Weighted analytic regularity in corner domains: The two-dimensional case.

Martin Costabel, Monique Dauge, Serge Nicaise

IRMAR, Université de Rennes 1, FRANCE

6th Singular Days on Asymptotic Methods for PDEs
WIAS Berlin, April 29 – May 1, 2010

Outline

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Elliptic boundary value problems

Ω: domain in $\mathbb{R}^n$ ($n \geq 2$), i.e. bounded and conected.
We consider

- *corner domains*, e.g. polygons if $n = 2$, polyhedra if $n = 3$,
- and also *smooth domains*

L: second order elliptic operator or system with smooth coefficients.
Example: $L = \Delta$ (Laplacian), $L =$ Lamé system (elasticity)

B: operator of order $k = 0$ or 1 with smooth coeff. which “covers” L on $\partial\Omega$
Example: $B = Id$ (Dirichlet, $k = 0$),
$B =$ conormal derivative associated with L (Neumann, $k = 1$)

Problem :

Given f, find u

$$(BVP) \qquad \begin{cases} Lu = f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Sobolev Regularity Shift for Smooth Domains

Sobolev spaces

$$\mathrm{H}^m(\Omega) = \{v \in \mathscr{D}'(\Omega) : \ \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} v \in \mathrm{L}^2(\Omega), \ |\boldsymbol{\alpha}| \leq m\}$$

Theorem: [AGMON-DOUGLIS-NIRENBERG 1959, 1964]

Let Ω be a smooth domain.
Let $m \geq 2$. If $u \in \mathrm{H}^2(\Omega)$ solves (BVP) with

$$f \in \mathrm{H}^{m-2}(\Omega)$$

then $u \in \mathrm{H}^m(\Omega)$ with estimates

$$\|u\|_{\mathrm{H}^m(\Omega)} \leq C \left\{ \|f\|_{\mathrm{H}^{m-2}(\Omega)} + \|u\|_{\mathrm{H}^1(\Omega)} \right\}.$$

Remark

If (BVP) has a coercive variational formulation in $\mathrm{V} \subset \mathrm{H}^1(\Omega)$, the above statement holds for $u \in \mathrm{H}^1(\Omega)$.

Case of corner domains

Let Ω be a domain with conical points.

The Sobolev regularity shift does not hold in general, due to the presence of singular functions.

Nevertheless, using Sobolev-Slobodeckii spaces $\mathrm{H}^s(\Omega)$ with real exponents:

Theorem: [KONDRAT'EV 1967] [DAUGE 1988]

Let (BVP) have a coercive variational formulation in $V \subset \mathrm{H}^1(\Omega)$.
Then there exists $s_{\Omega,L,B} > 0$ such that the following regularity holds:

$\forall s$, $\boxed{0 < s < s_{\Omega,L,B}}$, $s \neq \frac{1}{2}$, variational solutions u of (BVP) satisfy

$$f \in \mathrm{H}^{s-1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{H}^{s+1}(\Omega)$$

NB If $s < \frac{1}{2}$, the problem does not have the form (BVP) and the RHS has to be defined in variational form and set in the correct dual space.

Homogeneous Weighted Sobolev spaces

Let $\mathscr{C}$ be the set of corners the $\boldsymbol{c}$ of Ω.

- Weight := powers of $r(\boldsymbol{x}) = \min_{\boldsymbol{c}\in\mathscr{C}} |\boldsymbol{x} - \boldsymbol{c}|$
- Weight exponent := $\beta \in \mathbb{R}$
- **Homogeneous weighted Sobolev spaces**
  KONDRAT'EV, MAZ'YA-PLAMENEVSKII, NAZAROV, ROSSMANN

$$\mathrm{K}^m_\beta(\Omega) = \{v \in \mathscr{D}'(\Omega) : \underbrace{r(\boldsymbol{x})^{|\alpha|+\beta}}_{\text{depending on } \alpha} \partial^{\alpha}_{\boldsymbol{x}} v \in \mathrm{L}^2(\Omega),\ |\alpha| \le m\}$$

Theorem: [KONDRAT'EV 1967] + [Co-Da-Ni 2010]

Assume the coercive variational setting.

- If the variational space V is embedded in $\mathrm{K}^1_{-1}(\Omega)$, there exists $b_{\Omega,L,B} > 0$ such that the following regularity holds:
- $\forall b$, $\boxed{0 \le b < b_{\Omega,L,B}}$ and $\forall m \ge 1$

$$u \in \mathrm{V} \quad \text{and} \quad f \in \mathrm{K}^{m-1}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{K}^{m+1}_{-b-1}(\Omega)$$

Non-Homogeneous Weighted Sobolev spaces

- Weight := powers of $r(\boldsymbol{x}) = \min_{\boldsymbol{c}\in\mathscr{C}} |\boldsymbol{x} - \boldsymbol{c}|$
- Weight exponent := $\beta \in \mathbb{R}$
- **Non-Homogeneous weighted Sobolev spaces**
  MAZ'YA-PLAMENEVSKII, NAZAROV, ROSSMANN

$$\mathrm{J}^m_\beta(\Omega) = \{v \in \mathscr{D}'(\Omega) : \quad \underbrace{r(\boldsymbol{x})^{m+\beta}}_{\textit{independent of } \alpha} \quad \partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \in \mathrm{L}^2(\Omega), \ |\boldsymbol{\alpha}| \le m\}$$

Theorem: [MAZ'YA-PLAMENEVSKII 1984]

Assume the coercive variational setting.

- There exists $b^*_{\Omega,L,B} > 0$ such that the following regularity holds.
- $\forall b$, $\boxed{0 < b < b^*_{\Omega,L,B}}$ $\forall m \ge 1$, variational sol. u of (BVP) satisfy

$$f \in \mathrm{J}^{m-1}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{J}^{m+1}_{-b-1}(\Omega)$$

Compare results

1. The assumption $\mathrm{V} \subset \mathrm{K}^1_{-1}(\Omega)$ means that $u \in \mathrm{V} \Rightarrow u/r \in \mathrm{L}^2(\Omega)$. It is satisfied for any BC if $n \geq 3$ and for Dirichlet BC if $n = 2$.
2. The statement in J-spaces is valid for any BC, in particular Neumann BC for $n = 2$.
3. We have identity $\mathrm{J}^m_{-m}(\Omega) = \mathrm{H}^m(\Omega)$ and equality $s_{\Omega,L,B} = b^*_{\Omega,L,B}$.
4. Statements in K-spaces and J-spaces are valid for all $m \in \mathbb{N}$. Hence the possibility of statements with $m = +\infty$.
5. If Ω has an analytic boundary, the analytic regularity shift holds.
6. In cormer domains with analytic corners, the only hope for an analytic regularity shift is an analytic limit of K_β and J_β families.

Outline

Analytic regularity shift in smooth domains

Theorem: [Morrey-Nirenberg 1957]

Assume

- $\partial\Omega$ is analytic,
- the coefficients of L and B are analytic,
- the rhs f is analytic: $f \in \mathrm{A}(\Omega)$,

then u solution of (BVP) is analytic: $u \in \mathrm{A}(\Omega)$.

A recent improvement is the proof of *analytic estimates*

i.e. the Cauchy-type control of constants in the "standard" estimate

$$\|u\|_{\mathrm{H}^m(\Omega)} \leq C(m) \left\{ \|f\|_{\mathrm{H}^{m-2}(\Omega)} + \|u\|_{\mathrm{H}^1(\Omega)} \right\}$$

Global analytic estimates

Theorem: [COSTABEL-DAUGE-NICAISE 2010]

Assume

- $\partial\Omega$ is analytic,
- the coefficients of L and B are analytic,
- the rhs $f \in H^{m-2}(\Omega)$ for some $m \geq 2$.

Then u satisfies the a priori estimates of analytic type, $k = 0, 1, \ldots, m$

$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{L^2(\Omega)} \leq A^{k+1}\Big\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}\|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{L^2(\Omega)} + \sum_{|\boldsymbol{\alpha}|\leq 1}\|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{L^2(\Omega)}\Big\}$$

with a constant A independent of k, m and u.

Proof

- Nested open sets on local model problems, *see later*
- Faà di Bruno formula for local maps

Local analytic estimates

As usual the *global* statement is a consequence of a *local* statement.

With $\mathcal{U}$ and $\mathcal{U}'$ two open sets in $\mathbb{R}^n$ such that $\overline{\mathcal{U}} \subset \mathcal{U}'$, set

$$\mathcal{V} = \mathcal{U} \cap \Omega, \quad \mathcal{V}' = \mathcal{U}' \cap \Omega \quad \text{and} \quad \Gamma := \partial\mathcal{V}' \cap \partial\Omega$$

Main Proposition: [Costabel-Dauge-Nicaise 2010]

Assume

- each connected component of Γ is an analytic part in $\partial\Omega$.
- the coefficients of L and B are analytic.

Then u satisfies the *local a priori estimates* of analytic type, $k = 0, 1, \ldots$

$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\mathrm{L}^2(\mathcal{V})} \le A^{k+1}\Big\{\sum_{\ell=0}^{k-2} \frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{\mathrm{L}^2(\mathcal{V}')} + \sum_{|\boldsymbol{\alpha}|\le 1} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\mathrm{L}^2(\mathcal{V}')}\Big\}$$

with a constant A independent of k and u.

Interior estimates, preparation: ρ-estimates

$B_R = B(\mathbf{0}, R)$, ball centered at $\mathbf{0}$. Assume $\mathbf{0} \in \Omega$. For $R \leq R_0$, $B_R \subset \Omega$.

Lemma

L is assumed to be elliptic. Let $u \in \mathrm{H}^2(B_R)$ for $R < R_0$. Let $\rho \in (0, \frac{R}{2})$.
$\exists\, A > 0$ independent of u, R and ρ.

$$\sum_{|\alpha|\leq 2} \rho^{|\alpha|} \|\partial^\alpha u\|_{B_{R-|\alpha|\rho}} \leq A\Big(\rho^2 \|Lu\|_{B_{R-\rho}} + \sum_{|\alpha|\leq 1} \rho^{|\alpha|} \|\partial^\alpha u\|_{B_{R-|\alpha|\rho}}\Big)$$

Proof: Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $(-\infty, 0)$ and $\chi \equiv 0$ on $[1, +\infty)$

Define for $0 < \rho < R$, $\quad \chi_{R,\rho} : \quad \boldsymbol{x} \longmapsto \chi\left(\dfrac{|\boldsymbol{x}| - R + \rho}{\rho}\right)$

$\chi_{R,\rho} \equiv 1$ in $B_{R-\rho}$ and 0 outside B_R. Use elliptic estimates for $\chi_{R,\rho} u$

$$\|\chi_{R,\rho} u\|_{\mathrm{H}^2(B_R)} \leq C\left\{\|L(\chi_{R,\rho} u)\|_{\mathrm{L}^2(B_R)} + \|\chi_{R,\rho} u\|_{\mathrm{H}^1(B_R)}\right\}$$

with the control of derivatives

$$\forall \rho \in (0, R), \quad \forall \alpha,\ |\alpha| \leq 2, \quad |\partial^\alpha \chi_{R,\rho}| \leq C\, \rho^{-|\alpha|}$$

Interior estimates, nested balls (constant coeff case)

Proposition

Assume L elliptic with constant coefficients.
$\exists A \geq 1$ such that $\forall R \in (0, R_0]$, $\forall \rho \in (0, \frac{R}{k}]$ and $\forall k \geq 2$ there holds

$$\sum_{|\alpha|\leq k} \rho^{|\alpha|} \|\partial^\alpha u\|_{B_{R-|\alpha|\rho}} \leq A^k \Big\{ \sum_{|\beta|\leq k-2} A^{-|\beta|} \rho^{2+|\beta|} \|\partial^\beta Lu\|_{B_{R-\rho-|\beta|\rho}} + \sum_{|\alpha|\leq 1} \rho^{|\alpha|} \|\partial^\alpha u\|_{B_{R-|\alpha|\rho}} \Big\}$$

Proof: Recurrence. Use the Lemma for $\partial_{\boldsymbol{x}}^\beta u$ and that $L\,\partial_{\boldsymbol{x}}^\beta = \partial_{\boldsymbol{x}}^\beta L$

Proof of Main Proposition when $\Gamma = \varnothing$***:***
Estimates with factors $1/k!$ are obtained with the choice $\rho = \dfrac{R}{k}$

Proof of Main Proposition when $\Gamma \neq \varnothing$***:***
Combine with anisotropic estimates along the boundary
(tangential derivatives as above, then normal derivatives using the operator L)

Outline

Homogeneous Weighted Analytic classes

Recall

- Weight := powers of $r(\boldsymbol{x}) = \min_{\boldsymbol{c}\in\mathscr{C}} |\boldsymbol{x} - \boldsymbol{c}|$
- Weight exponent := $\beta \in \mathbb{R}$
- **Homogeneous weighted Sobolev spaces**

$$\mathrm{K}^m_\beta(\Omega) = \{v \in \mathscr{D}'(\Omega) : \ r(\boldsymbol{x})^{|\boldsymbol{\alpha}|+\beta}\partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \in \mathrm{L}^2(\Omega), \ |\boldsymbol{\alpha}| \leq m\}$$

Introduce

- **Analytic limit**

$$\mathrm{A}_\beta(\Omega) = \Big\{v \in \bigcap_{m\in\mathbb{N}} \mathrm{K}^m_\beta(\Omega) : \ \sum_{|\boldsymbol{\alpha}|=m} \|r(\boldsymbol{x})^{m+\beta}\partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v\|_{\mathrm{L}^2(\Omega)} \leq C^{m+1} m!\Big\}$$

Remark

Let Ω be a polygon. If $S = |\boldsymbol{x} - \boldsymbol{c}|^\lambda \varphi(\theta_{\boldsymbol{c}})$ is a singular function, then the angular function φ is *analytic*. Hence

$$\beta + \mathrm{Re}\,\lambda > -1 \quad \Longrightarrow \quad S \in \mathrm{K}^0_\beta(\Omega) \quad \Longrightarrow \quad S \in \mathrm{A}_\beta(\Omega)$$

Weighted analytic estimates – natural regularity shift

Theorem: [Costabel-Dauge-Nicaise 2010]

If

- Ω is an analytic corner domain (e.g., a polygon),
- L and B have analytic coefficients (e.g., constant coefficients),
- u solution of (BVP)

there exists a constant $C \geq 1$ indep. of u such that for all $k \in \mathbb{N}$,

$$\frac{1}{k!} \sum_{|\boldsymbol{\alpha}|=k} \| r^{\beta+|\boldsymbol{\alpha}|} \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u \|_{\Omega} \leq C^{k+1} \Big\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\boldsymbol{\alpha}|=\ell} \| r^{\beta+2+|\boldsymbol{\alpha}|} \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f \|_{\Omega} + \sum_{|\boldsymbol{\alpha}|\leq 1} \| r^{\beta+|\boldsymbol{\alpha}|} \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u \|_{\Omega} \Big\}$$

Corollary: Natural analytic regularity shift

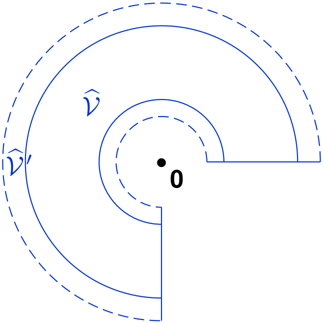
$$u \in \mathsf{K}^1_\beta(\Omega) \ \text{ and } \ f \in \mathsf{A}_{\beta+2}(\Omega) \quad \Longrightarrow \quad u \in \mathsf{A}_\beta(\Omega)$$

Outline

Proof of weighted analytic estimates

1. For simplicity:
   Ω polygon and L, B homogeneous with constant coeff.
2. Localization near a corner $\boldsymbol{c}$. Set $\boldsymbol{c} = \mathbf{0}$. We have $r = r(\boldsymbol{x}) = |\boldsymbol{x}|$
   Proof on a plane sector $\mathcal{K}$.
3. Regular reference configuration

$$\widehat{\mathcal{V}} = \{\boldsymbol{x} \in \mathcal{K},\ \tfrac{1}{2}-\varepsilon < r < 1\} \quad \& \quad \widehat{\mathcal{V}}' = \{\boldsymbol{x} \in \mathcal{K},\ \tfrac{1}{2}-2\varepsilon < r < 1+\varepsilon\}.$$



Proof of weighted analytic estimates

4. Unweighted reference estimate

$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k}\|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}} \le A_0^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}\|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{f}\|_{\widehat{\mathcal{V}}'} + \sum_{|\boldsymbol{\alpha}|\le 1}\|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}'}\right\}$$

5. Insert the weight ($\widehat{r} \simeq 1$ on $\mathcal{V}'$) $\Rightarrow$ weighted reference estimate

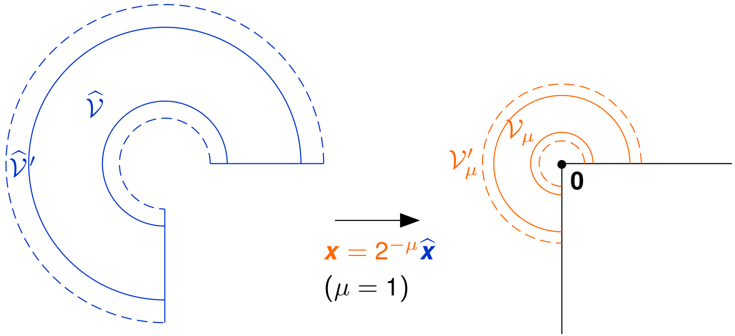
$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k}\|\widehat{r}^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}} \le A^{k+1}\left\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}\|\widehat{r}^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{f}\|_{\widehat{\mathcal{V}}'} + \sum_{|\boldsymbol{\alpha}|\le 1}\|\widehat{r}^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}'}\right\}$$

6. Locally finite covering $\mathcal{V}_\mu = 2^{-\mu}\widehat{\mathcal{V}}$ and $\mathcal{V}'_\mu = 2^{-\mu}\widehat{\mathcal{V}}'$, for $\mu = 1, 2, \dots$

$$\mathcal{V} := \mathcal{K}\cap\mathcal{B}(\mathbf{0},1) = \bigcup_{\mu\in\mathbb{N}}\mathcal{V}_\mu \quad\text{and}\quad \mathcal{V}' := \mathcal{K}\cap\mathcal{B}(\mathbf{0},1+\varepsilon) = \bigcup_{\mu\in\mathbb{N}}\mathcal{V}'_\mu\,.$$

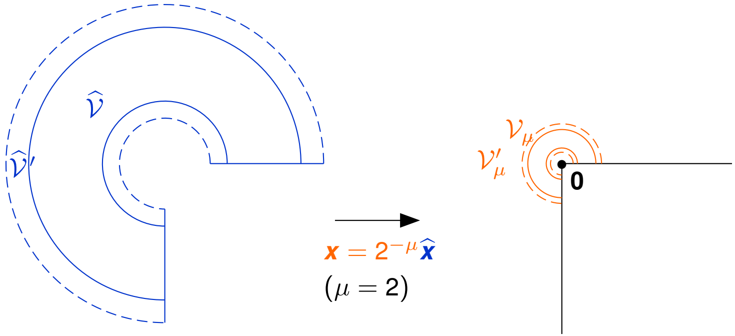
Proof of weighted analytic estimates

7. Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 1, \ldots$



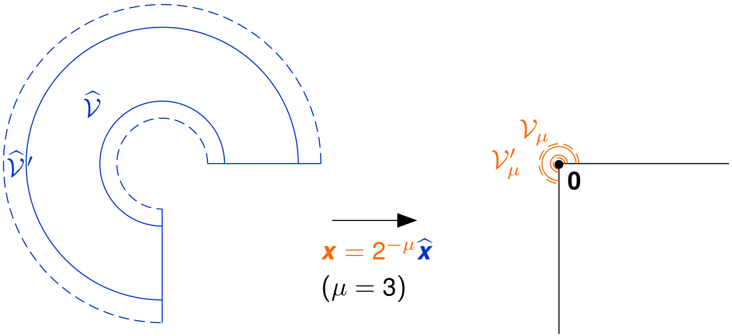
Proof of weighted analytic estimates

7. Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 2, \ldots$



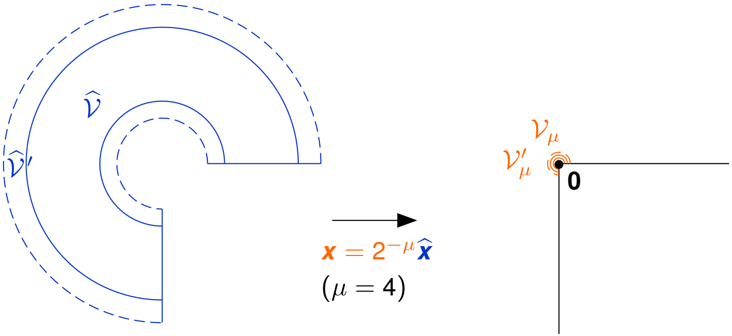
Proof of weighted analytic estimates

7. Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 3, \ldots$



Proof of weighted analytic estimates

7. Scale on $\mathcal{V}_\mu = 2^{-\mu}\mathcal{V}$ and $\mathcal{V}'_\mu = 2^{-\mu}\mathcal{V}'$, for $\mu = 4, \ldots$



Proof of weighted analytic estimates

8. To estimate u on $\mathcal{V}_\mu$ by $Lu = f$ on $\mathcal{V}'_\mu$ we set

$$\widehat{u}(\widehat{\boldsymbol{x}}) := u(\boldsymbol{x}) \quad \text{and} \quad \widehat{f}(\widehat{\boldsymbol{x}}) := L\widehat{u} \quad \text{which implies} \quad \widehat{f}(\widehat{\boldsymbol{x}}) = 2^{-2\mu} f(\boldsymbol{x}),$$

The reference estimate

$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k} \|\widehat{r}^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}} \le A^{k+1}\Big\{ \sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}\|\widehat{r}^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{f}\|_{\widehat{\mathcal{V}}'} + \sum_{|\boldsymbol{\alpha}|\le 1}\|\widehat{r}^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\widehat{u}\|_{\widehat{\mathcal{V}}'}\Big\}$$

becomes

$$\frac{1}{k!}\sum_{|\boldsymbol{\alpha}|=k} 2^{\mu\beta}\|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}u\|_{\mathcal{V}_\mu} \le A^{k+1}\Big\{ \sum_{\ell=0}^{k-2}\frac{1}{\ell!}\sum_{|\boldsymbol{\alpha}|=\ell}2^{\mu(\beta+2)}\|r^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}2^{-2\mu}f\|_{\mathcal{V}'_\mu} + \sum_{|\boldsymbol{\alpha}|\le 1}2^{\mu\beta}\|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}u\|_{\mathcal{V}'_\mu}\Big\}$$

Proof of weighted analytic estimates

9. Eliminate the common factor $2^{\mu\beta}$ and square:

$$\left(\frac{1}{k!}\right)^2 \sum_{|\boldsymbol{\alpha}|=k} \|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|^2_{\mathcal{V}_\mu} \le A_*^{2k+2}\Big\{ \sum_{\ell=0}^{k-2}\left(\frac{1}{\ell!}\right)^2 \sum_{|\boldsymbol{\alpha}|=\ell} \|r^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|^2_{\mathcal{V}'_\mu} + \sum_{|\boldsymbol{\alpha}|\le 1} \|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|^2_{\mathcal{V}'_\mu}\Big\}$$

10. Sum $\mu \in \mathbb{N}$ and use the finite covering property

$$\left(\frac{1}{k!}\right)^2 \sum_{|\boldsymbol{\alpha}|=k} \|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|^2_{\mathcal{V}} \le C A_*^{2k+2}\Big\{ \sum_{\ell=0}^{k-2}\left(\frac{1}{\ell!}\right)^2 \sum_{|\boldsymbol{\alpha}|=\ell} \|r^{\beta+2+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|^2_{\mathcal{V}'} + \sum_{|\boldsymbol{\alpha}|\le 1} \|r^{\beta+|\boldsymbol{\alpha}|}\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|^2_{\mathcal{V}'}\Big\}$$

11. QED

Outline

Weighted analytic regularity in polygons (Dirichlet)

- **Assume**: (BVP) has a coercive variational formulation and recall

Theorem: [Ko 1967]

$\exists\, b_{\Omega,L} > 0$ such that $\forall b$, $\boxed{0 \le b < b_{\Omega,L}}$ and $\forall m \ge 1$

$$u \in \mathrm{H}^1_0(\Omega) \quad \text{and} \quad f \in \mathrm{K}^{m-1}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{K}^{m+1}_{-b-1}(\Omega)$$

Fix $m = 1$ and combine with “Natural Analytic Regularity Shift” to obtain

Corollary: [Co-Da-Ni 2010]

$\forall b$, $\boxed{0 \le b < b_{\Omega,L}}$

$$u \in \mathrm{H}^1_0(\Omega) \quad \text{and} \quad f \in \mathrm{A}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{A}_{-b-1}(\Omega)$$

Note:
“NARS” does not require any singularity analysis, nor Mellin symbolic calculus.

Non-Homogeneous Weighted Analytic classes

Recall

- Weight := powers of $r(\boldsymbol{x}) = \min_{\boldsymbol{c}\in\mathscr{C}} |\boldsymbol{x}-\boldsymbol{c}|$
- Weight exponent := $\beta \in \mathbb{R}$
- **Non-homogeneous weighted Sobolev spaces**

$$\mathrm{J}^m_\beta(\Omega) = \{v \in \mathscr{D}'(\Omega) : \quad r(\boldsymbol{x})^{m+\beta}\partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v \in \mathrm{L}^2(\Omega),\ |\boldsymbol{\alpha}| \le m\}$$

Property

- **Embeddings**

$$m > -\beta - \frac{n}{2} \quad \Longrightarrow \quad \mathrm{J}^{m+1}_\beta(\Omega) \subset \mathrm{J}^m_\beta(\Omega)$$

Introduce

- **Analytic limit**

$$\mathrm{B}_\beta(\Omega) = \Big\{ v \in \bigcap_{m > -\beta - \frac{n}{2}} \mathrm{J}^m_\beta(\Omega) : \sum_{|\boldsymbol{\alpha}|=m} \|r(\boldsymbol{x})^{m+\beta}\partial^{\boldsymbol{\alpha}}_{\boldsymbol{x}} v\|_{\mathrm{L}^2(\Omega)} \le C^{m+1} m! \Big\}$$

Weighted analytic regularity in polygons (Neumann)

Recall

Theorem: [Ma-Pl 1984]

$\exists\, b_{\Omega,L,B} > 0$ such that $\forall b$, $\boxed{0 < b < b_{\Omega,L,B}}$ $\forall m \geq 1$,

$$u \in \mathrm{V} \quad \text{and} \quad f \in \mathrm{J}^{m-1}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{J}^{m+1}_{-b-1}(\Omega)$$

The “NARS” for J-spaces also holds / *see later*

Corollary: Natural analytic regularity shift

Let $\beta \in (-2,-1)$ (thus $m = 1 > -\beta - \frac{n}{2} = -\beta - 1$). Then

$$u \in \mathrm{J}^1_\beta(\Omega) \text{ and } f \in \mathrm{B}_{\beta+2}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{B}_\beta(\Omega)$$

Theorem: [Co-Da-Ni 2010] *Cf.* [Babuška-Guo 1988, 1989, 1993]

$\forall b$, $\boxed{0 < b < b_{\Omega,L,B}}$

$$u \in \mathrm{V} \quad \text{and} \quad f \in \mathrm{B}_{-b+1}(\Omega) \quad \Longrightarrow \quad u \in \mathrm{B}_{-b-1}(\Omega)$$

The trick for the proof of the ‘NARS” in J-spaces...

Replace the estimate in the smooth case

u satisfies the a priori estimates of analytic type, $k = 0, 1, 2, \ldots$

$$\frac{1}{k!} \sum_{|\boldsymbol{\alpha}|=k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\Omega} \le A^{k+1} \Big\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\boldsymbol{\alpha}|=\ell} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{\Omega} + \sum_{|\boldsymbol{\alpha}|\le 1} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\Omega} \Big\}$$

with a constant *A* independent of *k* and *u*.

...

... by

u satisfies the a priori estimates of analytic type, $k = 1, 2, \ldots$

$$\frac{1}{k!} \sum_{|\boldsymbol{\alpha}|=k} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\Omega} \leq A^{k+1} \Big\{ \sum_{\ell=0}^{k-2} \frac{1}{\ell!} \sum_{|\boldsymbol{\alpha}|=\ell} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f\|_{\Omega} + \sum_{|\boldsymbol{\alpha}|=1} \|\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} u\|_{\Omega} \Big\}$$

with a constant A independent of k and u.

Mathematical outcome

1. **The proof is much simpler** than in original papers by BABUŠKA-GUO because it clearly separates
   - *the issue of basic regularity* (e.g. in $K^2_\beta(\Omega)$ or $J^2_\beta(\Omega)$)
   - *the issue of analytic regularity* (natural regularity shift)

   These two independent modules can be assembled.
2. **The proof can be adapted** without much effort to
   - *homogeneous multi-degree elliptic systems* with constant coeff.
     e.g. Stokes,
   - *transmission problems*
     e.g. div $a(\boldsymbol{x})\nabla$, with $\boldsymbol{x} \mapsto a(\boldsymbol{x})$ piecewise constant on a polygonal decomposition of Ω
3. **The generalization** to non-zero boundary conditions, variable (analytic) coefficients, non-homogeneous operators is feasible with the same arguments.

Conclusion

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In preparation.

Thank you for your attention!