Diagonal elliptic systems

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(joint work with Miroslav Bulícek,...)

Wanted: regular solution $u = (u_1, \ldots, u_N)$ of

$$-\Delta u_\nu + \lambda u_\nu = H_\nu(x, u, \nabla u) \text{ in } \Omega \subset \mathbb{R}^n$$

- Ω bounded, $\partial\Omega$ Lipschitz
- $|H_\nu(x, \mu, p)| \leq K\,|p|^2 + K$
- structure conditions
- H = “Hamiltonian”

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Remark: Irregular unbounded solution may occur even in the scalar case

$$u = \ln|\ln|x||, \quad n = 2, -\Delta u = |\nabla u|^2$$

For bounded solutions: Theory of Ladyshenskaja-Uralceva

Aim: Existence of $W^{2,q}$ solutions,
i.e. essentially $C^\alpha \cap H^1$ solutions

$$-\Delta \leftrightarrow -\mathrm{D}_i(a_{ik}(x)\mathrm{D}_k u)$$

In this talk:

$$a_{ik}^\nu = a_{ik}^\mu.$$

Motivation of the structure conditions of the Hamiltonians: they come from stochastic differential games.

Well known:

$$|H(x,\mu,p)| \leq a\,|p|^2 + K\,,\;\; a < \|u\|_{L^\infty} \Rightarrow \;\text{regularity}$$

Exponentially dominable Hamiltonians
Bensoussan – JF.

$$|H_\nu(x,\mu,p)| \leq K\,|p_\nu|\,|p| + K\sum_{j=\nu}^{N} |p_j|^2 + K$$

Harmonic mappings (here on the sphere)

$$-\Delta u = u\,|\nabla u|^2 \text{ in } \Omega\,,\ u = g \text{ on } \partial\Omega$$

$n = 2$: Regularity for all $L^\infty \cap H^1$ – solutions
$n > 3$: $u = \frac{x}{|x|}$ counterexample

Regularity if Nöther condition holds
Test with

$$\varphi \cdot \nabla u \Rightarrow \int |\nabla u|^2 \operatorname{div}\varphi \mathrm{d}x - \sum \int \mathrm{D}_i u \mathrm{D}_k u \mathrm{D}_k \varphi_i \mathrm{d}x = 0$$

Wiegner's results: $u = (u_1, \dots, u_m)$

$$-\Delta u + uF_0(x, u, \nabla u) = f \text{ in } \Omega \subset \mathbb{R}^n$$
$$F_0 \approx |\nabla u|^2 \ , \ F_0 \geq 0$$

$+$ slight generalizations

$$-\Delta u_\nu + u_\nu F_0(x, u, \nabla u) + L(x, u, \nabla u)\nabla u_\nu = f_\nu$$
$$L \approx |\nabla u|$$
$$\Rightarrow \text{ Regularity}$$

Two dimensions: (old result of JF)
existence of $C^\alpha \cap H^1$-solutions for non diagonal elliptic systems

$$\begin{aligned} -\operatorname{div} a(x.u.\nabla u) + a_0(x.u.\nabla u) &= 0 \\ a_0 \cdot u &\geq 0 \end{aligned}$$

Remarkable: irregular solutions can occur:
Example

$$\begin{aligned} u_1 &= \cos(\alpha \ln|\ln|x||) \\ u_2 &= \sin(\alpha \ln|\ln|x||) \end{aligned}$$

Parabolic case ($n = 2$, nondiagonal case) has been treated by Specovius-N. & JF, if the elliptic part is an Euler operator. Wiegner's result in the parabolic case has not been published. (Wiegner, of course, knows how it works.)

These existence and regularity theorems are not sufficient for stochastic differential games!

Many examples:

- Discount control
- Stackelberg games
- Games with "complicated" non market interaction

$\Rightarrow$ "Zoo of diagonal elliptic systems"

Recently: Bensoussan + JF.
Bellman systems for stochastic differential games with discount control.

$$-\Delta u_\nu + \lambda u_\nu + u_\nu F_0(u, \nabla u) = H_\nu(x, u_\nu, \nabla u_\nu) + f_\nu(x)$$
$$F_0 \approx |\nabla u|^2 \ , \ F_0 \geq 0$$
$$H_\nu \approx |\nabla u_\nu|^2$$

Regular solutions for $n = 2$
Open: $n \geq 3$

Recently Bensoussan + JF + Vogelgesang

1. $\sum H_\nu(x, \nabla u) \geq c_0 \left|B_0 \nabla u\right|^2 - K \left|\sum_{\nu=1}^{N} \nabla u_\nu\right| \left|B_0 \nabla u\right| - K$
   $c_0 > 0$
2. $H_\nu(x, \nabla u) \leq K \left|B_0 \nabla u\right| \left|\nabla u_\nu\right| + K$

Contains a lot of standard games and Stackelberg games

JF + Bulícek:

$$1. \sum \gamma_\nu H_\nu(x, \nabla u) \geq c_0 \left|B_0 \nabla u\right|^2 - K \left|\sum_{\nu=1}^{N} \gamma_\nu \nabla u_\nu\right| \left|B_0 \nabla u\right| - K$$

$$c_0 > 0$$

$$2. H_\nu(x, \nabla u) \leq K \left|\nabla u_\nu\right|^2 + L(\nabla u) \cdot \nabla u_\nu + K$$

$$L \approx |\nabla u|$$

$\Rightarrow \exists$ regular solutions

Typical examples in the deterministic case

$$J_\nu(v) = \int_0^T l_\nu(x(t), v(t)) + f_\nu(x(t))\mathrm{d}t$$

$$\dot{x} = \sum A_\nu(x)v_\nu + b_0(x)\,,\ A_\nu\,,\ b_0 \text{ bounded}$$

$$l_\nu = \frac{1}{2}v_\nu Q_\nu(x)v_\nu + v_\nu B_\nu \bar{v}_\nu\,,\ \bar{v}_\nu = (v_1, \ldots, v_{\nu-1}, v_{\nu+1}, \ldots, v_N)$$

$$\text{say } l_\nu = \frac{1}{2}v_\nu^2 + v_\nu \Theta v_{\nu+1}$$

(zyclic game)

Nash point equilibrium:
Find v^* such that

$$J_\nu(v^*) \leq J_\nu(v_1^*, \ldots, v_{\nu-1}^*, v_\nu, v_{\nu+1}^*, \ldots, v_N^*)$$

Stochastic version:
Lagrange function

$$L_\nu(x,v,p) = l_\nu(x,v) + p_\nu \overbrace{\sum_{j=1}^{N} A_j(x)v_j}^{Av} + f_\nu(x)$$

Find Nash-Point for $L_\nu \Rightarrow$ solving $v^* = v^*(x,p)$ pointwise in x and p.
Define

$$H_\nu(x,p) = H_\nu(v^*(x,p))$$

The term Av comes from the stochastic ODE $\dot{X} = Av + d\omega$

Example:

$$L_\nu(x,v) = \frac{1}{2}v_\nu^2 + v_\nu \Theta v_{\nu+1} + p_\nu \sum_{j=1}^{N} A_j(x)v_j + f_\nu(x)$$

$$0 = \frac{\partial L}{\partial v_\nu} = v_\nu + v_{\nu+1}\Theta + \hat{p}_\nu\,, \quad \hat{p}_\nu = A_\nu^T p_\nu$$

$$L_\nu(x,v)_{v_\nu = v_\nu^*} = -\frac{1}{2}|\underbrace{v_{\nu+1}\Theta + \hat{p}_\nu}_{-v_\nu^*}|^2 + p_\nu \sum_{\substack{j=1 \\ j\neq\nu}}^{N} A_j(x)v_j + f_\nu(x)$$

$$\Rightarrow L_\nu = -\frac{1}{2}\left|v_\nu^*\right|^2 + p_\nu A_\nu v_\nu^* + p_\nu \sum_{j=1}^{N} A_j v_j + f_\nu(x)$$

$$\leq K\left|p_\nu\right|^2 + p_\nu \sum_{j=1}^{N} A_j v_j + f_\nu(x)$$

$\Rightarrow 1^{\mathrm{st}}$ structure condition is satisfied

2^{nd} structure condition

$$\sum_{\nu=1}^{N} L_\nu(x,v) = \underbrace{\frac{1}{2}|v|^2 + \Theta \sum_{\nu=1}^{N} v_\nu v_{\nu+1}}_{\text{assume } \geq |v|^2} + \sum p_\nu A v + \sum f_\nu$$

$|\Theta| < \frac{1}{2} \Rightarrow 2^{\mathrm{nd}}$ structure assumption is satisfied.

Dekuji vám za pozornost.

Vielen Dank für ihre Aufmerksamkeit.

Thank you for your attention.