On the sharp limit of the Van der Waals-Cahn-Hilliard model

Christiane Kraus, WIAS Berlin

We study liquid-vapor phase transitions of single substances at constant temperature T. A liquid and a vapor phase separated by a sharp interface satisfy in equilibrium the conditions

$$|p_L - p_V| = 2\sigma|k_m| \quad \text{and} \quad g_L(T, p_L) = g_V(T, p_V), \qquad \text{L: liquid,} \quad \text{V: vapor,}$$

where p is the pressure, σ the surface tension, g the specific Gibbs free energy and k_m the mean curvature of the interface. The first condition states mechanical equilibrium while the second condition establishes phase equilibrium which is described by the continuity of the specific Gibbs free energies g_L and g_V at the interface.

The sharp interface is represented by a smooth region of finite thickness in the Van der Waals–Cahn–Hilliard theory of phase transitions with the free energy density $E_\varepsilon = \varepsilon^2|\nabla u_\varepsilon|^2 + W(u_\varepsilon)$, where u_ε denotes the mass density. The continuity of the specific Gibbs free energies for phase equilibrium can be achieved by asymptotic studies of the free energy E_ε letting $\varepsilon \to 0$. However, there is no information on the condition for mechanical equilibrium in the literature.

We show that the equilibrium conditions can be obtained up to order $o(\varepsilon)$ in the sharp limit from the Van der Waals–Cahn–Hilliard model. Our proof is based on energy estimates and uniform convergence results for the density u_ε as $\varepsilon \to 0$.