PDEs and Variational Problems with random coefficients

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Overview

1. Introduction
   1. Setting: Random integrands/PDEs with random coeff.
   2. Motivation: Interface evolution in random media
2. Existence/Nonexistence: Nonnegative solutions for semilinear random PDE
3. Uniqueness: Unique minimizer for random functional with double-well structure.
4. Review of random homogenization

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PDEs with random coefficients

General form:

$$F(D^2u, Du, u, x, \omega) = 0,$$

where the random function

$$F : \mathbb{R}^{nxn}_{\mathrm{sym}} \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}^m$$

(here $m = 1$) satisfies **deterministic bounds/structural conditions.**
(E.g. continuous, uniformly elliptic etc.)
Probability measure $\mathbb{P}$ on all equations with these bounds
Not considered:

- Random initial conditions
- SPDEs

Usually: Law **translation invariant and ergodic**, so "almost sure" results for large-scale behaviour.
Homogenization: Behaviour of solns. for $F(D^2u, Du, u, x/\epsilon, \omega) = 0$, on bounded domain as $\epsilon \to 0$.

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Random Functionals

Find minimizer in a suitable function space (e.g. $H^{1,2}(D)$) of

$$u(x) \mapsto \int_D F(Du, u, x, \omega) dx$$

Minimizer will be random function.
$D = \mathbb{R}^n$: Minimizer under compact perturbations.

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Motivation: Randomly perturbed gradient flow

Key features of Models:

- Evolution decreases free energy
  - Free energy is surface energy, i.e. area of interface
- Heterogeneities influence free energy **locally** (on small scale)

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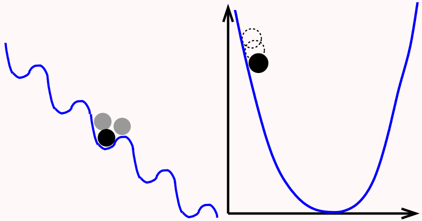

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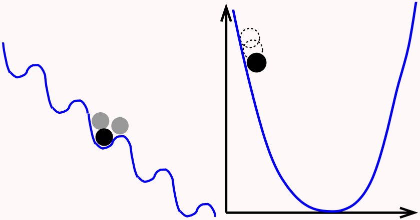

Zoom in on scale of heterogeneities

Perturbed Area Functional/Forced MCF

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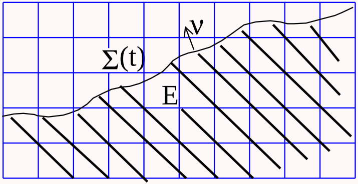
Liapunov functional (formal):

$$\mathrm{Area}(\Sigma) + \int_{\mathbb{R}^{n+1}\cap E} f(X)dX \quad \text{where } \Sigma = \partial E.$$



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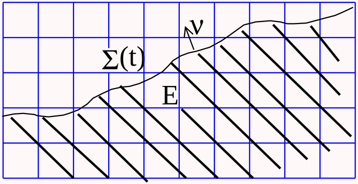

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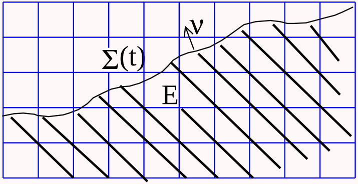

Gradient flow:

$$V_X = \kappa_X + f(X), \; X \in \Sigma(t) \subset \mathbb{R}^{n+1}$$

κ_X mean curvature of Σ at point X, V_X normal velocity at point X.

Behaviour on Large Scale:

("Undo zooming in")

$$V_X = \kappa_X + f(X, \omega), \; X \in \Sigma(t) \subset \mathbb{R}^{n+1}$$

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Questions:

1. **Effective Velocity** $(t = \epsilon^{-1} T, \; x = \epsilon^{-1} Y)$
2. **De-pinning Threshold** F_c
3. **Scaling law** for effective velocity as function of F and for "oscillation" of interface.

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Related Homogenization Problem

Interface as level set: $\Sigma(t) = \{x \in \mathbb{R}^{n+1} : \ w(x,t) = 0\}$

$$V = \epsilon\kappa + f\left(\frac{x}{\epsilon}\right) \Rightarrow \ w_t = \epsilon \,\mathrm{tr}\left[\left(I - \frac{1}{|\nabla w|^2}\nabla w \otimes \nabla w\right) D^2 w\right] + f\left(\frac{x}{\epsilon}\right)|\nabla u|$$

$$V = c(\nu) \ \Rightarrow \ \bar{w}_t = c\left(\frac{\nabla \bar{w}}{|\nabla \bar{w}|}\right)|\nabla \bar{w}|$$

"Singular" Homogenization: Averaging *and* singular limit.

- Degenerate, nonlinear
- $f(x)$ may change sign

Forcing $f(x)$ strictly positive (+additional conditions), not random:
P.-L. Lions, P.E. Souganidis, (2005),
Additional conditions: Caffarelli, Monneau
Connection: Level sets evolve by (forced) MCF (Chen-Giga-Goto)
Random case mostly open! Look for simplified model:
Random Obstacle Model.

Related work

- Physics: QEW ($\partial_t u = \Delta u + f(x, u, \omega)$)
  - S. Brazovsii, Th. Nattermann (et al.) (FRG)
- Material Science
  - K. Bhattacharya
  - B. Craciun, K. Bhattacharya (conditional result)
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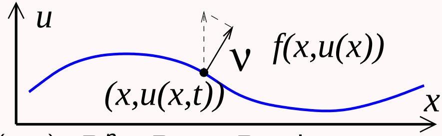
Forced Mean Curvature Flow/Semilinear "Approx."

Forced MCF (Gradient flow of perturbed surface energy):

$V_{x,u} = \kappa_{x,u} + f(x,u) + F$



If surface is graph $(x, u(x,t))$ then $u(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ solves

$$\partial_t u = \sqrt{1 + |\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + \sqrt{1+|\nabla u|^2}\,(f(x,u) + F).$$

gradient small, then (heuristic) approximation: **semilinear PDE**

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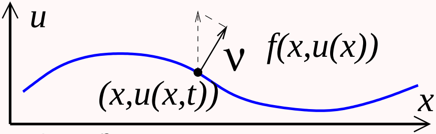

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Gradient flow with Lyapunov functional:

$$\int \left(\sqrt{1 + |\nabla u|^2} + \left[\int_0^{u(x)} f(x,s,\omega)ds\right] + Fu\right) dx$$

Direct Observation of Pinning and Bowing of a Single Ferroelectric Domain Wall, T. J. Yang, Venkatraman Gopalan, P. J. Swart, U. Mohideen, Physical Review Letters 82, 1999

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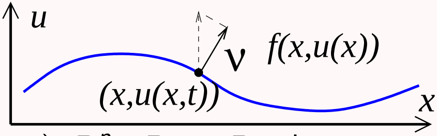

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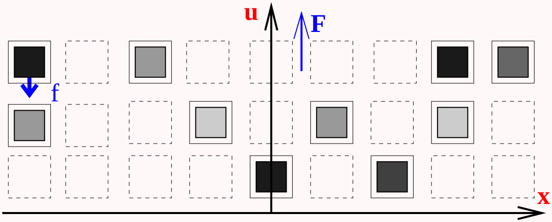
The Random Obstacle Model



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Quenched Edwards-Wilkinson Model (QEW)

Dynamic phase transitions in ferroic systems with pinned domain walls.
W. Kleemann. MFO Phasenübergänge, 20.-26. 06. 2004

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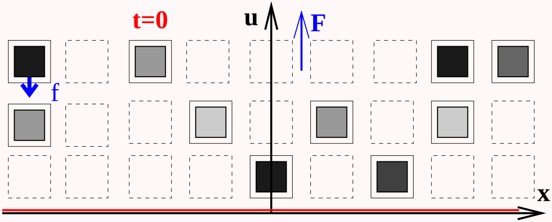

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Important: Comparison Principle. If u, v solns., $u(T) \leq v(T)$, (+b.c.) then $u(T+s) \leq v(T+s)$ for all $s \geq 0$.

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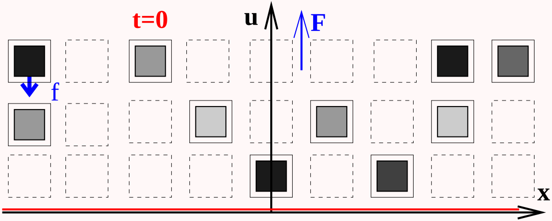

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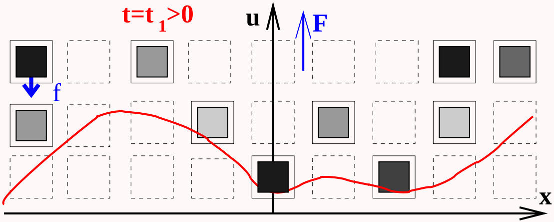

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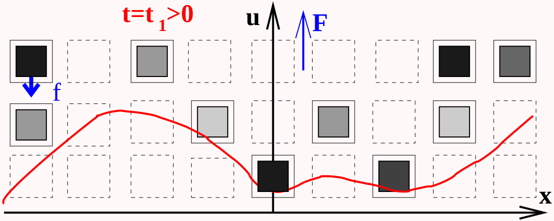

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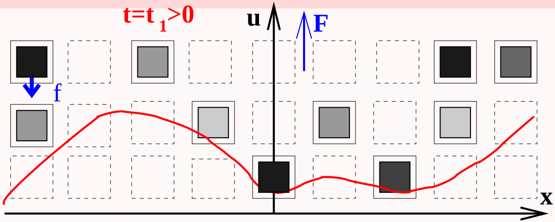

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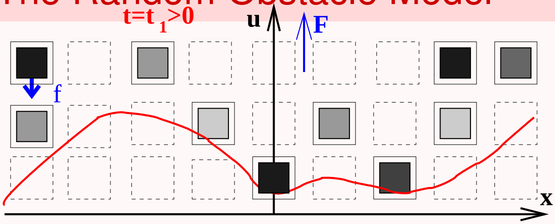

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- Periodic: Such F_* exists, velocity $\sim \sqrt{F - F_*}$
- $F = F^*$? Periodic: Stationary solution due to compactness (D.-Yip)

Random vs. Periodic: Loss of compactness and behaviour at critical forcing in zero dim.

What happens at $F = F*$?

• Periodic environment (compactness): Stationary solution exists as u.c. limit of stationary solutions for $F < F_*$.

• Random environment: Zero Velocity **AND** non-existence of stationary solution possible

$$\dot{X} = F + \sin(2\pi x)$$

$F_* = 1$

χ cut-off, $\chi = 1$ near $x = 0$, $\chi = 0$ on $\mathbb{R} \setminus [-1/8, 1/8]$.

Z_i i.i.d., $Z_i > 0$ a.s., $\mathbb{E}Z_0 = \infty$. (square-root behavior)

Time to cross obstacle at i: $\sim Z_i$

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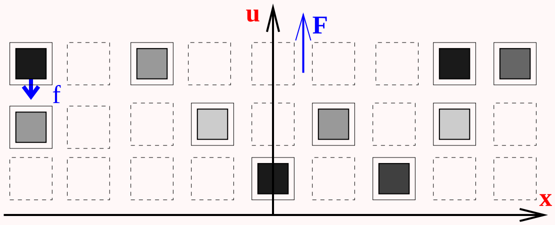
Random Obstacle Model: Precise Setting



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$F \geq 0$, (driving force), ϕ mollifier of $1_{[-\delta,\delta]^{n+1}}(x,u)$,

$$f(x,u) = \sum\nolimits_{(i,j)\in\mathbb{Z}^n\times(\mathbb{Z}+\frac{1}{2})} \left(\mathbb{E}(\ell_{ij}) - \ell_{i,j}(\omega)\right) \phi(x-i, u-j)$$

$(\ell_{i,j}(\omega))_{(i,j)\in\mathbb{Z}^n\times(\mathbb{Z}+\frac{1}{2})}$ are a family of i.i.d. exponential random variables.

Nonnegative Solutions for R. O. M.

$$0 = \Delta u(x,\omega) + f(x, u(x,\omega), \omega) + F \quad \text{on } \mathbb{R}^n, \qquad u(x) \geq 0 \quad (*)$$

Theorem (N.D., J. Coville, S. Luckhaus)

Let $n = 1$ and u solve $(*)$ Then there exist $F_0 > 0$ such that for $F > F_0$ there is almost surely no solution of $(*)$.

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Let $n = 1, 2$. There ex. $0 < F_1$ such that for $0 < F < F_1$, $(*)$ has almost surely a solution with $\mathbb{E}[u(x,\omega)] = c < \infty$ for all $x \in \mathbb{R}^n$.

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Barrier for/limit of

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Theorem (P. Dondl, M. Scheutzow)

- *Let $n = 1$. There ex. $0 < F_1$ and $C_1 > 0$ such that for $F < F_1$ there ex. almost surely a solution with ...*

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Let $n = 1$ and u solve $(*)$ Then there exist $F_0 > 0$ such that for $F > F_0$ there is almost surely **no** *solution of $(*)$.*

Theorem (N.D., P. Dondl, M. Scheutzow)

Let $n = 1, 2$. There ex. $0 < F_1$ such that for $0 < F < F_1$, $(*)$ has almost surely a solution with $\mathbb{E}[u(x,\omega)] = c < \infty$ for all $x \in \mathbb{R}^n$.

Theorem (P. Dondl, M. Scheutzow)

- *Let $n = 1$ There ex. $0 < F_1$ and $C_1 > 0$ such that for $F > F_1$, there ex. almost surely a solution with $\mathbb{E}[u(x,t,\omega)] \geq C_1 t$ for all $x \in \mathbb{R}$.*
- *Uniformly for x_1, x_2 in a fixed compact:*

$$\lim_{t\to\infty} t^{-1}\big(u(x_1,t,\omega) - u(x_2,t,\omega)\big) = 0$$

Non-Existence

$$\mathbb{P}\left\{\omega : \; u(x,\omega)) \geq KN - K|x| \right\} \geq 1 - Ce^{-\frac{N}{C}}$$

- Coarse-graining: Discretise $\rightarrow \bar{v}^{\delta}$, using that path between obstacles determined by values on boundary of obstacles.
- Estimate discrete Laplacian against obstacle: $\Delta_d(i) + F \leq C\ell_{i,[\bar{v}^{\delta}(i)[}(\omega)$
  Problem: Path may pass several obstacles above same integer
- Auxiliary random measure on paths: $\mathbb{P}(u(\omega)$ compatible with $\bar{v}^{\delta}(i)) \leq$ [illegible]
- Conclusion: Path crosses [illegible]
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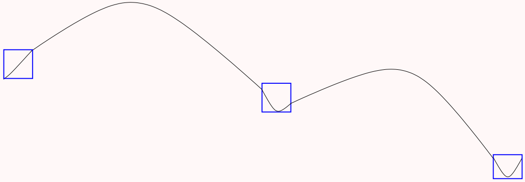

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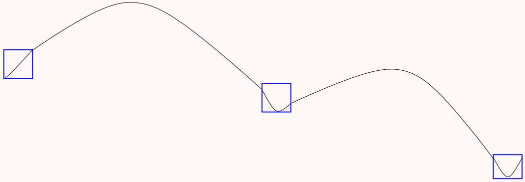

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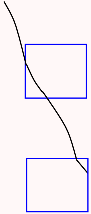

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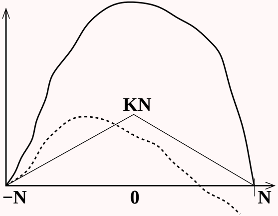

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Existence

Essential probabilistic step: Lipschitz percolation, i.e. a discrete 1-Lipschitz graph $w : \mathbb{Z}^n \to \mathbb{N}$ ex. with $(z, w(z))$ good for all $z \in \mathbb{Z}^n$ if $\mathbb{P}(bad)$ small, (D., Dondl, Grimmett, Holroyd, Scheutzow, Electr. J. of Prob.,)
Analyst's approach: Fixed point iteration

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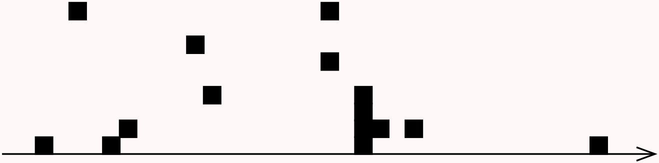

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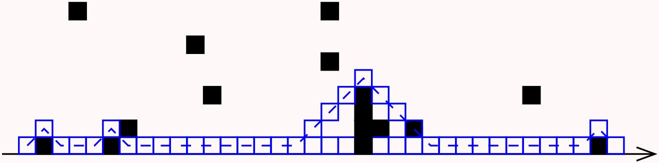

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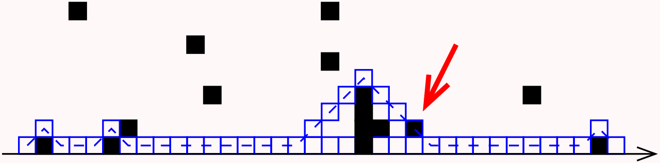

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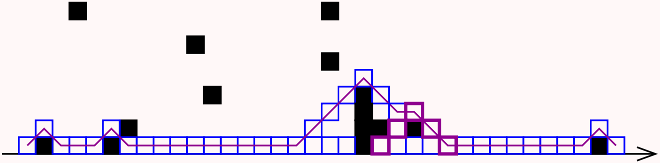

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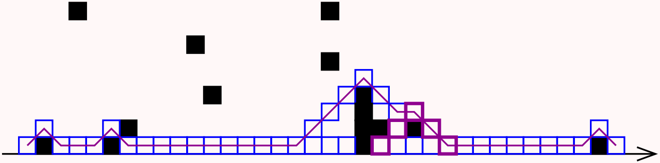
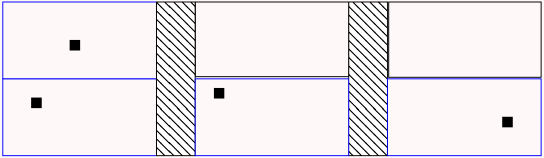


Branching process on cones: Dies out if "closed" cells rare.

Existence: Construction

Idea: Approximate Lipschitz graph by function s.t. $\Delta u + f(x, u, \omega) + F \leq 0$. ("convex" corners at obstacles)

- Discretization: Fix threshold R, call a box **open** if it contains obstacle with strength $> R$.



- Suppose: There exists Lipschitz graph $w \geq 1$ which is contained in the open set.
- From w construct function $v \geq 0$ with Lipschitz-constant $C(F)$ which solves $\Delta v = -F$ outside strong obstacles.
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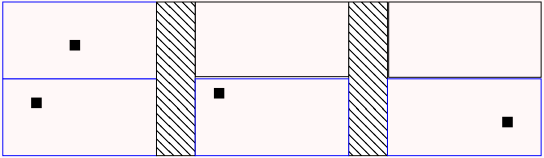

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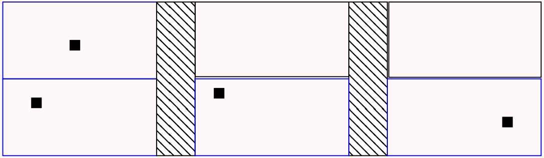

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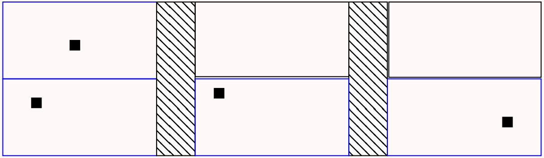

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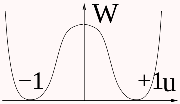
Minimisers of random of energy (with E. Orlandi)

$$\mathrm{Area}(\Sigma \cap \Lambda) + \int_{\Lambda \cap E} f(X) dX \quad \text{where } \Sigma = \partial E.$$

$$F_\epsilon(u) = \int_\Lambda \left(\frac{\epsilon}{2} |\nabla u(x)|^2 + \frac{1}{\epsilon} W(u(x)) + \frac{\alpha_\epsilon}{\epsilon} h\left(\frac{x}{\epsilon}, \omega\right) u(x) \right) dx$$

h bounded random field, short correlation length
W double-well potential, two minimizers ± 1.



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- Replace gradient term by nonlocal term

$$E_\Lambda(m, m_0) = \int_{\Lambda\times\Lambda} \mathrm{d}x\mathrm{d}y \frac{|\mathrm{m(x) - m(y)}|^2}{|\mathrm{x - y}|^{\mathrm{d+2s}}} + \underbrace{2\int_\Lambda \mathrm{d}x \int_{\mathbb{R}^{n+1}\setminus\Lambda} \mathrm{d}y \frac{|\mathrm{m(x) - m_0(y)}|^2}{|\mathrm{x - y}|^{\mathrm{d+2s}}}}_{\text{boundary cond. } \mathrm{m_0}}$$

$d = 2$, $s \in (\frac{1}{2}, 1)$ **or** $d = 1$, $s \in [\frac{1}{4}, 1)$: Unique minimiser (comp. pert.)

The functional

Randomness: $(g(z,\omega))_{z\in\mathbb{Z}^d}$, d space dimension family of uniformly bounded i.i.d. r.v. with mean zero and variance 1 and **Lebesgue-continuous** and symmetric distribution.

$$g(x,\omega) := \sum_{z\in\mathbb{Z}^d} g(z,\omega) 1_{(z+[-\frac{1}{2},\frac{1}{2}]^d)\cap\Lambda}(x),$$

Energy:

$$\mathcal{K}(v,\omega,\Lambda) = \int_\Lambda dx \int_\Lambda dy \frac{|v(x)-v(y)|^2}{|x-y|^{d+2s}} + \int_\Lambda W(v(x))\mathrm{dx} - \int_{\sim} \mathrm{g(x,\omega)v(x)dx}.$$

Boundary Cost:

$$\mathcal{W}((v,\Lambda),(u,\Lambda_1)) = 2\int_\Lambda \mathrm{dx} \int_{\sim_1} \mathrm{dy} \frac{|\mathrm{v(x)}-\mathrm{u(y)}|^2}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}}$$

$$G^{v_0}(v,\omega,\Lambda) = \mathcal{K}(v,\omega,\Lambda) + \mathcal{W}((v,\Lambda)(v_0,\Lambda^c))$$

Minimizer under compact perturbation

$u : \mathbb{R}^d \to \mathbb{R}$ **Minimizer under compact perturbations:** For any compact subdomain $U \subset$ we have

$$G^u(u, \omega, U) < \infty, \quad a.s.$$

and

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for any v which coincides with u in $\mathbb{R}^d \setminus U$.

$u : \Lambda \to \mathbb{R}$ is v^0-minimizer if it minimizes G^{v_0} among all functions which coincide with v^0 on $\mathbb{R}^d \setminus \Lambda$.
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Minimizers are ordered

u min. of $G^u(\cdot,\Lambda)$, v min. of $G^v(\cdot,\Lambda)$, then

- if $u = v$ on $\Lambda^c \Rightarrow u \leq v$ on Λ or $v \leq u$ on Λ
- if $u < v$ on open subset of Λ^c, then $u \leq v$ on Λ.

In general no uniqueness even on compact domains!
Idea:

$$G(u \vee v,\Lambda) + G(u \wedge v,\Lambda) \leq G(u,\Lambda) + G(v,\Lambda).$$

Extremal K-minimizers

On compact domain with b.c. in general no uniqueness, but there exists maximal and minimal minimizer.

Consider now constant b.c. $\pm K$ for $K \gg 1$ and let $u^{\pm,K,\Lambda_n}$ be the extremal min. with b.c. $\pm K$ on $\Lambda_n := (-n, n)^d$.

Define:

$$u^{\pm K}(x, \omega) := \lim_{n \to \infty} u^{\pm,K,\Lambda_n}(x, \omega)$$

Pointwise increasing bounded sequence, converges in better function spaces, consequence:

$u^{\pm K}(x, \omega)$ are min. under compact perturbations!

Moreover: Translation covariant
i.e. $u^{\pm K}(x, \omega)$ and $u^{\pm K}(y, \omega)$ are the same in law.

Extremal ergodic states

WANTED: Extremal min. under compact pert. on $\mathbb{R}^n$. If they are unique, all min. are equal.

Consequence of min. property of $u^{\pm K}$ and translation covariance: uniform bounds on $\|u^{\pm K}\|_\infty$ which do not depend on K.

Consequence:

$$u^{\pm}(x,\omega) := \lim_{K\to\infty} u^{\pm K}(x,\omega)$$

well defined, uniformly bounded and min. under compact pert.
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Bound on difference of optimal energies

$$\left| G^{v^+}(v^+, \Lambda) - G_1^{v^-}(v^-, \Lambda) \right| \leq C \begin{cases} |\Lambda|^{\frac{d-1}{d}} & \text{if } s \in (\frac{1}{2}, 1) \\ |\Lambda|^{\frac{d-2s}{d}} & \text{if } s \in (0, \frac{1}{2}) \\ |\Lambda|^{\frac{d-1}{d}} \log |\Lambda| & \text{if } s = \frac{1}{2} \end{cases} .$$

Note: $|\Lambda_n| \sim n^d$.

Idea: Interpolate on the boundary between u^+ and u^-, estimate "cost" by estimating singular integrals.

Central Limit Theorem: Set-up

Note: Minimal energy and minimizer depend in complicated way on all random variables $g(z,\omega)$.

σ-algebras:

• $\mathcal{B}_{n,i} = \sigma\left(\{g(z), z \in \Lambda_n,\ z \leq i\}\right)$ where $\leq$ refers to lexicographic ordering in $\mathbb{Z}^d$.

• $\mathcal{B}_{\Lambda_n} = \sigma\left(\{g(z), z \in \Lambda_n\}\right)$

• $\mathcal{B}(0) = \sigma\left(g(0)\right)$

Consider

$$\begin{aligned} F_n(\omega) &:= \mathbb{E}\left[\left\{G(v^+(\omega),\omega,\Lambda_n) - G(v^-(\omega),\omega,\Lambda_n)\right\}|\mathcal{B}_{\Lambda_n}\right] \\ &= \sum_{i\in\mathbb{Z}^d\cap\Lambda_n}\left(\mathbb{E}[F_n|\mathcal{B}_{n,i}] - \mathbb{E}[F_n|\mathcal{B}_{n,i-1}]\right) := \sum_{i\in\mathbb{Z}^d\cap\Lambda_n} Y_{n,i}. \end{aligned}$$

Martingale Difference: CLT $\Rightarrow F_n \sim \sqrt{|\Lambda|}N(0,D^2)$ where

$$D^2 = \mathbb{E}\left[\left(\mathbb{E}\left[F_n|\mathcal{B}(0)\right]\right)^2\right]$$

Central Limit Theorem: Result

Deterministic bound:

$$|F_n| \leq C \begin{cases} n^{d-1} & \text{if } s \in (\frac{1}{2}, 1) \\ n^{d-2s} & \text{if } s \in (0, \frac{1}{2}) \\ n^{d-1} \log n & \text{if } s = \frac{1}{2} \end{cases} .$$

Fluctuations: $n^{d/2}$ unless $D^2 = 0$.

Contradiction if $d = 2$, $s \in (\frac{1}{2}, 1)$ **or** $d = 1,\ s \in [\frac{1}{4}, 1)$ unless $D^2 = 0$.

"derivative" w.r.t. randomness

$$\omega(0) \mapsto \int_{Q(0)} v^+(\omega(0), \omega^{(0)}) \mathrm{d}x$$

is nondecreasing.

$$\frac{\partial G(v^\pm(\omega), \omega, \Lambda)}{\partial \omega(0)} = -\int_{(-1/2,1/2)^d} v^\pm(x, \omega) \mathrm{d}x.$$

Absolutely cont. random variables!

Heuristic: Suppose $u(\omega)$ minimises $F(u, \omega)$.

$$\frac{\partial F(u(\omega), \omega)}{\partial \omega}|_{(u(\omega),\omega)} = \frac{\partial F(u, \omega)}{\partial u}|_{(u(\omega),\omega)} + \frac{\partial F(u, \omega)}{\partial \omega}|_{(u(\omega),\omega)}$$

$$= \frac{\partial F(u, \omega)}{\partial \omega}|_{(u(\omega),\omega)}$$

$$G(u, \omega) = \ldots - \int_\Lambda g(x, \omega) u(x) \mathrm{d}x$$

Central Limit Theorem: Conclusion

$$0 = D^2 = \mathbb{E}\left[\left(\mathbb{E}\left[F_n|\mathcal{B}(0)\right]\right)^2\right] = \mathbb{E}\left[f^2(\omega(0))\right]$$

so $0 = f(s)$ a.s.

$$\begin{aligned} f'(s) &= \left.\frac{\partial G(v^+(\omega), \omega, \Lambda)}{\partial \omega(0)}\right|_{\omega(0)=s} - \left.\frac{\partial G(v^-(\omega), \omega, \Lambda)}{\partial \omega(0)}\right|_{\omega(0)=s} \\ &= \int_{(-1/2,1/2)^d} \left(v^+(x,\omega) - v^-(x,\omega)\right) \mathrm{d}x. \end{aligned}$$

$f(s) = 0 \Rightarrow$(mon.) $f'(s) = 0$ a.s. $\Rightarrow$ (ordered) $v^+ = v^-$ a.s.