Weierstrass Institute for
Applied Analysis and Stochastics

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Inhomogeneous and anisotropic phase-field quantities in the sharp interface limit

joint work with
Harald Garcke, University of Regensburg.

Outline

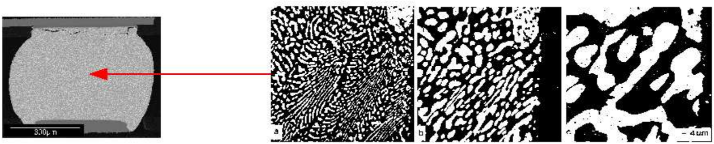
- ▷ Sharp interface and diffuse phase-field setting
- ▷ Modeling of surface quantities
- ▷ The Γ–limit of the phase-field energies
- ▷ Asymptotic behavior for minimizers
- ▷ Weak formulations of the Euler–Lagrange equations
- ▷ Convergence of the Lagrange multipliers
- ▷ Extended Gibbs–Thomson law in the sharp interface limit

Application: Morphology of solder materials

▷ Phase separation and coarsening in alloys

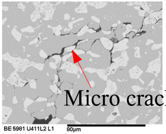
Solder ball

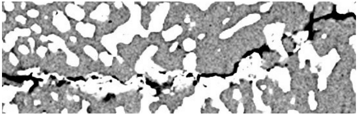
After solidification 3h. 300h.



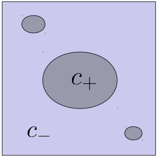
Tin-lead solder at 120° C

▷ Crack initiation and propagation

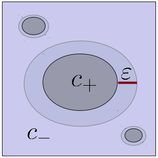




Sharp interface model



Diffuse phase-field model



Phase indicator function

$$c : \Omega \to \{c_-, c_+\}, \quad I = \partial\{c = c_-\} \cap \Omega$$

Surface energy

$$E(I) = \sigma_0 \int_I d\mathcal{H}^{n-1} = \frac{\sigma_0}{c_+ - c_-} \int_\Omega |\nabla c| = E(c)$$

First variation

$$\delta E(c)(\xi) = \sigma_0 \int_I \mathrm{div}_I \xi \, d\mathcal{H}^{n-1} = \sigma_0 \int_I H \, \xi \cdot \nu \, d\mathcal{H}^{n-1}$$

Smooth phase-field

$c_\varepsilon : \Omega \to \mathbb{R}$, I_ε: small transition region

Diffuse surface energy

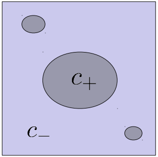
(van der Waals, Cahn-Hilliard)

$$E_\varepsilon(c_\varepsilon) = \int_\Omega \left(\varepsilon |\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} \psi(c_\varepsilon) \right) dx$$

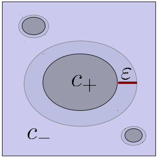
First variation

$$\delta E_\varepsilon(c_\varepsilon)(\xi) = \int_\Omega \left(-\varepsilon \triangle c_\varepsilon + \frac{1}{\varepsilon} \psi'(c_\varepsilon)\right) \xi \, dx$$

Sharp interface model



Diffuse phase-field model



$\varepsilon \longrightarrow 0$

Sharp interface limit

Phase indicator function

$$c : \Omega \to \{c_-, c_+\}, \quad I = \partial\{c = c_-\} \cap \Omega$$

Surface energy

$$E(I) = \sigma_0 \int_I d\mathcal{H}^{n-1} = \frac{\sigma_0}{c_+ - c_-} \int_\Omega |\nabla c|$$
$$= E(c)$$

First variation

$$\delta E(c)(\xi) = \sigma_0 \int_I \mathrm{div}_I \xi \, d\mathcal{H}^{n-1}$$
$$= \sigma_0 \int_I H \, \xi \cdot \nu \, d\mathcal{H}^{n-1}$$

Smooth phase-field

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First variation

$$\delta E_\varepsilon(c_\varepsilon)(\xi) = \int_\Omega \left(-\varepsilon \triangle c_\varepsilon + \frac{1}{\varepsilon} \psi'(c_\varepsilon)\right) \xi \, dx$$

Sharp interface limit

Minimizers (Modica & Mortola)

There exists a sequence $\{c_\varepsilon\}_{\varepsilon>0}$ of global minimizers c_ε for E_ε such that

$$c_\varepsilon \to c \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \to 0, \qquad c \in BV(\Omega, \{c_-, c_+\}).$$

Energies (Modica & Mortola)

$$E_\varepsilon \overset{\Gamma\text{–convergence}}{\longrightarrow} E$$

$$E_\varepsilon(c_\varepsilon) = \int_\Omega \Big(\varepsilon|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon}\psi(c_\varepsilon)\Big)dx \to E(c) = \frac{\sigma_0}{c_+ - c_-}\int_I |\nabla c|$$

$$\sigma_0 = 2\int_{c_-}^{c_+} \sqrt{\psi(s)}ds$$

Diffuse surface area measures

$$\mu_\varepsilon := \Big(\varepsilon|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon}\psi(c_\varepsilon)\Big)\mathcal{L}^n \to \mu := \frac{\sigma_0}{c_+ - c_-}|\nabla c| \quad \text{weakly}^*$$

Sharp interface limit

Structure of the interface (Modica)

$\Omega_- = \{x \in \Omega : c(x) = c_-\}$ has minimal perimeter $P_\Omega(\Omega_-)$, i. e.

$$P_\Omega(\Omega_-) = \min\left\{ P_\Omega(G) : G \subset \Omega, |G| = \frac{c_+|\Omega|}{c_+ - c_-} \right\}.$$

Theory of minimal surfaces $\Longrightarrow$ I: smooth hypersurface, minimal surface area

$\partial I \cap \partial\Omega$: 90° contact angle

Lagrange multipliers (Luckhaus $\&$ Modica)

$$\lambda_\varepsilon \to \lambda = \frac{\sigma_0 H}{c_+ - c_-}, \qquad H: \text{ mean curvature.}$$

Gibbs–Thomson law for minimizers

$$\lambda = \frac{\sigma_0 H}{c_+ - c_-}$$

Gibbs–Thomson law

▷ Isotropic case

$$E(c) = \frac{\sigma_0}{c_+ - c_-} \int_\Omega |\nabla c| = \sigma_0 \int_I d\mathcal{H}^{n-1}$$

Classical Gibbs–Thomson law

$$w = \sigma_0 H/(c_+ - c_-), \qquad H = \mathrm{div}_I \cdot \nu, \qquad \nu : \text{outer unit normal of } I.$$

Boundary condition for I

$$\partial I \cap \partial\Omega: \quad 90^\circ \text{ contact angle}$$

▷ Inhomogeneous, anisotropic, elastic case

$$E_0(c, u) = \int_I \sigma_0(x, \nu) d\mathcal{H}^{n-1} + \int_\Omega W(c, \mathcal{E}(u)) dx$$

Extended Gibbs–Thomson law

$$w = \Big(-\sigma_{0,x}(x,\nu) \cdot \nu - \nabla_I \cdot \sigma_{0,p}(x,\nu) + \nu [W Id - (\nabla u)^T W_{,\mathcal{E}}]_-^+ \nu \Big) / (c_+ - c_-)$$

Boundary condition for I

$$\partial I \cap \partial\Omega: \quad \sigma_{,p}(x,\nu) \cdot \nu_{\partial\Omega} = 0, \qquad \nu_{\partial\Omega} : \text{outer unit normal of } \partial\Omega$$

Gibbs–Thomson law

- I: not enough regularity is known in the singular limit

  $\Longrightarrow$ Weak formulation of the extended Gibbs–Thomson law

- Weak formulation (generalized BV–setting)

$$\int_\Omega \Big(\sigma_0(x,\nu)\nabla\cdot\xi + \sigma_{0,x}(x,\nu)\cdot\xi - \nu\cdot\nabla\xi\,\sigma_{0,p}(x,\nu)\Big)|\nabla\chi|$$
$$+\int_\Omega \Big(W(c,\mathcal{E}(u))Id - (\nabla u)^T W_{,\mathcal{E}}(c,\mathcal{E}(u))\Big) : \nabla\xi\; dx = \lambda\int_\Omega c\,\nabla\cdot\xi\,dx$$

$\xi \in C^1(\overline{\Omega},\mathbb{R}^n)$ with $\xi\cdot\nu_{\partial\Omega} = 0$ on $\partial\Omega$, $\nu_{\partial\Omega}$: outer unit normal of $\partial\Omega$,

$\nabla\chi$: distributional derivative of the characteristic function χ of Ω_-.

Phase-field approach

Two components

Concentrations: c_+, c_-, $c_+ + c_- = 1 \quad \Longrightarrow c = c_+ - c_-$

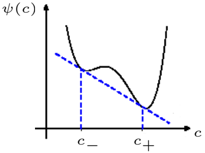
Mass conservation

$\int_\Omega c\,dx = m$

Total free energy

Ginzburg–Landau free energy

$$E(c,u) = \int_\Omega \Big(\varepsilon\sigma^2(x,c,\nabla c) + \frac{1}{\varepsilon}\psi(c) + W(c,\mathcal{E}(u)) \Big)\,dx$$



(Classical case: $\sigma(x,c,\nabla c) = |\nabla c|$, $W \equiv 0$)

Assumptions:

(A1) $\Omega \subset \mathbb{R}^n$: bounded domain with Lipschitz–boundary

(A2) $\psi \in C^1(\mathbb{R})$, $\psi \geq 0$ and $\psi(c) = 0 \Longleftrightarrow c \in \{c_-, c_+\}$,
$\psi(c) \geq d_1|c|^2 - d_2, \qquad d_1, d_2 > 0$ constants.

Phase-field approach

$$E(c,u) = \int_\Omega \Big(\varepsilon \sigma^2(x,c,\nabla c) + \frac{1}{\varepsilon}\psi(c) + W(c,\mathcal{E}(u)) \Big) dx$$

(A3) The anisotropic function $\sigma \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^+)$ satisfies the properties:

* $\sigma_{,x}, \sigma_{,p} \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \backslash \{0\})$, $\quad \sigma_{,pp} \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \backslash \{0\})$.
* σ is 1–homogenous, i.e. $\sigma(x,s,\lambda p) = \lambda \sigma(x,s,p)$ for all $p \in \mathbb{R}^n$ and $\lambda > 0$.
* There exist constants $\lambda_1 > 0$, $\lambda_2 > 0$ such that

$$\lambda_1 |p| \le \sigma(x,s,p) \le \lambda_2 |p| \quad \text{for all } x \in \overline{\Omega}, s \in \mathbb{R} \text{ and all } p \in \mathbb{R}^n .$$

* σ is strictly convex, i.e. there exists a constant $d_0 > 0$ such that

$$\sigma_{,pp}(x,s,p)\, q \cdot q \ge d_0 |q|^2$$

for all $x \in \Omega$, all $s \in \mathbb{R}$ and all $p, q \in \mathbb{R}^n$ with $p \cdot q = 0$, $|p| = 1$.

(A4) Elasticity

- u: displacement field
- $\mathcal{E}(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$: linearized strain tensor
- Elastic energy density $W \in C^1(\mathbb{R} \times \mathbb{R}^{n\times n}, \mathbb{R})$:

$$W(c, \mathcal{E}(u)) = \frac{1}{2}\big(\mathcal{E}(u) - \mathcal{E}^*(c)\big) : C(c)\big(\mathcal{E}(u) - \mathcal{E}^*(c)\big),$$

  $\mathcal{E}^*(c)$: eigenstrain, $C(c)$: stiffness tensor (symmetric, positive definite).

- There exists a constant $d_4 > 0$ such that for all $c \in \mathbb{R}$ and all $\mathcal{E} \in \mathbb{R}^{n\times n}$:

$$|W(c,\mathcal{E})| \le d_4\big(|c|^2 + |\mathcal{E}|^2 + 1\big),$$
$$|W_{,c}(c,\mathcal{E})| \le d_4\big(|c| + |\mathcal{E}|^2 + 1\big),$$
$$|W_{,\mathcal{E}}(c,\mathcal{E})| \le d_4\big(|c| + |\mathcal{E}| + 1\big).$$

Definition:

$X_{\mathrm{ird}}^{\perp} := \{u \in H^1(\Omega, \mathbb{R}^n) : (u,v)_{H^1} = 0$ for all $v \in X_{\mathrm{ird}}\}$

$X_{\mathrm{ird}} := \{v \in H^1(\Omega, \mathbb{R}^n)$: there exist $b \in \mathbb{R}^n$ and a skew symmetric $K \in \mathbb{R}^{n\times n}$ such that $v(x) = b + Kx\}$

Γ–limit of the Ginzburg–Landau energy

$E_\varepsilon : BV(\Omega) \times X_{\mathrm{ird}}^{\perp} \to \mathbb{R} \cup \{+\infty\}$

$$E_\varepsilon(c,u) = \begin{cases} \int_\Omega \big(\varepsilon\sigma^2(x,c,\nabla c) + \frac{1}{\varepsilon}\psi(c) + W(c,\mathcal{E}(u))\big)dx, & \text{if } c \in H^1(\Omega), \\ & \int_\Omega c\,dx = m, \\ \infty, & \text{else.} \end{cases}$$

$\big\downarrow \; \varepsilon \to 0$

$E_0 : BV(\Omega) \times X_{\mathrm{ird}}^{\perp} \to \mathbb{R} \cup \{+\infty\}$

$$E_0(c,u) = \begin{cases} \int_I \sigma_0(x,\nu)d\mathcal{H}^{n-1} + \int_\Omega W(c,\mathcal{E}(u))dx, & \text{if } c \in BV(\Omega,\{c_-,c_+\}), \\ & \int_\Omega c\,dx = m, \\ \infty, & \text{else.} \end{cases}$$

Definition:

$I = \partial^*\Omega_-,\ \Omega_- = \{x \in \Omega : c(x) = c_-\},\ \nu$: outer normal of $I = \partial^*\Omega_-$,

$\sigma_0(x,p) = 2\int_{c_-}^{c_+} \sqrt{\psi(s)}\sigma(x,s,p)ds\,;$

(Extension of results by Bouichetté 90, Owen and Sternberg 91 and Garcke 08)

Characterization of minimizers

Theorem [Garcke & Kraus 09]

Let the assumptions (A1) – (A4) be satisfied.

(i) For $\varepsilon > 0$ there exist minimizers $(c_\varepsilon, u_\varepsilon) \in H^1(\Omega) \times X_{\text{ird}}^{\perp}$ of E_ε.
In addition,

$$E_\varepsilon(c_\varepsilon, u_\varepsilon) \quad \text{is uniformly bounded as } \varepsilon \to 0.$$

(ii) For every sequence of minimizers $\{(c_{\varepsilon_k}, u_{\varepsilon_k})\}_{k\in\mathbb{N}} \subset H^1(\Omega) \times L^2(\Omega, \mathbb{R}^n)$ there exists a subsequence with

$$c_{\varepsilon_{k_j}} \to c \quad \text{in } L^2(\Omega), \quad c \in L^2(\Omega) \cap BV(\Omega, \{c_-, c_+\}),$$
$$u_{\varepsilon_{k_j}} \to u \quad \text{in } H^1(\Omega, \mathbb{R}^n).$$

(iii) (c, u) is a global minimizer of E_0.

Local surface quantities

Theorem [Garcke & Kraus 09]
Let the assumptions (A1) – (A4) be satisfied. Furthermore, let $(c_\varepsilon, u_\varepsilon)$ be a minimizer of E_ε. Then for each sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} \varepsilon_k = 0$, with

$$c_{\varepsilon_k} \to c \quad \text{in} \quad L^1(\Omega),$$

there exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that for a. e. $s \in [c_-, c_+]$:

* $\nabla\chi_{s,k} \to \nabla\chi_{\{c=c_-\}}$ weakly* in Ω
* $\int_\Omega \sigma(x, s, \nu_{s,k})|\nabla\chi_{s,k}| \to \int_\Omega \sigma(x, s, \nu)|\nabla\chi_{\{c=c_-\}}|$
* $|\nabla\chi_{s,k}| \to \mu$

$$\mu = |\nabla\chi_{\{c=c_-\}}|$$

$\chi_{s,k}$: characteristic function of $\Omega_{s,k} = \{x \in \Omega : c_{\varepsilon_k}(x) < s\}$,
$\nu_{s,k}$: outer unit normal of $\partial\Omega_{s,k}$.

Convergence of the Lagrange multipliers λ_{ε_k}

Theorem [Garcke & Kraus 09]
Let the assumptions (A2) – (A4) be satisfied and let Ω be with C^1–boundary.
Furthermore, let $(c_\varepsilon, u_\varepsilon)$ be a minimizer of E_ε.
Then for each sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} \varepsilon_k = 0$, with

$$c_{\varepsilon_k} \to c \quad \text{in} \quad L^1(\Omega),$$
$$u_{\varepsilon_k} \to u \quad \text{in} \quad L^2(\Omega, \mathbb{R}^n),$$

the corresponding sequence of Lagrange multipliers $\{\lambda_{\varepsilon_k}\}_{k\in\mathbb{N}}$ converges, i. e.

$$\lambda_{\varepsilon_k} \to \lambda,$$

where λ is a Lagrange multiplier of the minimum problem for E_0 with $\int_\Omega c\,dx = m$.

Euler–Lagarange equation for E_0:

$$\int_I \Big(\sigma_0(x,\nu)\nabla\cdot\xi + \sigma_{0,x}(x,\nu)\cdot\xi - \nu\cdot\nabla\xi\,\sigma_{0,p}(x,\nu)\Big)\,d\mathcal{H}^{n-1}$$
$$+\int_\Omega \Big(W(c,\mathcal{E}(u))Id - (\nabla u)^T W_{,\mathcal{E}}(c,\mathcal{E}(u))\Big) : \nabla\xi\;dx \quad = \lambda\int_\Omega c\,\nabla\cdot\xi\,dx$$

for all $\xi \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\xi\cdot\nu_{\partial\Omega} = 0$ on $\partial\Omega$.

Some aspects of the proof

Euler–Lagrange equation for E_ε:

$$\int_\Omega \Big(2\varepsilon\sigma(x,c_\varepsilon,\nabla c_\varepsilon)\sigma_{,x}(x,c_\varepsilon,\nabla c_\varepsilon)\cdot\xi \;-2\varepsilon\sigma(x,c_\varepsilon,\nabla c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\xi\,\sigma_{,p}(c_\varepsilon,\nabla c_\varepsilon)$$

$$+\Big(\varepsilon\,\sigma^2(x,c_\varepsilon,\nabla c_\varepsilon)+\frac{1}{\varepsilon}\psi(c_\varepsilon)\Big)\nabla\cdot\xi+\Big(W(c_\varepsilon,\mathcal{E}(u_\varepsilon))Id-(\nabla u_\varepsilon)^T W_{,\mathcal{E}}(c_\varepsilon,\mathcal{E}(u_\varepsilon))\Big):\nabla\xi\Big)dx$$

$$=\lambda_\varepsilon\int_\Omega c_\varepsilon\nabla\cdot\xi dx$$

Euler–Lagrange equation for E_0:

$$\int_I\Big(\sigma_{0,x}(x,\nu)\cdot\xi-\nu\cdot\nabla\xi\,\sigma_{0,p}(x,\nu)+\sigma_0(x,\nu)\nabla\cdot\xi\Big)d\mathcal{H}^{n-1}$$

$$+\int_\Omega\Big(W(c,\mathcal{E}(u))Id-(\nabla u)^T W_{,\mathcal{E}}(c,\mathcal{E}(u))\Big):\nabla\xi\;dx=\lambda\int_\Omega c\,\nabla\cdot\xi\,dx$$

Generalized total variation (Amar & Belletini 94/95): $f\in BV(\Omega)$

$\int_\Omega|\nabla f|_{\sigma(s)}=\sup\Big\{\int_\Omega f\,\mathrm{div}\eta\,dx:\eta\in C_c^1(\Omega),\,\sigma^*(x,s,\eta)\le 1\Big\}$, σ^*: dual function.

Integral formula $\int_\Omega|\nabla f|_{\sigma(s)}=\int_\Omega\sigma(x,s,\nu_f)|\nabla f|,\qquad \nu_f:=-\frac{\nabla f}{|\nabla f|}.$

Some aspects of the proof

▷ Equipartition of energy

$$\lim_{k\to\infty}\int_\Omega \left|\frac{1}{\varepsilon_k}\psi(c_{\varepsilon_k}) - \varepsilon_k\,\sigma^2(x, c_{\varepsilon_k}, \nabla c_{\varepsilon_k})\right| dx = 0$$

▷ Local convergence properties

There exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that for a. e. $s \in [c_-, c_+]$:

$$* \quad \int_\Omega \nu_{s,k} \cdot \nabla\xi\sigma_{,p}(x, s, \nu_{s,k})|\nabla\chi_{s,k}| \to \int_\Omega \nu \cdot \nabla\xi\sigma_{,p}(x, s, \nu)|\nabla\chi_{\{c=c_-\}}|$$

$$* \quad \int_\Omega \xi \cdot \sigma_{,x}(x, s, \nu_{s,k})|\nabla\chi_{s,k}| \to \int_\Omega \xi \cdot \sigma_{,x}(x, s, \nu)|\nabla\chi_{\{c=c_-\}}|$$

$\chi_{s,k}$: characteristic function of $\Omega_{s,k} := \{x \in \Omega : c_{\varepsilon_k}(x) < s\}$,

$\nu_{s,k}$: outer unit normal of $\partial\Omega_{s,k}$.

Some aspects of the proof

▷ Wulff–shapes and their geometric properties

For all $x \in \Omega$, $s \in [c_-, c_+]$, $\nu \in \mathbb{S}^{n-1}$ and all $p \in \mathbb{R}^n \backslash \{0\}$ with $\sigma^*(x, s, p) \leq 1$:

$$C \, |\sigma_{,\,p}(x, s, \nu) - p|^2 \leq \sigma(x, s, \nu) - p \cdot \nu$$

▷ Construction of smooth approximative functions g_s^δ

$$\text{suitable approximations } g_s^\delta \text{ for } \begin{cases} \text{phase–field quantities: } \sigma_{,\,p}(x, s, \nu_{s,k}) \\ \text{Cahn–Hoffman vector: } \sigma_{,\,p}(x, s, \nu) \end{cases}$$

■

Equilibrium conditions for E_0

$$E_0(c,u) = \int_I \sigma_0(x,\nu)\, d\mathcal{H}^{n-1} + \int_\Omega W(c,\mathcal{E}(u))\,dx$$

Assumptions:

* Ω domain with C^1–boundary
* $I = \partial^*\Omega_-$ is a C^2–hypersurface
* $\partial I \subset \partial\Omega$: finite number of C^1–(n-2) dim. hypersurfaces.
* $u|_{\Omega_-} \in H^2(\Omega_-,\mathbb{R}^n)$ and $u|_{\Omega_+} \in H^2(\Omega_+,\mathbb{R}^n)$

Necessary conditions:

▷ In Ω_- and Ω_+

$$\nabla\cdot W_{,\mathcal{E}}(c_-,\mathcal{E}(u_-)) = 0, \quad \nabla\cdot W_{,\mathcal{E}}(c_+,\mathcal{E}(u_+)) = 0$$

▷ On $\partial\Omega$

$$W_{,\mathcal{E}}\nu_{\partial\Omega} = 0$$

▷ On the interface I

$$[W_{,\mathcal{E}}\nu]_-^+ = 0\,, \qquad [u]_-^+ = 0$$

Gibbs–Thomson law:

$$-\sigma_{0,x}\cdot\nu_- - \nabla_I\cdot\sigma_{0,p}(x,\nu_-) + \nu[W Id - (\nabla u)^T W_{,\mathcal{E}}]_-^+\nu = \lambda[c]_-^+$$

▷ On $\partial I \cap \partial\Omega$

Force balance: $\sigma_{0,p}(x,\nu)\cdot\nu_{\partial\Omega} = 0$

Overview

Phase–field energy functional

$$E_\varepsilon(c,u) = \int_\Omega \Big(\varepsilon\sigma^2(x,c,\nabla c) + \frac{1}{\varepsilon}\psi(c) + W(c,\mathcal{E}(u)) \Big) dx$$

Chemical potential

$$w_\varepsilon = \frac{\delta E_\varepsilon}{\delta c} = -2\varepsilon\nabla\cdot\big(\sigma(x,c,\nabla c)\sigma_{,p}(x,c,\nabla c)\big) + 2\varepsilon\sigma(x,c,\nabla c)\sigma_{,c}(x,c,\nabla c) + \frac{1}{\varepsilon}\psi_{,c}(c) + W_{,c}(c,\mathcal{E}(u))$$

Sharp interface limit ↓ $\varepsilon \to 0$ for global minimizers

Sharp interface energy functional

$$E_0(c,u) = \int_I \sigma_0(x,\nu)\,d\mathcal{H}^{n-1} + \int_\Omega W(c,\mathcal{E}(u))\,dx$$

Chemical potential

Gibbs–Thomson law

$$w_0 = \frac{\delta E_0}{\delta c} = \big(-\sigma_{0,x}(x,\nu)\cdot\nu - \nabla_I\cdot\sigma_{0,p}(x,\nu) + \nu[WId - (\nabla u)^T W_{,\mathcal{E}}]^+_-\nu\big)/(c_+ - c_-)$$

on I