Propagation of acoustic waves in junction of thin slots

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Abstract

This work concerns the study of time domain wave propagation in thin domains that are junctions of thin slots. The idea is to reduce this model problem to wave equations on a one-dimensional graph with appropriated node conditions. We present here theoretical and numerical results.

1. Introduction - setting of the problem

1.1 Geometry and wave equation

Geometry Ω^ε: union of N slots whose length are L_i width are proportional to ε (see for instance figure 1). Limit geometry is $\mathcal{G}$.

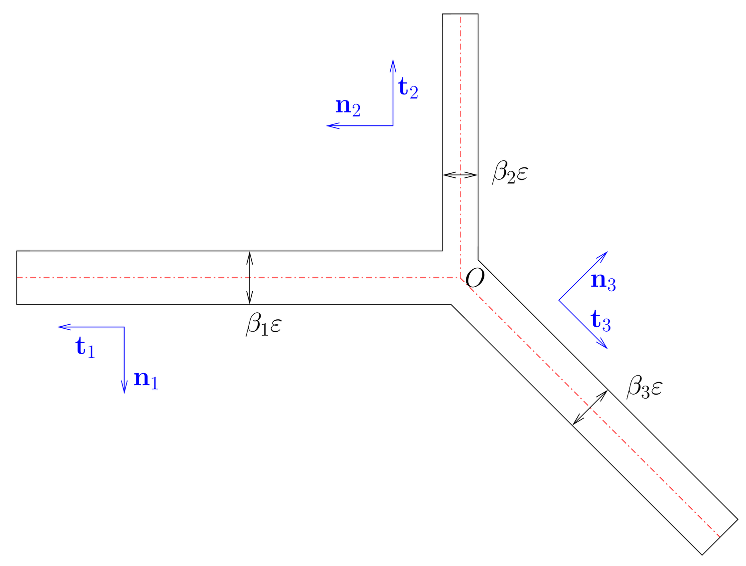

Figure 1: *Configuration of the domain* Ω^ε

Problem considered: find $u^\varepsilon_{\text{ex}}$ such that:

$$\left\{\begin{array}{rcll} \dfrac{\partial^2 u^\varepsilon_{\text{ex}}}{\partial t^2} - \Delta u^\varepsilon_{\text{ex}} & = & 0 & \text{in} \quad \mathbb{R}^*_+ \times \Omega^\varepsilon \\ \dfrac{\partial u^\varepsilon_{\text{ex}}}{\partial n} & = & 0 & \text{on} \quad \mathbb{R}^*_+ \times \partial\Omega^\varepsilon \end{array}\right.$$

with appropriated Cauchy data.

1.2 Limit model

Parametrization of $\mathcal{G}$: we parametrize $\mathcal{G}$ by

$$\mathcal{G} = \bigcup_{i=1}^{N} S_i \quad \text{with} \quad S_i = (0, L_i)$$

and we parametrize S_i by its curvilinear abscissa s_i (with $s_i = 0$ at point O).

Limit model: $u^\varepsilon_{\text{ex}}$ "converges" to u_{lim} defined on $\mathbb{R}^*_+ \times \mathcal{G}$ such that (denoting $u_{\text{lim},i}$ as restriction of u_{lim} to slot i):

$$\left\{\begin{array}{rcll} \dfrac{\partial^2 u_{\text{lim},i}}{\partial t^2} - \dfrac{\partial^2 u_{\text{lim},i}}{\partial s_i^2} & = & 0 & \text{in} \quad \mathbb{R}^*_+ \times S_i \\ u_{\text{lim},i}(t,0) & = & u_{\text{lim},j}(t,0) & \text{for} \quad i \neq j \\ \displaystyle\sum_{i=1}^{N} \beta_i \frac{\partial u_{\text{lim},i}}{\partial s_i}(t,0) & = & 0 & \end{array}\right.$$

One can find justification of this model for instance in [1] or [2]. One can also remark that limit model does only depend on topology of $\mathcal{G}$, not on geometry.

2. Matched asymptotic expansions

2.1 Overlaping domain decomposition

Cut-off function: $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous such that

$$\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0 \quad \text{et} \quad \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon)}{\varepsilon} = +\infty$$

i^{th} slot zone: domain Ω^ε_i (figure 2) defined by

$$\Omega^\varepsilon_i = \{\mathbf{x} \in \Omega^\varepsilon \quad \text{s.t.} \quad \mathbf{x} \cdot \mathbf{t}_i > \varphi(\varepsilon)\}$$

Variable change: $(s_i, \nu_i) = (\mathbf{x} \cdot \mathbf{t}_i, \varepsilon^{-1}\mathbf{x} \cdot \mathbf{n}_i)$

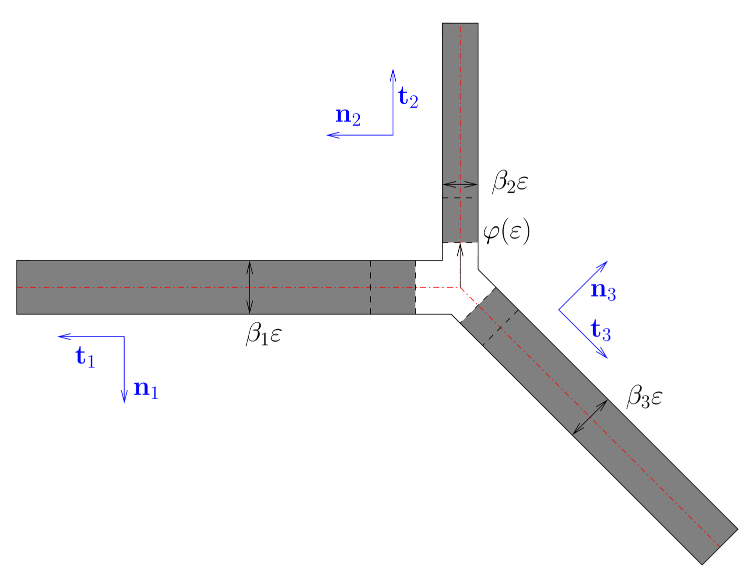

Figure 2: *Configuration of the slots* Ω^ε_i

Junction zone: domain J^ε (figure 3) defined by

$$J^\varepsilon = \bigcap_{i=1}^{N} \{\mathbf{x} \in \Omega^\varepsilon \quad \text{s.t.} \quad \mathbf{x} \cdot \mathbf{t}_i < 2\varphi(\varepsilon)\}$$

Variable change : $\widehat{\mathbf{x}} = \varepsilon^{-1}\mathbf{x}$. Note that $\varepsilon^{-1}J^\varepsilon$ tends to an unbounded domain J^∞ (figure 4), as ε tends to 0.

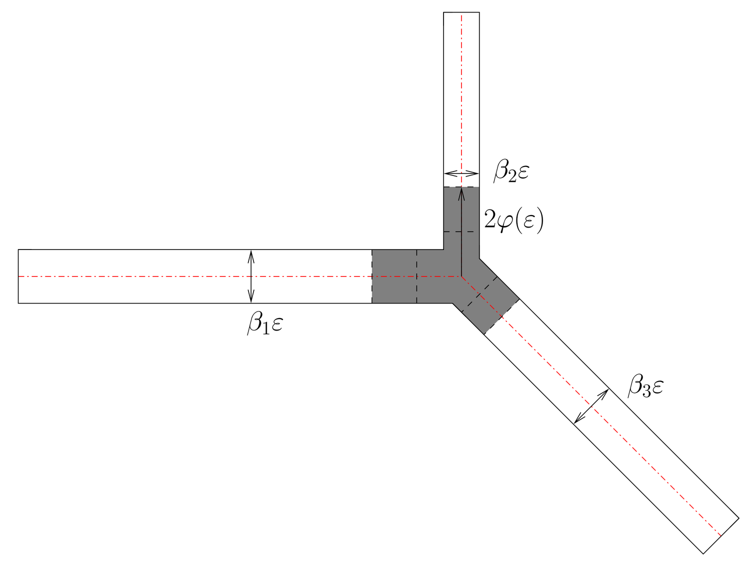

Figure 3: *Configuration of the junction* J^ε

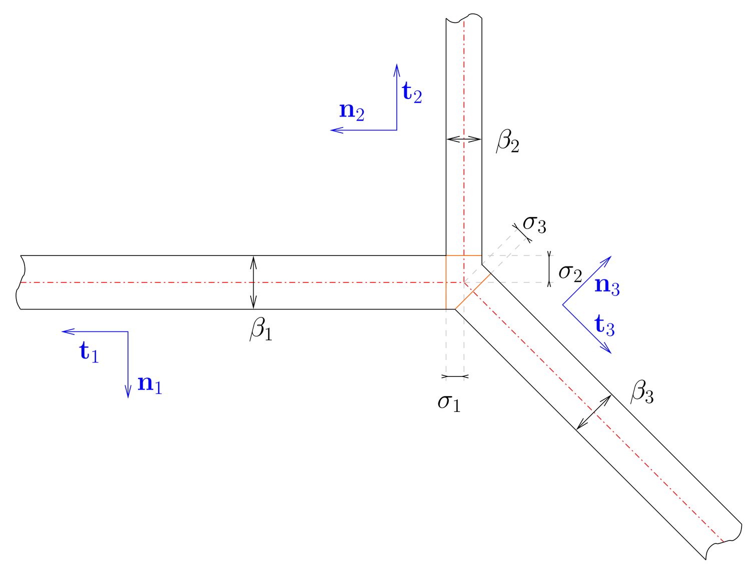

Figure 4: *Configuration of the infinite junction* J^∞

2.2 Basic equations

Our ansatz:

$$u^\varepsilon_{\text{ex}}(t,\mathbf{x}) = \sum_{k=0}^{\infty} \varepsilon^k u^k_i(t, s_i, \nu_i) \quad \text{in} \quad \mathbb{R}^*_+ \times \Omega^\varepsilon_i$$

$$u^\varepsilon_{\text{ex}}(t,\mathbf{x}) = \sum_{k=0}^{\infty} \varepsilon^k U^k(t, \widehat{\mathbf{x}}) \quad \text{in} \quad \mathbb{R}^*_+ \times J^\varepsilon$$

For the slots functions:

$$\begin{array}{rcll} u^k_i(t, s_i, \nu_i) & = & u^k_i(t, s_i), & (t, s_i) \in \mathbb{R}^*_+ \times (0, L_i) \\ \dfrac{\partial^2 u^k_i}{\partial t^2}(t, s_i) - \dfrac{\partial^2 u^k_i}{\partial s_i^2}(t, s_i) & = & 0, & (t, s_i) \in \mathbb{R}^*_+ \times (0, L_i) \end{array}$$

For the junctions functions

$$\begin{array}{rcll} \Delta U^k(t, \widehat{\mathbf{x}}) & = & \dfrac{\partial^2 U^{k-2}}{\partial t^2}(t, \widehat{\mathbf{x}}), & (t, \widehat{\mathbf{x}}) \in \mathbb{R}^*_+ \times J^\infty \\ \dfrac{\partial U^k}{\partial n}(t, \widehat{\mathbf{x}}) & = & 0, & (t, \widehat{\mathbf{x}}) \in \mathbb{R}^*_+ \times \partial J^\infty \end{array}$$

2.3 Matching conditions

On overlaping domain O^ε_i (figure 5), we have two ansatz.
$\Rightarrow$ we should have equality between these two ansatz.

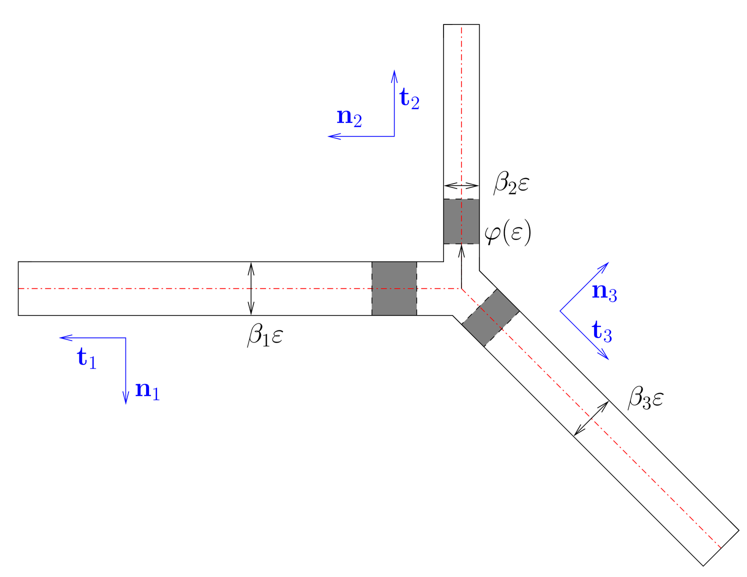

Figure 5: *Configuration of the overlaping domains* O^ε_i

Variable change: $(S_i, \nu_i) = (\widehat{\mathbf{x}} \cdot \mathbf{t}_i, \widehat{\mathbf{x}} \cdot \mathbf{n}_i)$
Identifying powers of ε gives

$$U^k(t, S_i, \nu_i) = \sum_{l=0} \frac{\partial^l u^{k-l}_i}{\partial s^l_i}(t,0)\frac{S^l_i}{l!} + O(S^k_i \exp(-\pi\beta_i^{-1}S_i))$$

2.4 Main results

Existence and uniqueness: the family of functions (u^k_i) and (U^k) are uniquely defined.
Convergence result: for any $\delta_i > 0$, one has, for ε small enough, and for any t:

$$\sqrt{\int_{-\beta_i/2}^{-\beta_i/2}\int_{\delta_i}^{L_i}\left|\frac{\partial(u^\varepsilon_{\text{ex}} - \sum_{l=0}^{k}\varepsilon^l u^l_i)}{\partial t}\right|^2 + \left|\nabla(u^\varepsilon_{\text{ex}} - \sum_{l=0}^{k}\varepsilon^l u^l_i)\right|^2} \leq C(t)\varepsilon^{k+1}$$

3. Improved Kirchoff conditions

3.1 Construction

Idea: use knowledge of (u^k_i) to build an approximate model defined on $\mathcal{G}$.

On J^∞, we introduce σ_i the distance between O and i^{th} canonical semi-strip (see figure 4). We call $\mathfrak{J}$ the domain given by

$$\mathfrak{J} = J^\infty \setminus \bigcup_{i=1}^{N} \left\{i^{\text{th}} \text{ canonical semi-strip}\right\}$$

and we call Γ_i, for $1 \leq i \leq N$, interface given by

$$\Gamma_i = \overline{\mathfrak{J}} \cap \overline{\left\{i^{\text{th}} \text{ canonical semi-strip}\right\}}$$

Next point: for $1 \leq i \leq N-1$, find $W_i \in \mathrm{H}^1_{loc}(J^\infty)$ s.t.

$$\left\{\begin{array}{rcll} \Delta W_i & = & 0 & \text{in } J^\infty \\ \dfrac{\partial W_i}{\partial n} & = & 0 & \text{on } \partial J^\infty \\ W_i & \sim & \beta_i^{-1} S_i & \text{on } i^{\text{th}} \text{ canonical semi-strip} \\ W_i & \sim & -\beta_{i+1}^{-1} S_{i+1} & \text{on } (i+1)^{\text{th}} \text{ canonical semi-strip} \\ W_i & = & O(1) & \text{on other ones} \\ \displaystyle\int_{\mathfrak{J}} W_i & = & 0 & \end{array}\right.$$

and we define the $N-1 \times N-1$ matrix K by

$$K_{i,j} = \frac{1}{\beta_i}\int_{\Gamma_i} W_j - \frac{1}{\beta_{i+1}}\int_{\Gamma_{i+1}} W_j$$

We also define the $N \times N-1$ matrix P by

$$P = \begin{pmatrix} -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & -1 & 1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 1 \end{pmatrix}$$

and the "jump" matrix $\mathcal{J}$ by $\mathcal{J} = P^T K^{-1} P$.

We define the $N \times N$ "average" matrix $\mathcal{A}$ by $\mathcal{A}_{i,j} = \text{area}(\mathfrak{J})/N^2$. We also define the "offbeat" vectors

$$\begin{array}{rcl} U^\varepsilon(t) & = & \left(u^\varepsilon_j(t, \varepsilon\sigma_j)\right)_{1 \leq j \leq N} \in \mathbb{R}^N \\ \partial_S U^\varepsilon(t) & = & \left(\partial_{s_j} u^\varepsilon_j(t, \varepsilon\sigma_j)\right)_{1 \leq j \leq N} \in \mathbb{R}^N \end{array}$$

Approximate model: find $u^\varepsilon_{\text{app}}$ such that

$$\left\{\begin{array}{rcll} \dfrac{\partial^2 u^\varepsilon_{\text{app},i}}{\partial t^2} - \dfrac{\partial^2 u^\varepsilon_{\text{app},i}}{\partial s_i^2} & = & 0 & \text{in} \quad \mathbb{R}^*_+ \times (\varepsilon\sigma_i, L_i) \\ \partial_S U^\varepsilon_{\text{app}}(t) & = & \left(\dfrac{1}{\varepsilon}\mathcal{J} + \varepsilon\mathcal{A}\dfrac{\partial^2}{\partial t^2}\right) U^\varepsilon_{\text{app}}(t) & \end{array}\right.$$

Convergence result (proved in [3]): for any $\delta_i > 0$, one has, for ε small enough, and for any t:

$$\sqrt{\int_{-\beta_i/2}^{-\beta_i/2}\int_{\delta_i}^{L_i}\left|\frac{\partial(u^\varepsilon_{\text{ex}} - u^\varepsilon_{\text{app}})}{\partial t}\right|^2 + \left|\nabla(u^\varepsilon_{\text{ex}} - u^\varepsilon_{\text{app}})\right|^2} \leq C(t)\varepsilon^2$$

References

[1] P. Kuchment, *Graph models for waves in thin structures*, Waves Random Media, **12** (2002), 4, R1–R24

[2] J. Rubinstein and M. Schatzman, *Variational problems on multiply connected thin strips. I. Basic estimates and convergence of the Laplacian spectrum*, Arch. Ration. Mech. Anal.,

[3] P. Joly and A. Semin, *Study of propagation of acoustic waves in junction of thin slots*, INRIA Research Report 7265, April 2010.