Interactions between moderately close inclusions for the Laplace equation and applications in mechanics

V. Bonnaillie-Noël
with D. Brancherie, M. Dambrine, F. Hérau, S. Tordeux and G. Vial

IRMAR, CNRS, ENS Cachan Bretagne and Univ. Rennes 1

http://w3.bretagne.ens-cachan.fr/math/macadam/

6th Singular Days on Asymptotic Methods for PDEs

WIAS, Berlin, April 30, 2010

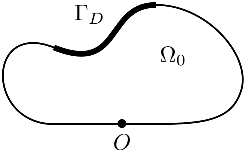
Motivations

Multiscale Asymptotics and Computational Approximation for surface Defects and Applications in Mechanics

- take into account the surface and volume microdefects of materials
  $\rightsquigarrow$ multiscale asymptotic analysis
- give a numerical method with a reasonable computation cost
  $\rightsquigarrow$ superposition method
- application in mechanics
  $\rightsquigarrow$ crack initiation and propagation

Non perturbed problem

Assumption



- $O \in \partial\Omega_0 \setminus \Gamma_D$
- $\partial\Omega_0$ flat around O
- $f \in C_0^\infty(\Omega_0)$

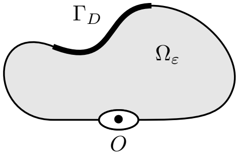
Solution of the unperturbed problem

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega_0 \\ u_0 = 0 & \text{on } \Gamma_D \\ \partial_{\mathbf{n}} u_0 = 0 & \text{on } \partial\Omega_0 \setminus \Gamma_D \end{cases}$$

A single defect

[Mazja, Nazarov, Plamenevskii], *[Dambrine, Vial]*

Assumption



- $\Omega_\varepsilon = \Omega_0 \setminus \varepsilon\omega$
  ω start-shaped with respect to O
- $f \in C_0^\infty(\Omega_0)$

Solution of the perturbed problem

$$\begin{cases} -\Delta u_\varepsilon &= f & \text{in } \Omega_\varepsilon \\ u_\varepsilon &= 0 & \text{on } \Gamma_D \\ \partial_{\mathbf{n}} u_\varepsilon &= 0 & \text{on } \partial\Omega_\varepsilon \setminus \Gamma_D \end{cases}$$

Compare u_ε and u_0

Main ideas of the asymptotic analysis

- use multiscale expansion
  - *slow variable* x with the scale of the domain
  - *fast variable* x/ε with the scale of the perturbation

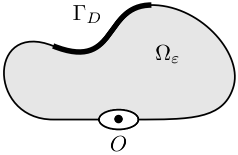
- compare u_ε to the limit u_0
  $\Rightarrow$ correctors to *compensate* the Taylor expansion of u_0 at 0

- make the correctors stick on Ω_ε
  (through a cutoff function for boundary inclusions)
  $\Rightarrow$ generate correctors in slow variables

One inclusion

First term

$r_\varepsilon^0 = u_\varepsilon - u_0$ satisfies



$$\begin{cases} -\Delta r_\varepsilon^0 = 0 & \text{in } \Omega_\varepsilon \\ r_\varepsilon^0 = 0 & \text{on } \Gamma_D \\ \partial_{\mathbf{n}} r_\varepsilon^0 = 0 & \text{on } \partial\Omega_\varepsilon \setminus (\varepsilon\partial\omega \cup \Gamma_D) \\ \partial_{\mathbf{n}} r_\varepsilon^0 = -\partial_{\mathbf{n}} u_0 & \text{on } \varepsilon\partial\omega \end{cases}$$

For $x = \varepsilon X \in \varepsilon\partial\omega$, then

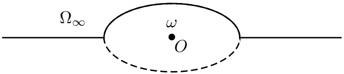
$$\partial_{\mathbf{n}} u_0(x) = \nabla u_0(0) \cdot \mathbf{n} + \mathcal{O}(\varepsilon)$$

We lift this term to obtain a better approximation of u_ε

One inclusion

Profile

Solution of a problem in an infinite domain $\Omega_\infty = \lim_{\varepsilon \to 0} \Omega_\varepsilon / \varepsilon$



$$\begin{cases} -\Delta V^1 & = & 0 & \text{in } \Omega_\infty \\ \partial_n V^1 & = & g_1 & \text{on } \partial\Omega_\infty \\ V^1 & \to & 0 & \text{at infinity} \end{cases}$$

$$g_1 = -\nabla u_0(0) \cdot \mathbf{n}$$

Theorem

There exists a unique weak solution V^1 in the variational space

$$\left\{ V \, ; \, \nabla V \in \mathrm{L}^2(\Omega_\infty) \quad \text{and} \quad \frac{V}{(1 + |X|) \log(2 + |X|)} \in \mathrm{L}^2(\Omega_\infty) \right\}$$

Behavior at infinity

$$V^1(X) = \mathcal{O}(|X|^{-1}) \text{ and } \nabla V^1(X) = \mathcal{O}(|X|^{-2}) \quad \text{as } |X| \to \infty$$

One inclusion

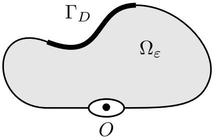
Second term (1)

We write

$$u_\varepsilon(x) = u_0(x) + \varepsilon\chi(x) V^1\left(\frac{x}{\varepsilon}\right) + r_\varepsilon^1(x)$$

with χ regular, radial, with support in $\mathcal{B}(0, r^*)$ and $\chi = 1$ in $\mathcal{B}(0, \frac{r^*}{2})$

The remainder r_ε^1 satisfies



$$\left\{\begin{array}{rcll} -\Delta r_\varepsilon^1 & = & \Delta[\chi(\cdot)\varepsilon V^1(\frac{\cdot}{\varepsilon})] & \text{in } \Omega_\varepsilon \\ r_\varepsilon^1 & = & 0 & \text{on } \Gamma_D \\ \partial_{\mathbf{n}} r_\varepsilon^1 & = & 0 & \text{on } \partial\Omega_\varepsilon \setminus (\varepsilon\omega \cup \Gamma_D) \\ \partial_{\mathbf{n}} r_\varepsilon^1 & = & \psi_\varepsilon^0 & \text{on } \varepsilon\partial\omega \end{array}\right.$$

$\psi_\varepsilon^0 = -\partial_{\mathbf{n}}(u_0 + \varepsilon\chi V^1(\frac{\cdot}{\varepsilon}))$ comes from the Taylor expansion of u_0 at 0

2 terms have now to be lifted

One inclusion

Second term (2)

Let w^1 be such that

$$\begin{cases} -\Delta w^1 = \varphi_1 & \text{in } \Omega_0 \\ w^1 = 0 & \text{on } \Gamma_D \\ \partial_{\mathbf{n}} w^1 = 0 & \text{on } \partial\Omega_0 \setminus \Gamma_D \end{cases}$$

with φ_1 deducing from the behavior of V^1

Let V^2 be such that

$$\begin{cases} -\Delta V^2 = 0 & \text{in } \Omega_\infty \\ \partial_n V^2 = g_1 & \text{on } \partial\Omega_\infty \\ V^2 \to 0 & \text{at infinity} \end{cases}$$

with g_1 coming from the Taylor expansion of u_0 and the trace of w^1

One inclusion

A model result: expansion at order N

For all N, the solution u_ε of

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \Gamma_D \\ \partial_{\mathbf{n}} u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \backslash \Gamma_D \end{cases}$$

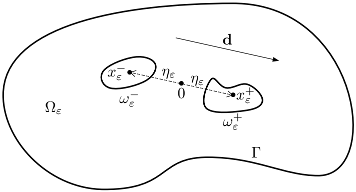
writes

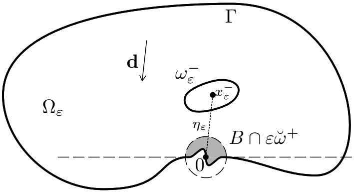
$$u_\varepsilon(x) = u_0(x) + \chi(x) \sum_{i=1}^{N} \varepsilon^i V^i(\tfrac{x}{\varepsilon}) + \sum_{i=2}^{N} \varepsilon^i w^i(x) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^{N+1})$$

- the profiles V^i compensate the ith term u^i of Taylor expansion of u_0 and of w^j for $j < i$
- the correctors w^i compensate the error generated by the cutoff $\|w^i\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}(1)$

Two inclusions

[Bonnaillie-Noël, Dambrine, Tordeux, Vial, 2009]





$$\begin{cases} -\Delta u_\varepsilon = f & \text{on } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{in } \Gamma \\ \partial_{\mathbf{n}} u_\varepsilon = 0 & \text{in } \partial\Omega_\varepsilon^\pm \end{cases}$$

Two inclusions

Several situations

- $\eta_\varepsilon = \mathcal{O}(1)$: no interaction

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^+\left(\frac{x-x^+}{\varepsilon}\right) + V_0^-\left(\frac{x-x^-}{\varepsilon}\right)\right] + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^2)$$

- $\eta_\varepsilon = \mathcal{O}(\varepsilon)$: total interaction

$$u_\varepsilon(x) = u_0(x) + \varepsilon W_0\left(\frac{x}{\varepsilon}\right) + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}(\varepsilon^2)$$

W_0: profile associated with $\omega = \omega^+ \cup \omega^-$

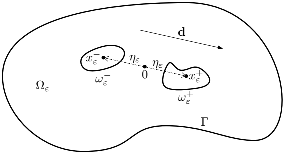
- $\eta_\varepsilon = \mathcal{O}(\varepsilon^\alpha), 0 < \alpha < 1$: medium case

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] + \mathcal{O}(???)$$

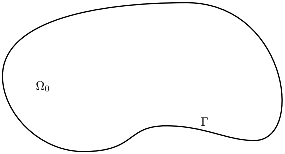
Two inclusions

Asymptotic expansion for two interior inclusions

$$\Omega_\varepsilon = \Omega_0 \backslash \overline{\left(\omega_\varepsilon^- \cup \omega_\varepsilon^+\right)}, \text{ with } \omega_\varepsilon^\pm = x_\varepsilon^\pm + \varepsilon\omega^\pm, \quad x_\varepsilon^\pm = \pm\varepsilon^\alpha \mathbf{d}$$



$$\begin{cases} -\Delta u_\varepsilon &= f \quad \text{in } \Omega_\varepsilon \\ u_\varepsilon &= 0 \quad \text{on } \Gamma = \partial\Omega_0 \\ \partial_{\mathbf{n}} u_\varepsilon &= 0 \quad \text{on } \partial\omega_\varepsilon^\pm \end{cases}$$



$$\begin{cases} -\Delta u_0 &= f \quad \text{in } \Omega_0 \\ u_0 &= 0 \quad \text{on } \Gamma = \partial\Omega_0 \end{cases}$$

Two interior inclusions

First remainder $r_\varepsilon^0 = u_\varepsilon - u_0$

$$\begin{cases} -\Delta r_\varepsilon^0 & = & 0 & \text{in } \Omega_\varepsilon \\ r_\varepsilon^0 & = & 0 & \text{on } \partial\Omega_0 \\ \partial_{\mathbf{n}} r_\varepsilon^0 & = & -\partial_{\mathbf{n}} u_0 & \text{on } \partial\omega_\varepsilon^+ \cup \partial\omega_\varepsilon^- \end{cases}$$

Profiles

$$\begin{cases} -\Delta V_0^\pm & = & 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega^\pm} \\ \partial_{\mathbf{n}} V_0^\pm & = & -\mathbf{n} \cdot \nabla u_0(0) & \text{on } \partial\omega^\pm \\ V_0^\pm & \to & 0 & \text{at infinity} \end{cases}$$

$$V_0^\pm(X) = \sum_{k=1}^{N-1} V_{0,k}^\pm(X) + \mathcal{O}_\infty(|X|^{-N}) \quad \text{with } V_{0,k}^\pm \in \mathcal{O}_\infty(|X|^{-k})$$

Two interior inclusions

Second remainder r_ε^1

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] + r_\varepsilon^1(x)$$

$$\begin{cases} -\Delta r_\varepsilon^1 &= 0 & \text{in } \Omega_\varepsilon \\ r_\varepsilon^1(x) &= -\varepsilon\left[V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] & \text{on } \partial\Omega_0 \\ \partial_{\mathbf{n}} r_\varepsilon^1(x) &= \mathbf{n}\cdot\nabla u_0(0) - \mathbf{n}\cdot\nabla u_0(x) - \mathbf{n}\cdot\nabla V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) & \text{on } \partial\omega_\varepsilon^+ \\ \partial_{\mathbf{n}} r_\varepsilon^1(x) &= \mathbf{n}\cdot\nabla u_0(0) - \mathbf{n}\cdot\nabla u_0(x) - \mathbf{n}\cdot\nabla V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right) & \text{on } \partial\omega_\varepsilon^- \end{cases}$$

$$x \in \partial\Omega_0, \qquad r_\varepsilon^1(x) = \sum_{\substack{j\geq 1, k\geq 0,\\ j+\alpha k\leq N}} \varepsilon^{j+\alpha k} f_{j,k}(x) + o(\varepsilon^N)$$

$$x = \pm\varepsilon^\alpha \mathbf{d} + \varepsilon X \in \partial\omega_\varepsilon^\pm,$$

$$\begin{aligned} \partial_{\mathbf{n}} r_\varepsilon^1(x) &= \mathbf{n}\cdot\left(\nabla u_0(0) - \nabla u_0(\pm\varepsilon^\alpha\mathbf{d} + \varepsilon X) - \nabla V_0^\mp(\pm 2\varepsilon^{\alpha-1}\mathbf{d} + X)\right) \\ &= \sum_{\substack{j\geq 0, k\geq 0,\\ 0<j+\alpha k\leq N}} \varepsilon^{j+\alpha k} g_{j,k}^\pm(X) + \sum_{2\leq j\leq \frac{N}{1-\alpha}} \varepsilon^{j(1-\alpha)} h_j^\mp(X) + o(\varepsilon^N) \end{aligned}$$

Two interior inclusions

Two scale expansion

Theorem

The solution u_ε admits the expansion at order N

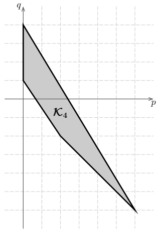
$$u_\varepsilon(x) = u_0(x) + \varepsilon \left[V_0^- \left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+ \left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right) \right]$$

$$+ \sum_{(p,q)\in\mathcal{K}_N} \varepsilon^{p+\alpha q} \left(v_{p+\alpha q}(x) + \varepsilon \left[V_{p+\alpha q}^- \left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + V_{p+\alpha q}^+ \left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right) \right] \right) + r_\varepsilon^N(x)$$

with

$$\mathcal{K}_N = \{(p,q) \in \mathbb{Z}^2 \mid p \geq 0,$$
$$q \geq -\tfrac{3}{2}p + 1,\ q \geq -p \text{ and } p + \alpha q \leq N\}$$

$$\|r_\varepsilon^N\|_{\mathrm{H}^1(\Omega_\varepsilon)} = o(\varepsilon^N)$$



$\mathcal{K}_4$ for $\alpha = \frac{3}{5}$

Two interior inclusions

Interpretation of the first terms

$$u_\varepsilon(x) = u_0(x) + \varepsilon \left[V_0^- \left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+ \left(\frac{x - x_\varepsilon^+}{\varepsilon}\right) \right] + r_\varepsilon^1(x)$$

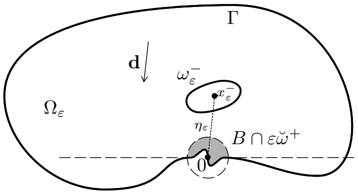
with

$$\|r_\varepsilon^1\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}\big(\varepsilon^{\min(1+\alpha, 3-2\alpha)}\big)$$

The first remainder r_ε^1 contains information about higher-order influence

- ◊ if $\alpha < 2/3$: inclusions relatively far-away from each other
  the leading term arises from the Taylor expansion of u_0 at O
- ◊ if $2/3 < \alpha < 1$: rather close inclusions
  the remainder mainly consists in the *interaction* between the profiles V_0^- and V_0^+
- ◊ if $\alpha = 2/3$:
  the two contributions are equally balanced

One interior inclusion and one at the boundary



Theorem

$$u_\varepsilon(x) = \zeta(|\tfrac{x}{\varepsilon}|)u_0(x) + \varepsilon\left[V_0^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_0^+\left(\tfrac{x}{\varepsilon}\right)\right]$$

$$+ \sum_{(p,q)\in\mathcal{K}_N} \varepsilon^{p+\alpha q}\left(\zeta(|\tfrac{x}{\varepsilon}|)v_{p+\alpha q}(x) + \varepsilon\left[V_{p+\alpha q}^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_{p+\alpha q}^+\left(\tfrac{x}{\varepsilon}\right)\right]\right)$$

$$+ r_\varepsilon^N(x)$$

Applications for numerical computations

One inclusion

Idea

Approximate u_ε by the first order expansion

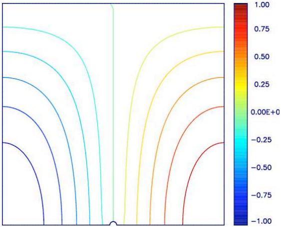
$$u_\varepsilon(x) \approx u_1(x) = u_0(x) + \varepsilon\chi(x)V(\tfrac{x}{\varepsilon})$$

Compute

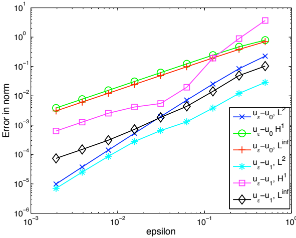
- u_0 : the solution of the problem set in the unperturbed domain
- V : the profile defined in the unbounded domain

Numerical computation - one inclusion

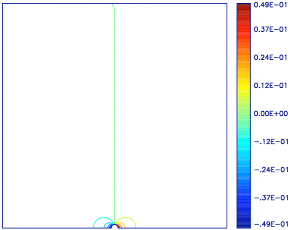
Computations for the Neumann problem



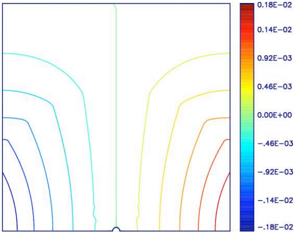
$u_\varepsilon \ (\varepsilon = \frac{1}{32})$



Errors norms



$u_\varepsilon - u_0 \ (\varepsilon = \frac{1}{32})$



$u_\varepsilon - u_1 \ (\varepsilon = \frac{1}{32})$

Numerical computation - one inclusion

Computation of the profile

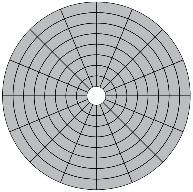
$$\begin{cases} -\Delta V = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega} \\ \partial_{\mathbf{n}} V = g & \text{on } \partial\omega \\ V \to 0 & \text{at infinity} \end{cases}$$

- Strategy 1 : absorbing conditions on $|x| = R$

Dirichlet: $V = 0$

Robin: $V + R\partial_{\mathbf{n}} V = 0$

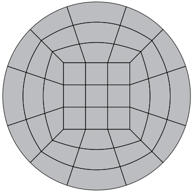
Ventcel: $V + \frac{3R}{2}\partial_{\mathbf{n}} V - \frac{R^2}{2}\partial_\tau^2 V = 0$



- Strategy 2 : inversion $\varphi : z \mapsto 1/z$

$W = V \circ \varphi$ solution of

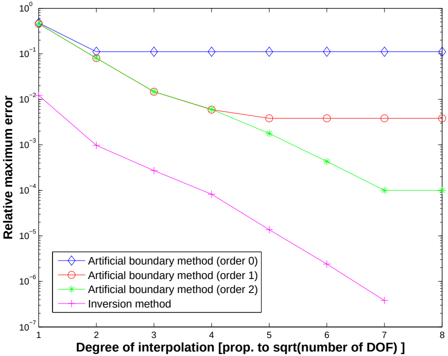
$$\begin{cases} -\Delta W = 0 & \text{in } \varphi(\omega) \\ \partial_{\mathbf{n}} W = \partial_{\mathbf{s}}\varphi\,(g \circ \varphi) & \text{on } \partial\varphi(\omega) \\ W(0) = 0 & \end{cases}$$



Numerical computation - one inclusion

Comparison of strategies with a fixed number of d.o.f.

- ω disk
- $R = 10$
- known profile: $V(x) = \frac{\cos\theta}{r}$
  inversion : $W(x) = x_1$
  boundary condition : $g(x) = \cos\theta - 2\cos(2\theta) - 3\cos(3\theta)$

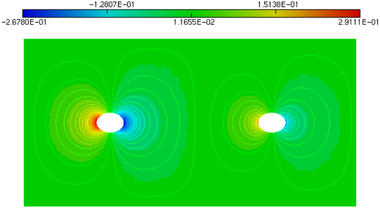


$\mathbb{Q}_8$ mesh and variable interpolation degree

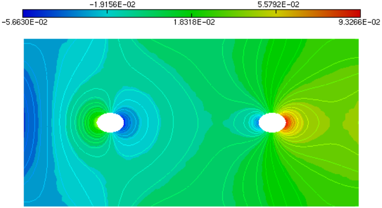
Numerical computation - two inclusions

Numerical simulations: $\alpha = 0.5$, $\varepsilon = 0.05$

$$u_1(x) = u_0(x) + \varepsilon \left[\nabla u_0(0) \cdot \mathbf{V}_{\omega^-}\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \nabla u_0(0) \cdot \mathbf{V}_{\omega^+}\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right) \right]$$



$u_\varepsilon - u_0$



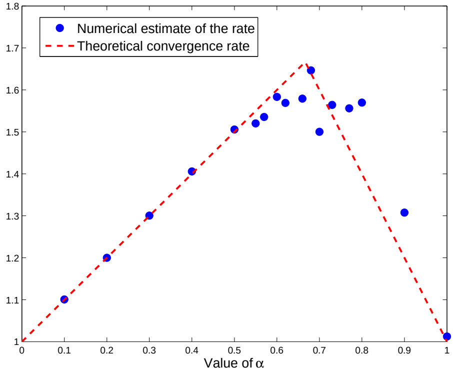
$u_\varepsilon - u_1$

Numerical computation - two inclusions

Error order with respect to α

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^-\left(\frac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x-x_\varepsilon^+}{\varepsilon}\right)\right] + \mathcal{O}_{\mathrm{H}^1(\Omega_\varepsilon)}\left(\varepsilon^{\min(1+\alpha,3-2\alpha)}\right)$$

Circular inclusions with analytic profiles



Applications in mechanics

Context

Description of the behavior till rupture of complex structure

- evaluation of ultimate load
- evaluation of dissipated energy

Objectives

Computation of limit load by taking into account

- the influence of geometrical surface defects
- the influence of localization zones

without fine geometrical description

2 tools:

- asymptotic analysis of the Navier equations (first step)
- continuum damage model and strong discontinuity approach (second and third phases)

Applications in mechanics

Asymptotic analysis *[Brancherie, Dambrine, Vial, Villon 2007]*, *[Bonnaillie-Noël, Brancherie, Dambrine, Tordeux, Vial 2010]*

Navier equations

$$\begin{cases} -\mu\Delta \mathbf{u}_\varepsilon - (\lambda+\mu)\,\nabla\,\operatorname{div}\mathbf{u}_\varepsilon = \mathbf{0} & \text{on } \Omega_\varepsilon \\ \mathbf{u}_\varepsilon = \mathbf{u}^d & \text{in } \Gamma_d \\ \sigma\cdot\mathbf{n} = \mathbf{g} & \text{in } \Gamma_t \end{cases}$$

Profiles basis $(\mathbf{v}_\ell)_{\ell=1,2}$

$$\begin{cases} -\mu\Delta \mathbf{v}_\ell - (\lambda+\mu)\nabla\operatorname{div}(\mathbf{v}_\ell) = \mathbf{0} & \text{on } \Omega_\infty \\ \displaystyle\sum_{j=1}^{2}\sigma_{ij}(\mathbf{v}_\ell)\mathbf{n}_j = \mathrm{G}_{\ell,i} & \text{in } \partial\Omega_\infty \\ \mathbf{v}_\ell \to 0 & \text{at infinity} \end{cases}$$

with $\mathbf{G}_1 = (\mathbf{n}_1, 0)$ and $\mathbf{G}_2 = (0, \mathbf{n}_1)$

$\mathbf{n}_1$: first component of the outer normal on $\partial\Omega_\infty$

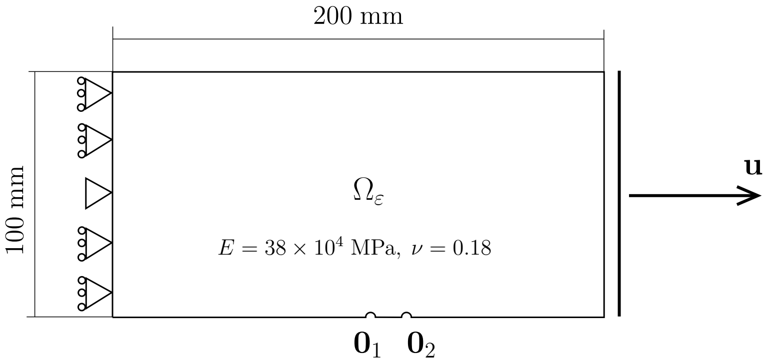
Asymptotic expansion

For 2 defects of size $\sim\varepsilon$, at distance $\sim\varepsilon^\alpha$ $(0<\alpha<1)$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) - \varepsilon\sum_{j=1}^{2}\left[\alpha_1^j\mathbf{v}_1^j\left(\frac{x-x_\varepsilon^j}{\varepsilon}\right) + \alpha_2^j\mathbf{v}_2^j\left(\frac{x-x_\varepsilon^j}{\varepsilon}\right)\right] + \mathcal{O}\left(\varepsilon^{\min(1+\alpha,3-2\alpha)}\right)$$

Applications in mechanics - simulations

Traction test

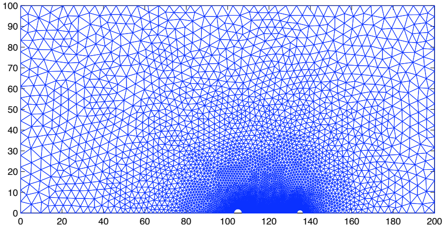


$\varepsilon_1 = 2\,\mathrm{mm}, \qquad \varepsilon_2 = 1\,\mathrm{mm}, \qquad d(O_1, O_2) = 30\,\mathrm{mm}$

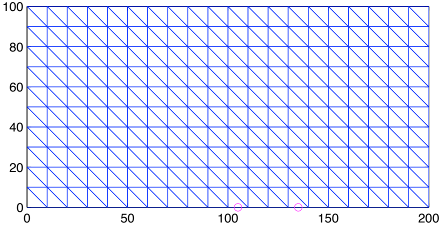
Applications in mechanics - simulations

Meshes

- Fine mesh *(reference computation)*

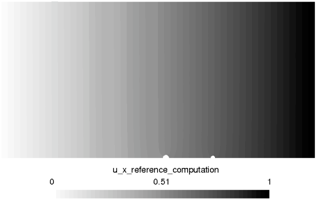


- Coarse mesh *(for the asymptotic expansion)*

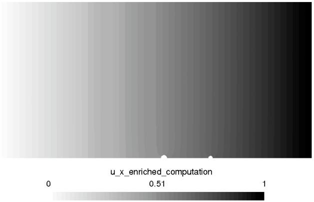


Applications in mechanics - simulations

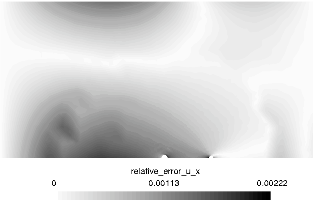
Displacement



Reference



Enriched computation



Relative error ($\sim 0.25\%$)

Applications in mechanics

Computation of the profile for planar linear elasticity

$$\begin{cases} -\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\,(\mathbf{u}) & = & 0 & \text{in } \mathbb{R}^2_+ \setminus \overline{\omega} \\ \sigma(\mathbf{u}).\mathbf{n} & = & g & \text{on } \partial\omega \\ \mathbf{u} & \to & 0 & \text{at infinity} \end{cases}$$

Compute the leading terms at infinity in the upper-half plane
Seek an algebraic expression cancelling then

$\Rightarrow$ absorbing boundary conditions on $|x| = R$

$$\sigma(\mathbf{u})\mathbf{n} + \frac{1}{R}\,\frac{E}{1+\nu}\begin{bmatrix} \frac{1}{1-\nu} & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u} + \frac{1}{R}\,\frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\Delta_\tau\mathbf{u} = 0$$

Difficulty: $\frac{E(1-\nu)}{2(1+\nu)(1-2\nu)} > 0$ since $E > 0$ and $\nu \in (-1, 0.5)$

$\Rightarrow$ Degenerate Ventcel boundary value conditions

Ventcel conditions

[Bonnaillie-Noël, Dambrine, Hérau, Vial 2010]

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \\ \partial_n u + \alpha u + \beta \Delta_\tau u & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

with Ω bounded regular domain in $\mathbb{R}^d$, $d \geq 2$ and $\alpha, \beta \in \mathbb{R}$

$\alpha > 0$ and $\beta < 0$: variational approach

Find $u \in \mathcal{H}(\Omega)$ such that $\forall v \in \mathcal{H}(\Omega),\ A(u,v) = B(v)$

with

$$B(v) = \int_\Omega fv \quad \text{and} \quad A(u,v) = \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} \alpha\, uv - \beta\, \nabla_\tau u . \nabla_\tau v$$

on the Hilbert space

$$\mathcal{H}(\Omega) = \left\{ u \in \mathrm{H}^1(\Omega),\ u_{|\partial\Omega} \in \mathrm{H}^1(\partial\Omega) \right\}$$

If $\alpha > 0$ and $\beta < 0$, the bilinear form A is coercive

Ventcel conditions with bad sign

$\alpha \in \mathbb{R}$ and $\beta > 0$: no variational apporach

1. Existence and uniqueness

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \partial_n u + \alpha u + \beta \Delta_\tau u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

   if $\alpha \notin \{\alpha_n\}$

2. Continuity with respect to the domain and convergence when $\omega \to \emptyset$

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \setminus \overline{\omega} \\ \partial_n u + \alpha u + \beta \Delta_\tau u &= 0 \quad \text{on } \partial\Omega \\ u &= g \quad \text{on } \partial\omega \end{cases}$$

Ventcel conditions

Numerical illustrations

$$\left\{\begin{array}{rcll} -\Delta u & = & 0 & \text{in } B(0,R)\setminus\overline{\omega} \\ R\partial_n u + \alpha u + \beta\Delta_\tau u & = & g & \text{on } \partial B(0,R) \\ u & = & 0 & \text{on } \partial\omega \end{array}\right.$$

and

$$Pu = -\beta\Delta_\tau u - \alpha u - R\Lambda u \quad \text{on} \quad \partial B(0,R)$$

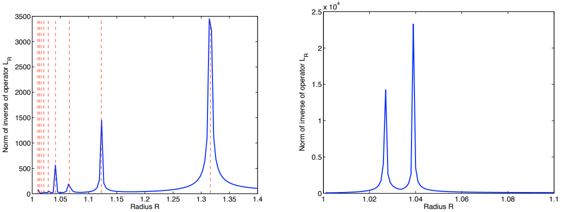
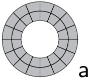
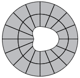


Figure: Norm of inverse with respect to R :  and 

Perspectives

- Asymptotic expansion for Dirichlet conditions
  term in $1/\ln\varepsilon$
- Asymptotic expansion when the distance between the inclusions is ε^α with $\alpha > 1$
- Computation of the profile for the linear elasticity
  Ventcel conditions