. The right–hand sides were taken as functions of $\mathcal{P}$, denoted by $\mathcal{R}(\mathcal{P})$. As a result (26) takes on the form

$$\mathcal{P} = \mathcal{S}^{-1}[\mathcal{R}(\mathcal{P})]$$

where $\mathcal{S}^{-1}$ represents solving the system with given $\mathcal{R}$. This justifies using fixed point method. The Ishikawa algorithm for finding x such that $x = \Phi(x)$ may be written as the following iteration:

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n\Phi(x_n) \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\Phi(y_n) \end{aligned}$$

where $0 \le \alpha_n, \beta_n < 1$,

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$$

and

$$\sum_{n=0}^{\infty} \alpha_n = \infty.$$

In our case it was taken $\alpha_n = \beta_n = 1/\sqrt{n+1}$.
For the range of flow parameters used in computations the convergence was quite quick. The same procedure and approximation was used for solving the system (34), since it has the same structure. It was convenient, because even if $\mathbf{w}, \omega, \xi, \psi$ enter the right-hand side linearly, the expression for n makes iterations necessary. In the weak formulation the second derivatives of $\mathbf{u}$ disappear and the particular form of D (t_i harmonic in Ω) could be exploited.

As it is easily seen, the shape derivative of the drag (35) is computed for the particular transformation field **T** and resulting matrix D. The general movement of the curve ∂S was expressed as linear combination of ”bump” deformations, which were constructed in the following way.

First, the boundary ∂S was approximated by the closed, smooth (C^2) spline passing through all the discretization nodes on ∂S and parametrized by arclength s

$$\gamma = \gamma(s), \quad s \in [0, L], \quad \gamma(s_k) = \mathbf{p}_k, \quad k = 1 \dots, K.$$

Next at each point $\mathbf{p}_k = \gamma(s_k)$ the outer normal vector was computed

$$\mathbf{n}_k = \frac{N\gamma'(s_k)}{\|\gamma'(s_k)\|}, \qquad N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which indicated the direction of movement for this point. Finally, the "bump" function was defined

$$b_k(s) = \exp\big[-\big(\frac{dist(s, s_k)}{d_0}\big)^2\big],$$

where

$$dist(s, s_k) = \min[|s - s_k|, L - |s - s_k|]$$

is the minimal distance from s to s_k (remember that γ is closed) and d_0 represents the width of the "bump". Using this function and taking $\mathbf{h}(\mathbf{p}_j) = b_k(s_j)\mathbf{n}_j$, $j = 1, \dots, K$ one can compute the corresponding $\mathbf{T}_k = \mathbf{T}(\mathbf{h})$ and D_k.
Having $D := D_k$ it was possible to solve the non linear system and obtain the shape derivative. This procedure had to be repeated K times, for each vertex on ∂S.
After performing small movements of boundary points along $\mathbf{n}_k$ it has been observed that in the regions of bigger curvature the points $\mathbf{p}_k$ tended to converge to each other, causing even the overlap of triangles after several iteration steps. To remedy this behavior the following procedure was used after each step.
Taking new positions of points $\mathbf{p}'_k$ as nodes the new spline $\gamma_1(s)$ was computed, $\gamma_1(s'_k) = \mathbf{p}'_k$. Then the parameters s'_k were slightly shifted, so that the distances between neighboring points along γ_1 were equal on all the new boundary, i.e. the

new nodes were uniformly distributed. This prevented spoiling the quality of triangulation.

In numerical computations we considered the problem of drag minimization and, for illustration purposes only, drag maximization. We describe briefly the numerical results given in Figures 2-6. The results are only preliminary, since they are obtained with few steps of the simple gradient method, with the shape gradient numerically evaluated. Triangulation and computational domain is shown in Fig.1. The flow is from the left, Reynolds number is $R = 0.01$, viscosity ratio $\lambda = 100$, the flow velocity is $U_1 = 1, U_2 = 0$ on outer boundary. The coefficient in gas law is $\gamma = 5/3$. In order to prevent moving the obstacle toward the boundary of the computational region, it is assumed that its gravity center is fixed at the origin. The total volume of the obstacle is kept constant.

The optimized shapes after few iterations are shown. The computations in case of drag minimization seem to converge to some shape, in case of drag maximization the situation is different, because the optimal shape cannot exist. The results shown are raw, in the sense that there was no attempt to exploit the symmetry of the problem. In view of this remark they look satisfactory.

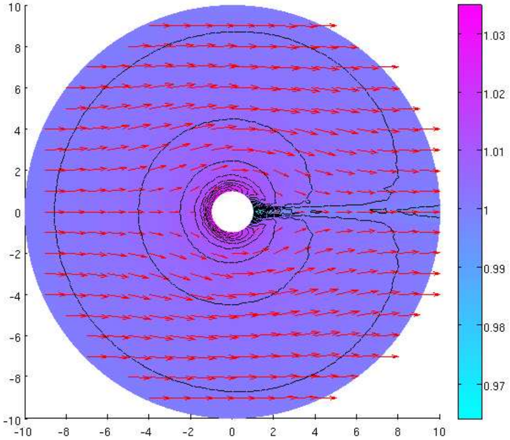


Figure: Initial flow **u** and pressure p.

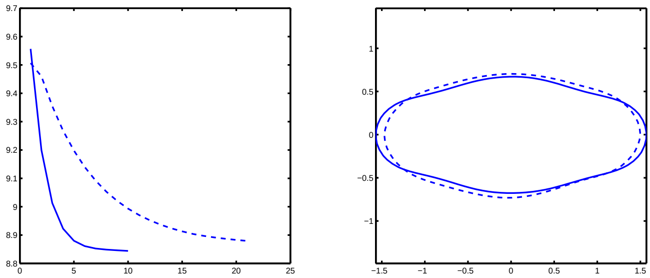


Figure: Shape of minimal drag for rough (dashed line) and finer discretizations. On the left history of optimization.

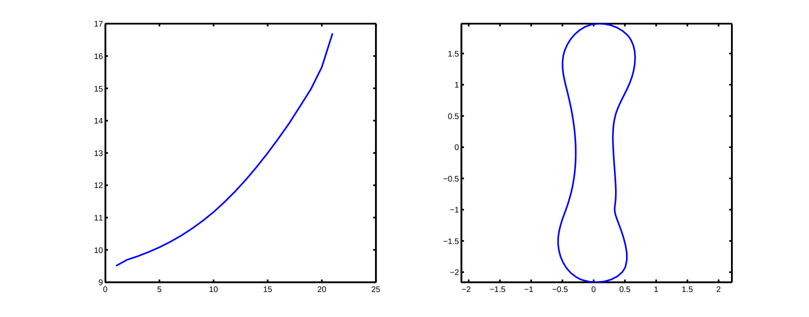


Figure: Shape after few steps of drag maximization and the history of drag values.

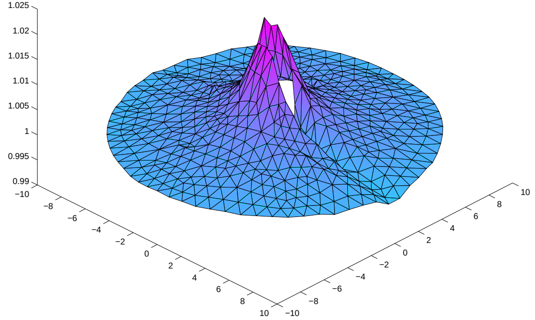


Figure: Pressure distribution around shapes of minimal drag.

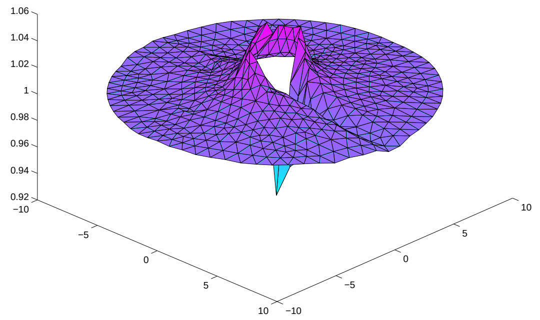


Figure: Pressure distribution around shapes of maximal drag.

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Conclusions

The drag functional for compressible Navier Stokes equations is shape differentiable. Therefore, the numerical methods of shape optimization can be applied in order to solve such optimal design problems like minimization of the drag and/or maximization of the lift.
The same result can be obtained for the complete system with the equation for the temperature, this is the subject of the current studies since even the existence of the solutions is an open problem.

Conclusions

- Shape optimization problem for inhomogeneous non stationary compressible Navier-Stokes equations is well posed.
- Shape gradient of the work functional to be constructed.
- Complete model including the energy balance is still to be investigated for shape optimization.