Finite element error estimates for boundary control problems

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Corner singularities in 2D

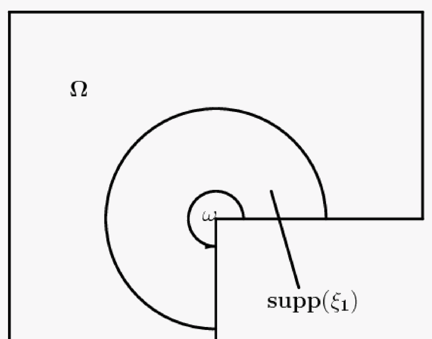
The solution y of

$$-\Delta y + y = f \text{ in } \Omega, \quad \partial_n y = g \text{ on } \Gamma$$

is not contained in $W^{2,2}(\Omega)$, if $\omega > \pi$.
Instead one has $y = y_r + y_s$ with $y_r \in W^{2,2}(\Omega)$ and

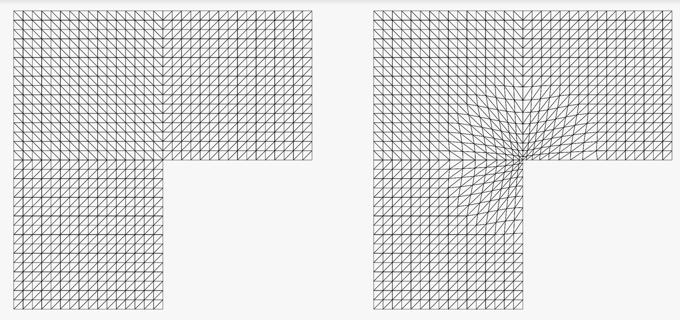
$$y_s = \xi_1(r)\gamma r^\lambda \cos(\lambda\phi) \quad \text{with } \lambda = \frac{\pi}{\omega}.$$

$\xi_1(r)$ is a smooth cut-off function and γ a coefficient.



Mesh grading and finite element error estimates

$$h_T \sim \begin{cases} h^{1/\mu} & \text{for } r_T = 0 \\ h r_T^{1-\mu} & \text{for } R \geq r_T > 0 \\ h & \text{for } r_T > R \end{cases}$$



The finite element error can be estimated by

$$\|y - y_h\|_{L^2(\Omega)} + h^{1/2}\|y - y_h\|_{L^2(\Gamma)} + h\|y - y_h\|_{H^1(\Omega)} \leq ch^2,$$

if $\mu < \lambda$ and using piecewise linear and continuous ansatz functions.

Optimal control problem

Neumann boundary control problem

$$\min F(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)},$$
$$-\Delta y + y = 0 \quad \text{in } \Omega, \quad \partial_n y = u \quad \text{on } \Gamma,$$
$$a \leq u(x) \leq b \quad \text{for a.a. } x \in \Gamma$$

- Ω is a nonconvex polygonal domain
- $u \in L^2(\Gamma)$ and $y_d \in L^2(\Omega)$

Reduced formulation

$$J(\bar u) = \min_{u \in U^{ad}} F(Su, u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}$$

- $S : L^2(\Gamma) \to L^2(\Omega)$ control-to-state mapping
- $U^{ad} := \{u \in L^2(\Gamma) : a \leq u(x) \leq b \text{ a.e.on } \Gamma\}$

First order optimality condition

$$(\bar p_{|\Gamma} + \nu \bar u, u - \bar u)_{L^2(\Gamma)} \geq 0 \quad \forall u \in U^{ad}$$
$$\Leftrightarrow \bar u = \Pi_{[a,b]}\left(-\bar p_{|\Gamma}/\nu\right)$$

with $\bar p = P(\bar y - y_d)$ and $\bar y = S\bar u$ defined by

$$-\Delta \bar p + \bar p = \bar y - y_d \quad \text{in } \Omega, \quad \partial_n \bar p = 0 \quad \text{on } \Gamma,$$
$$-\Delta \bar y + \bar y = 0 \quad \text{in } \Omega, \quad \partial_n \bar y = \bar u \quad \text{on } \Gamma.$$

Discretization of the optimal control problem

Fully discrete approach

Find $\bar u_h \in U_h^{ad}$ such that

$$J_h(\bar u_h) = \min_{u_h \in U_h^{ad}} J_h(u_h) := \frac{1}{2}\|S_h u_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Gamma)}$$

with

$$V_h = \{y_h \in C(\bar\Omega) : y_h|_T \in \mathcal{P}_1 \; \forall T \in \mathcal{T}_h\},$$
$$U_h = \{u_h \in L^\infty(\Gamma) : u_h|_G \in \mathcal{P}_0 \; \forall G \in \mathcal{G}_h\},$$
$$U_h^{ad} = U_h \cap U^{ad}.$$

Discrete optimality system for $\bar y_h = S_h \bar u_h$, $\bar p_h = P_h(\bar y_h - y_d)$ and $\bar u_h$:

$$a(\bar y_h, v_h) = (\bar u_h, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h$$
$$a(\bar p_h, v_h) = (\bar y_h - y_d, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h$$
$$(\bar p_{h|\Gamma} + \nu \bar u_h, u_h - \bar u_h)_{L^2(\Gamma)} \geq 0 \quad \forall u \in U_h^{ad}$$

Projection Operator R_h

The operator R_h projects continuous functions in the space of piecewise constant functions,

$$(R_h f)(x) := f(S_G) \quad \text{if } x \in G$$

where S_G denotes the midpoint of the element G.

Error estimates for the fully discrete approach

Assume that $\text{meas}(\bigcup_{G \in \mathcal{G}_h : \bar u \notin W^{3/2,2}_{3(1-\mu)/2}(G)} G) < ch$ is satisfied. Then there holds

$$\|S_h \bar u - S_h R_h \bar u\|_{L^2(\Omega)} \leq ch^{3/2} \quad \text{for } \mu < 1/3 + 2\lambda/3,$$
$$\text{(Supercloseness)} \quad \|\bar u_h - R_h \bar u\|_{L^2(\Gamma)} \leq ch^{3/2} \quad \text{for } \mu < \lambda < 1/3 + 2\lambda/3,$$
$$\|\bar y - \bar y_h\|_{L^2(\Omega)} + \|\bar p - \bar p_h\|_{L^2(\Gamma)} \leq ch^{3/2} \quad \text{for } \mu < \lambda.$$

Postprocessing step [Meyer, Rösch 2004]: $\tilde u_h := \Pi_{[a,b]}\left(-\bar p_{h|\Gamma}/\nu\right)$

Postprocessing for the control

On a family of meshes with mesh grading parameter $\mu < \lambda$ the inequality

$$\nu\|\bar u - \tilde u_h\|_{L^2(\Gamma)} \leq ch^{3/2}$$

is satisfied under the assumption $\text{meas}(\bigcup_{G \in \mathcal{G}_h : \bar u \notin W^{3/2,2}_{3(1-\mu)/2}(G)} G) < ch$.

Variational approach

Find $\bar u_* \in U^{ad}$ such that

$$J_h(\bar u_*) = \min_{u \in U^{ad}} J_h(u) := \frac{1}{2}\|S_h u - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}$$

with

$$V_h = \{y_h \in C(\bar\Omega) : y_h|_T \in \mathcal{P}_1 \; \forall T \in \mathcal{T}_h\}.$$

Discrete optimality system for $\bar y_h = S_h \bar u_*$, $\bar p_h = P_h(\bar y_h - y_d)$ and $\bar u_*$:

$$a(\bar y_h, v_h) = (\bar u_*, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h$$
$$a(\bar p_h, v_h) = (\bar y_h - y_d, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h$$
$$(\bar p_{h|\Gamma} + \nu \bar u_*, u - \bar u_*)_{L^2(\Gamma)} \geq 0 \quad \forall u \in U^{ad}$$

Error estimates for the variational approach

On a family of admissible meshes the estimate

$$\nu\|\bar u - \bar u_*\|_{L^2(\Gamma)} \leq \|S^*(S\bar u - y_d) - S_h^*(S\bar u - y_d)\|_{L^2(\Gamma)} + c\|S\bar u - S_h \bar u\|_{L^2(\Omega)}$$

is valid.
On a family of meshes with mesh grading parameter $\mu < \lambda$ the inequality

$$\nu\|\bar u - \bar u_*\|_{L^2(\Gamma)} \leq ch^{3/2}$$

is satisfied.

Numerical experiment for the fully discrete approach

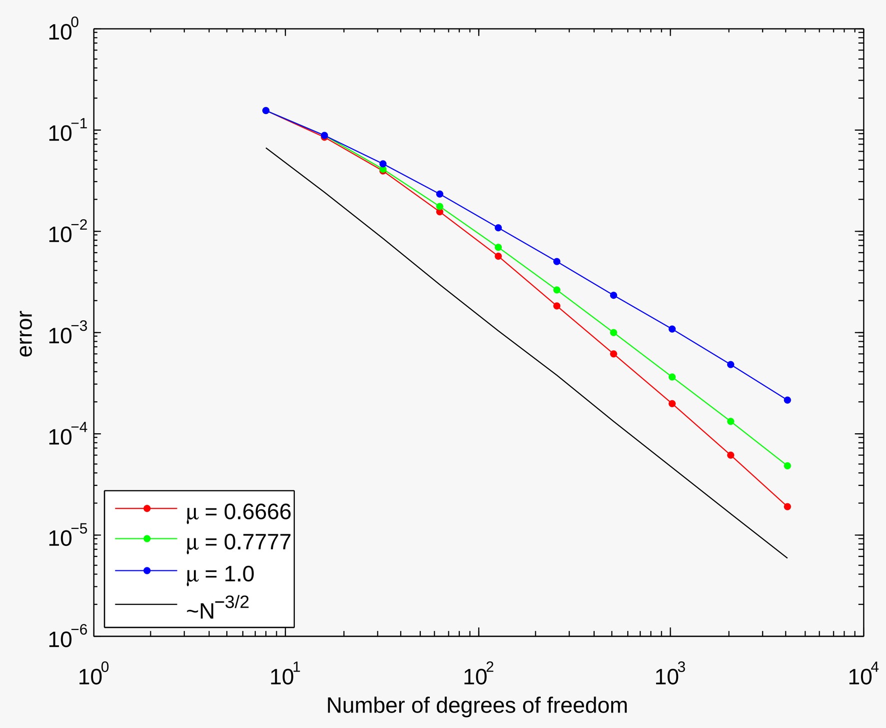
Example

$$-\Delta y + y = 0 \quad \text{in } \Omega,$$
$$\partial_n y = u + g_2 \quad \text{on } \Gamma,$$
$$-\Delta p + p = y - y_d \quad \text{in } \Omega,$$
$$\partial_n p = g_1 \quad \text{on } \Gamma,$$
$$u = \Pi_{[-0.5,0.5]}\left(-p_{|\Gamma}\right) \quad \text{on } \Gamma.$$

The data are chosen such that

$$\bar y = 0 \quad \text{in } \Omega,$$
$$\bar p = -r^\lambda \cos(\lambda\phi) \quad \text{in } \Omega,$$
$$\bar u = \Pi_{[-0.5,0.5]}(y_d) \quad \text{on } \Gamma.$$

Error in the control



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