Theory of function and distribution spaces

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Contents

Introduction

The aim of these notes is to give a self-contained precise introduction into the spaces of distributions, Sobolev and in particular Besov spaces. For the latter we rely on the Fourier transform on tempered distributions. Those spaces are commonly used in the theory of partial differential equations, hence also some sections are devoted to applications to partial differential equations.

The first part of these notes concentrate on introducing the essentials of distribution spaces; their topologies, convolution and mollifiers and the Fourier transformation on tempered distributions. There is a lot of good literature on the theory of distributions, for example [Don69] (quite an old fashioned way of writing and no inner references, though quite complete), [DK10] [Fri98] (both good introduction to distribution spaces, comparable to the first part of these notes, approach the theory without requiring knowledge on topology), [Str03] (similar the previous two, though different from the taste of the author, very sparse on topological issues), [Hor66] (is more an introduction into topological vector spaces, with distributions a final section), [Leo17] (a brief introduction to distributions in order to be able to introduce Sobolev spaces and also Besov spaces, but without using the Fourier transformation).

The second part of these notes concentrates on Besov spaces. The notes differ from the literature in that the proofs contain more details and rigour. Other references in which Besov spaces are introduced are [BCD11] (these notes have a lot of overlap with Section 2 of that book, which focusses less on the details but contains more content on the applications), [Gra14] (contains a brief introduction into Besov spaces, but also contains many more function spaces), [Saw18] (contains most of the contents of these notes, though written in a different style), [ST87] , [Tri83] , [Tri92] , [Tri78] , [Trè06] (all very extensive books on function spaces, containing also Besov spaces, also the style here differs from the one in these notes).

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Conventions and notation

- $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_{-1} = \{-1, 0\} \cup \mathbb{N}$.
- d is an element of $\mathbb{N}$.
- Ω is a nonempty open subset of $\mathbb{R}^d$.
- $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- For $x \in \mathbb{R}^d$ or $x \in \mathbb{C}^d$ we write $|x|$ for its Euclidean norm $\sqrt{\sum_{i=1}^d |x_i|^2}$, $|x|_1 = \sum_{i=1}^d |x_i|$ and $|x|_\infty = \max_{i \in \{1,\dots,d\}} |x_i|$.
- For $a, b \in \mathbb{R}$ we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.
- For $x \in \mathbb{R}^d$, $r > 0$ we write $B(x, r)$ for the (Euclidean) ball in $\mathbb{R}^d$ with centre x and radius r:

$$B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}.$$

  (See also 1.7.)
- For a set $A \subset \Omega$ we write A° for its interior and $\overline{A}$ for its closure, see also 3.1.
- We write ∇ for the gradient of a function, which is the vector consisting of first derivatives

$$\nabla f = (\partial_1 f, \cdots, \partial_d f).$$

- For a set $A \subset \Omega$ we write $\mathbb{1}_A$ for the indicator function of A, see Definition 2.4.
- We write $\| \cdot \|_{L^p}$ for the norm on the L^p spaces. See Section A. As is common, we do not distinguish between a function in $\mathcal{L}^p$ and its corresponding equivalence class in L^p.
- For the inner product on L^2 we write $\langle \cdot, \cdot \rangle_{L^2}$ (to avoid confusion with the notation $\langle \cdot, \cdot \rangle$ for the pairing between distributions and test functions). So

$$\langle f, g \rangle_{L^2} = \int f \overline{g}.$$

- (Notation of limits of partial sums) Suppose $(v_n)_{n \in \mathbb{N}}$ is a sequence in a topological vector space X. We say that $\sum_{n=1}^\infty v_n$ exists if $\sum_{n=1}^N v_n \xrightarrow{N \to \infty} v$ for some $v \in X$ and write $\sum_{n=1}^\infty v_n$ for v.

  If $\mathbb{I}$ is a countable set and $v_i \in X$ for $i \in \mathbb{I}$, then we say that

$$\sum_{i \in \mathbb{I}} v_i$$

  exists, if there exists a $v \in X$ such that for each bijection $q : \mathbb{N} \to \mathbb{I}$, $\sum_{n=1}^\infty v_{q(n)} = v$ in X, and write $\sum_{i \in \mathbb{I}} v_i$ for v.

1 Spaces of differentiable functions and testfunctions

In the next section, Section 2, we introduce the objects called *distributions*, which play the central role of this text. A distribution can be viewed as a sort of generalised function. As we will see, many functions like for example all continuous functions “are” or “can be viewed” as distributions. Moreover, the distributions, like the differentiable functions, form a vector space on which operations like translation, multiplication with differentiable functions and differentiation are defined and follow the usual formal rules of calculus. With the important difference that all distributions are differentiable, in the sense that a derivative of a distribution always exists as a distribution.

This is a huge advantage which makes the theory of distributions very suitable as a tool for (partial) differential equations, of which we will see a little bit for example in Section 11.

In this section we consider the space $\mathcal{D}(\Omega)$ of testfunctions on Ω, which are smooth functions with compact support. A distribution, as defined later in Definition 2.1, is a linear function $\mathcal{D}(\Omega) \to \mathbb{F}$ with certain continuity properties. In this section we show that the space $\mathcal{D}(\Omega)$ is not empty, and on the contrary, is large enough in the sense that a compact set K and a closed set F with $K \cap F = \emptyset$ can be separated by a testfunction in the sense that there exists a testfunction which equals 1 on K and equals 0 on F (Lemma 1.13). Moreover, the notion of a partition of unity will be introduced which will show its use multiple times (Definition 1.10).

Remember that Ω is a nonempty open subset of $\mathbb{R}^d$ and that the underlying field $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Before we define the notion of a testfunction in Definition 1.5, we introduce some definitions and recall a fact about the space of k-times continuously differentiable functions.

Definition 1.1. Let $f : \Omega \to \mathbb{F}$ be a continuous function. We define the *support* of f, $\operatorname{supp} f$, to be the closure in Ω of the set

$$\{x \in \Omega : f(x) \neq 0\}. \tag{1.1}$$

This means that it is the set of all x such that for all neighbourhoods V of x (open set that contains x) there exists an element y in that neighbourhood such that $f(y) \neq 0$.

Let $A \subset \Omega$. We say that f *vanishes* on A if $f = 0$ on A, i.e., $f(x) = 0$ for all $x \in A$. Let $\mathcal{U}$ be the collection of all open subsets of Ω on which f vanishes. Then $U := \bigcup \mathcal{U}$ is the largest open subset of Ω on which f vanishes, and $\operatorname{supp} f = \Omega \setminus U$.

If F is a set of functions $\Omega \to \mathbb{F}$, then we write F_{c} for the subset of compactly supported functions in F, i.e., $F_{\mathrm{c}} = \{f \in F : \operatorname{supp} f \text{ is compact}\}$.

Definition 1.2.
- We write e_i for the basis vector in $\mathbb{R}^d$ in the i-th direction, for $i \in \{1, \ldots, d\}$.

- For $\alpha \in \mathbb{N}_0^d$, we write $|\alpha| = \sum_{i=1}^d \alpha_i$.
- For $\alpha \in \mathbb{N}_0^d$ and any $|\alpha|$-times continuously differentiable function $f : \Omega \to \mathbb{F}$, we write
$$\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f,$$
where ∂_i is the partial derivation with respect to the i-th coordinate, i.e., for a differentiable function $f : \Omega \to \mathbb{F}$,
$$\partial_i f(x) = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h} \qquad (x \in \Omega).$$
- For a one-dimensional differentiable function $f : \mathbb{R} \to \mathbb{F}$ we will also write ∂f or f' to denote its derivative, and $\partial^k f$ for its k-th derivative (the letter D will *not* be used for a derivative, but will be reserved for Fourier multipliers, see Section 19).

Definition 1.3.
- We write $C(\Omega, \mathbb{F})$ or $C(\Omega)$ for the set of continuous functions $\Omega \to \mathbb{F}$. We will also write $C^0(\Omega) = C(\Omega)$ and
$$\|\varphi\|_{C^0(\Omega)} = \sup_{x \in \Omega} |\varphi(x)| \qquad (\varphi \in C(\Omega)),$$
observe that $\|\varphi\|_{C^0(\Omega)} = \|\varphi\|_{L^\infty(\Omega)}$ for $\varphi \in C(\Omega)$.
In general, we will write $\|\cdot\|_{C^0}$ and $\|\cdot\|_{L^p}$ instead of $\|\cdot\|_{C^0(\Omega)}$ and $\|\cdot\|_{L^p(\Omega)}$.
- For $k \in \mathbb{N}$ we write $C^k(\Omega, \mathbb{F})$ or $C^k(\Omega)$ for the k-times continuously differentiable functions $\Omega \to \mathbb{F}$, and $\|\cdot\|_{C^k(\Omega)} : C^k(\Omega) \to [0, \infty]$ for
$$\|f\|_{C^k(\Omega)} = \max_{\beta \in \mathbb{N}_0^d : |\beta| \le k} \|\partial^\beta f\|_{L^\infty} \qquad (f \in C^k(\Omega, \mathbb{F})). \tag{1.2}$$
In general, we will write $\|\cdot\|_{C^k}$ instead of $\|\cdot\|_{C^k(\Omega)}$.
- $C^\infty(\Omega, \mathbb{F})$ or $C^\infty(\Omega)$ is the set of ∞-times continuously differentiable functions $\Omega \to \mathbb{F}$, i.e.,
$$C^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} C^k(\Omega).$$
A function $f : \Omega \to \mathbb{F}$ is called *smooth* if it is in $C^\infty(\Omega)$.
- For a subset $A \subset \mathbb{F}$ and for $k \in \mathbb{N}_0 \cup \{\infty\}$ we write $C^k(\Omega, A)$ for the set of functions in $C^k(\Omega, \mathbb{F})$ which take their values in A, i.e., those $f \in C^k(\Omega, \mathbb{F})$ for which $f(\Omega) \subset A$.

1.4. As on $\mathbb{R}^n$ for some $n \in \mathbb{N}$ the norm $|\cdot|_1$ and $|\cdot|_\infty$ defined by $|x|_1 = \sum_{i=1}^n |x_i|$ and $|x|_\infty = \max_{i=1}^n |x_i|$ for $x \in \mathbb{R}^n$ are equivalent, or more specifically:
$$|x|_\infty \le |x|_1 \le n|x|_\infty \qquad (x \in \mathbb{R}^n),$$
it is easy to see that there exists a $C > 0$ such that
$$\|f\|_{C^k} \le \sum_{\beta \in \mathbb{N}_0^d : |\beta| \le k} \|\partial^\beta f\|_{L^\infty} \le C\|f\|_{C^k} \qquad (f \in C^k(\Omega, \mathbb{F})).$$

Definition 1.5 (Testfunctions)**.** $\mathcal{D}(\Omega)$ is defined to be the vector space $C_c^\infty(\Omega)$. An element of $\mathcal{D}(\Omega)$ is called a *testfunction.*

Later, in Definition 4.1, we equip $\mathcal{D}(\Omega)$ with a topology.

The next lemma shows there exist many testfunctions, namely one can separate points from closed sets that do not contain that point.

Lemma 1.6. *Let $x \in \mathbb{R}^d$ and U be an open subset of $\mathbb{R}^d$ such that $x \in U$. There exists a testfunction $\varphi : \mathbb{R}^d \to [0,1]$ such that $\varphi(x) = 1$ and $\operatorname{supp}\varphi \subset U$.*

Proof. We take $x = 0$ and show that for every $\varepsilon > 0$ there exists a function ψ_ε such that $\psi_\varepsilon(0) = 1$ and $\operatorname{supp}\psi_\varepsilon \subset \overline{B(0,\varepsilon)}$. Consider the function $\psi_\varepsilon : \mathbb{R}^d \to [0,\infty)$ defined by

$$\psi_\varepsilon(y) = \begin{cases} e^{\frac{1}{|y|^2-\varepsilon^2}} & \text{if } |y| < \varepsilon, \\ 0 & \text{if } |y| \geq \varepsilon. \end{cases}$$

One can prove that this function is C^∞ by using that $\lim_{t\to\infty} p(t)e^{-t} = 0$ for any polynomial p. Then ψ_ε is strictly positive at 0 with support in $\overline{B(0,\varepsilon)}$. ☐

Exercise 1.A. Prove that the function $f : \mathbb{R} \to [0,1]$ defined by

$$f(t) = \begin{cases} 0 & t \leq 0, \\ e^{-\frac{1}{t}} & t > 0. \end{cases}$$

is smooth.

Exercise 1.B. Let $d = 1$ and φ be a nonzero testfunction. How that φ' is nonzero as well and conclude that $\partial^k\varphi$ is nonzero for all $k \in \mathbb{N}_0$.

Exercise 1.C. Prove the following statement. *For any sequence $(x_n)_{n\in\mathbb{N}}$ in Ω of distinct elements such that for each compact $K \subset \Omega$ there are only finitely many elements of the sequence in K (that is, no subsequence converges), and for any sequence $(\lambda_n)_{n\in\mathbb{N}}$ there exists a smooth function ψ on Ω with $\psi(x_n) = \lambda_n$ for all $n \in \mathbb{N}$.*

We will now prepare ourselves to show that there exist so called partitions of unity (Definition 1.10). They will be used often in the following. First we recall a definition to show that Ω can be written as a union of compact sets.

1.7 (Notation)**.** For $x \in \mathbb{R}^d$, $r > 0$ we write $B(x,r)$ for the (Euclidean) ball in $\mathbb{R}^d$ with centre x and radius r:

$$B(x,r) = \{y \in \mathbb{R}^d : |x-y| < r\}.$$

Theorem 1.8. *There exists an increasing sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ such that $K_n \subset K_{n+1}^\circ$ for all $n \in \mathbb{N}$ and*

$$\Omega = \bigcup_{n\in\mathbb{N}} K_n.$$

Consequently, for each compact set $K \subset \Omega$ there exists an $n \in \mathbb{N}$ such that $K \subset K_n$.

Proof. Observe that if $\Omega = \mathbb{R}^d$, then we can take K_n to be the closure of the ball around 0 with radius n: $\overline{B(0,n)}$.

Let us now assume that $\Omega \neq \mathbb{R}^d$. We first prove that Ω is the union of countably many closed sets in $\mathbb{R}^d$. Let $f : \Omega \to [0,\infty)$ be such that $f(x)$ is the distance from x to $\mathbb{R}^d \setminus \Omega$, i.e.,

$$f(x) = \inf\{|x-y| : y \in \mathbb{R}^d \setminus \Omega\} \qquad (x \in \Omega).$$

Then f is a continuous function and therefore $A_n = f^{-1}[\frac{1}{n},\infty)$ is a closed subset of Ω, $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} A_n$.

Now it is straightforward to check that $K_n = A_n \cap \overline{B(0,n)}$ satisfies the conditions.

The consequence follows by using the fact that $\Omega = \bigcup_{n\in\mathbb{N}} K_n^\circ$. □

Definition 1.9. Let E be an open subset of Ω. A collection of subsets of E, $\mathcal{U}$, is called a *covering* of E if $\bigcup \mathcal{U} = E$. It is called an *open covering* if each element in $\mathcal{U}$ is an open set. If $\mathcal{U}$ and $\mathcal{V}$ are covers of E, then $\mathcal{V}$ is called a *refinement* of $\mathcal{U}$ or *finer* than $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ with $V \subset U$. A covering $\mathcal{U}$ is called *locally finite* if for all $x \in E$ there exists a neighbourhood V of x such that V intersects only finitely many elements of $\mathcal{U}$. If $\mathcal{V}$ and $\mathcal{U}$ are coverings of E and $\mathcal{V} \subset \mathcal{U}$, then $\mathcal{V}$ is called a *subcovering* of $\mathcal{U}$.

With the help of Theorem 1.8 one can show that there exists an open locally finite covering of Ω. We will see this in the proof of Theorem 1.11.

Definition 1.10. Let $\mathcal{U}$ be a covering of Ω. A *partition of unity on Ω subordinated to $\mathcal{U}$*, is a sequence $(\chi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ with

$$0 \le \chi_n(x) \le 1, \qquad \sum_{n\in\mathbb{N}} \chi_n(x) = 1 \qquad (x \in \Omega),$$

for each $n \in \mathbb{N}$ there exists a $U \in \mathcal{U}$ with $\operatorname{supp} \chi_n \subset U$.

Let us show that partitions of unity exist.

Theorem 1.11. *Let $\mathcal{U}$ be an open covering of Ω. Then there exists a partition of unity on Ω subordinated to $\mathcal{U}$, $(\chi_n)_{n\in\mathbb{N}}$, such that the sets $\{x \in \Omega : \chi_n(x) > 0\}$ form a locally finite covering of Ω. Consequently, for each $\varphi \in \mathcal{D}(\Omega)$ there exists an $N \in \mathbb{N}$ such that $\varphi = \sum_{n=1}^N \chi_n \varphi$.*

Proof. In this proof, let us call an element φ of $\mathcal{D}$ "small" if $\operatorname{supp}\varphi$ is contained in an element of $\mathcal{U}$.

Step 1 Let A be a compact subset of an open set $W \subset \Omega$. Let $x \in A$ and let $U \in \mathcal{U}$ be such that $x \in U$. By applying Lemma 1.6 to the compact set $\{x\}$ and the open set $U \cap W$, we find a small $\varphi \in \mathcal{D}$ with $\varphi(x) > 0$ and $\operatorname{supp}\varphi \subset U \cap W$.

It follows from the compactness of A that there exist $N \in \mathbb{N}$ and small $\varphi_1, \dots, \varphi_N \in \mathcal{D}$ with $\varphi_i \geq 0$ and $\operatorname{supp} \varphi_i \subset W$ for each i and $A \subset \bigcup_{i \in \mathbb{N}} \{x \in \Omega : \varphi_i(x) > 0\} \subset W$.

Step 2 Let $(K_n)_{n \in \mathbb{N}}$ be as in Theorem 1.8 and put $K_0 = \emptyset$. For every $n \in \mathbb{N}$, $A_n := K_n \setminus K_{n-1}^\circ$ is a compact set, contained in the open set $W_n := K_{n+1}^\circ \setminus K_{n-1}$. By applying, for each $n \in \mathbb{N}$, Step 1, to A_n and W_n, one obtains a sequence $(\varphi_i)_{i \in \mathbb{N}}$ of small elements of $\mathcal{D}$ and a sequence $1 = N_1 < N_2 < \cdots$ in $\mathbb{N}$ such that for all $n \in \mathbb{N}$

$$
\begin{aligned}
&0 \leq \varphi_i, \ \operatorname{supp} \varphi_i \subset W_n \text{ if } N_n \leq i < N_{n+1}, \\
&A_n \subset \bigcup_{i=N_n}^{N_{n+1}} U_i, \ \text{where } U_i := \{x \in \Omega : \varphi_i(x) > 0\}.
\end{aligned}
$$

The collection of open sets $\{U_i : i \in \mathbb{N}\}$ forms a covering of Ω. We prove it to be locally finite. To that end, let $n \in \mathbb{N}$; it suffices to show that K_n° intersects only finitely many of the sets U_i, which will be the case if $U_i \subset \Omega \setminus K_n$ for all $i > N_{n+1}$. Take $i \in \mathbb{N}$, $i > N_{n+1}$. There is an $m \in \mathbb{N}$ with $N_m \leq i < N_{m+1}$. Then $m \geq n+1$, whence $U_i \subset W_m = K_{m+1}^\circ \setminus K_{m-1} \subset \Omega \setminus K_{m-1} \subset \Omega \setminus K_n$.

Step 3 It follows that we can define a function $\varphi : \Omega \to (0, \infty)$ by $\varphi(x) = \sum_{n \in \mathbb{N}} \varphi_n(x)$ for $x \in \Omega$, and that $\varphi \in C^\infty(\Omega)$. By setting $\chi_n := \frac{\varphi_n}{\varphi}$ for $n \in \mathbb{N}$ one obtains the desired partition of unity on Ω. □

Remark 1.12. Observe that for $(\chi_n)_{n \in \mathbb{N}}$ as in Theorem 1.11: For any sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $\mathbb{F}$, the formula $\psi(x) = \sum_{n=1}^\infty \lambda_n \chi_n(x)$ for $x \in \Omega$ defines a C^∞ function ψ on Ω.

With the help of the partition of unity of Theorem 1.11 we can extend the statement of Lemma 1.6 in such a way that we can find a testfunction that equals 1 on a compact set:

Lemma 1.13. *Let $K \subset \Omega$ be a compact set and U be an open subset of $\mathbb{R}^d$ such that $K \subset U$. There exists a testfunction $\varphi : \mathbb{R}^d \to [0,1]$ such that $\varphi = 1$ on K and* $\operatorname{supp} \varphi \subset U$.

Exercise **1.D.** Prove Lemma 1.13.

Let us recall the Leibniz' differentiation rule.

1.14 (Leibniz' rule)**.** If $k \in \mathbb{N}_0$, $f, g \in C^k(\Omega)$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, then

$$
\partial^\alpha (fg) = \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \leq \alpha}} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g), \tag{1.3}
$$

where $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for all $i \in \{1, \dots, d\}$ and with $\alpha! = \prod_{i=1}^d \alpha_i!$,

$$
\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha - \beta)! \beta!} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}.
$$

Exercise **1.E.** Let $\alpha \in \mathbb{N}_0^d$.

(a) Show there exists a smooth function $\psi \in C^\infty(\mathbb{R}^d)$ such that $\partial^\alpha \psi(0) = 1$.

(b) Let $\psi \in C^\infty$. Let $\varphi \in \mathcal{D}(\Omega)$ be such that $\varphi = 1$ on a neighbourhood of 0 (i.e., on $B(0, r)$ for some $r > 0$). Show that

$$\partial^\alpha(\psi\varphi)(0) = \partial^\alpha \psi(0).$$

(c) Prove that for all $\alpha \in \mathbb{N}_0^d$ there exists a testfunction $\varphi : \mathbb{R}^d \to [0, 1]$ with $\partial^\alpha \varphi(0) = 1$.

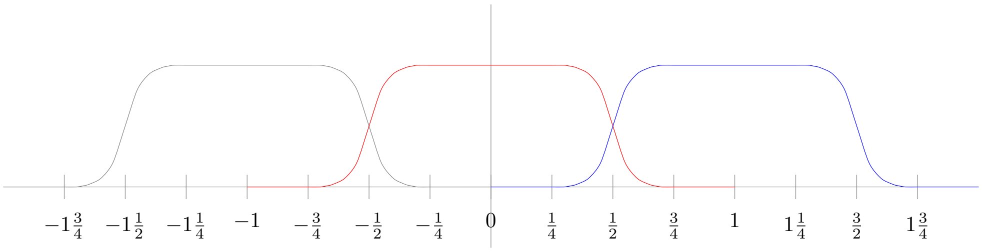

Figure 1: An example of a function χ as in Exercise 1.F.

Exercise **1.F.** See also Figure 1.

(a) Show that there exists a smooth function $\chi : \mathbb{R}^d \to [0, 1]$ such that $\chi = 1$ on $[-\frac{1}{4}, \frac{1}{4}]^d$ and $\chi = 0$ outside $(-\frac{3}{4}, \frac{3}{4})^d$ and such that

$$\sum_{k \in \mathbb{Z}^d} \chi(x - k) = 1 \qquad (x \in \mathbb{R}^d).$$

(b) Show that one can find such a function χ such that it is of the form $\chi(x) = \prod_{i=1}^d \eta(x_i)$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ for a testfunction $\eta : \mathbb{R} \to [0, 1]$ with $\eta = 1$ on $[-\frac{1}{4}, \frac{1}{4}]^d$ and $\eta = 0$ outside $(-\frac{3}{4}, \frac{3}{4})^d$.

(c) Let $n \in \mathbb{N}$ and $\psi : \mathbb{R} \to [0, 1]$ be the function given by $\psi(x) = \sum_{k=-n}^n \eta(x - k)$, with η as in (b). Show that $\|\psi\|_{C^m} = \|\eta\|_{C^m}$ for all $m \in \mathbb{N}_0^d$.

(d) Let $n \in \mathbb{N}$ and $\varphi : \mathbb{R}^d \to [0, 1]$ be the function given by $\varphi(x) = \sum_{k \in [-n,n]^d \cap \mathbb{Z}^d} \chi(x - k)$, which χ as in (b). Show that $\|\psi\|_{C^m} = \|\chi\|_{C^m}$ for all $m \in \mathbb{N}_0^d$.

2 Distributions

In this section we introduce the notion of a distribution and show that all locally integrable functions can be viewed as distributions. Motivated by formulae that hold for those functions, we define operations on distributions like derivation and multiplication with smooth functions. Then we consider the order of a distribution and Radon measures, which also can be viewed as distributions.

Definition 2.1 (Distributions)**.** A linear function $u : \mathcal{D}(\Omega) \to \mathbb{F}$, is called a *distribution* if for all compact sets $K \subset \Omega$, there exist $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|u(\varphi)| \leq C\|\varphi\|_{C^k} \qquad (\varphi \in \mathcal{D}(\Omega),\ \operatorname{supp}\varphi \subset K). \tag{2.1}$$

2.2. Observe that if u and v are distributions (on Ω) and $\lambda, \mu \in \mathbb{F}$, then $w : \mathcal{D}(\Omega) \to \mathbb{F}$ defined by $w(\varphi) = \lambda u(\varphi) + \mu v(\varphi)$ is a distribution.

Definition 2.3. We define $\mathcal{D}'(\Omega)$ to be the vector space of distributions.

Let us first consider some examples of distributions. A large class of distributions is given by locally integrable functions:

Definition 2.4. For a set $A \subset \Omega$ we define its *indicator function*, $\mathbb{1}_A$, by

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

Definition 2.5. We say that a function $f : \Omega \to \mathbb{F}$ is *locally integrable* if $f\mathbb{1}_K$ is an integrable function for all compact sets $K \subset \Omega$. We write $\mathcal{L}^1_{\text{loc}}(\Omega)$ for the space of all locally integrable functions and $L^1_{\text{loc}}(\Omega)$ for the space of their equivalence classes similarly as for $\mathcal{L}^p$ and L^p (see Section A). Similarly, $\mathcal{L}^p_{\text{loc}}(\Omega)$ is written for those functions $f : \Omega \to \Omega$ for which $f\mathbb{1}_K \in \mathcal{L}^p$ for all compact sets $K \subset \Omega$ and $L^p_{\text{loc}}(\Omega)$ for the space of equivalence classes in $\mathcal{L}^p_{\text{loc}}(\Omega)$.

Of course all continuous functions are locally integrable, but also all elements of $\mathcal{L}^p(\Omega)$:

Exercise 2.A. Prove that every function in $\mathcal{L}^p(\Omega)$ is locally integrable, where p is an element of $[1, \infty]$. Conclude that $\mathcal{L}^p_{\text{loc}}(\Omega) \subset \mathcal{L}^1_{\text{loc}}(\Omega)$.

2.6 (Locally integrable functions as distributions)**.** Let f be a locally integrable function on Ω. Define $u_f : \mathcal{D}(\Omega) \to \mathbb{F}$ by

$$u_f(\varphi) = \int_\Omega f\varphi = \int_\Omega f(x)\varphi(x)\ \mathrm{d}x \qquad (\varphi \in \mathcal{D}(\Omega)). \tag{2.2}$$

It is straightforward to verify that u_f is a distribution.

Similarly, if $f \in L^1_{\text{loc}}(\Omega)$ then we define u_f also by (2.2) (this is well-defined as if g and h are locally integrable functions which are equal a.e. (almost everywhere), then $u_f = u_g$).

Definition 2.7. Let V and W be vector spaces. We say that V *is embedded in* W if there exists a linear injection $V \to W$, which will also be called an *embedding*.

For Lemma 2.9 we recall Lebesgue's differentiation theorem:

Theorem 2.8 (Lebesgue's differentiation theorem)**.** [HvNVW16, Theorem 2.3.4] *For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ almost every point in $\mathbb{R}^d$ is a Lebesgue point, i.e., for almost all points x,*

$$\varepsilon^{-d} \int_{B(x,\varepsilon)} |f(y) - f(x)| \,\mathrm{d}y \xrightarrow{\varepsilon \downarrow 0} 0. \tag{2.3}$$

Lemma 2.9. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$, $u_f = 0$. Then $f = 0$ almost everywhere. In other words, the function $L^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$ given by $f \mapsto u_f$ is an embedding.*

Proof. Let $B \subset \Omega$ be a ball. Then by Theorem 1.11 there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of positive functions in $\mathcal{D}(\Omega)$ with $\varphi_n \uparrow \mathbb{1}_B$, indeed take $\varphi_n = \sum_{i=1}^n \chi_i$ where $(\chi_n)_{n\in\mathbb{N}}$ is a partition of unity on B. Then, by Lebesgue's Dominated Convergence Theorem, $\int_B f = \lim_{n\to\infty} \int f\varphi_n = \lim_{n\to\infty} u_f(\varphi_n) = 0$. As, by Theorem 2.8, almost every point of Ω is a density point of f, $f = 0$ almost everywhere on Ω. □

2.10 (Convention/Notation)**.** It is customary to identify locally integrable functions with their equivalence classes in L^1_{loc} and regard such an equivalence class as a function. Similarly, it is customary to identify the distribution u_f with the function f for any $f \in \mathcal{L}^1_{\mathrm{loc}}$ or $f \in L^1_{\mathrm{loc}}$, and to say that u_f "is" a function. We mostly follow this habit, but tend to be careful.

A distribution is determined by its "local behaviour", in the sense that if it is equal to zero around each point in Ω, then it is equal to zero, in the sense of the following theorem.

Theorem 2.11. *If $u, v \in \mathcal{D}'(\Omega)$ are such that for all $x \in \Omega$ there exists an open neighbourhood U of x such that $u(\varphi) = v(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi \subset U$, then $u = v$.*

Exercise 2.B. Prove Theorem 2.11.

Before we define operations on distributions, we motivate these by identities for locally integrable functions and their corresponding distributions in 2.13.

2.12 (Notation)**.** Let $f : \Omega \to \mathbb{F}$ and $y \in \mathbb{R}^d$. We define

$$-\Omega = \{-x : x \in \Omega\},$$
$$\Omega + y = \{x + y : x \in \Omega\},$$

and define the functions $\check{f} : -\Omega \to \mathbb{F}$ and $\mathcal{T}_y f : \Omega + y \to \mathbb{F}$ by

$$\check{f}(x) = f(-x), \qquad \mathcal{T}_y f(x) = f(x - y) \qquad (x \in \mathbb{R}^d). \tag{2.4}$$

We also write $\mathcal{R}f = \check{f}$.

2.13. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. The following statements follow by applying the change of variables formulae and integration by parts. (For the notation u_f see (2.2).)

(a) $u_{\check{f}}$ is a distribution on $-\Omega$ and for $\varphi \in \mathcal{D}(-\Omega)$

$$u_{\check{f}}(\varphi) = \int_{-\Omega} f(-x)\varphi(x)\ \mathrm{d}x = \int_{\Omega} f(x)\varphi(-x)\ \mathrm{d}x = u_f(\check{\varphi}) \tag{2.5}$$

(b) $u_{\mathcal{T}_y f}$ is a distribution on $\Omega + y$ and for $\varphi \in \mathcal{D}(\Omega + y)$

$$u_{\mathcal{T}_y f}(\varphi) = \int_{\Omega+y} f(x-y)\varphi(x)\ \mathrm{d}x = \int_{\Omega} f(x)\varphi(x+y)\ \mathrm{d}x = u_f(\mathcal{T}_{-y}\varphi). \tag{2.6}$$

(c) Suppose $f \in C^k(\Omega)$ for some $k \in \mathbb{N}$. Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$. Then $u_{\partial^\alpha f}$ is a distribution and for $\varphi \in \mathcal{D}(\Omega)$

$$u_{\partial^\alpha f}(\varphi) = \int_{\Omega} \partial^\alpha f(x)\varphi(x)\ \mathrm{d}x = (-1)^{|\alpha|}\int_{\Omega} f(x)\partial^\alpha\varphi(x)\ \mathrm{d}x = (-1)^{|\alpha|}u_f(\partial^\alpha\varphi). \tag{2.7}$$

(d) Let $\psi \in C^\infty(\Omega)$. Then $u_{\psi f}$ is a distribution and for $\varphi \in \mathcal{D}(\Omega)$

$$u_{\psi f}(\varphi) = \int_{\Omega} \psi(x)f(x)\varphi(x)\ \mathrm{d}x = u_f(\psi\varphi). \tag{2.8}$$

(e) Let $l : \mathbb{R}^d \to \mathbb{R}^d$ be linear and bijective. Then $f \circ l$ is locally integrable and $u_{f\circ l}$ is a distribution on $l^{-1}(\Omega)$ and for $\varphi \in \mathcal{D}(l^{-1}(\Omega))$

$$\begin{aligned} u_{f\circ l}(\varphi) &= \int_{l^{-1}(\Omega)} f\circ l(x)\varphi(x)\ \mathrm{d}x = \frac{1}{|\det l|}\int_{\Omega} f(x)\varphi\circ l^{-1}(x)\ \mathrm{d}x \\ &= \frac{1}{|\det l|}u_f(\varphi\circ l^{-1}). \end{aligned} \tag{2.9}$$

Exercise 2.C. Let $u \in \mathcal{D}'(\Omega)$. Define $w : \mathcal{D}(-\Omega) \to \mathbb{F}$ as follows. For $\varphi \in \mathcal{D}(-\Omega)$ define $w(\varphi)$ to be equal to the right-hand side of (2.5) with "u" instead of "u_f", i.e., $w(\varphi) = u(\check{\varphi})$. Check that w is a distribution (on $-\Omega$). Do the same for (2.6), (2.7), (2.8) and (2.9).

The analogous operations for distributions generalise the previous relations.

Definition 2.14. Let $y \in \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d, \psi \in C^\infty(\Omega)$ and $l : \mathbb{R}^d \to \mathbb{R}^d$ linear and bijective. For a distribution $u \in \mathcal{D}'(\Omega)$ we define the following distributions (it is easy to check that these are indeed distributions, see Exercise 2.C)

(a) $\check{u} \in \mathcal{D}'(-\Omega)$ by

$$\check{u}(\varphi) = u(\check{\varphi}) \qquad (\varphi \in \mathcal{D}(-\Omega)),$$

(b) $\mathcal{T}_y u \in \mathcal{D}'(\Omega + y)$ by

$$\mathcal{T}_y u(\varphi) = u(\mathcal{T}_{-y}\varphi) \qquad (\varphi \in \mathcal{D}(\Omega + y)),$$

(c) $\partial^\alpha u \in \mathcal{D}'(\Omega)$ by

$$\partial^\alpha u(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi) \qquad (\varphi \in \mathcal{D}(\Omega)),$$

(d) $\psi u \in \mathcal{D}'(\Omega)$ by

$$\psi u(\varphi) = u(\psi\varphi) \qquad (\varphi \in \mathcal{D}(\Omega)),$$

(e) $u \circ l \in \mathcal{D}'(l(\Omega))$ by

$$u \circ l(\varphi) = \frac{1}{|\det l|} u(\varphi \circ l^{-1}) \qquad (\varphi \in \mathcal{D}(l(\Omega))).$$

Observe that all the above operations are "linear in u". Moreover, observe that

$$\mathcal{T}_y \partial^\alpha u = \partial^\alpha \mathcal{T}_y u \qquad (y \in \mathbb{R}^d, \alpha \in \mathbb{N}_0^d).$$

Example 2.15. Let $d = 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function: $f(x) = |x|$ for $x \in \mathbb{R}$. Then one can compute (see Exercise 2.D)

$$\begin{aligned} \partial u_f &= u_g, \text{ for } g = \mathbb{1}_{(0,\infty)} - \mathbb{1}_{(-\infty,0)}, \\ \partial^2 u_f(\varphi) &= \partial u_g(\varphi) = 2\varphi(0). \end{aligned}$$

Of course for $x \neq 0$ we have $g(x) = f'(x)$. On the other hand, the second derivative $\partial^2 u_f$, which equals ∂u_g is not equal to u_h for any function h for which $h(x) = g'(x)$ for all $x \neq 0$. Actually, $\partial^2 u_f$ is given by a distribution corresponding to a specific Radon measure, namely the so-called Dirac δ-measure. We will now turn to such measures and their corresponding distributions.

Exercise 2.D. For f as in Example 2.15 check that $\partial u_f(\varphi) = u_g(\varphi)$ and $\partial^2 u_f(\varphi) = 2\varphi(0)$ for $\varphi \in \mathcal{D}(\Omega)$.

Exercise 2.E. In this exercise we consider dimension one and want to consider the function $x \mapsto \frac{1}{x}$ as a distribution. However, there is a problem of defining the integral by testing it against a testfunction and integrating around zero. Therefore we define the distribution differently.
(1) First prove that for all $\varphi \in \mathcal{D}(\mathbb{R})$ the limit

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{\varphi(x)}{x} \, \mathrm{d}x$$

exists, and equals $-\int_{\mathbb{R}} \varphi'(x) f(x) \, \mathrm{d}x$, where $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} \log|x| & x \neq 0, \\ 0 & x = 0. \end{cases}$$

For this check that f is integrable around zero and conclude that it is locally integrable.
(2) Prove that $u : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ defined by

$$u(\varphi) := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x} \, \mathrm{d}x \qquad (\varphi \in \mathcal{D}(\mathbb{R})),$$

defines a distribution and $u = \partial u_f$, where f is as in (1).

Exercise 2.F. (a) Prove that for any $u \in \mathcal{D}'(\mathbb{R})$, the following are equivalent

(1) $\partial u = 0$,
(2) there exists a $c \in \mathbb{F}$ such that $u = c u_{\mathbb{1}}$.

(Hint: Let $\psi \in \mathcal{D}$ with $\int \psi = 1$. Prove that for all $\varphi \in \mathcal{D}$, $\varphi - (\int \varphi)\psi$ has a primitive in $\mathcal{D}$.)

(b) Prove that for all $u \in \mathcal{D}'(\mathbb{R})$ there exists a $v \in \mathcal{D}'(\mathbb{R})$ such that $u = \partial v$.

Besides locally integrable functions, we can also interpret Radon measures as distributions. We introduce the notion of a Radon measure first, for some basic definitions from measure theory we refer the reader to Appendix A.

Definition 2.16. We write Borel(Ω) for the smallest σ-algebra that contains all open subsets of Ω. Borel(Ω) is called the *Borel-σ-algebra on* Ω. A *Radon measure on* Ω is a measure on Borel(Ω) (i.e., a countably additive function $\mu : \mathrm{Borel}(\Omega) \to [0, \infty]$ with $\mu(\emptyset) = 0$) such that $\mu(K) < \infty$ for all compact sets $K \subset \Omega$.

2.17 (Radon measures as distributions)**.** Let μ be a Radon measure on Ω. Then $u_\mu : \mathcal{D}(\Omega) \to \mathbb{F}$ defined by

$$u_\mu(\varphi) = \int_\Omega \varphi \, \mathrm{d}\mu \qquad (\varphi \in \mathcal{D}(\Omega)) \tag{2.10}$$

is a distribution.

Let us write λ for the Lebesgue measure. If $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ and $f \geq 0$, then $f\lambda$ is a Radon measure, where $f\lambda(A) = \int \mathbb{1}_A f$. Then u_f defined as in 2.6 equals $u_{f\lambda}$. Not all Radon measures are of the form $f\lambda$ with $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$, see for example the Dirac δ-measure:

Definition 2.18. For $x \in \mathbb{R}^d$ define the measure δ_x as follows; for a Borel measurable set A set

$$\delta_x(A) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

Then δ_x is a Radon measure, which is called the *Dirac δ-measure centered at x*. We call δ_0 the *Dirac δ-measure*. A *point measure* is a Dirac δ-measure centered at some point.

Observe that $\delta_x(\varphi) = \varphi(x)$ for any Borel measurable function φ.

Theorem 2.19. *If μ is a Radon measure on Ω, then for all open sets $U \subset \Omega$*

$$\mu(U) = \sup\{\mu(K) : K \subset U, K \text{ is compact}\} \tag{2.11}$$
$$= \sup\{\int \varphi \, \mathrm{d}\mu : \varphi \in C_c^\infty(\Omega, [0,1]), \operatorname{supp} \varphi \subset U\}. \tag{2.12}$$

Consequently, if $u_\mu(\varphi) = \int \varphi \, \mathrm{d}\mu = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, then $\mu = 0$. In other words, the map from the space of Radon measures into the space of distributions, $\mu \mapsto u_\mu$, is injective.

Proof. (2.11) follows from Theorem 1.8. (2.12) follows similarly as in the proof of Lemma 2.9; by Theorem 1.11 there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ with $\varphi_n \uparrow \mathbb{1}_U$. Then $\int \varphi_n \, \mathrm{d}\mu \uparrow \int \mathbb{1}_U \mu = \mu(U)$. □

2.20. Now the space of Radon measures is not a vector space, but it is closed under additions and multiplication with positive scalars. As the map $\mu \mapsto u_\mu$ preserves addition and multiplication with positive scalars, one could also say that the space of Radon measures is "embedded" into the space of distributions. As Radon measures can attain the value ∞ on a set, there is not a straightforward way of making sense of the difference of two Radon measures as a function on the σ-algebra (take for example λ the Lebesgue measure on $\mathbb{R}$ and let $\mu = \sum_{z\in\mathbb{Z}} \delta_z$).

We introduce the order of a distribution and then discuss how Radon measures correspond to distributions with order 0.

Definition 2.21. If u is a distribution and there exist a $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|u(\varphi)| \le C\|\varphi\|_{C^k} \qquad (\varphi \in \mathcal{D}(\Omega)), \tag{2.13}$$

then u is said to be of *order at most* k. In other words, for this C and k (2.1) holds for all compact sets K. If u is of order at most 0, we also say that u is of *order* 0. If $k \in \mathbb{N}$ and u is of order at most k but not of order at most $k-1$, then u is said to be of *order* k.

2.22. Every distribution of the form u_f for an integrable function f and every distribution of the form u_μ for a Radon measure μ with $\mu(\Omega) < \infty$ is of order 0.

On the other hand, the distribution $u_{\mathbb{1}}$ corresponding to the function $\mathbb{1}$ which is equal to 1 everywhere is not of any finite order: Let χ be as in Exercise 1.F and define $\varphi_n(x) = \sum_{k\in[-n,n]\cap\mathbb{Z}^d} \chi(x-k)$. Then $\|\varphi_n\|_{C^m} = \|\varphi_1\|_{C^m}$ for all $n \ge 2$ and $m \in \mathbb{N}_0^d$. On the other hand $u_{\mathbb{1}}(\varphi_n) \ge \int \mathbb{1}_{[-(n-1),n-1]} \ge (2(n-1))^d$ as $\varphi_n \ge \mathbb{1}_{[-(n-1),n-1]}$ for all $n \in \mathbb{N}$.

Exercise 2.G.

(a) Let $k \in \mathbb{N}_0$, $d = 1$. Show that $\partial^k \delta$ is a distribution of order at most k.

(b) Show that $\partial^k \delta$ is not of order at most $k-1$, i.e., show that it is of order k. (Hint: Test it against $\varphi_\varepsilon(x) = x^k \varphi(\frac{x}{\varepsilon})$ for $\varepsilon > 0$ with $\varphi \in \mathcal{D}(\Omega)$ with $\varphi = 1$ on a neighbourhood of 0.)

(c) What is the order of $\partial^\alpha \delta$ for $\alpha \in \mathbb{N}_0^d$?

(d) Construct a distribution which is not represented by a locally integrable function and not of any finite order.

Actually, all distributions of order 0 correspond to linear combinations of finite Radon measures (a Radon measure μ is called finite if $\mu(\Omega)$ is finite), see Theorem 2.28. Such linear combinations are called signed Radon measures if $\mathbb{F} = \mathbb{R}$ and complex Radon measures if $\mathbb{F} = \mathbb{C}$, see Definition 2.23 and Theorem 2.24. For the proof of Theorem 2.28, besides the Riesz representation theorem (Theorem 2.30) we will use convolutions. As convolutions are treated in Section 7, we postpone the proof of Theorem 2.28 to Section 9 (see below Corollary 9.8).

Definition 2.23. A Radon measure μ is called *finite* if $\mu(\Omega) < \infty$. We write $\mathcal{M}(\Omega, \mathbb{F})$ or $\mathcal{M}(\Omega)$ for the set of countably additive functions $\mu : \text{Borel}(\Omega) \to \mathbb{F}$. Elements of $\mathcal{M}(\Omega, \mathbb{R})$ are called *signed measures* and elements of $\mathcal{M}(\Omega, \mathbb{C})$ are called *complex measures.* We say that a signed or complex Radon measure μ is *positive*, if $\mu(A) \ge 0$ for all $A \in \text{Borel}(\Omega)$. We define $\|\cdot\|_{\mathcal{M}} : \mathcal{M}(\Omega, \mathbb{F}) \to [0, \infty]$ by

$$\|\mu\|_{\mathcal{M}} = \sup \left\{ \sum_{i=1}^n |\mu(A_i)| : A_1, \dots, A_n \text{ is a partition of } \Omega \text{ in } \text{Borel}(\Omega) \right\},$$

where a partition of Ω in $\text{Borel}(\Omega)$ is a finite number of pairwise disjoint sets $A_1, \dots, A_n$ in $\text{Borel}(\Omega)$ such that $\bigcup_{i=1}^n A_i = \Omega$.

We state the Hahn–Jordan Decomposition Theorem without proof; a proof can be found in [Con90, Theorem C.1] .

Theorem 2.24 (Hahn–Jordan Decomposition)**.** *For each $\mu \in \mathcal{M}(\Omega, \mathbb{R})$ there exist exactly one pair of disjunct positive finite Radon measures μ_+, μ_- such that there exist measurable sets $E_+, E_- \in \text{Borel}(\Omega)$ such that $E_+ \cap E_- = \emptyset$ and $E_+ \cup E_- = X$, $\mu_+(E_-) = 0$, $\mu_-(E_+) = 0$ and*

$$\mu = \mu_+ - \mu_-.$$

Moreover,

$$\|\mu\|_{\mathcal{M}} = \mu_+(\Omega) + \mu_-(\Omega).$$

For a μ in $\mathcal{M}(\Omega, \mathbb{C})$ we can define the functions $\Re\mu, \Im\mu : \text{Borel}(\Omega) \to \mathbb{R}$ defined by

$$(\Re\mu)(A) = \Re(\mu(A)) \qquad (\Im\mu)(A) = \Im(\mu(A)) \qquad (A \in \text{Borel}(\Omega)),$$

are signed measures, or in other words, elements of $\mathcal{M}(\Omega, \mathbb{R})$. By applying the previous theorem to $\Re\mu$ and $\Im\mu$ we obtain the following.

Corollary 2.25. *For each $\mu \in \mathcal{M}(\Omega, \mathbb{C})$ there exist four positive finite Radon measures $\mu_1, \mu_2, \mu_3, \mu_4$ such that*

$$\mu = \mu_1 - \mu_2 + \mathrm{i}[\mu_3 - \mu_4],$$

and

$$\|\mu\|_{\mathcal{M}} = \mu_1(\Omega) + \mu_2(\Omega) + \mu_3(\Omega) + \mu_4(\Omega).$$

The Hahn–Jordan decomposition in particular implies:

Lemma 2.26. *$\|\cdot\|_{\mathcal{M}}$ is a norm on $\mathcal{M}(\Omega)$. $L^1(\Omega, \mathbb{F})$ is embedded in $\mathcal{M}(\Omega, \mathbb{F})$ by the function $L^1(\Omega, \mathbb{F}) \to \mathcal{M}(\Omega, \mathbb{F})$, $f \mapsto f\lambda$, where λ is the Lebesgue measure on Ω. Moreover,*

$$\|f\|_{L^1} = \|f\lambda\|_{\mathcal{M}} \qquad (f \in L^1(\Omega, \mathbb{F})). \tag{2.14}$$

Proof. That $\|\cdot\|_{\mathcal{M}}$ is a norm follows by the Hahn–Jordan decompositions. In case $\mathbb{F} = \mathbb{R}$ (2.14) follows as $\|f\|_{L^1} = \int f^+ + \int f^-$, where $f^+(x) = f(x) \vee 0$ and $f^-(x) = -f(x) \vee 0$ for $x \in \Omega$. The case $\mathbb{F} = \mathbb{C}$ is similar. □

2.27. For a signed or complex Radon measure $\mu \in \mathcal{M}(\Omega, \mathbb{F})$ we define $u_\mu : \mathcal{D}(\Omega) \to \mathbb{F}$ by (2.10). As a consequence of Theorem 2.24 and Theorem 2.19 $\mathcal{M}(\Omega, \mathbb{F})$ is embedded in $\mathcal{D}'(\Omega)$ by the embedding $\mu \mapsto u_\mu$.

Theorem 2.28. *A distribution u is of order 0 if and only if $u = u_\mu$ for a $\mu \in \mathcal{M}(\Omega, \mathbb{F})$.*

The following theorem, the Riesz representation theorem, will be used for the proof of Theorem 2.28 in Section 9 (it is given below Corollary 9.8). For a proof of Theorem 2.30 see [Con90, Theorem C.18] .

Definition 2.29. We write $C_0(\Omega, \mathbb{F})$ or $C_0(\Omega)$ for the continuous functions $\Omega \to \mathbb{F}$ such that for all $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $|f| < \varepsilon$ on $\Omega \setminus K$. $C_0(X, \mathbb{F})$ is equipped with the norm $\|\cdot\|_{C^0}$.

Theorem 2.30 (Riesz representation theorem)**.** *For $\mu \in \mathcal{M}(\Omega, \mathbb{F})$ define $\Psi_\mu : C_0(\Omega, \mathbb{F}) \to \mathbb{F}$ by*

$$\Psi_\mu(f) = \int f \, \mathrm{d}\mu \qquad (f \in C_0(\Omega, \mathbb{F})).$$

Then $\psi_\mu \in C_0(\Omega, \mathbb{F})'$ and the map $\mathcal{M}(\Omega, \mathbb{F}) \to C_0(\Omega, \mathbb{F})'$, $\mu \mapsto \Psi_\mu$ is an isometric isomorphism.

Corollary 2.31. *$\mathcal{M}(\Omega, \mathbb{F})$ is a Banach space under the norm $\|\cdot\|_{\mathcal{M}}$.*

Proof. This follows by Theorem 2.30 as the dual of a normed space, in this case $C_0(\Omega, \mathbb{F})'$, is a Banach space. See for example [Con90, Proposition III.5.4] or [Rud91, Theorem 4.1]. □

3 Topological vector spaces

In this section we introduce some topological notions like topological vector spaces, locally convex spaces and weak topologies on dual pairs. We discuss a few properties like metrizability and separation properties. These notions will be used in Section 4 to define the topologies on $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$. We recall some definitions of topology in 3.1, but will not require the reader to be familiar with nets; see 3.2 for some comments.

3.1 (Topological vocabulary)**.** Let X be a set. A *topology* τ on X is a set of subsets of X, called *open* sets, such that $\emptyset, X \in \tau$, if $A, B \in \tau$ then $A \cap B \in \tau$ and if $\mathcal{U} \subset \tau$, then $\bigcup \mathcal{U} \in \tau$. Then the pair (X, τ) is called a *topological* space; but it is also common to say that X is a topological space itself.

If X is a topological space with topology τ, then we use the following vocabulary. A set $F \subset X$ is called *closed* if its complement $X \setminus F$ is open. The *closure* $\overline{A}$ of a set $A \subset X$ is the smallest closed set that contains A; it is the intersection of all closed sets that contain A. The *interior* A° of a set $A \subset X$ is the largest open set contained in A; it is the union of all open subsets of A. A set $K \subset X$ is *compact* if each open covering of K has a finite subcovering. A *neighbourhood* of a point $x \in X$ is a set that contains an open set that contains x. For a set $A \subset X$ we call an set that contains an open set that contains A also a *neighbourhood* of A. If $Y \subset X$ and $\sigma = \{U \cap Y : U \in \tau\}$, then σ is called the *relative topology* on Y. If σ is a collection of subsets of X, then τ is said to be *generated* by σ, or we also say, σ *generates* the topology τ if τ is the smallest topology that contains σ. One can show that τ consists of those sets which are unions of finite intersections of elements in σ. If $x \in X$ and $\mathcal{U}$ is a collection of subsets of X with $x \in U$ for all $U \in \mathcal{U}$, then $\mathcal{U}$ is called a *local base* for x if for each neighbourhood V of x there exists a $U \in \mathcal{U}$ with $U \subset V$. X is called *connected* if it is not the union of two disjoint non-empty open subsets of X. X is called *metrizable* if there exists a metric d on X such that the topology is generated by the balls $\{x \in X : d(x, y) < r\}$ with $y \in X$ and $r > 0$; in that case one says that d is *compatible* with the topology on X. A metric d on a vector space X is called *translation invariant* if $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$.

Let X and Y be topological spaces. A function $f : X \to Y$ is called *continuous* if $f^{-1}(U)$ is open in X for each open set in Y. The *product topology* on $X \times Y$ is the topology generated by the collection of sets $U \times V$, with U being an open set in X and Y an open set in Y. Equivalently, the product topology on $X \times Y$ is the smallest topology such that the projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous. If not mentioned otherwise, for two topological spaces X and Y the space $X \times Y$ is equipped with the product topology.

3.2 (Sequences and nets)**.** For metric spaces X and Y we have that $f : X \to Y$ is continuous if and only if $x_n \to x$ implies $f(x_n) \to f(x)$ for each sequence $(x_n)_{n \in \mathbb{N}}$ and x in X. Moreover, a set F in X is closed if and only if each sequence in F that converges in X has its limit in F. This does not hold in general for any topological space. But one can replace the usage of sequences by nets. In order to keep the text readable for those who are not so familiar with that notion, we avoid it. Moreover, for the theory of distributions, as we will see, it is often enough to consider convergence of sequences.

Definition 3.3. A vector space X equipped with a topology τ is called a *topological vector space* if both the operations addition and scalar multiplication, that is the functions

$$X \times X \to X, \quad (x, y) \mapsto x + y,$$
$$\mathbb{F} \times X \to X, \quad (\lambda, X) \mapsto \lambda x,$$

are continuous.

It is easy to see that each normed vector space is a topological vector space.

Observe that if $\mathcal{U}$ is a local base for 0, then the topology of X is generated by the sets $x + U$ with $x \in X$ and $U \in \mathcal{U}$.

We recall the definition of a seminorm.

Definition 3.4. Let X be a vector space. A *seminorm* on X is a function $p : X \to [0, \infty)$ such that

$$p(x + y) \leq p(x) + p(y) \qquad (x, y \in X),$$
$$p(\lambda x) = |\lambda| p(x) \qquad (\lambda \in \mathbb{F}, x \in X).$$

A seminorm p is a *norm* if $p(x) = 0$ implies $x = 0$.

Definition 3.5. Let $\mathcal{P}$ be a collection of seminorms on a vector space X. Let τ be the topology generated by the sets

$$\{x \in X : p(x - y) < r\} \qquad (y \in X, r > 0).$$

Then τ is called the *topology generated by* $\mathcal{P}$.

A set U is open in X, i.e., $U \in \tau$ if and only if for each $y \in U$ there exist $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$, $r_, \ldots, r_n > 0$ such that

$$\bigcap_{i=1}^{n} \{x \in X : p_i(x - y) < r_i\} \subset U.$$

Moreover, X equipped with the topology generated by $\mathcal{P}$ is a topological vector space (the proof of this is rather straightforward and left to the reader).

Definition 3.6. Let $\mathcal{P}$ be a collection of seminorms. A vector space X equipped with the topology generated by $\mathcal{P}$ is called a *locally convex space* if

$$\bigcap_{p \in \mathcal{P}} \{x \in X : p(x) = 0\} = \{0\}. \tag{3.1}$$

If $\mathcal{P}$ is such that (3.1) holds, then $\mathcal{P}$ is called a *separating family of seminorms* and the topology generated by $\mathcal{P}$ is called a *locally convex topology.*

A locally convex space is Hausdorff, that is, if $x, y \in X$ and $x \neq y$, then there exist open sets U, V such that $U \cap V = \emptyset$, $x \in U$ and $y \in V$: By (3.1) there exists a $p \in \mathcal{P}$ such that $p(x-y) \neq 0$. Let $\varepsilon > 0$ be such that $p(x-y) > 2\varepsilon$. Take $U = \{z \in X : p(z-x) < \varepsilon\}$ and $V = \{z \in X : p(z-y) < \varepsilon\}$.

Of course, every normed space is a locally convex space.

3.7 (Why is it called "locally convex"?)**.** From this definition it might not be clear why this is called "locally convex". This is due to the fact that the topology is such that for all x and open neighbourhood U of x there exists a convex open neighbourhood V with $x \in V \subset U$, see for example [Rud91, Theorem 1.37].

As we will see in 3.11 the most obvious choice of equipping the testfunctions with the $\|\cdot\|_{C^k}$ norms with $k \in \mathbb{N}$, leads to a metrizable locally convex topology with is not complete. That it is metrizable follows from the following theorem.

Theorem 3.8. *A locally convex vector space X is metrizable if and only if it is generated by countably many seminorms. Moreover, if the topology of X is generated by a countable number of seminorms $p_1, p_2, \dots,$ then the topology is compatible with the translation invariant metric d on X defined by*

$$d(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} \wedge p_n(x-y) \qquad (x, y \in X). \tag{3.2}$$

Proof. (Optional) If X is generated by countably many seminorms $p_1, p_2, \dots,$ then it is easy to check that the metric defined in (3.2) is compatible with the topology.

Suppose X is metrizable and $\mathfrak{d}$ be a metric on X that is compatible with the topology. Then it is easy to check that $d(x,y) := \mathfrak{d}(0, y-x)$ for $x, y \in X$ defines a translation invariant metric on X. Moreover, as the neighbourhoods of 0 are the same for $\mathfrak{d}$ and d, also d is compatible with the topology on X (as a translation by x of a local base for 0 is a local base for x).

X has a countable local base for each $y \in X$ (such X are called "first countable"), namely $\mathcal{U}_y = \{V_n : n \in \mathbb{N}\}$, where

$$V_n := \{x \in X : d(x,y) < \frac{1}{n}\} \qquad (n \in \mathbb{N}).$$

As d is translation invariant we have $\mathcal{U}_y = x + \mathcal{U}_0$, indeed $\{x \in X : d(x,y) < \frac{1}{n}\} = y + \{x \in X : d(0, y-x) < \frac{1}{n}\}$ for all $n \in \mathbb{N}$.

Let $\mathcal{P}$ be a collection of seminorms that generates the topology on X. Define

$$U_{p,m} := \{x \in X : p(x) < \frac{1}{m}\} \qquad (p \in \mathcal{P}, m \in \mathbb{N}).$$

Then $\{U_{p,m} : p \in \mathcal{P}, m \in \mathbb{N}\}$ forms a local base for 0. Therefore, for each $n \in \mathbb{N}$, we can choose a $p_n \in \mathcal{P}$ and $m_n \in \mathbb{N}$ such that

$$U_{p_n, m_n} \subset V_n.$$

This implies that $\{U_{p_n,m_n} : n \in \mathbb{N}\}$ is a local base for 0. Therefore the topology is generated by the countable set of seminorms $\{p_n : n \in \mathbb{N}\}$. $\square$

Let us introduce the notion of a Cauchy sequence and completeness for locally convex spaces. In 3.10 we comment on how this notion agrees with the notions of Cauchy sequences and completeness for metric spaces.

Definition 3.9. Let X be a locally convex topological space and $\mathcal{P}$ be a collection of seminorms that generate the topology of X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called $\mathcal{P}$-*Cauchy* or just *Cauchy*, if for each $p \in \mathcal{P}$ it is Cauchy with respect to p:

$$\text{for all } \varepsilon > 0 \text{ there exists an } N \in \mathbb{N} \text{ such that } p(x_n - x_m) < \varepsilon \text{ for all } n, m \geq N.$$

If each $\mathcal{P}$-Cauchy sequence in X converges, X is called $\mathcal{P}$-*complete* or just *complete*.

3.10 ($\mathcal{P}$-Cauchy and Cauchy with respect to a metric)**.** Let X and $\mathcal{P}$ be as in Definition 3.9. If X is a normed space, which means that its topology is generated by one norm, then a sequence is $\mathcal{P}$-Cauchy if and only if it is Cauchy with respect to the norm (and thus distance). Consequently, the space is $\mathcal{P}$-complete if and only if it is complete with respect to the norm.

If X is metrizable, which means that one can take $\mathcal{P}$ to be countable, then a sequence is $\mathcal{P}$-Cauchy if and only if it is d-Cauchy with d the translation invariant metric as in (3.2).

It is important to have such a translation invariant metric, as the following example illustrates: Take $X = \mathbb{R}$ and let d and $\mathfrak{d}$ be metrics on $\mathbb{R}$ given by $d(x,y) = |x-y|$ and $\mathfrak{d}(x,y) = |e^{-x} - e^{-y}|$ for $x, y \in \mathbb{R}$. Then both metrics generate the same topology, however the sequence $1, 2, 3, \dots$ is $\mathfrak{d}$-Cauchy but not d-Cauchy, and $\mathbb{R}$ is complete under d but not under $\mathfrak{d}$.

3.11 (A metrizable topology on $\mathcal{D}(\Omega)$ which is not complete)**.**
On $\mathcal{D}(\Omega)$ the function $\|\cdot\|_{C^k}$ is a (semi)norm for each $k \in \mathbb{N}_0$ and therefore these seminorms generate a locally convex topology on $\mathcal{D}(\Omega)$. This topology is metrizable, see Theorem 3.8, however the topology is not complete: Let χ be a testfunction which equals 1 at 0 with support in $(-\frac{1}{2}, \frac{1}{2})$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be in $(0,\infty)$ and $\varepsilon_n \downarrow 0$ and let $\chi_n = \mathcal{T}_n\chi$. Take $\varphi_n = \sum_{i=1}^n \varepsilon_i \chi_i$. Then $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric d of Theorem 3.8 with $p_k = \|\cdot\|_{C^k}$ but does not have a limit in $\mathcal{D}(\mathbb{R}^d)$. One can adapt this argument to show that also $\mathcal{D}(\Omega)$ is not complete for any open set $\Omega \subset \mathbb{R}^d$ (for example in the spirit of Exercise 1.C or with the use of a partition of unity).

Exercise **3.A.** Show that (for a general open set $\Omega \subset \mathbb{R}^d$) the space $\mathcal{D}(\Omega)$ is not complete in the metrizable topology generated by the seminorms $\|\cdot\|_{C^k}$ for $k \in \mathbb{N}$.

The topology mentioned in 3.11 is not the topology which $\mathcal{D}(\Omega)$ is equipped with. In order to introduce that topology, we make some notions and introduce some definitions.

First, we observe that distributions define seminorms on the space of testfunctions:

3.12. Each distribution u defines a seminorm on $\mathcal{D}(\Omega)$ by

$$\varphi \mapsto |u(\varphi)|. \tag{3.3}$$

By Lemma 2.9 applied to $f \in \mathcal{D}(\Omega)$ it follows that if $u_f(\varphi) = u_\varphi(f) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, then $f = 0$. Therefore the collection of seminorms generated by the distributions, that is the seminorms (3.3) for $u \in \mathcal{D}'(\Omega)$ form a separating family.

In the notation $\mathcal{D}'(\Omega)$ we have used the symbol " $'$ ", which is commonly used to denote the topological dual of a space, as in Definition 3.13. The definition of the topologies of $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are given in Definition 4.1 in terms of the topologies defined in Definition 3.16. The topology on $\mathcal{D}(\Omega)$ is exactly the one generated by the seminorms defined by the distributions as in 3.12. In the following definition we introduce the notion of a topological dual of a topological vector space. It turns out that $\mathcal{D}'(\Omega)$ is indeed the topological dual of $\mathcal{D}(\Omega)$ with the topology that we consider. This relies on Theorem 3.19, as we mention in Definition 4.1. First we introduce the notion of a dual and use the Hahn–Banach theorem to show Lemma 3.15, which shows that for a locally convex space X with its dual X' forms a pair as defined in Definition 3.16. As those theorems are not essential for the further theory of distributions, the reader may skip those theorems which are indicated by ($\diamond \diamond \diamond$).

Definition 3.13. Let X be a topological vector space over $\mathbb{F}$. The space of linear continuous maps $X \to \mathbb{F}$ is called the *dual* or *topological dual* of X. We write X' for the dual of X. Each element of $x \in X$ determines a seminorm on X' by $f \mapsto |f(x)|$. We equip X' with the locally convex topology generated by these seminorms, which is also called the *weak* topology*.

The elements of X' for a locally convex space X separate the points in X, see Lemma 3.15. To prove this lemma we use the Hahn–Banach theorem, which we state without a proof (for a proof see for example [Con90, Corollary III.6.4] or [Rud91, Theorem 3.3]).

Theorem 3.14 (Hahn–Banach)**.** ($\diamond\diamond\diamond$) *Let X be a vector space, M be a linear subspace of X and p be a seminorm on X. If $f : M \to \mathbb{F}$ is a linear function such that $|f(x)| \le p(x)$ for all $x \in M$, then there is a linear extension of f to X, $F : X \to \mathbb{F}$ ($F|_M = f$) with $|F(x)| \le p(x)$ for all $x \in X$.*

Lemma 3.15. ($\diamond \diamond \diamond$) *Let X be a locally convex vector space. Let M be a linear subspace of X. Let $f : M \to \mathbb{F}$ be linear and continuous (with respect to the relative topology), then f has a continuous linear extension to X.*

Consequently, for each $x, y \in X$ with $x \neq y$ and $x \neq 0$ there exists an $F \in X'$ such that $F(x) = 1$ and $F(y) = 0$.

Proof. Suppose that $\mathcal{P}$ is a collection of seminorms that generates the topology of X. As

f is continuous on M there exist seminorms $p_1, \dots, p_n \in \mathcal{P}$ and $r_1, \dots, r_n > 0$ such that

$$\bigcap_{i=1}^{n} \{x \in X : p_i(x) < r_i\} \subset f^{-1}(B(0,1)).$$

Let $p = \max_{i=1}^n \frac{p_i}{r_i}$. It is easy to check that p is a continuous seminorm on X. As $p(x) < 1$ implies $f(x) < 1$, by linearity we have $|f(x)| \leq p(x)$. By the Hahn–Banach theorem there exists an exists a linear extension $F : X \to \mathbb{F}$ of f, i.e., $F(x) = f(x)$ for $x \in M$, and $|F(x)| \leq p(x)$. The latter implies that F is continuous. □

Definition 3.16. Let X and Y be vector spaces over $\mathbb{F}$ and $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{F}$ be a bilinear form that satisfies the separation axioms:

$$\begin{aligned} \langle x, y \rangle = 0 \text{ for all } y \in Y \quad &\text{implies} \quad x = 0, \\ \langle x, y \rangle = 0 \text{ for all } x \in X \quad &\text{implies} \quad y = 0. \end{aligned}$$

Such a pair (X, Y) is called a *dual pair.*

The *weak topology* $\sigma(X, Y)$ on X is the coarsest topology on X such that all maps $\langle \cdot, y \rangle$ with $y \in Y$ are continuous. In other words, this topology is generated by the seminorms $x \mapsto |\langle x, y \rangle|$ for $y \in Y$ and therefore a locally convex topology. Similarly one defines the weak topology $\sigma(Y, X)$ on Y; it is the coarsest topology on Y such that the maps $\langle x, \cdot \rangle$ with $x \in X$ are continuous.

Example 3.17.

(a) Let V be a vector space and $V^\#$ be its algebraic dual. The pair $(V, V^\#)$ forms a dual pair under the bilinear form $V \times V^\# \to \mathbb{F}$ given by $(x, f) \mapsto f(x)$.
(b) Let X be a locally convex vector space. By Lemma 3.15 the pair (X, X') forms a dual pair under the bilinear form $X \times X' \to \mathbb{F}$ given by $(x, f) \mapsto f(x)$.
(c) As the distributions separate the testfunctions as we have observed in 3.12, the pair $(\mathcal{D}(\Omega), \mathcal{D}'(\Omega))$ forms a dual pair under the bilinear form $\mathcal{D}(\Omega) \times \mathcal{D}'(\Omega) \to \mathbb{F}$ given by $(\varphi, u) \mapsto u(\varphi)$.

We will show that for X and Y as in Definition 3.16 Y is the dual of X equipped with $\sigma(X, Y)$ in Theorem 3.19, so that the name "dual pair" now actually makes sense. The next lemma is a preparation for the proof.

Lemma 3.18. ($\diamond \diamond \diamond$) *Let X and Y be a dual pair. Let $n \in \mathbb{N}$ and $y_1, \dots, y_n \in Y$ be linearly independent. Then there exist linearly independent $x_1, \dots, x_n \in X$ such that $\langle x_i, y_j \rangle = \delta_{i,j}$ for all $i, j \in \{1, \dots, n\}$, where $\delta_{i,j}$ is the Kronecker symbol, which equals* 1 *if $i = j$ and* 0 *otherwise.*

Proof. For $n = 1$ this follows immediately by the separation axioms. We continue by induction. Let $n \in \mathbb{N}$ and $y_1, \dots, y_{n+1} \in Y$ be linearly independent. Let $\tilde{x}_1, \dots, \tilde{x}_n$ be such that $\langle \tilde{x}_j, y_i \rangle = \delta_{i,j}$ for $i, j \in \{1, \dots, n\}$. Let M_n be the linear span of $\tilde{x}_1, \dots, \tilde{x}_n$ and let $F_n = \{x \in X : \langle x, y_i \rangle = 0, i \in \{1, \dots, n\}\}$. Then X is the direct sum of F_n and M_n, i.e., $X = F_n + M_n$. The map $\langle \cdot, y_{n+1} \rangle$ cannot vanish on F_n, because if it would, then y_{n+1} would be a linear combination of $y_1, \dots, y_n$. Therefore there exists an $x_{n+1} \in F_n$ with $\langle x_{n+1}, y_{n+1} \rangle = 1$. By defining $x_i = \tilde{x}_i - \langle \tilde{x}_i, y_{n+1} \rangle x_{n+1}$ we have $\langle x_i, y_j \rangle = \delta_{i,j}$ for all $i, j \in \{1, \dots, n+1\}$, because of this the x_i have to be independent. □

Theorem 3.19. ($\diamond \diamond \diamond$) *Let X and Y be a dual pair. The dual of the topological vector space $(Y, \sigma(Y, X))$ is X. This means that if $f : Y \to \mathbb{F}$ is continuous and linear, then there exists a unique $x \in X$ such that $f(y) = \langle x, y \rangle$ for $y \in Y$.*

Proof. As $f^{-1}(B(0,1))$ is an open set, there exist $x_1, \dots, x_n \in X$ such that

$$\bigcap_{i=1}^{n} \{y \in Y : |\langle x_i, y \rangle| < 1\} \subset f^{-1}(B(0,1)).$$

So $\max_{i=1}^{n} |\langle x_i, y \rangle| < 1$ implies $f(y) < 1$ for all $y \in Y$. As f is linear, we obtain

$$|f(y)| \le \max_{i=1}^{n} |\langle x_i, y \rangle| \qquad (y \in Y). \tag{3.4}$$

We may assume that $x_1, \dots, x_n$ are linearly independent. Let $f_i \in Y^{\#}$ be defined by $f_i(y) = \langle x_i, y \rangle$ for $y \in Y$. Then $f_1, \dots, f_n$ are linearly independent. By Lemma 3.18 f cannot be linearly independent from $f_1, \dots, f_n$ because of (3.4). Hence f is a linear combination of $f_1, \dots, f_n$, from which it follows that there exists an x (which is a linear combination of $x_1, \dots, x_n$) such that $f(y) = \langle x, y \rangle$ for all $y \in Y$. □

4 Topologies on the spaces of testfunctions and distributions

In this section we equip $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ with topologies.

Definition 4.1 (Pairing and topologies on $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$)**.**
We define $\langle \cdot, \cdot \rangle : \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{F}$ by

$$\langle u, \varphi \rangle = u(\varphi) \qquad ((u, \varphi) \in \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)). \tag{4.1}$$

We equip the space of testfunctions $\mathcal{D}(\Omega)$ with the weak topology $\sigma(\mathcal{D}(\Omega), \mathcal{D}'(\Omega))$ and the space of distributions $\mathcal{D}'(\Omega)$ with the weak* topology, that is, with the weak topology $\sigma(\mathcal{D}(\Omega)', \mathcal{D}(\Omega))$.

By Theorem 3.19 it follows that $\mathcal{D}'(\Omega)$ is the dual space of $\mathcal{D}(\Omega)$ (equipped with the $\sigma(\mathcal{D}(\Omega), \mathcal{D}'(\Omega))$ topology).

4.2 (Convention/Notation)**.** As we mentioned already in 2.10, it is customary to identify locally integrable functions with their corresponding distribution. That is, one may write

"f" instead of "u_f" for a locally integrable function (or equivalence class) f. However, we still prefer not to write "$f(\varphi)$" instead of "$u_f(\varphi)$" so we will write "$\langle f, \varphi\rangle$" instead.

The notation "$\langle\cdot,\cdot\rangle$" is also commonly used for inner products and this might cause confusion. Indeed, say we take $f, g \in \mathcal{D}$ and as mentioned above, view f as the distribution u_f. Then $\langle f, g\rangle$ is $\int fg$ which is not the same (at least not for general $\mathbb{C}$-valued functions) as $\int f\overline{g}$, the latter is the inner product of f and g, for which we write $\langle f, g\rangle_{L^2}$.

Remark 4.3 (Another way to introduce the topology on $\mathcal{D}(\Omega)$)**.** In the literature there are basically two approaches to the topologies on testfunctions and the distributions. The one presented here, where first the distributions are defined and then the topology as a weak topology. Or one where one first defines a topology on the testfunctions via an inductive limit approach, also called .... Then the space of distributions is defined to be the dual of this space, i.e., the space of linear functionals on the testfunctions that are continuous with respect to the topology (this approach is followed by the books on functional analysis [Con90] and [Rud91]). The topologies differ slightly (we comment on this in ...), but the convergence of sequences is the same, and, as we will see, the space of distributions is determined by that (and therefore is the same for both approaches).

4.4. Let $y \in \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $\psi \in C^\infty(\Omega)$ and $l : \mathbb{R}^d \to \mathbb{R}^d$ be a linear bijection. Observe that the operations $\check{}$, $\mathcal{T}_y, \partial^\alpha$, multiplication by ψ and composition with l, i.e.,

$$\begin{aligned}
&\mathcal{D}(\Omega) \to \mathcal{D}(-\Omega), && \varphi \mapsto \check{\varphi}, && \mathcal{D}'(\Omega) \to \mathcal{D}'(-\Omega), && u \mapsto \check{u},\\
&\mathcal{D}(\Omega) \to \mathcal{D}(\Omega+y), && \varphi \mapsto \mathcal{T}_y\varphi, && \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega+y), && u \mapsto \mathcal{T}_y u,\\
&\mathcal{D}(\Omega) \to \mathcal{D}(\Omega), && \varphi \mapsto \partial^\alpha\varphi, && \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega), && u \mapsto \partial^\alpha u,\\
&\mathcal{D}(\Omega) \to \mathcal{D}(\Omega), && \varphi \mapsto \psi\varphi, && \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega), && u \mapsto \psi u,\\
&\mathcal{D}(\Omega) \to \mathcal{D}(l(\Omega)), && \varphi \mapsto \varphi\circ l, && \mathcal{D}'(\Omega) \to \mathcal{D}'(l(\Omega)), && u \mapsto u\circ l,
\end{aligned}$$

are continuous. As an example we consider multiplication with ψ. Let us write $\Pi_\psi : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ for $\varphi \mapsto \psi\varphi$:
As the operations is linear, it is sufficient to show continuity at zero. Instead of showing that the preimage of any open neighbourhood of 0 under the multiplication is an open set, it is sufficient to consider the preimage of neighbourhoods generated by the distributions. The following sets form a local base for 0 in the topology on $\mathcal{D}(\Omega)$ (Exercise 4.A):

$$\bigcap_{i=1}^n \{\varphi \in \mathcal{D}(\Omega) : |\langle u_i, \varphi\rangle| < 1\} \qquad (n \in \mathbb{N}, u_1, \dots, u_n \in \mathcal{D}'(\Omega)). \tag{4.2}$$

Fix $u_1, \dots, u_n$ in $\mathcal{D}'(\Omega)$ and let $U = \bigcap_{i=1}^n \{\varphi \in \mathcal{D}(\Omega) : |\langle u_i, \varphi\rangle| < 1\}$. Then

$$\begin{aligned}
\Pi_\psi^{-1}(U) = \{\varphi \in \mathcal{D}(\Omega) : \psi\varphi \in U\} &= \bigcap_{i=1}^n \{\varphi \in \mathcal{D}(\Omega) : |\langle u_i, \psi\varphi\rangle| < 1\}\\
&= \bigcap_{i=1}^n \{\varphi \in \mathcal{D}(\Omega) : |\langle \psi u_i, \varphi\rangle| < 1\},
\end{aligned}$$

which is again a set of the form (4.2), hence open. With similar arguments the other operations can be shown to be continuous; this is left to the reader.

Exercise **4.A.** Show that indeed the open sets

$$\bigcap_{i=1}^{n}\{\varphi \in \mathcal{D} : |\langle u_i, \varphi\rangle| < 1\}$$

for $u_1, \ldots, u_n \in \mathcal{D}'(\Omega)$ form a local base at 0 in $\mathcal{D}(\Omega)$.

The characterisation of convergence of sequences is given in Theorem 4.11. Before some auxiliary lemmas are given. First we show that we can embed $\mathcal{D}(U)$ in $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ in $\mathcal{D}'(U)$ whenever U is an open subset of Ω.

Definition 4.5. Let X and Y be topological vector spaces. We say that X is *continuously embedded* in Y, and write $X \hookrightarrow Y$, if there exists a continuous embedding $X \to Y$.

If Z is another topological vector space, then we will write "$X \hookrightarrow Y \hookrightarrow Z$" instead of "$X \hookrightarrow Y$ and $Y \hookrightarrow Z$", etc.

4.6 (Restriction of a distribution to a smaller set)**.** Suppose U is an open subset of Ω. As every compact set in U is compact in Ω, there exists a linear injection

$$\iota : \mathcal{D}(U) \to \mathcal{D}(\Omega),$$

with $\iota(\varphi)(x) = \varphi(x)$ for $x \in U$ and $\iota(\varphi)(x) = 0$ for $x \in \Omega \setminus U$, and $\varphi \in \mathcal{D}(U)$. On the other hand, for $u \in \mathcal{D}'(\Omega)$ we define $\rho(u) : \mathcal{D}(U) \to \mathbb{F}$ by

$$\langle \rho(u), \varphi\rangle = \langle u, \iota(\varphi)\rangle \qquad (\varphi \in \mathcal{D}(U)).$$

Then $\rho(u)$ is a distribution, and so ρ forms a map $\mathcal{D}'(\Omega) \to \mathcal{D}'(U)$. It follows that both ρ and ι are linear and continuous. So $\mathcal{D}(U)$ can be continuously embedded in $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ can be continuously embedded in $\mathcal{D}'(U)$, i.e.,

$$\mathcal{D}(U) \hookrightarrow \mathcal{D}(\Omega), \qquad \mathcal{D}'(\Omega) \hookrightarrow \mathcal{D}'(U).$$

For this reason, we will view $\rho(u)$ as the restriction of u to $\mathcal{D}(U)$. Therefore, when $v \in \mathcal{D}'(U)$ we will say "$u = v$ on U" instead of "$\rho(u) = v$". Moreover, if $u \in \mathcal{D}'(U)$ and $\varphi \in \mathcal{D}(\Omega)$, then we will write "$u(\varphi)$" instead of "$u(\varphi|_U)$".

Observe, moreover, that ι is also continuous with respect to the seminorms $\|\cdot\|_{C^k}$ with $k \in \mathbb{N}_0$, and

$$\|\iota(\varphi)\|_{C^k(\Omega)} = \|\varphi\|_{C^k(U)} \qquad (\varphi \in \mathcal{D}(U)). \tag{4.3}$$

The following lemma is an application of the mean-value theorem to higher dimensions.

Lemma 4.7. *Let $V, U \subset \mathbb{R}^d$, V be convex, U be open and $V \subset U$. If $\psi \in C^1(U)$ and*

$$M = \max_{i=1}^{d} \sup_{x\in V} |\partial_i\psi(x)| < \infty,$$

then ψ is Lipschitz continuous on V with Lipschitz constant M.

Proof. Let $x, y \in U$. For $t \in [0,1]$ we have that $\frac{\mathrm{d}}{\mathrm{d}t}\psi(tx+(1-t)y)$ equals the directional derivative of ψ in the direction $x-y$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(tx+(1-t)y) = \nabla\psi(tx+(1-t)y)\cdot(x-y).$$

Therefore by the mean-value theorem

$$|\psi(x)-\psi(y)| = \Big|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\psi(tx+(1-t)y)\;\mathrm{d}t\Big| \le \max_{i=1}^{d}\|\partial_i\psi\|_{L^\infty}|x-y|.$$

$\square$

Observe that if Ω is a convex open subset of $\mathbb{R}^d$ and $\psi \in C^1(\Omega)$, $\|\psi\|_{C^1(\Omega)} < \infty$, then ψ is Lipschitz continuous. However, if Ω is not convex, this need not be true.

Exercise 4.B. Construct a function ψ which is C^1 on $(0,1)\cup(1,2)$ and $\|\psi'\|_{L^\infty} < \infty$, but which is not Lipschitz continuous.

We will use the previous lemma in combination with the following lemma.

Lemma 4.8. *Let $X \subset \mathbb{R}^d$ be compact. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of uniformly Lipschitz continuous functions, i.e., there exists an $M > 0$ such that*

$$|f_n(x)-f_n(y)| \le M|x-y| \qquad (n\in\mathbb{N}, x, y \in X).$$

If $(f_n)_{n\in\mathbb{N}}$ converges pointwise to zero, i.e., $f_n(x)\to 0$ for all $x \in X$, then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to zero, i.e., $\|f_n\|_{L^\infty}\to 0$.

Proof. Let $\varepsilon > 0$. As X is compact, there exist $x_1,\dots,x_k \in X$ such that $X \subset \bigcup_{i=1}^k B(x_i, \frac{\varepsilon}{2M})$. Let $N \in \mathbb{N}$ be such that $|f_n(x_i)| < \frac{\varepsilon}{2}$ for all $i \in \{1,\dots,k\}$ and all $n \ge N$. By the Lipschitz continuity we have that for all $y \in X$ that there exists a i such that $y \in B(x_i, \frac{\varepsilon}{2M})$ and thus for all $n \ge N$

$$|f_n(y)| \le |f_n(y)-f_n(x_i)| + |f_n(x_i)| < M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Definition 4.9. Let $K \subset \Omega$ be compact and $m \in \mathbb{N}_0$. We define the seminorm $\|\cdot\|_{C^m,K}$ on $C^m(\Omega)$ by

$$\|f\|_{C^m,K} = \|f|_K\|_{C^m(K)} = \max_{\beta\in\mathbb{N}_0^d:|\beta|\le m}\sup_{x\in K}|\partial^\beta f(x)| \qquad (f \in C^\infty(\Omega)). \tag{4.4}$$

Lemma 4.10. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\Omega)$. If for all $\alpha \in \mathbb{N}_0^d$, $(\partial^\alpha\varphi_n)_{n\in\mathbb{N}}$ is uniformly bounded and if $\partial^\alpha\varphi_n$ converges pointwise to zero, that is, if*

$$\sup_{n\in\mathbb{N}}\|\partial^\alpha\varphi_n\|_{L^\infty} < \infty \qquad (\alpha \in \mathbb{N}_0^d),$$
$$\partial^\alpha\varphi_n(x) \xrightarrow{n\to\infty} 0 \qquad (x \in \Omega),$$

then for all $m \in \mathbb{N}_0$ and compact sets $K \subset \Omega$

$$\|\varphi_n\|_{C^m,K} \xrightarrow{n\to\infty} 0.$$

Proof. Let ι be the continuous embedding $\mathcal{D}(\Omega) \to \mathcal{D}(\mathbb{R}^d)$, see 4.6. Then $\partial^\alpha \iota(\varphi_k) = \iota(\partial^\alpha \varphi_k)$ and so $(\partial^\alpha \iota(\varphi_k))_{k\in\mathbb{N}}$ is uniformly bounded for any $\alpha \in \mathbb{N}_0^d$. By an application of Lemma 4.7 it follows that $(\partial^\alpha \iota(\varphi_k))_{k\in\mathbb{N}}$ and thus $(\partial^\alpha \varphi_k)_{k\in\mathbb{N}}$ are uniformly Lipschitz. Hence by Lemma 4.8 it follows that $\|\partial^\alpha \varphi_k\|_{L^\infty} \to 0$ for all $\alpha \in \mathbb{N}_0^d$. □

Theorem 4.11. *A sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to a φ in $\mathcal{D}(\Omega)$ if and only if* (a) *and* (b)*:*

(a) *There exists a compact set $K \subset \Omega$ such that the supports of φ_n and φ lies within K for all $n \in \mathbb{N}$.*

(b) $\|\varphi_n - \varphi\|_{C^m} \to 0$ *for all $m \in \mathbb{N}$.*

Exercise 4.C. Prove the "if" part of Theorem 4.11: that (a) and (b) imply $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$.

Proof of the "only if" part of Theorem 4.11. Suppose that $\varphi_n \to 0$ in $\mathcal{D}(\Omega)$. We deduce (a) and (b), arguing by contradiction.

Suppose (a) is not satisfied. Then no compact subset of Ω contains the supports of all functions $\varphi_1, \varphi_2, \dots$. Let $(K_n)_{n\in\mathbb{N}}$ be as in Theorem 1.8. Inductively, choose $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that for all $i \in \mathbb{N}$

$$\bigcup_{j=1}^{i-1} \operatorname{supp} \varphi_{n_j} \subset K_i, \quad \operatorname{supp} \varphi_{n_i} \not\subset K_i.$$

For $i \in \mathbb{N}$ choose x_i in $\Omega \setminus K_i$ with $\varphi_{n_i}(x_i) \neq 0$. If $i, j \in \mathbb{N}$ and $j < i$, then $\varphi_{n_j}(x_i) = 0$ since $x_i \notin K_j$ and $\operatorname{supp} \varphi_{n_i} \subset K_j$.

Now let us define a measure with support being equal to the set of x_i's as follows. We let $\mu = \sum_{i\in\mathbb{N}} a_i \delta_{x_i}$, where the a_i's are chosen such that $\sum_{i=1}^k a_i \varphi_{n_k}(x_i) = 1$; this can always be done inductively. By assumption on the sequence $(x_k)_{k\in\mathbb{N}}$, this measure is a Radon measure, as any compact set $K \subset \Omega$ contains only finitely many x_k's. Therefore it defines a distribution. But $\int \varphi_n \, d\mu = 1$ for all $n \in \mathbb{N}$, which contradicts the hypothesis that $\varphi_n \to 0$ in $\mathcal{D}(\Omega)$.

In order to show (b) we use Lemma 4.10. As $\varphi_n \to 0$ in $\mathcal{D}(\Omega)$ and $\partial^\alpha \delta_x \in \mathcal{D}'(\Omega)$ for all $x \in \Omega$ and $\alpha \in \mathbb{N}_0^d$, we have $\partial^\alpha \varphi_n(x) \xrightarrow{n\to\infty} 0$ for all $x \in \Omega$ and $\alpha \in \mathbb{N}_0^d$. Hence by Lemma 4.10 it is sufficient to show

$$\text{for all } \alpha \in \mathbb{N}_0^d \quad (\partial^\alpha \varphi_n)_{n\in\mathbb{N}} \text{ is uniformly bounded.}$$

As ∂^α is a continuous function $\mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ for all $\alpha \in \mathbb{N}_0^d$, it is sufficient to show

$$\psi_n \to 0 \text{ in } \mathcal{D}(\Omega) \quad \Longrightarrow \quad (\psi_n)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{4.5}$$

To prove the statement (4.5) let us assume that $\psi_n \to 0$ in $\mathcal{D}(\Omega)$ and that ψ_n is not uniformly bounded. Therefore, by possibly passing to a subsequence, we may assume that $\|\psi_n\|_{L^\infty} > 3^n$ for all $n \in \mathbb{N}$. Then we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω such that

$$|\psi_n(x_n)| = \|\psi_n\|_{L^\infty}.$$

As ψ_n converges pointwise to zero, we may and do assume –by possibly passing to a subsequence– that $\sum_{i=1}^{n-1} |\psi_n(x_i)| < 1$ for all $n \in \mathbb{N}$. As we did before let us construct a Radon measure. We let

$$\mu = \sum_{i\in\mathbb{N}} a_i \delta_{x_i}, \qquad a_i = 3^{-i} \frac{\psi_i(x_i)}{|\psi_i(x_i)|} \quad (i \in \mathbb{N}).$$

Then

$$\int \psi_n \,\mathrm{d}\mu = \sum_{i=1}^{n-1} a_i \psi_n(x_i) + a_n \psi_n(x_n) + \sum_{i=n+1}^{\infty} a_i \psi_n(x_i).$$

As $|a_i| \le \frac{1}{3}$ for all i and $\sum_{i=n+1}^{\infty} 3^{-i} = \frac{1}{2} 3^{-n}$, by the assumptions

$$\begin{aligned} \left|\int \psi_n \,\mathrm{d}\mu\right| &\ge -\frac{1}{3} \sum_{i=1}^{n-1} |\psi_n(x_i)| + 3^{-n} \|\psi_n\|_{L^\infty} - \|\psi_n\|_{L^\infty} \sum_{i=n+1}^{\infty} 3^{-i} \\ &\ge -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} > 0. \end{aligned}$$

Therefore $\int \psi_n \,\mathrm{d}\mu$ does not converge to zero, which contradicts our hypothesis. ☐

Corollary 4.12. *Let $\varphi_\varepsilon \in \mathcal{D}(\Omega)$ for all $\varepsilon > 0$ and $\varphi \in \mathcal{D}(\Omega)$. Then $\varphi_\varepsilon \xrightarrow{\varepsilon\downarrow 0} \varphi$ in $\mathcal{D}(\Omega)$ if and only if* (a) *and* (b)*:*

(a) *There exists a compact set $K \subset \Omega$ such that the supports of φ_ε and φ lies within K for all $\varepsilon > 0$.*

(b) $\|\varphi_\varepsilon - \varphi\|_{C^m} \xrightarrow{\varepsilon\downarrow 0} 0$ *for all $m \in \mathbb{N}$.*

Proof. Again the "if" part is left as an exercise. If (a) does not hold, then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_n \downarrow 0$ such that (a) of Theorem 4.11 does not hold, and thus $\varphi_{\varepsilon_n} \not\to \varphi$ and so $\varphi_\varepsilon \not\to \varphi$ in $\mathcal{D}(\Omega)$. Similarly, if (b) does not hold, then one can conclude that $\varphi_\varepsilon \not\to \varphi$ in $\mathcal{D}(\Omega)$. ☐

Definition 4.13. Let X and Y be topological vector spaces. A function $f : X \to Y$ is called *sequentially continuous* if for any sequence $(x_n)_{n\in\mathbb{N}}$ and x in X:

$$x_n \to x \implies f(x_n) \to f(x).$$

We say that X is *sequentially continuously embedded* in Y, and write $X \hookrightarrow_{\text{seq}} Y$ if there exists a sequentially continuous embedding $X \to Y$.

As a continuous map is sequentially continuous, if X is continuously embedded in Y, then X is sequentially continuously embedded in Y.

In general (topological spaces), sequential continuity does not imply continuity. (For those who are familiar to the notion of nets, continuity of a function $f : X \to Y$, with X and Y topological vector spaces, means $f(x_\iota) \to u(x)$ for any net $(x_\iota)_{\iota \in \mathbb{I}}$ with $x_\iota \to x$ in X.) However, as the next theorem implies, a linear function on testfunctions with values in $\mathbb{F}$ is continuous if and only if it is sequentially continuity.

Theorem 4.14. *A linear function $u : \mathcal{D}(\Omega) \to \mathbb{F}$ is a distribution if and only if it is sequentially continuous, i.e., $\varphi_n \to \varphi$ implies $u(\varphi_n) \to u(\varphi)$ for all sequences $(\varphi_n)_{n\in\mathbb{N}}$ and φ in $\mathcal{D}(\Omega)$.*

Exercise 4.D. Prove Theorem 4.14. (Prove that a linear function $\mathcal{D}(\Omega) \to \mathbb{F}$ which is not a distribution is not sequentially continuous.)

4.15 ($\mathcal{D}$ is not metrizable)**.** Let us show that $\mathcal{D}(\Omega)$ is not metrizable. We show that if there is a metric on $\mathcal{D}(\Omega)$, then it generates a different topology. Suppose d is a metric on $\mathcal{D}(\Omega)$, such that under the topology of d, $\mathcal{D}(\Omega)$ is a topological vector space. We can find a increasing sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ who's union equals Ω. For $n \in \mathbb{N}$, let χ_n be a test function that equals 1 on K_n. We can and do choose $\lambda_n \in \mathbb{R}$ such that $d(\lambda_n\chi_n, 0) \le 2^{-n}$. Then $\lambda_n\chi_n$ converges to 0 but (a) of Theorem 4.11 is not satisfied, which means that $\lambda_n\chi_n$ converges in the topology generated by d but not in the weak topology $\sigma(\mathcal{D}, \mathcal{D}')$.

Remember the seminorms $\|\cdot\|_{C^m,K}$ defined in (4.4).

Definition 4.16. We define $\mathcal{E}(\Omega)$ to be the set $C^\infty(\Omega)$ equipped with the topology generated by the seminorms $\|\cdot\|_{C^m,K}$ with $K \subset \Omega$ compact and $m \in \mathbb{N}_0$.

By Theorem 4.11 the space $\mathcal{D}(\Omega)$ is sequentially continuously embedded in $\mathcal{E}(\Omega)$;

$$\mathcal{D}(\Omega) \hookrightarrow_{\text{seq}} \mathcal{E}(\Omega).$$

Exercise 4.E. Justify the statement $\mathcal{D}(\Omega) \hookrightarrow_{\text{seq}} \mathcal{E}(\Omega)$.

Definition 4.17. Let $K \subset \Omega$ be compact. We define $\mathcal{D}_K(\Omega)$ to be the space

$$\{\varphi \in \mathcal{D}(\Omega) : \operatorname{supp}\varphi \subset K\},$$

equipped with the topology generated by the seminorms $\|\cdot\|_{C^m}$ with $m \in \mathbb{N}_0$.

Definition 4.18. A locally convex space is called a *Fréchet space* if it is complete and metrizable with a translation invariant metric.

Theorem 4.19.

(a) *For each $m \in \mathbb{N}_0$ the space $C^m(\Omega)$, equipped with the seminorms $\|\cdot\|_{C^m,K}$ with $K \subset \Omega$ compact, is a Fréchet space.*

(b) *$\mathcal{E}(\Omega)$ is a Fréchet space.*

(c) *$\mathcal{D}_K(\Omega)$ is a Fréchet space for each compact set $K \subset \Omega$.*

Proof. (a) Let $(K_n)_{n\in\mathbb{N}}$ be the increasing sequence of compact sets such that $\bigcup_{n\in\mathbb{N}} K_n = \Omega$ as in Theorem 1.8. Then the topology of $C^m(\Omega)$ is generated by the seminorms $\|\cdot\|_{C^m,K_n}$ with $n \in \mathbb{N}$, so that by Theorem 3.8 we see that $C^m(\Omega)$ is metrizable with a translation invariant metric. Let us show that $C^m(\Omega)$ is complete.

For $m = 0$ this follows from the fact that for any compact set $K \subset \Omega$ the space $C(K)$ is complete under the supremum norm. Indeed, if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0(\Omega)$, then for each compact $K \subset \Omega$ there exists a continuous function f_K such that $f_n \to f_K$ uniformly on K. As f_K equals $f_{\tilde{K}}$ for a compact $\tilde{K} \subset \Omega$ with $K \subset \tilde{K}$. Then there exists a continuous function $f \in C(\Omega)$ such that $\|f_n - f\|_{C^0,K} \to 0$ for each compact $K \subset \Omega$ (take $f(x) = f_{K_n}(x)$ for $x \in K_n \setminus K_{n-1}^\circ$ with $K_0 = \emptyset$).

Suppose $m \in \mathbb{N}$ and $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^m(\Omega)$. Then for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \le m$, the sequence $(\partial^\beta f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(\Omega)$; hence there exists a g_β in $C(\Omega)$ such that $\partial^\beta f_n \to g_\beta$ in $C^0(\Omega)$. Let us write f for g_0. It is sufficient to show that $\partial^\beta f$ exists and equals g_β. By performing an induction argument, we may as well assume that $|\beta| = 1$, i.e., $\beta = e_i$ for some $i \in \{1, \dots, d\}$ (where e_i is the i-th unit vector in $\mathbb{R}^d$). For all $x \in \mathbb{R}^d$ and $h \in \mathbb{R}$ we have

$$\begin{aligned} f(x + he_i) - f(x) &= \lim_{n\to\infty} f_n(x + he_i) - f_n(x) \\ &= \lim_{n\to\infty} \int_0^h \partial_i f_n(x + te_i)\,\mathrm{d}t = \int_0^h g_{e_i}(x + te_i)\,\mathrm{d}t, \end{aligned}$$

and thus $g_{e_i} = \partial_i f$.

(b) As in (a), there are countably many seminorms that generate the topology, hence $\mathcal{E}(\Omega)$ is metrizable with a translation invariant metric. The completeness basically follows from the completeness of $C^m(\Omega)$: Suppose $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}(\Omega)$. Then, for $m \in \mathbb{N}$, it is a Cauchy sequence in $C^m(\Omega)$ and therefore there exists a g_m such that $\lim_{n\to\infty} f_n = g_m$ in $C^m(\Omega)$. As convergence in $C^{m+1}(\Omega)$ implies convergence in $C^m(\Omega)$ for each $m \in \mathbb{N}$, we have $g_{m+1} = g_m$ and thus $g_m = g_0$ for all $m \in \mathbb{N}$. Therefore $f := g_0 \in C^m(\Omega)$ for each $m \in \mathbb{N}$ and thus $f \in \mathcal{E}(\Omega)$ and $f_n \to f$ in $\mathcal{E}(\Omega)$.

(c) By (b) it is sufficient to show that $\mathcal{D}_K(\Omega)$ is closed in $\mathcal{E}(\Omega)$. Suppose $\varphi \in \mathcal{E}$ and $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}_K(\Omega)$ that converges to φ in $\mathcal{E}$. Then for $x \in \Omega \setminus K$

$$|\varphi(x)| \le |\varphi_n(x)| + |\varphi(x) - \varphi_n(x)| \le \|\varphi - \varphi_n\|_{C^0} \qquad (n \in \mathbb{N}).$$

By taking $n \to \infty$ on the right-hand side, we see that $\varphi = 0$ outside K, i.e., $\varphi \in \mathcal{D}_K(\Omega)$. ☐

For the proof that $\mathcal{D}'(\Omega)$ is weak* complete, see Theorem 4.26, we use Baire's Category Theorem. A proof can be found for example in [Rud91, Theorem 2.2] or [dPvR13, Theorem 11.1] .

Theorem 4.20 (Baire's Category Theorem)**.** *Let X be a complete nonempty metric space and let $U_1, U_2, \ldots$ be dense open subsets of X. Then the intersection of those sets, $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.*

4.21. Equivalent to the above statement of Baire's Category Theorem, one obtains the following statement by taking complements: *If X is a complete nonemtpy metric space and $A_1, A_2, \ldots$ are closed subsets of X such that the interior of $\bigcup_{n\in\mathbb{N}} A_n$ is nonempty, then there exists an n such that the interior of A_n is nonempty.*

Lemma 4.22. *Let X be a topological vector space whose topology is compatible with a translation invariant metric and assume that X is complete with respect to this metric. Let q be a seminorm on X such that the set $\{x \in \Omega : q(x) \le 1\}$ is closed. Then q is continuous.*

Proof. For $t \in [0, \infty)$ put $[q \le t] = \{x \in \Omega : q(x) \le t\}$. As $X = \bigcup_{m\in\mathbb{N}}[q \le m]$ and every $[q \le m]$ is closed, the Baire Category Theorem tells us that for some m the set $[q \le m]$ has an interior point, $a \in [q \le m]^\circ$, say. Then $0 = a - a \in [q \le m]^\circ + [q \le m]^\circ = [q \le 2m]^\circ$. Then every set $[q \le t]$ for $t > 0$ is a neighbourhood of 0, so that q is continuous at 0. As $|q(x) - q(y)| \le q(x - y)$ for all $x, y \in X$, it follows that q is continuous (Exercise 4.F). □

Exercise **4.F.** Let q be a seminorm on a topological vector space X. Prove that the following statements are equivalent:

(a) q is continuous.

(b) $\{x \in X : q(x) < 1\}$ is open.

(c) $0 \in \{x \in X : q(x) < 1\}^\circ$.

(d) $0 \in \{x \in X : q(x) \le 1\}^\circ$.

(e) q is continuous at 0.

(f) There exists a continuous seminorm p on X such that $q \le p$ (i.e., $q(x) \le p(x)$ for all $x \in X$).

Exercise **4.G.** (a) Suppose that $\mathcal{P}$ is a collection of seminorms on a vector space X such that

$$q(x) = \sup_{p\in\mathcal{P}} p(x) < \infty \qquad (x \in X).$$

Show that q is a seminorm.

(b) Suppose that $\mathcal{F}$ is a collection of lower semicontinuous functions on a topological space X such that

$$g(x) = \sup_{f\in\mathcal{F}} f(x) < \infty \qquad (x \in X).$$

Show that g is lower semicontinuous (which means that $[g \le c] = \{x \in X : g(x) \le c\}$ is closed for all $c > 0$).

Exercise **4.H.** Prove: *Suppose X is a locally convex vector space whose topology is generated by countably many seminorms $p_1, p_2, \dots$ with $p_1 \le p_2 \le \cdots$. A linear function $f : X \to \mathbb{F}$ is continuous if and only if there exist $C > 0$ and $n \in \mathbb{N}$ such that*

$$|f(x)| \le C p_n(x) \qquad (x \in X).$$

Definition 4.23. We write $\mathcal{E}'(\Omega)$ for the space of continuous linear functions $u : \mathcal{E}(\Omega) \to \mathbb{F}$. That is, $u \in \mathcal{E}'(\Omega)$ if and only if there exist a compact set K, a $m \in \mathbb{N}_0$ and a $C > 0$ such that

$$|u(\varphi)| \le C\|\varphi\|_{C^m,K} \qquad (\varphi \in \mathcal{E}(\Omega)). \tag{4.6}$$

We equip $\mathcal{E}'(\Omega)$ with the weak* topology $\sigma(\mathcal{E}'(\Omega), \mathcal{E}(\Omega))$.

The following theorem is a *principle of uniform boundedness.* For a more general statement, see [Rud91, Theorem 2.6], which is a consequence of the Banach–Steinhaus Theorem.

Theorem 4.24. (a) *Let $\mathcal{U} \subset \mathcal{D}'(\Omega)$ and assume*

$$\sup_{u \in \mathcal{U}} |u(\varphi)| < \infty \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Then, for each compact $K \subset \Omega$ there exists a $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|u(\varphi)| \le C\|\varphi\|_{C^m} \qquad (\varphi \in \mathcal{D}_K(\Omega), u \in \mathcal{U}). \tag{4.7}$$

(b) *Let $\mathcal{U} \subset \mathcal{E}'(\Omega)$ and assume*

$$\sup_{u \in \mathcal{U}} |u(\varphi)| < \infty \qquad (\varphi \in \mathcal{E}(\Omega)).$$

Then, there exists a compact $K \subset \Omega$, $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|u(\varphi)| \le C\|\varphi\|_{C^m,K} \qquad (\varphi \in \mathcal{E}(\Omega), u \in \mathcal{U}). \tag{4.8}$$

Proof. (a) Let $K \subset \Omega$ be compact. Define the function $q : \mathcal{D}_K(\Omega) \to [0, \infty)$ by

$$q(\varphi) := \sup_{u \in \mathcal{U}} |u(\varphi)|.$$

This defines a seminorm as it is the supremum of a family of seminorms and it is lower semicontinuous as it is the supremum of continuous functions (see Exercise 4.G). Therefore $\{\varphi \in \mathcal{D}_K(\Omega) : q(\varphi) \le 1\}$ is closed in $\mathcal{D}_K(\Omega)$. Therefore, by Lemma 4.22 q is continuous on $\mathcal{D}_K(\Omega)$. As the topology of $\mathcal{D}_K(\Omega)$ is generated by the seminorms $\|\cdot\|_{C^m}$ for $m \in \mathbb{N}$ and $\|\cdot\|_{C^m} \le \|\cdot\|_{C^k}$ for $k \ge m$, there exists a $C > 0$ and $m \in \mathbb{N}$ such that (4.7) (see Exercise 4.H).

(b) follows by the above argument but with "$\mathcal{D}(\Omega)$" and "$\mathcal{D}_K(\Omega)$" both replaced by "$\mathcal{E}(\Omega)$" and using that the topology on $\mathcal{E}(\Omega)$ is generated by the seminorms $\|\cdot\|_{C^m,K}$ for $m \in \mathbb{N}$ and $K \subset \Omega$ compact. $\square$

Proposition 4.25. *The pairing maps* $\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{F}$, $(u, \varphi) \mapsto u(\varphi) = \langle u, \varphi \rangle$ *and* $\mathcal{E}'(\Omega) \times \mathcal{E}(\Omega) \to \mathbb{F}$, $(v, \psi) \mapsto v(\psi) = \langle v, \psi \rangle$ *are sequentially continuous.*

Proof. Let $(u_n, \varphi_n)_{n\in\mathbb{N}}$ and (u, φ) be in $\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)$ such that $(u_n, \varphi_n) \to (u, \varphi)$ in $\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)$, i.e., $u_n \to u$ in $\mathcal{D}'(\Omega)$ and $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$. By Theorem 4.11 (a) there exists a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_n, \operatorname{supp} \varphi \subset K$ for all $n \in \mathbb{N}$. By Theorem 4.24 there exists a $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|u_n(\varphi)| \leq C\|\varphi\|_{C^m} \qquad (\varphi \in \mathcal{D}_K(\Omega), n \in \mathbb{N}).$$

Therefore

$$|u_n(\varphi_n) - u(\varphi)| \leq |u_n(\varphi_n - \varphi)| + |u_n(\varphi) - u(\varphi)| \leq C\|\varphi_n - \varphi\|_{C^m} + |u_n(\varphi) - u(\varphi)|.$$

The latter converges to zero by Theorem 4.11 (b).

The sequential continuity of the map $\mathcal{E}'(\Omega) \times \mathcal{E}(\Omega) \to \mathbb{F}$, $(v, \psi) \mapsto v(\psi) = \langle v, \psi \rangle$ follows similarly and the proof is left to the reader. □

Theorem 4.26. $\mathcal{D}'(\Omega)$ *and* $\mathcal{E}'(\Omega)$ *are weak* sequentially complete.*

Proof. First we prove that $\mathcal{D}'(\Omega)$ is weak* complete. The completeness of $\mathcal{E}'(\Omega)$ can be proved similarly; we comment on this at the end of the proof.

Suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}'(\Omega)$ such that $(\langle u_n, \varphi \rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence for all $\varphi \in \mathcal{D}(\Omega)$. It will be clear what the limit should be: We define $u : \mathcal{D}(\Omega) \to \mathbb{F}$ such that $\langle u, \varphi \rangle = \lim_{n\to\infty} \langle u_n, \varphi \rangle$ for any $\varphi \in \mathcal{D}(\Omega)$. Clearly u is linear, so let us show that it is a distribution. By Theorem 4.24, for each compact $K \subset \Omega$, there exist a $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|u(\varphi)| \leq \sup_{n\in\mathbb{N}} |u_n(\varphi)| \leq C\|\varphi\|_{C^m} \qquad (\varphi \in \mathcal{D}_K(\Omega)).$$

Therefore u is a distribution.

If $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}'(\Omega)$, then one can follow the above prove with "$\mathcal{D}(\Omega)$" and "$\mathcal{D}_K(\Omega)$" both replaced by "$\mathcal{E}(\Omega)$". □

4.27. We equip the space of locally integrable functions on Ω with the topology defined by the seminorms $\|\cdot\|_{L^1,K}$ with $K \subset \Omega$ being compact, where

$$\|\varphi\|_{L^1,K} := \|\varphi \mathbb{1}_K\|_{L^1} = \int_K |\varphi| \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Similarly, for $p \in [1, \infty]$, $L^p_{\text{loc}}(\Omega)$ is equipped with the seminorms $\|\cdot\|_{L^p,K}$ with $K \subset \Omega$ being compact, defined by $\|\varphi\|_{L^p,K} := \|\varphi \mathbb{1}_K\|_{L^p}$.

It is rather straightforward to check for $p \in [1, \infty]$, $L^p(\Omega) \hookrightarrow L^p_{\text{loc}}(\Omega)$ and

$$\mathcal{E}(\Omega) \hookrightarrow L^p_{\text{loc}}(\Omega) \hookrightarrow L^1_{\text{loc}}(\Omega) \qquad (p \in [1, \infty]). \tag{4.9}$$

Exercise **4.I.** Show (4.9). (Hint: Exercise 2.A.)

Theorem 4.28. *Let $p \in [1, \infty]$. The embedding $\mathcal{D}(\Omega) \to L^p(\Omega)$ is sequentially continuous. Moreover, the function $L^p_{\text{loc}}(\Omega) \to \mathcal{D}'(\Omega)$, $f \mapsto u_f$ is a continuous embedding:*

$$\mathcal{D}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^p_{\text{loc}}(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \qquad (p \in [1, \infty]).$$

Proof. With Theorem 4.11 it follows that $\mathcal{D}(\Omega) \to L^p(\Omega)$ is sequentially continuous. By (4.9) it is sufficient to show that the map $L^p_{\text{loc}}(\Omega) \to \mathcal{D}'(\Omega)$, $f \mapsto u_f$ is a continuous embedding for $p = 1$. The injectivity follows from Lemma 2.9. The continuity is left as an exercise (see Exercise 4.J). □

Exercise **4.J.** Prove the continuity of the functions $L^p_{\text{loc}}(\Omega) \to \mathcal{D}'(\Omega)$, $f \mapsto u_f$ and $\mathcal{M}(\Omega) \to \mathcal{D}'(\Omega)$, $\mu \mapsto u_\mu$ (see Definition 2.23 for $\mathcal{M}(\omega)$).

The following theorem is a kind of counterpart to Theorem 2.11.

Theorem 4.29. *Let $\mathcal{U}$ be an open covering of Ω. For each $U \in \mathcal{U}$ let u_U be a distribution on U. Suppose that, if $U, V \in \mathcal{U}$, then $u_U = u_V$ on $U \cap V$, in the sense that $u_U(\varphi) = u_V(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\operatorname{supp} \varphi \subset U \cap V$. Then there exists a unique distribution u on Ω such that $u = u_U$ on U for all $U \in \mathcal{U}$.*

Proof. Choose a partition $(\chi_n)_{n \in \mathbb{N}}$ of unity subordinated to $\mathcal{U}$ as in Theorem 1.11. As for every $\varphi \in \mathcal{D}(\Omega)$ there is an $N \in \mathbb{N}$ with $\varphi = \sum_{n=1}^N \chi_n \varphi$, there is a unique linear $u : \mathcal{D}(\Omega) \to \mathbb{F}$ described by

$$u(\varphi) = \sum_{n=1}^{\infty} u_{U_n}(\chi_n \varphi) \qquad (\varphi \in \mathcal{D}(\Omega)),$$

where $U_n = \{x \in \Omega : \chi_n(x) > 0\}$. We are done if this u is a distribution.

That u is a linear function on $\mathcal{D}(\Omega)$ is straightforward to check. For the continuity we use Theorem 4.14 to restrict to sequential continuity. By Theorem 4.11 we know that if $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, then there exists a compact set K that contains the supports of all φ_n's. Therefore, there are only finitely many k such that $u_{U_k}(\chi_k \varphi_n)$ is nonzero for some n. That is, there exists a $L \in \mathbb{N}$ such that $u(\varphi_n) = \sum_{k=1}^L u_{U_k}(\chi_k \varphi_n)$ for all $n \in \mathbb{N}$. As for all k we have $\chi_k \varphi_n \to \chi_k \varphi$, we have $u_{U_k}(\chi_k \varphi_n) \to u_{U_k}(\chi_k \varphi)$. From this we conclude the continuity of u. □

5 Compactly supported distributions

In Definition 4.16 we have introduced the topological space $\mathcal{E}(\Omega)$ consisting of all smooth functions on Ω and in Definition 4.23 we defined its topological dual $\mathcal{E}'(\Omega)$. In this section we will further study $\mathcal{E}'(\Omega)$, which is in one to one correspondence to the set of compactly supported distributions, see 5.2.

Definition 5.1. Let u be a distribution on Ω. We call a (relatively) closed subset A of Ω a *carrier* of u if $u(\varphi) = 0$ for every $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp} \varphi \cap A = \emptyset$. The intersection of all carriers of u is defined to be the *support* of u,

$$\operatorname{supp} u.$$

We prove $\operatorname{supp} u$ to be a carrier of u (hence, the smallest carrier of u). Let $\mathcal{U}$ be the collection of all complements (in Ω) of the carriers of u; take $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp} \varphi \subset \bigcup \mathcal{U}$. We wish to prove $u(\varphi) = 0$. By Theorem 1.11 there exist $\chi_1, \dots, \chi_N \in \mathcal{D}(\Omega)$ with $\varphi = \sum_{n=1}^N \chi_n \varphi$, where for each n the support of χ_n is contained in an element of $\mathcal{U}$. Then $\operatorname{supp} \chi_n \varphi \subset \operatorname{supp} \chi_n$, whence $u(\chi_n \varphi) = 0$ for each n, and $u(\varphi) = 0$.

We say that u *vanishes* on an open set $U \subset \Omega$ if $u(\varphi) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp} \varphi \subset U$. Then $\Omega \setminus U$ is a carrier of u. Moreover, let $\mathcal{U}$ be the collection of all open sets $U \subset \Omega$ on which u vanishes. Then u vanishes on $\bigcup \mathcal{U}$ and $\operatorname{supp} u$ is the complement of $\bigcup \mathcal{U}$.

From this the following statements are immediate

$$\begin{aligned}
&\varphi \in \mathcal{D}(\Omega), \operatorname{supp} \varphi \cap \operatorname{supp} u = \emptyset \implies u(\varphi) = 0, \\
&\chi \in C^\infty(\Omega), \chi = 1 \text{ on a neighbourhood of } \operatorname{supp} u \implies \chi u = u.
\end{aligned}$$

Note that $\operatorname{supp} \varphi \cap \operatorname{supp} u = \emptyset$ means that φ equals 0 on an open set that contains $\operatorname{supp} u$. If φ equals 0 on $\operatorname{supp} u$, then the evaluation $u(\varphi)$ might not equal zero as the following example illustrates: Take $u = \partial \delta_0$ and let φ be a testfunction such that $\varphi(x) = x$ around 0 (see Exercise 1.E).

Observe moreover that for $\alpha \in \mathbb{N}_0^d$ and $\psi \in C^\infty(\Omega)$

$$\operatorname{supp} \partial^\alpha u \subset \operatorname{supp} u, \qquad \operatorname{supp} \psi u \subset \operatorname{supp} \psi \cap \operatorname{supp} u.$$

Exercise 5.A. Show that each compactly supported distribution is of finite order.

5.2 (Each element of $\mathcal{E}'$ defines a compactly supported distribution)**.** Let $u \in \mathcal{E}'(\Omega)$ and let $K \subset \Omega$ be compact, $m \in \mathbb{N}_0$ and $C > 0$ be such that

$$|u(\varphi)| \le C \|\varphi\|_{C^m, K} \qquad (\varphi \in \mathcal{E}(\Omega)).$$

If $\varphi \in \mathcal{E}(\Omega)$ and $\operatorname{supp} \varphi \cap K = \emptyset$ then $\|\varphi\|_{C^m, K} = 0$ and thus $u(\varphi) = 0$. Hence $\operatorname{supp} u \subset K$ and so an element of $\mathcal{E}'(\Omega)$ defines a distribution with compact support. In 5.6 we will prove that a distribution with compact support can be extended to an element of $\mathcal{E}'(\Omega)$.

Proposition 5.3. *For each $m \in \mathbb{N}$ there exists a $C > 0$ such that*

$$\|fg\|_{C^m} \le C \|f\|_{C^m} \|g\|_{C^m} \qquad (f, g \in C^m(\Omega)) \tag{5.1}$$

$$\|fg\|_{C^m, K} \le C \|f\|_{C^m, K} \|g\|_{C^m, K} \qquad (f, g \in C^m(\Omega), K \subset \Omega \text{ compact}). \tag{5.2}$$

Consequently, the functions

$$\begin{aligned}
&\mathcal{E}(\Omega) \times \mathcal{D}(\Omega) \to \mathcal{D}(\Omega), && (\psi, \varphi) \mapsto \psi\varphi, \\
&\mathcal{D}'(\Omega) \times \mathcal{E}(\Omega) \to \mathcal{D}'(\Omega), && (u, \varphi) \mapsto \varphi u, \\
&\mathcal{E}'(\Omega) \times \mathcal{E}(\Omega) \to \mathcal{E}'(\Omega), && (v, \psi) \mapsto \psi v,
\end{aligned}$$

are sequentially continuous.

Proof. By Leibniz' differentiation rule (see 1.14) we have for $x \in \Omega$

$$\begin{aligned}
\max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} |\partial^\alpha (fg)(x)| &\le \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le \alpha}} \binom{\alpha}{\beta} |\partial^\beta f(x)| |\partial^\gamma g(x)| \\
&\le \left(\max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le \alpha}} \binom{\alpha}{\beta} \right) \max_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| \le k}} \max_{\substack{\gamma \in \mathbb{N}_0^d \\ |\gamma| \le k}} |\partial^\beta f(x)| |\partial^\gamma g(x)|. \qquad (5.3)
\end{aligned}$$

Hence with $C = \max_{\alpha \in \mathbb{N}_0^d : |\alpha| \le k} \sum_{\beta \in \mathbb{N}_0^d : \beta \le \alpha} \binom{\alpha}{\beta}$ one has (5.1) and (5.2).

The continuity of the product maps $\mathcal{D}'(\Omega) \times \mathcal{E}(\Omega) \to \mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega) \times \mathcal{E}(\Omega) \to \mathcal{E}'(\Omega)$ follow by Proposition 4.25. □

5.4. Observe that Leibniz' rule (1.3) extends to the product of a distribution with a smooth function. That is, if $u \in \mathcal{D}(\Omega)$ and $\psi \in C^\infty(\Omega)$, then

$$\partial^\alpha (\psi u) = \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le \alpha}} \binom{\alpha}{\beta} (\partial^\beta \psi)(\partial^{\alpha - \beta} u),$$

5.5. Observe that for each compact set $K \subset \Omega$ there exists a $\chi \in C_c^\infty(\Omega, [0,1])$ such that $\chi = 1$ on a neighbourhood of K: Let $\mathfrak{K} \subset \Omega$ be compact and such that $K \subset \mathfrak{K}^\circ$ (which exists by Theorem 1.8). By Lemma 1.13 there exists such a χ which equals 1 on $\mathfrak{K}$ and thus on $\mathfrak{K}^\circ$ which is an open set that contains K, i.e., $\mathfrak{K}$ is a neighbourhood of K.

5.6 (Each compactly supported distribution extends to an element of $\mathcal{E}'$)**.**
Let u be a distribution on Ω with compact support K. We will show that there exists exactly one $v \in \mathcal{E}'$ such that $u(\varphi) = v(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$.

We have already seen in Definition 5.1 that if $\varphi \in \mathcal{D}(\Omega)$ and $\operatorname{supp} \varphi \subset \Omega \setminus K$, then $u(\varphi) = 0$. Let χ be a testfunction that is equal to 1 on a neighbourhood of K. Then $\operatorname{supp}(\varphi - \chi\varphi) \subset \Omega \setminus K$ and thus

$$u(\varphi) = u(\chi\varphi) \qquad (\varphi \in \mathcal{D}(\Omega)). \qquad (5.4)$$

Let $K_0 = \operatorname{supp} \chi$. As u is a distribution, there exist $C_1 > 0$ and $m \in \mathbb{N}_0$ such that $|u(\varphi)| \le C_1 \|\varphi\|_{C^m}$ for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp} \varphi \subset K_0$. This implies

$$|u(\varphi)| = |u(\chi\varphi)| \le C_1 \|\chi\varphi\|_{C^m} \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Let $C > 0$ be such that (5.1) holds, then with $C' = C_1 C \|\chi\|_{C^m}$,

$$|u(\chi\varphi)| \le C' \|\varphi\|_{C^m, K_0} \qquad (\varphi \in \mathcal{E}(\Omega)). \tag{5.5}$$

Therefore $v : \mathcal{E}(\Omega) \to \mathbb{F}$ defined by $v(\varphi) = u(\chi\varphi)$ for $\varphi \in \mathcal{E}$ is an element of $\mathcal{E}'$.

Let us show that this v is the only element of $\mathcal{E}'$ such that $v(\varphi) = u(\varphi)$ for $\varphi \in \mathcal{D}(\Omega)$. Suppose $w \in \mathcal{E}'$ is such that $w(\varphi) = u(\varphi)$ for $\varphi \in \mathcal{D}(\Omega)$. Let $L \subset \Omega$ be compact, $m \in \mathbb{N}$ and $M > 0$ such that

$$|w(\varphi)| \le M \|\varphi\|_{C^m, L} \qquad (\varphi \in \mathcal{E}(\Omega)).$$

Let $\eta \in \mathcal{D}(\Omega)$ be equal to 1 on a neighbourhood of L. Then $w(\eta\psi) = w(\psi)$ for all $\psi \in \mathcal{E}(\Omega)$. Therefore

$$w(\psi) = w(\eta\psi) = u(\eta\psi) = u(\chi\eta\psi) = u(\chi\psi) = v(\psi) \qquad (\psi \in \mathcal{E}(\Omega)).$$

With a partition of unity we have the following approximations.

Lemma 5.7. *Let $(\chi_n)_{n\in\mathbb{N}}$ be a partition of unity as in Theorem 1.11. Then*

$$\sum_{n=1}^{N} \chi_n \varphi \xrightarrow{N\to\infty} \varphi \quad \text{in } \mathcal{D}(\Omega) \qquad (\varphi \in \mathcal{D}(\Omega)), \tag{5.6}$$

$$\sum_{n=1}^{N} \chi_n u \xrightarrow{N\to\infty} u \quad \text{in } \mathcal{D}'(\Omega) \qquad (u \in \mathcal{D}'(\Omega)), \tag{5.7}$$

$$\sum_{n=1}^{N} \chi_n \varphi \xrightarrow{N\to\infty} \varphi \quad \text{in } \mathcal{E}(\Omega) \qquad (\varphi \in \mathcal{E}(\Omega)), \tag{5.8}$$

$$\sum_{n=1}^{N} \chi_n u \xrightarrow{N\to\infty} u \quad \text{in } \mathcal{E}'(\Omega) \qquad (u \in \mathcal{E}'(\Omega)). \tag{5.9}$$

Proof. As for each $\varphi \in \mathcal{D}(\Omega)$ there exists an $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} \chi_n \varphi = \varphi$, (5.6) and (5.7) follow immediately. Let $K \subset \Omega$ be compact. Let $\mathfrak{K} \subset \Omega$ be a compact set such that $K \subset \mathfrak{K}^\circ$ and let $N \in \mathbb{N}$ be such that $\sum_{n=1}^{N} \chi_n(x) = 1$ for all $x \in \mathfrak{K}$. Then

$$\left\| \psi - \sum_{n=1}^{N} \chi_n \psi \right\|_{C^m, K} = 0 \qquad (\psi \in \mathcal{E}(\Omega), m \in \mathbb{N}_0),$$

so that (5.8) and (5.9) follow. □

Exercise 5.B. Show that if $u \in \mathcal{E}'(\Omega)$ and $u \neq 0$, then there exists a $\varphi \in \mathcal{D}(\Omega)$ such that $u(\varphi) \neq 0$. Moreover, show that $u(\psi) = 0$ for every $\psi \in \mathcal{E}(\Omega)$ with $\operatorname{supp} \psi \cap \operatorname{supp}(u|_{\mathcal{D}}) = \emptyset$.

Definition 5.8. Let X be a topological space. We call a set $A \subset X$ *sequentially dense* in X if for each $x \in X$ there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A that converges to x.

If there exists a countable sequentially dense subset of X, we call X *sequentially separable.*

Any sequentially dense set is also dense. If X is a metric space, then any dense set is also sequentially dense. Not every dense set is sequentially dense:

Example 5.9. Let $X = \mathbb{R}$ (or a uncountable set) and call a set $U \subset X$ open if its complement is either countable or equal to X. With this topology the set $X \setminus \{0\}$ is dense in X but not sequentially dense as if a sequence $(x_n)_{n\in\mathbb{N}}$ converges to some x in X, then there exists an $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

Exercise 5.C. Verify the statement in Example 5.9.

Theorem 5.10. *The embedding $\mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ is a sequential continuous embedding; the map $\iota : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ defined by $\iota(u) = u|_{\mathcal{D}(\Omega)}$ is a continuous embedding and its image is the set of compactly supported distributions and the map $\zeta : \mathcal{E}'(\Omega) \to \mathcal{E}'(\mathbb{R}^d)$ defined by $(\zeta u)(\psi) = u(\psi|_\Omega)$ is a continuous embedding:*

$$\mathcal{D}(\Omega) \hookrightarrow_{\text{seq}} \mathcal{E}(\Omega), \qquad \mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega), \qquad \mathcal{E}'(\Omega) \hookrightarrow \mathcal{E}'(\mathbb{R}^d).$$

Moreover, $\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{E}(\Omega)$ and $\mathcal{E}'(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$.

Proof. That $\mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ is sequentially continuous follows from Theorem 4.11. That ι is continuous follows from the fact that $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$. That ι forms a bijection on to the set of compactly supported distributions follows from 5.2 and 5.6. That the image of $\mathcal{E}'(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$ follows from Lemma 5.7.

That the map ζ is continuous follows as $\psi|_\Omega$ is in $\mathcal{E}(\Omega)$ for each $\psi \in \mathcal{E}(\mathbb{R}^d)$.

That $\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{E}(\Omega)$ and that $\mathcal{E}'(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$ follows from Lemma 5.7. □

Definition 5.11. Let $y \in \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d, \psi \in C^\infty(\Omega)$ and $l : \mathbb{R}^d \to \mathbb{R}^d$ linear and bijective. For a $u \in \mathcal{E}'(\Omega)$ we define $\check{u}, \mathcal{T}_y u, \partial^\alpha u, \psi u$ and $u \circ l$ by the formulas as in Definition 2.14 but replacing "$\mathcal{D}$" everywhere by "$\mathcal{E}$".

Again, it is straightforward to check that $\check{u}, \mathcal{T}_y u, \partial^\alpha u, \psi u$ and $u \circ l$ are all in $\mathcal{E}'(\Omega)$ and moreover that with $\iota : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ as in Theorem 5.10,

$$\begin{aligned}
\iota(\check{u}) &= \iota(u)\check{}, \\
\iota(\mathcal{T}_y u) &= \mathcal{T}_y \iota(u), \\
\iota(\partial^\alpha u) &= \partial^\alpha \iota(u), \\
\iota(\psi u) &= \psi \iota(u), \\
\iota(u \circ l) &= \iota(u) \circ l.
\end{aligned}$$

Definition 5.12. For $v \in \mathcal{E}'(\Omega)$ we define

$$\operatorname{supp} v = \operatorname{supp}(v|_{\mathcal{D}}).$$

By Exercise 5.B, $\operatorname{supp} v$ is the smallest closed set A such that

$$v(\psi) = 0 \text{ for all } \psi \in \mathcal{E}(\Omega) \text{ with } \operatorname{supp} \psi \cap A = \emptyset.$$

In 5.6 the inequality (5.5) holds for K_0 which is larger than K. The next exercise illustrates that (5.5) may not hold for $K_0 = \operatorname{supp} u$.

Exercise 5.D. [DK10, Exercise 8.3] Let $d = 1$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements and x be in $\mathbb{R}$ such that $x_n \to x$ and such that $x_n \neq x$ for all $n \in \mathbb{N}$.

(a) Show that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $(0, \infty)$ such that

$$\sum_{n\in\mathbb{N}} a_n = \infty, \qquad \sum_{n\in\mathbb{N}} a_n |x_n - x| < \infty.$$

(b) Prove that the formula

$$u(\varphi) = \sum_{n\in\mathbb{N}} a_n(\varphi(x_n) - \varphi(x)) \qquad (\varphi \in \mathcal{D}(\mathbb{R}))$$

defines a distribution u of order ≤ 1. Prove that the support of u is the compact set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$.

(c) Show that for all $n \in \mathbb{N}$ there exists a $\varphi_n \in \mathcal{D}$ such that $\varphi_n = 1$ on a neighbourhood of x_i for all $i \in \{1, \ldots, n\}$ and $\varphi_n = 0$ on a neighbourhood of x_j for all $j > n$ and $\varphi_n = 0$ on a neighbourhood of x. Prove that for all $m \in \mathbb{N}$

$$\|\varphi_n\|_{C^m, \operatorname{supp} u} = 1, \qquad u(\varphi_n) = \sum_{i=1}^{n} a_i.$$

(d) Conclude that for $K = \operatorname{supp} u$, (4.6) does not hold for any $k \in \mathbb{N}$.

The following example illustrates that the embeddings $\mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$ and $\mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ from Theorem 5.10 are not homeomorphisms on their images.

5.13. We show that (a) the relative topology of $\mathcal{D}(\Omega)$ as a subspace of $\mathcal{E}(\Omega)$ is different from the topology on $\mathcal{D}(\Omega)$, namely $\sigma(\mathcal{D}(\Omega), \mathcal{D}'(\Omega))$; (b) $\mathcal{E}'(\Omega)$ does not have the same topology as $\iota(\mathcal{E}'(\Omega))$ (However, $(\mathcal{E}'(\Omega), \sigma(\mathcal{E}'(\Omega), \mathcal{D}(\Omega)))$ is homeomorphic to $\iota(\mathcal{E}'(\Omega))$.) and (c) $\mathcal{E}'(\Omega)$ is not metrizable.

Choose $(\chi_n)_{n\in\mathbb{N}}$ as in Theorem 1.11, letting $\mathcal{U}$ consist of all open subsets of Ω (we assume $\chi_n \neq 0$ for all $n \in \mathbb{N}$). For each $n \in \mathbb{N}$ choose x_n such that $\chi_n(x_n) \neq 0$.

(a) Let $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ be such that $\int \lambda_n\chi_n = 1$ for all $n \in \mathbb{N}$. Let $\varphi_n = \lambda_n\chi_n$ for all $n \in \mathbb{N}$. For all compact sets K there exists an $N \in \mathbb{N}$ such that $\operatorname{supp}\varphi_n \cap K = \operatorname{supp}\chi_n \cap K = \emptyset$ for all $n \geq N$. Therefore $\|\varphi_n\|_{C^m,K} \xrightarrow{n\to\infty} 0$ for all $m \in \mathbb{N}$ and compact $K \subset \Omega$, i.e., $\varphi_n \to 0$ in $\mathcal{E}'(\Omega)$. However, for u the distribution corresponding to the Lebesgue measure, or equivalently to the constant function $\mathbb{1}$, we have $u(\varphi_n) = 1$ for all n, whence $(\varphi_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{D}(\Omega)$.

(b) δ_{x_n} is an element of $\mathcal{D}'(\Omega)$ and of $\mathcal{E}'(\Omega)$ for all $n \in \mathbb{N}$. We have $\delta_{x_n} \to 0$ in $\mathcal{D}'(\Omega)$ but not in $\mathcal{E}'(\Omega)$, as we have $\delta_n(\mathbb{1}) = 1$ for all $n \in \mathbb{N}$.

(c) $\mathcal{E}'(\Omega)$ is not metrizable, as we proceed to show. Suppose its topology is given by a metric, d. For every $n \in \mathbb{N}$ we have $\lim_{\lambda\downarrow 0}\lambda\delta_{x_n} = 0$ in $\mathcal{E}'(\Omega)$, so there is a $\lambda_n > 0$ with $d(\lambda_n\delta_{x_n}, 0) < \frac{1}{n}$. By Remark 1.12 there exists a $\psi \in C^\infty(\Omega) = \mathcal{E}(\Omega)$ with $\psi(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n\chi_n(x_n)}\chi_n(x)$; then $\lambda_n\delta_{x_n}(\psi) \geq 1$ for all n. But $\lambda_n\delta_{x_n} \to 0$ in $\mathcal{E}'(\Omega)$ since $d(\lambda_n\delta_{x_n}, 0) \to 0$. Contradiction.

6 ◇ Structure theorems

In this section we show that every distribution is a linear combination of derivatives of continuous functions, that is, we describe the global structure of distributions. These theorems are often called “structure theorems”. In this section we write $\mathcal{D}_K$ instead of $\mathcal{D}_K(\Omega)$ or $\mathcal{D}_K(\mathbb{R}^d)$, where the identification for a compact $K \subset \Omega$ of an element $\mathcal{D}_K(\mathbb{R}^d)$ with an element $\mathcal{D}_K(\Omega)$ is the obvious one.

We start by describing a distribution on $\mathcal{D}_K$.

Theorem 6.1. *Let $u \in \mathcal{D}'(\Omega)$ and $K \subset \Omega$ be compact. Then there exists an $f \in C(\Omega)$ and an $\alpha \in \mathbb{N}_0^d$,*

$$u(\varphi) = \partial^\alpha u_f(\varphi) \qquad (\varphi \in \mathcal{D}_K). \tag{6.1}$$

α can be chosen to be $(N+2,\ldots,N+2)$, where $N \in \mathbb{N}$ is such that there exists a $C > 0$ such that $|u(\varphi)| \leq C\|\varphi\|_{C^N}$ for all $\varphi \in \mathcal{D}_K$.

Proof. By performing a rescaling and a translation, we may as well assume that the support of u lies within the unit cube $Q = [0,1]^d$ (which itself does not need to be included in Ω). By the mean value theorem we have

$$\|\psi\|_{L^\infty} \leq \max_{x\in Q}|\partial_i\psi(x)| \qquad (\psi \in \mathcal{D}_Q, i \in \{1,\ldots,n\}). \tag{6.2}$$

Let $T = \partial_1\partial_2\cdots\partial_d$, i.e., $T = \partial^{(1,\ldots,1)}$. For $y \in Q$ let $Q(y) = \prod_{i=1}^d [0, y_i]$. Then

$$\psi(y) = \int_{Q(y)} (T\psi)(x)\,\mathrm{d}x \qquad (\psi \in \mathcal{D}_Q). \tag{6.3}$$

For $N \in \mathbb{N}$ we have by (6.2) and (6.3)

$$\|\psi\|_{C^N} \le \max_{x\in Q} |T^N\psi(x)| \le \int_Q |T^{N+1}\psi(x)| \,\mathrm{d}x \qquad (\psi \in \mathcal{D}_Q).$$

Let $C > 0$ and $N \in \mathbb{N}$ be such that

$$|u(\varphi)| \le C\|\varphi\|_{C^N} \qquad (\varphi \in \mathcal{D}_K),$$

so that

$$|u(\varphi)| \le C \max_{x\in Q} |T^N\varphi(x)| \le C \int_Q |T^{N+1}\varphi(x)| \,\mathrm{d}x \qquad (\varphi \in \mathcal{D}_K). \tag{6.4}$$

By (6.3) it follows that T is injective on $\mathcal{D}_K$, hence T^{N+1} is injective on $\mathcal{D}_K$. Let $Y = \{T^{N+1}\varphi : \varphi \in \mathcal{D}_K\}$. Define $u_1 : Y \to \mathbb{F}$ by $u_1 = u \circ (T^{N+1})^{-1}$, i.e.,

$$u_1(T^{N+1}\varphi) = u(\varphi) \qquad (\varphi \in \mathcal{D}_K).$$

By (6.4) we have

$$|u_1(\psi)| \le C \int_K |\psi(x)| \,\mathrm{d}x \qquad (\psi \in Y).$$

By the Hahn–Banach Theorem (Theorem 3.14) u_1 extends to a bounded linear functional on $L^1(K)$, i.e., an element of $L^1(K)'$ which can be represented by a bounded Borel–measurable function g (in $L^\infty(K)$):

$$u(\varphi) = u_1(T^{N+1}\varphi) = \int_Q g(x)(T^{N+1}\varphi)(x) \,\mathrm{d}x \qquad (\varphi \in \mathcal{D}_K). \tag{6.5}$$

Extend g as a function on $\mathbb{R}^d$ by setting $g = 0$ on $\mathbb{R}^d \setminus K$ and put

$$f(y) = (-1)^{(N+1)d} \int_{\prod_{i=1}^d (-\infty, y_i]} g \qquad (y \in \mathbb{R}^d).$$

Then f is continuous, and by applying integration by parts d times, (6.5) gives

$$u(\varphi) = (-1)^{(N+2)d} \int_\Omega f(x) T^{N+2}\varphi(x) \,\mathrm{d}x \qquad (\varphi \in \mathcal{D}_K).$$

This implies (6.1). □

We use the previous theorem to represent compactly supported distributions:

Theorem 6.2. *Let $u \in \mathcal{D}'(\Omega)$ be compactly supported with support K. Let $U \subset \Omega$ be open such that $K \subset U$. Suppose u has order N. Then there exist finitely many continuous functions $f_\beta \in C(\Omega)$ with* $\operatorname{supp} f_\beta \subset U$ *for $\beta \in \mathbb{N}_0^d$ with $\beta \le (N+2, \dots, N+2)$, such that*

$$u = \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le (N+2,\dots,N+2)}} \partial^\beta u_{f_\beta}.$$

Proof. Let $V, W \subset \Omega$ be open sets such that $\overline{V}$ and $\overline{W}$ are compact and $K \subset V \subset W \subset U$. (use Theorem 1.8 with "U" (or "W") instead of "Ω"). Let $\alpha = (N+2, \cdots, N+2)$. By Theorem 6.1 applied with $K = \overline{W}$ there exists a $f \in C(\Omega)$ such that

$$u(\varphi) = u_{\partial^\alpha f}(\varphi) \qquad (\varphi \in \mathcal{D}(W)).$$

Let $\chi \in \mathcal{D}(\Omega)$ be supported in W and equal to 1 on $\overline{V}$. Then by Leibniz' formula (see 1.14)

$$\begin{aligned} u(\varphi) = u(\chi\varphi) &= (-1)^{|\alpha|} \int f \cdot \partial^\alpha(\chi\varphi) \\ &= (-1)^{|\alpha|} \int f \cdot \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le \alpha}} \binom{\alpha}{\beta} (\partial^{\alpha-\beta}\chi)(\partial^\beta \varphi) \\ &= \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le \alpha}} (-1)^{|\beta|} \int f_\beta (\partial^\beta \varphi), \end{aligned}$$

for

$$f_\beta = (-1)^{|\alpha-\beta|} f \cdot \binom{\alpha}{\beta} (\partial^{\alpha-\beta}\chi) \qquad (\beta \in \mathbb{N}_0^d : \beta \le \alpha).$$

□

By using the partition of unity to represent a distribution by a sum of compactly supported distributions, we obtain the following representation of general distributions.

Theorem 6.3. *Let $u \in \mathcal{D}'(\Omega)$. There exist $(g_\alpha)_{\alpha \in \mathbb{N}_0^d}$ in $C(\Omega)$ such that for each compact set $K \subset \Omega$,* $\operatorname{supp} g_\alpha \cap K \ne \emptyset$ *for only finitely many α and*

$$u = \sum_{\alpha \in \mathbb{N}_0^d} \partial^\alpha u_{g_\alpha}.$$

If u has finite order N, one can choose the g_α such that $g_\alpha = 0$ for $\alpha \not\le (N+2, \dots, N+2)$.

Proof. Let $(\chi_n)_{n \in \mathbb{N}}$ be a partition of unity as in Theorem 1.11. By Theorem 6.2 for each $n \in \mathbb{N}$ there exists an $N_n \in \mathbb{N}$ and continuous functions $f_{n,\beta}$ for $\beta \in \mathbb{N}_0^d$ with $\beta \le (N_n, \dots, N_n)$ with supports in $U_n = \{x \in \Omega : \chi_n(x) > 0\}$, such that

$$\chi_n u = \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le (N_n, \dots, N_n)}} \partial^\beta u_{f_{n,\beta}}.$$

For $\beta \in \mathbb{N}_0^d$ with $\beta \not\le (N_n, \dots, N_n)$ set $f_{n,\beta} = 0$. Define

$$g_\alpha = \sum_{n=1}^\infty f_{n,\alpha}$$

By the fact that the sets U_n form a locally finite cover of Ω it follows that for all compact $K \subset \Omega$ the intersection $\operatorname{supp} f_{n,\beta} \cap K$ being a subset of $V_n \cap K$ is nonempty only for finitely many β and n. Therefore $\operatorname{supp} g_\alpha \cap K \neq \emptyset$ for only finitely many α. Furthermore

$$u = \sum_{n \in \mathbb{N}} \chi_n u = \sum_{n \in \mathbb{N}} \sum_{\beta \in \mathbb{N}_0^d} \partial^\beta u_{f_{n,\beta}} = \sum_{\alpha \in \mathbb{N}_0^d} \partial^\alpha u_{g_\alpha}.$$

If u has order N, then one can choose $N_n = N + 2$ for all $n \in \mathbb{N}$ by Theorem 6.2. □

7 Intermezzo: Convolutions of functions

We still consider Ω to be an open subset of $\mathbb{R}^d$, though most statements are about functions on $\mathbb{R}^d$.

Before we define the convolution of two functions, we recall some measure theoretic statements. With "measurable" in this section we mean "Borel measurable".

7.1. Because the operations of addition and multiplication are measurable, the following statement holds: *If $f, g : \mathbb{R}^d \to \mathbb{F}$ are measurable and $z \in \mathbb{R}^d$, then the following functions are measurable*

$$\begin{aligned}
&\mathbb{R}^d \to \mathbb{F}, && x \mapsto f(x - z),\\
&\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{F}, && (x, y) \mapsto f(x - y),\\
&\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{F}, && (x, y) \mapsto f(x)g(y),\\
&\mathbb{R}^d \to \mathbb{F}, && x \mapsto f(x)g(x).
\end{aligned}$$

We write fg for the function $x \mapsto f(x)g(x)$.

We recall Fubini's theorem (for the product space $\mathbb{R}^d \times \mathbb{R}^d$ only). For a proof see for example [Bog07, Theorem 3.4.4] or [Hal74, Theorem 36.C].

Theorem 7.2 (Fubini's Theorem)**.** *Let $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{F}$ be integrable. Then for almost all $x \in \mathbb{R}^d$ the functions $y \mapsto f(x, y)$ and $y \mapsto f(y, x)$ are integrable. If $g : \mathbb{R}^d \to \mathbb{F}$ is such that $g(x) = \int_{\mathbb{R}^d} f(x, y)\, \mathrm{d}y$ for almost all $x \in \mathbb{R}^d$, then g is integrable and $\int_{\mathbb{R}^d} g = \int_{\mathbb{R}^d \times \mathbb{R}^d} f$.*

Definition 7.3. Let f and g be measurable functions on $\mathbb{R}^d$. The function $(x, y) \mapsto f(x)g(y - x)$ is measurable, and for almost every $y \in \mathbb{R}^d$ the function $x \mapsto f(x)g(y - x)$ is measurable (see 7.1). Then we define a function $f * g$ on $\mathbb{R}^d$ by

$$f * g(y) = \begin{cases} \int f(x)g(y - x)\, \mathrm{d}x & \text{if } x \mapsto f(x)g(y - x) \text{ is integrable,} \\ 0 & \text{otherwise} \end{cases}$$

$f * g$ is called the *convolution* of f and g.

7.4. (1) Putting it differently: If $f\mathcal{T}_y\check{g}$ is integrable for almost every y, then $f * g$ exists, and $f * g(y) = \int f\mathcal{T}_y\check{g}$ for almost every y.
(2) If $f * g$ exists and if $\tilde{f}, \tilde{g}$ are functions with $\tilde{f} = f$ a.e. and $\tilde{g} = g$ a.e., then $\tilde{f} * \tilde{g}$ exists and equals $f * g$. Thus, we can see $*$ as an operation on equivalence classes of functions. Also, "$f * g$" is often viewed as an equivalence class.
(3) If $f_1 * g$ and $f_2 * g$ exist, then $(f_1 + f_2) * g$ exists and is equal (a.e.) to $f_1 * g + f_2 * g$.
(4) If $f * g$ exists, then $g * f$ exists an equals $f * g$. Indeed, for almost every y the function $f\mathcal{T}_y\check{g}$ is integrable; then so are $(\mathcal{T}_{-y}g)\check{f}$ and $g\mathcal{T}_y\check{f}$. The rest is easy.

Let us also recall the following theorem, which is sometimes also referred to as the Fubini theorem (the statement of Theorem 7.2 is most commonly known under the name Fubini's theorem).

Theorem 7.5. *If $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is measurable, then*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, y) \, \mathrm{d}y \, \mathrm{d}x.$$

Theorem 7.6. *Let f and g be integrable functions on $\mathbb{R}^d$. Then for almost every y the function $x \mapsto f(x)g(y-x)$ is integrable, $f * g$ is integrable, and $\int_{\mathbb{R}^d}(f * g) = \int_{\mathbb{R}^d} f \cdot \int_{\mathbb{R}^d} g$, $\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$.*

Proof. The function $F : (x, y) \mapsto f(x)g(y-x)$ on $\mathbb{R}^d \times \mathbb{R}^d$, being the product of two measurable functions, is measurable (see 7.1). For every x the function $y \mapsto F(x, y)$ is integrable and its integral is $f(x) \int_{\mathbb{R}^d} g$. That F is integrable follows from Theorem 7.5. Therefore, by Fubini's theorem $f*g$ is defined and integrable and its integral is $\int_{\mathbb{R}^d} f \int_{\mathbb{R}^d} g$. □

The following theorem will be used often later on. It generalises the inequality $\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$.

Theorem 7.7 (Young's inequality)**.** *Let $p, q, r \in [1, \infty]$ be such that*

$$\tfrac{1}{p} + \tfrac{1}{q} = 1 + \tfrac{1}{r}.$$

For $f \in \mathcal{L}^p(\mathbb{R}^d)$, $g \in \mathcal{L}^q(\mathbb{R}^d)$ we have $f * g \in \mathcal{L}^r(\mathbb{R}^d)$ and

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}.$$

Proof. We assume $f, g \ge 0$. Observing that p and q play symmetrical roles, we consider four cases.

Case 1: $r = 1$ and thus $p = q = 1$. This case is covered by Theorem 7.6.

Case 2: $r = \infty$ and thus $\frac{1}{p} + \frac{1}{q} = 1$. Then for every $y \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} f(x)g(y-x) \, \mathrm{d}x$ exists and is at most $\|f\|_{L^p}\|g\|_{L^q}$ by Hölder's inequality (see Theorem A.4). Consequently, $f * g$ exists and $\|f * g\|_{L^\infty} \le \|f\|_{L^p}\|g\|_{L^q}$.

Case 3: $1 < r < \infty$. As f^p and g^q are integrable, by Theorem 7.5 the function $x \mapsto f^p(x)g^q(y-x)$ is integrable for almost all $y \in \mathbb{R}^d$.

Fix such a y. Put $h(x) = g(y-x)$ for $x \in \mathbb{R}^d$. As $fh = f^{\frac{p}{r}} h^{\frac{q}{r}} \cdot f^{\frac{r-p}{r}} \cdot h^{\frac{r-q}{r}}$, an application of the Generalized Hölder inequality (see Theorem A.5) with $n = 3$,

$$p_1 = r, \quad p_2 = \frac{pr}{r-p}, \quad p_3 = \frac{qr}{r-q},$$

(with the convention that $\frac{1}{0} = \infty$) so that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, shows that fh is integrable and

$$\int fh \le \left(\int f^p h^q \right)^{\frac{1}{r}} \left(\int f^p \right)^{\frac{r-p}{pr}} \left(\int h^q \right)^{\frac{r-q}{qr}} = \left(\int f(x)^p g(y-x)^q \, \mathrm{d}x \right)^{\frac{1}{r}} A,$$

with $A = (\int f^p)^{\frac{r-p}{pr}} (\int h^q)^{\frac{r-q}{qr}}$, which is independent of y.

The above implies that the function $x \mapsto f(x)g(y-x)$ is integrable for almost every y, so that $f*g$ exists, and $(f*g)^r \le (f^p * g^q)A^r$. By Theorem 7.6 it follows that $f*g \in L^r$ and

$$\begin{aligned}
(\|f*g\|_{L^r})^r &\le \left(\int f^p * g^q \right) A^r \le \int f^p \cdot \int g^q \cdot A^r \\
&= \int f^p \cdot \int g^q \cdot \left(\int f^p \right)^{\frac{r-p}{p}} \left(\int g^q \right)^{\frac{r-q}{q}} = \|f\|_{L^p}^r \|g\|_{L^q}^r.
\end{aligned}$$

□

The following is a consequence of Young's inequality.

Corollary 7.8. *Let $p, q, r \in [1, \infty]$ be such that*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

For $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ and $h \in L^r(\mathbb{R}^d)$ we have that $(f*g)h$ is integrable and

$$\int (f*g)h = \int f(g * \check{h}) = \int f(\check{g} * h) \tag{7.1}$$

$$\|(f*g)h\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}. \tag{7.2}$$

Exercise **7.A.** Prove Corollary 7.8.

In the following definition we define the essential support of a measurable function. This agrees with the support of a distribution when it is given by a locally integrable function. We consider the essential support of a convolution in Theorem 7.10.

Definition 7.9. Let $f : \Omega \to \mathbb{F}$. We say that an open set $U \subset \Omega$ is *f-null* if $f = 0$ almost everywhere on U. We define the *essential support* of f,

$$\operatorname{ess\,supp} f,$$

to be the complement of the union of all f-null open sets.

Actually, this union itself is f-null, hence is the largest f-null open set. To see that, observe that there exist a countable number of f-null open sets $(U_n)_{n\in\mathbb{N}}$ such that $U = \bigcup_{n\in\mathbb{N}} U_n$ by Theorem 1.11.

As the essential support of f is equal to the one of g if f and g are equal almost everywhere, one can make sense of the essential support for equivalence classes of locally integrable functions in the usual way by identifying an equivalence class with an element in it: If $f \in L^1_{\mathrm{loc}}(\Omega)$, then $\operatorname{ess\,supp} f$ is defined by the essential support of any function representing f.

Of course, for a continuous function f we have

$$\operatorname{supp} f = \operatorname{ess\,sup} f.$$

If $f \in L^1_{\mathrm{loc}}$, then by Lemma 2.9 it follows for an open set $U \subset \Omega$ that $\Omega \backslash U$ is a carrier for u_f if and only if U is f-null, hence

$$\operatorname{ess\,sup} f = \operatorname{supp} u_f.$$

Theorem 7.10. *For any two measurable functions f, g on $\mathbb{R}^d$ such that $f * g$ exists, we have*

$$\operatorname{ess\,supp} f * g \subset \overline{\operatorname{ess\,supp} f + \operatorname{ess\,supp} g}.$$

Proof. Let $A = \operatorname{ess\,supp} f$, $B = \operatorname{ess\,supp} g$, and take $c \in \mathbb{R}^d$ such that $f * g(c) \neq 0$. It suffices to prove $c \in A + B$. Now $f = f\mathbb{1}_A$ a.e. and $g = g\mathbb{1}_B$ a.e., so $0 \neq f * g(c) = (f\mathbb{1}_A) * (g\mathbb{1}_B)(c) = \int (f\mathbb{1}_A)(x)(g\mathbb{1}_B)(c-x)\, \mathrm{d}x$. Therefore there is an x such that $(f\mathbb{1}_A)(x)(g\mathbb{1}_B)(c-x) \neq 0$: Then $\mathbb{1}_A(x) \neq 0$ and $\mathbb{1}_B(c-x) \neq 0$, whence $c = c - x + x \in A + B$. ☐

Theorem 7.10 states that the support is included in the closure of the sum of two closed sets. It is necessary to take the closure as Example 7.13 illustrates. We first recall the following facts about closedness of sums of closed sets in Lemma 7.11 and Example 7.12.

Lemma 7.11. *Let $A, B \subset \mathbb{R}^d$ and A be compact and B closed. Then $A + B$ is closed. If, moreover, B is compact then $A + B$ is compact.*

Proof. Let $(d_n)_{n\in\mathbb{N}}$ be a sequence in $A + B$ that converges to an element d in $\mathbb{R}^d$. We prove that $d \in A + B$. By definition, for each n there exist $a_n \in A$ and $b_n \in B$ such that

$d_n = a_n + b_n$. As A is compact, $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Let us assume $(a_n)_{n\in\mathbb{N}}$ itself converges in A to an element a. Then $d_n - a_n \to d - a$ and as $d_n - a_n \in B$ for all n and B is closed, $d - a \in B$, which implies $d = a + d - a \in A + B$.

If B is compact, then $A + B$ is the image of the compact set $A \times B$ of the addition function $+ : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $(x, y) \mapsto x + y$, which is continuous, hence the image is compact. □

The assumption that A is not only closed, but also bounded (which together is the same as compact for subsets of $\mathbb{R}^d$) is essential as the following example illustrates.

Example 7.12. Let $A = \mathbb{N}$ and $B = \{-m + \frac{1}{m} : m \in \mathbb{N}, m \geq 2\}$. Then $A + B$ is not closed as $\frac{1}{m}$ is an element of $A + B$ for all $m \in 1 + \mathbb{N}$ but 0 is not.

Example 7.13. $[\operatorname{ess\,supp} f + \operatorname{ess\,supp} g \subsetneq \operatorname{supp} f * g = \overline{\operatorname{ess\,supp} f + \operatorname{ess\,supp} g}]$
We adapt Example 7.12 to obtain two measurable functions f and g which are not almost everywhere equal to zero. We define the sets $A, B \subset \mathbb{R}$ by

$$A = \bigcup_{n=2}^{\infty} \Big[n, n + \frac{1}{n}\Big], \qquad B = \bigcup_{m=2}^{\infty} \Big[-m + \frac{1}{m}, -m + \frac{2}{m}\Big].$$

We define $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = |x|^{-2}\mathbb{1}_A(x), \qquad g(x) = |x|^{-2}\mathbb{1}_B(x) \qquad (x \in \mathbb{R}).$$

Then f and g are integrable functions and so $f * g$ exists (and is integrable). Moreover, $\operatorname{supp} f = A$, $\operatorname{supp} g = B$,

$$A + B = \bigcup_{n,m=2}^{\infty} \Big[n - m + \frac{1}{m}, n - m + \frac{2}{m} + \frac{1}{n}\Big].$$

As in Example 7.12, the set $A + B$ is not closed as 0 is not in $A + B$ but $\frac{1}{m}$ is for all $m \in 1 + \mathbb{N}$. For each $n, m \in 1 + \mathbb{N}$ and $z \in (n - m + \frac{1}{m}, n - m + \frac{2}{m} + \frac{1}{n})$ we can show that $f * g(z) \neq 0$, so that as the support of a function is closed, $\overline{A + B} \subset \operatorname{supp} f * g$. And thus in this case $A + B \subsetneq \operatorname{supp} f * g = \overline{A + B}$

Exercise 7.B. (a) Choose $f, g \in L^1(\mathbb{R})$ such that f and g are not continuous but $f * g$ is.

(b) Let $\alpha > -1$. Let $f(x) = g(x) = x^\alpha$ for $x \in (0, 1)$ and $f(x) = g(x) = 0$ for $x \notin (0, 1)$. Show that there exists a $c > 0$ such that $f * g(y) = cy^{2\alpha+1}$. Conclude that $f * g$ is not continuous for $\alpha \in (-1, -\frac{1}{2})$.

7.14 (Notation)**.** For any closed set $A \subset \mathbb{R}^d$ we write $[A]_\varepsilon$ for the set of those points in $\mathbb{R}^d$ that are at distance at most ε from A, so that

$$[A]_\varepsilon = A + \overline{B(0, \varepsilon)} = \{y \in \mathbb{R}^d : \inf_{x \in A} |x - y| \leq \varepsilon\}.$$

By Lemma 7.11 $[A]_\varepsilon$ is closed.

Theorem 7.15. *Let $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\psi \in C_c(\mathbb{R}^d)$. For $\varepsilon > 0$ we write ψ_ε for the function defined by $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(\varepsilon^{-1}x)$. Then the following statements hold.*

(a) $f * \psi_\varepsilon(x) \xrightarrow{\varepsilon \downarrow 0} (\int \psi) f(x)$ *for all Lebesgue points $x \in \mathbb{R}^d$ of f.*

(b) *If f is continuous on an open set $U \subset \mathbb{R}^d$, then $f * \psi_\varepsilon \to (\int \psi) f$ uniformly on all compact subsets of U.*

(c) *If $p \in [1, \infty)$ and $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, then $f * \psi_\varepsilon \to (\int \psi) f$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$.*

(d) *If $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^d)$, then $f * \psi_\varepsilon \to (\int \psi) f$ in $L^p(\mathbb{R}^d)$.*

Proof. As $\int \psi_\varepsilon = \int \psi$ for all $\varepsilon > 0$, we have

$$f * \psi_\varepsilon(x) - (\int \psi) f(x) = \int \psi_\varepsilon(x - y)\Big(f(y) - f(x)\Big)\,\mathrm{d}y.$$

As we can find an $\varepsilon > 0$ such that $\operatorname{supp}\psi_\varepsilon \subset B(0,1)$, we may without loss of generality assume that $\operatorname{supp}\psi \subset B(0,1)$. Then

$$|f * \psi_\varepsilon(x) - (\int \psi) f(x)| \le \|\psi\|_{L^\infty} \varepsilon^{-d} \int_{B(x,\varepsilon)} |f(y) - f(x)|\,\mathrm{d}y. \tag{7.3}$$

From this (a) follows. Suppose f is continuous on an open set U and $K \subset U$ is compact. Let $\delta > 0$ be such that $[K]_\delta \subset U$. As f is uniformly continuous on $[K]_\delta$, the convergence in (2.3) is valid uniformly for $x \in K$. Hence (b) also follows from (7.3).

Let us turn to the proof of (c). Let $K \subset \mathbb{R}^d$ be compact. We will show

$$\|f * \psi_\varepsilon - (\textstyle\int \psi) f\|_{L^p, K} \to 0.$$

But first we observe that for all $h \in L^p_{\mathrm{loc}}$ we have

$$|h * \psi_\varepsilon(x)| \le \int |h(y)\psi_\varepsilon(x-y)|\,\mathrm{d}y \le \int |(h\mathbb{1}_{[K]_\varepsilon})(y)\psi_\varepsilon(x-y)|\,\mathrm{d}y \qquad (x \in K),$$

so that with Young's inequality and as $\|\psi_\varepsilon\|_{L^1} = \|\psi\|_{L^1}$

$$\|(h * \psi_\varepsilon)\mathbb{1}_K\|_{L^p} \le \|\psi\|_{L^1}\|h\mathbb{1}_{[K]_\varepsilon}\|_{L^p} = \|\psi\|_{L^1}\|h\|_{L^p,[K]_\varepsilon}. \tag{7.4}$$

Let $\delta > 0$. There exists a function g (by Lemma A.14) that is continuous on $[K]_1$ and equals 0 outside $[K]_1$ such that

$$\|f - g\|_{L^p([K]_1)} < \delta.$$

Then, as $|f * \psi_\varepsilon - (\int \psi) f| \le |f * \psi_\varepsilon - g * \psi_\varepsilon| + |g * \psi_\varepsilon - (\int \psi) g| + |(\int \psi) g - (\int \psi) f|$, we obtain for $\varepsilon \in (0,1)$ by using (7.4)

$$\begin{aligned}\|f * \psi_\varepsilon - (\textstyle\int \psi) f\|_{L^p,K} &\le \|(f-g) * \psi_\varepsilon\|_{L^p,K} + \|g * \psi_\varepsilon - (\textstyle\int \psi) g\|_{L^p,K} \\ &\quad + \|\psi\|_{L^1}\|g - f\|_{L^p,K} \\ &\le 2\delta\|\psi\|_{L^1} + (\int \mathbb{1}_K)^{\frac{1}{p}} \|g * \psi_\varepsilon - (\textstyle\int \psi) g\|_{L^\infty,K}.\end{aligned}$$

As g is continuous on the set $U = [K]_1^\circ$, (b) implies $\|g * \psi_\varepsilon - (\int \psi)g\|_{L^\infty,K} \xrightarrow{\varepsilon \downarrow 0} 0$, which in turn implies (c).

(d) follows similarly as (c), because $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ (Lemma A.14), so that for all $\delta > 0$ there exists a $g \in C_c(\mathbb{R}^d)$ such that $\|f - g\|_{L^p} < \delta$. □

8 Convolution of distributions with testfunctions

In this section we consider the convolution of a distribution with a testfunction. As we have seen in 7.4: For $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ we have

$$f * \varphi(y) = \int f \mathcal{T}_y \check{\varphi} = u_f(\mathcal{T}_x \check{\varphi}) \qquad (y \in \mathbb{R}^d),$$

see 7.4. This equality motivates the following generalisation of the notion of convolution between functions to convolution between distributions and testfunctions:

Definition 8.1. Let (u, φ) be in $\mathcal{D}'(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d)$ or in $\mathcal{E}'(\mathbb{R}^d) \times \mathcal{E}(\mathbb{R}^d)$. We define the *convolution* of u with φ to be the function $\mathbb{R}^d \to \mathbb{F}$ defined by

$$u * \varphi(x) = u(\mathcal{T}_x \check{\varphi}) \qquad (x \in \mathbb{R}^d).$$

It is easy (see Exercise 8.A) to check the following properties.

Lemma 8.2. *Let (u, φ) be in $\mathcal{D}'(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d)$ or in $\mathcal{E}'(\mathbb{R}^d) \times \mathcal{E}(\mathbb{R}^d)$. Then*

$$\begin{aligned} \delta_0 * \varphi &= \varphi, \\ \delta_y * \varphi &= \mathcal{T}_y \varphi \qquad (y \in \mathbb{R}^d), \\ \mathcal{R}(u * \varphi) &= \mathcal{R}u * \mathcal{R}\varphi, \\ \mathcal{T}_y(u * \varphi) &= (\mathcal{T}_y u) * \varphi = u * (\mathcal{T}_y \varphi), \\ u(\varphi) &= u * \check{\varphi}(0). \end{aligned}$$

Exercise 8.A. Prove Lemma 8.2.

Before we turn to the differentiability of the convolution $u * \varphi$, in the following lemma we will show that the convergence of difference quotients of testfunctions hold in the topology of $\mathcal{D}$.

Lemma 8.3. *Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\psi \in \mathcal{E}(\mathbb{R}^d)$ and $i \in \{1, \ldots, d\}$. Then*

$$\left(\frac{\mathcal{T}_0 - \mathcal{T}_{he_i}}{h} \right) \varphi \xrightarrow{h \to 0} \partial_i \varphi \quad \text{in } \mathcal{D}(\mathbb{R}^d), \tag{8.1}$$

$$\left(\frac{\mathcal{T}_0 - \mathcal{T}_{he_i}}{h} \right) \psi \xrightarrow{h \to 0} \partial_i \psi \quad \text{in } \mathcal{E}(\mathbb{R}^d). \tag{8.2}$$

Consequently, for $u \in \mathcal{D}'(\mathbb{R}^d)$ *and* $v \in \mathcal{E}'(\mathbb{R}^d)$,

$$\left(\frac{\mathcal{T}_0 - \mathcal{T}_{he_i}}{h}\right) u \xrightarrow{h\to 0} \partial_i u \qquad \text{in } \mathcal{D}'(\mathbb{R}^d),$$
$$\left(\frac{\mathcal{T}_0 - \mathcal{T}_{he_i}}{h}\right) v \xrightarrow{h\to 0} \partial_i v \qquad \text{in } \mathcal{E}'(\mathbb{R}^d).$$

Proof. Let us first show (8.1). A simple argument by contradiction shows that it is sufficient to show that

$$\lim_{n\to\infty} \left(\frac{\mathcal{T}_0 - \mathcal{T}_{h_n e_i}}{h_n}\right) \varphi = \partial_i \varphi$$

for every sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\setminus\{0\}$ that converges to 0. Let $(h_n)_{n\in\mathbb{N}}$ be such a sequence and, for $n \in \mathbb{N}$, put

$$\psi_n(x) := \left(\frac{\mathcal{T}_0 - \mathcal{T}_{h_n e_i}}{h_n}\right) \varphi(x) - \partial_i \varphi(x) \qquad (x \in \mathbb{R}^d). \tag{8.3}$$

We apply Theorem 4.11 to prove $\psi_n \to 0$ in $\mathcal{D}(\mathbb{R}^d)$. That (a) of Theorem 4.11 is satisfied can be easily seen from the compactness of $\operatorname{supp}\varphi$ (see also Lemma 7.11). It follows from Taylor's Theorem (see Theorem B.4) that

$$\left|\frac{f(0) - f(-h)}{h} - f'(0)\right| \le |h| \|f''\|_{L^\infty} \qquad (h \in \mathbb{R}\setminus\{0\}),$$

for every smooth function f on $\mathbb{R}$ for which f'' is bounded. By choosing $f(h) = \partial^\alpha \varphi(x + he_i)$, where $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$, we obtain

$$\left|\frac{\partial^\alpha \varphi(x) - \partial^\alpha \varphi(x - he_i)}{h} - \partial_i \partial^\alpha \varphi(x)\right| \le |h| \|\partial_i^2 \partial^\alpha \varphi\|_{L^\infty}. \tag{8.4}$$

Therefore, for each $\alpha \in \mathbb{N}_0^d$,

$$\|\partial^\alpha \psi_n\|_{L^\infty} = \sup_{x\in\mathbb{R}^d} \left|\frac{\partial^\alpha \varphi(x) - \partial^\alpha \varphi(x - h_n e_i)}{h_n} - \partial_i \partial^\alpha \varphi(x)\right| \to 0,$$

which implies (b) of Theorem 4.11.

For (8.2): Let $K \subset \mathbb{R}^d$ be compact. As the set $K_1 := K + \overline{B(0,1)}$ is compact, $\|\partial_i^2 \partial^\alpha \varphi\|_{L^\infty(K_1)} < \infty$. Also by using Taylor's Theorem, similarly as how we obtained (8.4), one obtains

$$\left|\frac{\partial^\alpha \psi(x) - \partial^\alpha \psi(x - he_i)}{h} - \partial_i \partial^\alpha \psi(x)\right| \le |h| \|\partial_i^2 \partial^\alpha \psi\|_{L^\infty(K_1)} \qquad (h \in \overline{B(0,1)}, x \in K).$$

Therefore, for each $\alpha \in \mathbb{N}_0^d$,

$$\left\|\partial^\alpha \left[\left(\frac{\mathcal{T}_0 - \mathcal{T}_{he_i}}{h}\right) \varphi - \partial_i \varphi\right]\right\|_{L^\infty(K)} = \sup_{x\in K} \left|\frac{\partial^\alpha \varphi(x) - \partial^\alpha \varphi(x - he_i)}{h} - \partial_i \partial^\alpha \varphi(x)\right| \to 0,$$

and thus $\left\|\left(\frac{\mathcal{T}_0 - \mathcal{T}_{he_i}}{h}\right) \varphi - \partial_i \varphi\right\|_{C^m,K} \to 0$ for all $m \in \mathbb{N}$. □

Theorem 8.4. *Let* (u,φ) *be in* $\mathcal{D}'(\mathbb{R}^d)\times\mathcal{D}(\mathbb{R}^d)$ *or in* $\mathcal{E}'(\mathbb{R}^d)\times\mathcal{E}(\mathbb{R}^d)$. *Then* $u*\varphi\in C^\infty(\mathbb{R}^d)$ *and*

$$\partial^\alpha(u*\varphi)=u*(\partial^\alpha\varphi)=(\partial^\alpha u)*\varphi \qquad (\alpha\in\mathbb{N}_0^d). \tag{8.5}$$

Proof. Let $\in\{1,\dots,d\}$. It is enough to prove that $\partial_i(u*\varphi)=u*(\partial_i\varphi)$. Let us write $\mathcal{R}$ here for the reflector operator $\mathcal{E}(\mathbb{R}^d)\to\mathcal{E}(\mathbb{R}^d)$, $\varphi\mapsto\check{\varphi}$. Then $\partial_i\mathcal{R}=-\mathcal{R}\partial_i$ and $u*\varphi(x)=u(\mathcal{T}_y\mathcal{R}\varphi)$ for $x\in\mathbb{R}^d$. Then for every $x\in\mathbb{R}^d$ we obtain by Lemma 8.3

$$\begin{aligned}
\partial_i(u*\varphi)(x)&=\lim_{h\to0}\frac{u*\varphi(x+he_i)-u*\varphi(x)}{h}\\
&=\lim_{h\to0}\frac{u(\mathcal{T}_{x+he_i}\mathcal{R}\varphi)-u(\mathcal{T}_x\mathcal{R}\varphi)}{h}\\
&=\lim_{h\to0}u\left(\mathcal{T}_x\mathcal{R}\left(\frac{\mathcal{T}_{-he_i}\varphi-\varphi}{h}\right)\right)\\
&=u\left(\mathcal{T}_x\mathcal{R}\left(\lim_{h\to0}\frac{\mathcal{T}_{-he_i}\varphi-\varphi}{h}\right)\right)\\
&=u(\mathcal{T}_x\mathcal{R}\partial_i\varphi)=u*(\partial_i\varphi)(x).
\end{aligned}$$

Moreover, $u(\mathcal{T}_x\mathcal{R}\partial_i\varphi)=-u(\partial_i\mathcal{T}_x\mathcal{R}\varphi)=\partial_iu(\mathcal{T}_x\mathcal{R}\varphi)=(\partial_iu)*\varphi$. □

The statement of Theorem 7.10, which states that the support of the convolution of two functions is included in the closure of the sum of the supports, extends to distributions (see Theorem 8.6).

8.5 (Convention for the notation "supp")**.** As we have observed in Definition 7.9, we have

$$\begin{aligned}
\operatorname{supp}u_f&=\operatorname{supp}f \qquad (f\in C(\Omega)),\\
\operatorname{supp}u_g&=\operatorname{ess\,sup}g \qquad (f\in L^1_{\mathrm{loc}}(\Omega).
\end{aligned}$$

As we in general do not distinguish between a locally integrable function f and its corresponding distribution u_f, we will also write "$\operatorname{supp}f$" instead of "$\operatorname{supp}u_f$", which then corresponds to "$\operatorname{ess\,sup}f$" or if f is continuous with "$\operatorname{supp}f$" (of course $\operatorname{supp}f$ as in Definition 1.1 also appears in the literature for functions that are not continuous, but as our focus is on distributions, we assume that no confusion arises).

Theorem 8.6. *Let* (u,φ) *be in* $\mathcal{D}'(\mathbb{R}^d)\times\mathcal{D}(\mathbb{R}^d)$ *or in* $\mathcal{E}'(\mathbb{R}^d)\times\mathcal{E}(\mathbb{R}^d)$. *Then*

$$\operatorname{supp}u*\varphi\subset\operatorname{supp}u+\operatorname{supp}\varphi.$$

Exercise **8.B.** Prove Theorem 8.6.

Theorem 8.7. *Convolution is a sequentially continuous operation: The maps*

$$\begin{aligned}
\mathcal{D}'\times\mathcal{D}&\to\mathcal{E}, &&(u,\varphi)\mapsto u*\varphi, &(8.6)\\
\mathcal{E}'\times\mathcal{D}&\to\mathcal{D}, &&(v,\varphi)\mapsto v*\varphi, &(8.7)\\
\mathcal{E}'\times\mathcal{E}&\to\mathcal{E}, &&(v,\psi)\mapsto v*\psi, &(8.8)
\end{aligned}$$

are sequentially continuous.

Proof. We prove sequential continuity of (8.6) and (8.8); the sequential continuity of (8.7) follows from (8.6) by Theorem 4.11.

Proof of the sequential continuity of (8.6).

Suppose $(u_n, \varphi_n) \to (u, \varphi)$ in $\mathcal{D}' \times \mathcal{D}$, i.e., $u_n \to u$ in $\mathcal{D}'$ and $\varphi_n \to \varphi$ in $\mathcal{D}$. Let $K \subset \mathbb{R}^d$ be compact and $m \in \mathbb{N}_0$. We show that

$$\|u_n * \varphi_n - u * \varphi\|_{C^m,K} \to 0,$$

by showing

$$\|u_n * (\varphi_n - \varphi)\|_{C^m,K} \to 0, \tag{8.9}$$

$$\|(u_n - u) * \varphi\|_{C^m,K} \to 0. \tag{8.10}$$

By Theorem 4.11 (a) there exists a compact set L such that the supports of $\check{\varphi}_n$ and $\check{\varphi}$ are contained in L for all $n \in \mathbb{N}$. Write $u_\infty = u$. As the set $K + L$ is compact (see Lemma 7.11), by Theorem 4.24 (a), there exist $C > 0$ and $k \in \mathbb{N}$ such that

$$|\partial^\alpha u_n(\eta)| \le C\|\eta\|_{C^k} \qquad (\eta \in \mathcal{D}_{K+L}, n \in \mathbb{N} \cup \{\infty\}, \alpha \in \mathbb{N}_0^d, |\alpha| \le m+1). \tag{8.11}$$

Then

$$\operatorname{supp} \mathcal{T}_x \check{\varphi}_n,\ \operatorname{supp} \mathcal{T}_x \check{\varphi} \subset K + L \qquad (n \in \mathbb{N}, x \in K),$$

hence, using Theorem 8.4

$$\begin{aligned}\|u_n * (\varphi_n - \varphi)\|_{C^m,K} &\le \sup_{x \in K} \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le m}} |\partial^\alpha u_n(\mathcal{T}_x \mathcal{R}(\varphi_n - \varphi))| \\ &\le C\|\varphi_n - \varphi\|_{C^k} \to 0,\end{aligned}$$

which means (8.9) holds.

To prove (8.10) we use Lemma 4.8. Without loss of generality we may assume K to be convex (as we can always choose a larger compact convex set). First observe, that as $u_n \to u$ in $\mathcal{D}'$, we have $\partial^\alpha(u_n - u) * \varphi(x) \to 0$ for all $x \in K$ and $\alpha \in \mathbb{N}_0^d, |\alpha| \le m$. Therefore, by Lemma 4.8, (8.10) follows when $(\partial^\alpha(u_n - u) * \varphi)_{n \in \mathbb{N}}$ is a sequence of uniformly Lipschitz continuous functions on K for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$. For this, by Lemma 4.7, it is sufficient to show that $\sup_{n \in \mathbb{N} \cup \{\infty\}} \|u_n * \varphi\|_{C^{m+1},K} < \infty$. But by (8.11) for all $n \in \mathbb{N} \cup \{\infty\}$ we have

$$\begin{aligned}\|u_n * \varphi\|_{C^{m+1},K} &\le \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le m}} \sup_{x \in K} |\partial^\alpha u_n * \varphi(x)| \\ &\le C \sup_{x \in K} \|\mathcal{T}_x \check{\varphi}\|_{C^k} = C\|\varphi\|_{C^k}.\end{aligned}$$

This finishes the proof of the sequential continuity of (8.6).

Proof of the sequential continuity of (8.8). Suppose $(u_n, \varphi_n) \to (u, \varphi)$ in $\mathcal{E}' \times \mathcal{E}$. Let $K \subset \mathbb{R}^d$ be compact and $m \in \mathbb{N}_0$. We show both (8.9) and (8.10). Write $u_\infty = u$. By Theorem 4.24 (b) there exist a compact set $L \subset \mathbb{R}^d$, $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|\partial^\alpha u_n(\eta)| \le C\|\eta\|_{C^k,L} \qquad (\eta \in \mathcal{E}, n \in \mathbb{N} \cup \{\infty\}, \alpha \in \mathbb{N}_0^d, |\alpha| \le m). \tag{8.12}$$

Then

$$\begin{aligned}\|u_n * (\varphi_n - \varphi)\|_{C^m,K} &\le \sup_{x\in L} \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le m}} |\partial^\alpha u_n(\mathcal{T}_x \mathcal{R}(\varphi_n - \varphi))| \\ &\le C\|\varphi_n - \varphi\|_{C^k,K+L} \to 0,\end{aligned}$$

so that (8.9) holds.

To prove (8.10) by the same reasoning (using Lemma 4.8 and Lemma 4.7) it is sufficient to show $\sup_{n\in\mathbb{N}\cup\{\infty\}} \|u_n * \varphi\|_{C^{m+1},K} < \infty$. By (8.12) we have for all $n \in \mathbb{N} \cup \{\infty\}$ we have

$$\begin{aligned}\|u_n * \varphi\|_{C^{m+1},K} &\le \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le m}} \sup_{x\in K} |\partial^\alpha u_n * \varphi(x)| \\ &\le C \sup_{x\in K} \|\mathcal{T}_x \check{\varphi}\|_{C^k,L} = C\|\varphi\|_{C^k,K+L}.\end{aligned}$$

□

Let $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$ and $u \in \mathcal{D}'(\mathbb{R}^d)$. The convolution $\eta = \varphi * \psi$ is a smooth function with compact support by Theorem 8.6 (and Lemma 7.11). Hence it is a testfunction and $u * \eta = u * (\varphi * \psi)$ is a smooth function. On the other hand, $v = u * \varphi$ is a smooth function and therefore can be identified with its corresponding distribution, moreover, $v*\psi = (u*\varphi)*\psi$ is a smooth function. In Theorem 8.9 we will see that these functions are equal, that is, the convolution obeys an associativity rule. Before, we prove an auxiliary lemma that considers an approximation of $\varphi * \psi$. This lemma will be used later on for example to prove that $\mathcal{D}$ and $\mathcal{D}'$ are sequentially separable (in Theorem 8.15). We give two proofs of Theorem 8.9, one which relies on the Structure Theorem 6.2 and one which relies on Lemma 8.8 (hence one may postpone reading Lemma 8.8 if one is happy to use Theorem 6.2).

Lemma 8.8. *Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then there is a sequence $(w_j)_{j\in\mathbb{N}}$ in $\mathcal{D}'$ consisting of finite linear combinations of point masses, supported in* $\operatorname{supp}\varphi$*, such that $w_j * \psi \to \varphi * \psi$ in $\mathcal{D}(\mathbb{R}^d)$ for all $\psi \in \mathcal{D}(\mathbb{R}^d)$.*

Proof. For $j \in \mathbb{N}$, take a locally finite partition of unity $(\chi_n)_{n\in\mathbb{N}}$ as in Theorem 1.11 with $\operatorname{diam}\operatorname{supp}\chi_n < \frac{1}{j}$ for every $n \in \mathbb{N}$, and define

$$w_j = \sum_{n\in\mathbb{N}} (\int \varphi\chi_n)\delta_{a_n},$$

where, for $n \in \mathbb{N}$, $a_n \in \operatorname{supp} \chi_n$ and if $\varphi\chi_n \neq 0$ then also $a_n \in \operatorname{supp} \varphi$.

By definition we have $\operatorname{supp} w_j \subset \operatorname{supp} \varphi$, so that $\operatorname{supp} w_j * \psi \subset \operatorname{supp} \varphi + \operatorname{supp} \psi$. Therefore, by Theorem 4.11, for $\psi \in \mathcal{D}(\mathbb{R}^d)$, $w_j * \psi \to \varphi * \psi$ in $\mathcal{D}(\mathbb{R}^d)$ when $\partial^\alpha(w_j * \psi) \to \partial^\alpha(\varphi * \psi)$ uniformly for all $\alpha \in \mathbb{N}_0^d$. As $\partial^\alpha(w_j * \psi) = w_j * (\partial^\alpha \psi)$ and $\partial^\alpha(\varphi * \psi) = \varphi * (\partial^\alpha \psi)$ and $\partial^\alpha \psi$ is a testfunction, it is sufficient to show:

Claim: *If $\psi \in \mathcal{D}(\mathbb{R}^d)$, then $w_j * \psi \to \varphi * \psi$ uniformly.*

Let $\varepsilon > 0$. Take $j \in \mathbb{N}$ such that $|\psi(x) - \psi(y)| < \varepsilon$ whenever $|x - y| < \frac{1}{j}$. It is sufficient to show that

$$|(w_j * \psi)(a) - (\varphi * \psi)(a)| \leq \varepsilon \int |\varphi| \qquad (a \in \mathbb{R}^d). \tag{8.13}$$

Fix $a \in \mathbb{R}^d$ and let $\eta = \mathcal{T}_a \check{\psi}$. As $w_j * \psi(a) = w_j(\eta)$ and $\varphi * \psi(a) = \int \varphi\eta$, instead of (8.13) we may as well show

$$|w_j(\eta) - \int \varphi\eta| \leq \varepsilon \int |\varphi|. \tag{8.14}$$

Observe that η satisfies, like ψ, $|\eta(x) - \eta(y)| < \varepsilon$ whenever $|x - y| < \frac{1}{j}$. We have

$$w_j(\eta) - \int \varphi\eta = \sum_{n \in \mathbb{N}} \left(\int \varphi\chi_n \right) \eta(a_n) - \int \varphi\eta = \sum_{n \in \mathbb{N}} \int \varphi\chi_n(\eta(a_n) - \eta).$$

As $a_n \in \operatorname{supp} \chi_n$ it follows that $|w_j(\eta) - \int \varphi\eta| \leq \sum_{n \in \mathbb{N}} \int |\varphi|\chi_n \varepsilon = \varepsilon \int |\varphi|$, which implies (8.13). □

Theorem 8.9. (a) *Let $u \in \mathcal{D}'$ and $\varphi, \psi \in \mathcal{D}$. Then*

$$(u * \psi) * \varphi = u * (\psi * \varphi). \tag{8.15}$$

(b) *Let $v \in \mathcal{E}'$, $\varphi \in \mathcal{D}$ and $\eta \in \mathcal{E}$. Then*

$$v * (\varphi * \eta) = (v * \varphi) * \eta = (v * \eta) * \varphi \tag{8.16}$$

Proof. (a) It suffices to prove $(u * \psi) * \varphi(0) = u * (\psi * \varphi)(0)$ for any $u \in \mathcal{D}'$, $\varphi, \psi \in \mathcal{D}$ (take translations and use the rules in Lemma 8.2). Choose $\chi \in C_c^\infty(\mathbb{R}^d, [0, 1])$ such that $\chi = 1$ on an open set that contains $-\operatorname{supp} \psi - \operatorname{supp} \varphi$. Both $((1 - \chi)u * \psi) * \varphi$ and $((1 - \chi)u) * (\psi * \varphi)$ have supports in $\operatorname{supp}(1 - \chi) + \operatorname{supp} \psi + \operatorname{supp} \varphi$, so both vanish at 0.

Continuation of the proof using Theorem 6.2. χu has compact support, therefore it is equal to a finite linear combination of derivatives of continuous functions by Theorem 6.2. As for any continuous function $f \in C(\Omega)$ and $\alpha \in \mathbb{N}_0^d$

$$(\partial^\alpha f * \psi) * \varphi = \partial^\alpha((f * \psi) * \varphi) = \partial^\alpha(f * (\psi * \varphi)) = \partial^\alpha f * (\psi * \varphi),$$

it follows that $(\chi u * \psi) * \varphi = \chi u * (\psi * \varphi)$.

Continuation of the proof using Lemma 8.8 Let $(w_j)_{j\in\mathbb{N}}$ be as in Lemma 8.8, so that $w_j * \eta \to \varphi * \eta$ in $\mathcal{D}(\mathbb{R}^d)$ for all $\eta \in \mathcal{D}(\mathbb{R}^d)$. Write $v = \chi u$. As $(v * \psi)$ and φ are testfunctions, we have $(v * \psi) * \varphi = \varphi * (v * \psi)$ (see 7.4(4)). Therefore, by the properties of $(w_j)_{i\in\mathbb{N}}$ and the continuity as in Theorem 8.7, we are done if we can show that

$$w_j * (v * \psi) = v * (w_j * \psi) \qquad (j \in \mathbb{N}).$$

But this follows as w_j is a linear combination of point measures for each $i \in \mathbb{N}$, and

$$\begin{aligned}\delta_y * (v * \psi)(x) &= \mathcal{T}_y(v * \psi)(x) = v(\mathcal{T}_{x-y}\check{\psi}) = v(\mathcal{T}_x(\mathcal{T}_y\psi)\check{}) = v * (\mathcal{T}_y\psi)(x) \\ &= v * (\delta_y * \psi)(x).\end{aligned}$$

(b) If $\eta \in \mathcal{D}$, then this follows by (a) (and the fact that $\varphi * \eta = \eta * \varphi$). As $\mathcal{D}$ is sequentially dense in $\mathcal{E}$ (Theorem 5.10), (8.16) follows by the sequential continuity of the convolution map $\mathcal{E} \to \mathcal{E}$, $\psi \mapsto w * \psi$ with w being the element in $\mathcal{E}'$ given by either v, $v * \varphi$ or φ. □

8.10. As a direct consequence of Theorem 8.9 we have for $u \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$

$$\langle u * \psi, \varphi\rangle = (u * \psi) * \check{\varphi}(0) = u * (\psi * \check{\varphi})(0) = \langle u, \check{\psi} * \varphi\rangle,$$

and for $v \in \mathcal{E}'(\mathbb{R}^d)$ and $\eta \in \mathcal{E}(\mathbb{R}^d)$

$$\langle v * \psi, \eta\rangle = (v * \psi) * \check{\eta}(0) = v * (\psi * \check{\eta})(0) = \langle v, \check{\psi} * \eta\rangle.$$

Observe that if $u = u_f$ for an $f \in L^1(\mathbb{R}^d)$, then the above relation agrees with $\int (f * \psi)\varphi = \int f(\check{\psi} * \varphi)$ as in Corollary 7.8.

Definition 8.11. Let ψ be a testfunction such that $\operatorname{supp}\psi \subset B(0,1)$ and $\int \psi = 1$ (the existence is guaranteed by Lemma 1.6). Such a function is called a *mollifier.* For a mollifier ψ and for $\varepsilon > 0$ we define ψ_ε to be the function on $\mathbb{R}^d$ defined by

$$\psi_\varepsilon(x) = \varepsilon^{-d}\psi(\tfrac{x}{\varepsilon}) \qquad (x \in \mathbb{R}^d).$$

Then $\operatorname{supp}\psi_\varepsilon \subset B(0,\varepsilon)$ and $\int \psi_\varepsilon = 1$ for all $\varepsilon > 0$. For a distribution u we call

$$u * \psi_\varepsilon \tag{8.17}$$

a *mollification* of u (with respect to ψ of order ε).

Let ψ be a mollifier. By Theorem 8.4 we know that $u_\varepsilon = u * \psi_\varepsilon$ is a smooth function for all $\varepsilon > 0$. For a function f in L^p_{loc} we also know that $f_\varepsilon := f * \psi_\varepsilon \to f$ in L^p_{loc}, by Theorem 7.15. So in particular,

$$\int f_\varepsilon\varphi \to \int f\varphi \qquad (\varphi \in \mathcal{D}(\mathbb{R}^d)),$$

which implies that $f_\varepsilon \to f$ in $\mathcal{D}'(\mathbb{R}^d)$. This “extends” to any distribution, see the following theorem. This theorem follows by Theorem 7.15.

Theorem 8.12. *Let ψ be a mollifier, $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $u \in \mathcal{D}'(\mathbb{R}^d)$, $\eta \in \mathcal{E}(\mathbb{R}^d)$ and $v \in \mathcal{E}'(\mathbb{R}^d)$. Then $u_\varepsilon = u * \psi_\varepsilon \in \mathcal{E}(\mathbb{R}^d)$, $v_\varepsilon = v * \psi_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$,*

$$\begin{aligned} &\operatorname{supp} u_\varepsilon \subset [\operatorname{supp} u]_\varepsilon, && \operatorname{supp} v_\varepsilon \subset [\operatorname{supp} v]_\varepsilon, \\ &\varphi * \psi_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \varphi \quad \text{in } \mathcal{D}(\mathbb{R}^d), && \eta * \psi_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \eta \quad \text{in } \mathcal{E}(\mathbb{R}^d), \\ &u_\varepsilon \xrightarrow{\varepsilon \downarrow 0} u \quad \text{in } \mathcal{D}'(\mathbb{R}^d) && v_\varepsilon \xrightarrow{\varepsilon \downarrow 0} v \quad \text{in } \mathcal{E}'(\mathbb{R}^d). \end{aligned}$$

Proof. $\operatorname{supp} u_\varepsilon \subset [\operatorname{supp} u]_\varepsilon$ follows from Theorem 8.6.

For the convergence $\varphi * \psi_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \varphi$ in $\mathcal{D}(\mathbb{R}^d)$ we use Corollary 4.12. As for $\varepsilon \le 1$ the supports $\operatorname{supp} \varphi * \psi_\varepsilon$ are contained in the compact set $[\operatorname{supp} \varphi]_1$ it is sufficient to show that $\|\varphi * \psi_\varepsilon - \varphi\|_{C^m} \to 0$ for all $m \in \mathbb{N}_0$. As $\partial^\alpha \eta \in \mathcal{D}(\mathbb{R}^d)$ and $\partial^\alpha(\eta * \psi_\varepsilon) = (\partial^\alpha \eta) * \psi$ for all $\eta \in \mathcal{D}(\mathbb{R}^d)$, it is sufficient to show that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ $\varphi * \psi_\varepsilon \to \varphi$ uniformly. But this follows from Theorem 7.15 (b)

The convergence $u_\varepsilon \xrightarrow{\varepsilon \downarrow 0} u$ in $\mathcal{D}'(\mathbb{R}^d)$ follows from the identity in 8.10 and the convergence $\check{\psi}_\varepsilon * \varphi \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

The convergences $\eta * \psi_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \eta$ and $v_\varepsilon \xrightarrow{\varepsilon \downarrow 0} v$ follow in a similar fashion, their proof is left to the reader. □

The next example illustrates that the inclusion $\operatorname{supp} u_\varepsilon \subset [\operatorname{supp} u]_\varepsilon$ can be a strict inclusion.

Example 8.13 ($\operatorname{supp} u_\varepsilon \neq [\operatorname{supp} u]_\varepsilon$)**.** Consider $d = 1$. By choosing a mollifier ψ (for example the one of Lemma 1.6) which is supported in $(-1, 1)$ we can define another mollifier $\tilde{\psi}$ that is supported in $(-\frac{3}{4}, -\frac{1}{4}) \cup (\frac{1}{4}, \frac{3}{4})$ as follows:

$$\tilde{\psi} = \tfrac{1}{2}(\mathcal{T}_{-\frac{1}{2}} \psi_{\frac{1}{4}} + \mathcal{T}_{\frac{1}{2}} \psi_{\frac{1}{4}}).$$

Then $\delta * \tilde{\psi}$ equals $\tilde{\psi}$, which is zero around zero. Hence $0 \notin \operatorname{supp} \delta * \tilde{\psi}$ but $0 \in (\operatorname{supp} \delta)_1$, i.e., the inclusion in Theorem 8.12 may be strict.

By using the previous theorem it is immediate that $\mathcal{E}(\mathbb{R}^d)$ is sequentially dense in $\mathcal{D}'(\mathbb{R}^d)$. As $\mathcal{D}(\mathbb{R}^d)$ is sequentially dense in $\mathcal{E}(\mathbb{R}^d)$, also $\mathcal{D}(\mathbb{R}^d)$ is sequentially dense in $\mathcal{D}'(\mathbb{R}^d)$ as $\mathcal{E}(\mathbb{R}^d)$ is continuously embedded in $\mathcal{D}'(\mathbb{R}^d)$ (see Theorem 5.10, that $\mathcal{E}(\mathbb{R}^d)$ is continuously embedded in $\mathcal{D}'(\mathbb{R}^d)$ follows for example by Theorem 4.28). Moreover, we have the following:

Lemma 8.14. *Let $\Omega \subset \mathbb{R}^d$ be open. $\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$ and in $\mathcal{E}'(\Omega)$.*

Proof. We will show that there exist $(\varphi_\varepsilon)_{\varepsilon>0}$ in $\mathcal{D}(\Omega)$ that converge to u in $\mathcal{D}'(\Omega)$. As the compactly supported distributions are sequentially dense in $\mathcal{D}'(\Omega)$ (see Theorem 5.10), we may assume u has compact support. Let $\chi \in \mathcal{D}(\Omega)$ be equal to 1 on a neighbourhood of $\operatorname{supp} u$, so that $\chi u = u$. Let $\overline{u} \in \mathcal{D}'(\mathbb{R}^d)$ be the distribution given by $\overline{u}(\varphi) = u(\chi \varphi|_\Omega)$.

Observe $\operatorname{supp} \overline{u} = \operatorname{supp} u$. Let ψ be a mollifier. Then $\overline{u} * \psi_\varepsilon \to \overline{u}$ in $\mathcal{D}'(\mathbb{R}^d)$ by Theorem 8.12. Moreover $\overline{u} * \psi_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ and there exists a $\delta > 0$ such that $[\operatorname{supp} u]_\delta \subset \Omega$, and thus $\operatorname{supp}(\overline{u} * \psi_\varepsilon) \subset \Omega$ for all $\varepsilon \in (0, \delta)$. Therefore $\varphi_\varepsilon := \overline{u} * \psi_\varepsilon|_\Omega$ is an element of $\mathcal{D}(\Omega)$ and by continuity of the embedding $\rho : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{D}'(\Omega)$ as in 4.6 is follows that $\varphi_\varepsilon \to u$ in $\mathcal{D}'(\Omega)$.

That $\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{E}'(\Omega)$ follows in a similar fashion, this proof is left to the reader. □

Theorem 8.15. *Let $\Omega \subset \mathbb{R}^d$ be open. Then $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ are sequentially separable.*

Proof. By Lemma 8.14 it is sufficient to show that $\mathcal{D}(\Omega)$ is sequentially separable.

We invoke Lemma 8.8. Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ be a mollifier. By Theorem 8.12 we know that the set of functions

$$\{\varphi * \psi_\varepsilon : \varphi \in \mathcal{D}, \varepsilon > 0, [\operatorname{supp} \varphi]_\varepsilon \subset \Omega\} \tag{8.18}$$

is dense in $\mathcal{D}(\Omega)$. Let $\mathbb{A} \subset \mathcal{D}(\Omega)$ be the collection of functions of the form

$$\sum_{k=1}^{m} \lambda_k \delta_{x_k} * \psi_\varepsilon$$

for some $m \in \mathbb{N}$, $\lambda_k \in \mathbb{Q} + \mathrm{i}\mathbb{Q}$, $x_k \in \mathbb{Q}^d \cap \Omega$ for all $k \in \{1, \dots, m\}$ and with $\varepsilon \in (0, \infty) \cap \mathbb{Q}$ such that

$$[\{x_1, \dots, x_m\}]_\varepsilon \subset \Omega.$$

Then $\mathbb{A}$ is a countable set. By Lemma 8.8 it follows that $\mathbb{A}$ is dense in (8.18) and thus in $\mathcal{D}(\Omega)$. □

Similarly to the proof that $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}'(\Omega)$ we can (and will) prove that $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for finite p. For this we use the following lemma to show that the compactly supported functions in $L^p(\Omega)$ are dense in $L^p(\Omega)$. For this reason we do not need to include the "∂^α" in the lemma, but we do so as this will be used later on.

Lemma 8.16. *Let $p \in [1, \infty)$. Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ be equal to 1 a neighbourhood of 0. For $R > 0$ write $\chi_R = l_{\frac{1}{R}} \chi = \chi(\frac{1}{R} \cdot)$. Then*

$$\|u \partial^\alpha (\chi_R - \mathbb{1})\|_{L^p} \xrightarrow{R \to \infty} 0 \qquad (\alpha \in \mathbb{N}_0^d, u \in L^p(\mathbb{R}^d)).$$

Proof. Observe that for $R \geq 1$,

$$\|\partial^\alpha (\chi_R - \mathbb{1})\|_{L^\infty} \leq 1 + R^{-|\alpha|} \|\partial^\alpha \chi\|_{L^\infty} \leq 1 + \|\partial^\alpha \chi\|_{L^\infty}.$$

Therefore the convergence follows by Lebesgue's dominated convergence theorem, because $\partial^\alpha(\chi_R - \mathbb{1})$ converges pointwise to 0 as $R \to \infty$. □

Theorem 8.17. *Let $p \in [1, \infty)$. $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$. Consequently, $L^p(\Omega)$ is separable.*

Proof. By Lemma 8.16 it is sufficient to show that for all $f \in L^p(\Omega)$ with compact support there exist $f_\varepsilon \in \mathcal{D}(\Omega)$ for $\varepsilon > 0$ with $f_\varepsilon \xrightarrow{\varepsilon \downarrow 0} f$ in $L^p(\Omega)$. Let $f \in L^p(\Omega)$ be compactly supported and let ψ be a mollifier. Let $\overline{f}$ be the function in $L^p(\mathbb{R}^d)$ that equals f on Ω and equals 0 elsewhere. Then $\overline{f} * \psi_\varepsilon \in \mathcal{D}(\Omega)$ for all $\varepsilon > 0$. As in the proof of Lemma 8.14, let $\delta > 0$ be such that $[\operatorname{supp} f]_\delta \subset \Omega$ and thus $\operatorname{supp} \overline{f} * \psi_\varepsilon \subset \Omega$ for all $\varepsilon \in (0, \delta)$. By Theorem 8.12 $\overline{f} * \psi_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ and thus $f_\varepsilon := (\overline{f} * \psi_\varepsilon)|_\Omega \in \mathcal{D}(\Omega)$. By Theorem 7.15 (c) we have $\overline{f} * \psi_\varepsilon \to f$ in $L^p(\mathbb{R}^d)$ and therefore conclude $f_\varepsilon \to f$ in $L^p(\Omega)$.

As $\mathcal{D}(\Omega)$ is separable (Theorem 8.15) and is continuously embedded in $L^p(\Omega)$ (Theorem 4.28), it follows that $L^p(\Omega)$ is separable. □

Exercise **8.C.** Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and $F \subset \mathbb{R}^d$ be a closed set. Show that $\psi * \mathbb{1}_F$ is a smooth function and that all of its derivatives are bounded.

Exercise **8.D.** For each of the following cases, find $u \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$ such that:

(a) $u * \varphi(x) = 0$ for all $x \in \mathbb{R}$, but $u \neq 0$ and $\varphi \neq 0$,

(b) $u * \varphi(x) = 1$ for all $x \in \mathbb{R}$,

(c) $u * \varphi(x) = x$ for all $x \in \mathbb{R}$,

(d) $u * \varphi(x) = \sin x$ for all $x \in \mathbb{R}$.

Exercise **8.E.** Consider the distribution on $\mathbb{R}$ given by $h = \mathbb{1}_{[0,\infty)}$, also called the Heaviside function. For $\varphi \in \mathcal{D}(\mathbb{R})$ calculate $h * \varphi'$, where φ' denotes the derivative of φ. Calculate the derivative of the distribution corresponding to h, i.e., calculate ∂u_h. Validate by these calculations that $(h * \varphi)' = h * \varphi' = \partial u_h * \varphi$.

9 Distributions of finite order

In this section we prove additional properties of distributions of finite order. First, we have seen that for a distribution u and a testfunction φ one has $u(\varphi) = 0$ as soon as $\varphi = 0$ on a neighbourhood of the support of u. And that in general, the condition $\varphi = 0$ on $\operatorname{supp} u$ does not imply $u(\varphi) = 0$, see Definition 5.1. In Theorem 9.4 we will see that if a distribution u is of order k, then

$$\partial^\alpha \psi = 0 \text{ on } \operatorname{supp} u \text{ for all } \alpha \in \mathbb{N}_0 \text{ with } |\alpha| \leq k,$$

implies $u(\psi) = 0$. With this theorem we prove that a distribution supported in a point is given by a linear combination of derivatives of the point measure at the supporting point, in Theorem 9.5. Moreover, we show that distributions of order k can be extended to continuous linear functions $C_0^k(\Omega) \to \mathbb{F}$ and with that prove Theorem 2.28.

First we consider some auxiliary lemmas to prove Theorem 9.4.

Lemma 9.1. *Let $\varphi \in \mathcal{D}(\Omega)$, $a \in \Omega$ and $k \in \mathbb{N}_0$. Suppose that that $\partial^\alpha \varphi(a) = 0$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$. Then, for all $\varepsilon > 0$ with $B(a,\varepsilon) \subset \Omega$,*

$$|\partial^\alpha \varphi(x)| \le \sum_{\beta \in \mathbb{N}_0^d : |\beta| \le k+1} \|\partial^\beta \varphi\|_{L^\infty} \varepsilon^{k+1-|\alpha|} \qquad (x \in B(a,\varepsilon), \alpha \in \mathbb{N}_0^d, |\alpha| \le k). \tag{9.1}$$

Proof. By Taylor's formula (see Theorem B.7) we know that φ equals its remainder of order k at a as its Taylor polynomial of order k at a equals zero. Hence by (B.1) with $l = k+1$ we see that for $M = \sum_{\beta \in \mathbb{N}_0^d : |\beta| \le k+1} \|\partial^\beta \varphi\|_{L^\infty}$

$$|\varphi(x)| \le M|x-a|^{k+1} \le M\varepsilon^{k+1} \qquad (x \in B(a,\varepsilon)).$$

By a repetition of the above argument for the derivatives of φ, we obtain (9.1). □

Definition 9.2. For $k \in \mathbb{N}_0$ we define $C_\mathrm{b}^k(\Omega)$ to be the space

$$C_\mathrm{b}^k(\Omega) = \{f \in C^k(\Omega) : \|f\|_{C^k} < \infty\}, \tag{9.2}$$

equipped with the norm $\|\cdot\|_{C^k}$. Thus $C_\mathrm{b}^k(\Omega)$ is the space of smooth functions whose derivatives up to order k are bounded.

$C_\mathrm{b}^\infty(\Omega)$ is the space

$$C_\mathrm{b}^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} C_\mathrm{b}^k(\Omega), \tag{9.3}$$

equipped with the seminorms $\|\cdot\|_{C^k}$ for $k \in \mathbb{N}_0$.

For $k \in \mathbb{N}_0$, $C_\mathrm{b}^k(\Omega)$ is a Banach space and $C_\mathrm{b}^\infty(\Omega)$ is a Fréchet space; this follows from Theorem 4.19 (see Exercise 9.A). Moreover, as for any compact set $K \subset \Omega$ and $k \in \mathbb{N}_0$

$$\|f\|_{C^k,K} \le \|f\|_{C^k} \qquad (f \in C_\mathrm{b}^\infty(\Omega)),$$

$C_\mathrm{b}^\infty(\Omega)$ is continuously embedded in $\mathcal{E}(\Omega)$;

$$C_\mathrm{b}^\infty(\Omega) \hookrightarrow \mathcal{E}(\Omega).$$

Exercise 9.A. Prove that $C_\mathrm{b}^k(\Omega)$ is a Banach space for $k \in \mathbb{N}_0$ and $C_\mathrm{b}^\infty(\Omega)$ a Fréchet space.

In 5.6 we have used that for any compact set we can find a testfunction that equals 1 on that compact set. Now with the tools of convolution, we can also construct such functions by a convolution of a mollifier. Moreover, for any closed set F we can also find smooth functions which are equal to 1 on a neighbourhood of F with a control on the growth of the derivatives:

Lemma 9.3. *Let $F \subset \mathbb{R}^d$ be a closed set. For each $\varepsilon > 0$ there exists a $\eta_\varepsilon \in C_{\mathrm{b}}^\infty(\mathbb{R}^d)$ such that*

$$\eta_\varepsilon = 1 \text{ on } [F]_\varepsilon, \quad \operatorname{supp} \eta_\varepsilon \subset [F]_{3\varepsilon}, \tag{9.4}$$

and for each $k \in \mathbb{N}$ there exists a $C > 0$ such that

$$\|\partial^\alpha \eta_\varepsilon\|_{L^\infty} \le C\varepsilon^{-|\alpha|} \qquad (\alpha \in \mathbb{N}_0^d, |\alpha| \le k, \varepsilon > 0). \tag{9.5}$$

Consequently, if F is a compact set in Ω, then for each $\varepsilon > 0$ such that $[F]_{3\varepsilon} \subset \Omega$, there exists a $\eta_\varepsilon \in \mathcal{D}(\Omega)$ with (9.4) *and* (9.5)*.*

Proof. Let ψ be a positive mollifier. For $\varepsilon > 0$ define $\eta_\varepsilon = \mathbb{1}_{[F]_{2\varepsilon}} * \psi_\varepsilon$. Then $\operatorname{supp} \eta_\varepsilon \subset [F]_{3\varepsilon}$. As $\mathbb{1}_{[F]_{2\varepsilon}}(x - y) = 1$ for all $y \in B(0, \varepsilon)$ and $x \in [F]_\varepsilon$, and as ψ_ε integrates to 1 on $B(0, \varepsilon)$ and has support in $B(0, \varepsilon)$:

$$\eta_\varepsilon(x) = \int \mathbb{1}_{[F]_{2\varepsilon}}(x - y)\psi_\varepsilon(y) \, \mathrm{d}y = \int_{B(0,\varepsilon)} \psi_\varepsilon(y) \, \mathrm{d}y = 1 \qquad (x \in [F]_\varepsilon),$$

i.e., $\eta_\varepsilon = 1$ on $[F]_\varepsilon$. Let $\alpha \in \mathbb{N}_0^d$. We have $\partial^\alpha \eta_\varepsilon = \mathbb{1}_{[F]_{2\varepsilon}} * \partial^\alpha \psi_\varepsilon$ and $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}^d$. Therefore, by Young's inequality, for $C = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| \le k} \|\partial^\alpha \psi\|_{L^1}$

$$\|\partial^\alpha \eta_\varepsilon\|_{L^\infty} \le \|\mathbb{1}_{[F]_{2\varepsilon}}\|_{L^\infty} \|\partial^\alpha \psi_\varepsilon\|_{L^1} \le C\varepsilon^{-|\alpha|}.$$

□

Theorem 9.4. *Let $k \in \mathbb{N}_0$ and $u \in \mathcal{D}'(\Omega)$ be a distribution of order k. Suppose that $\psi \in \mathcal{D}(\Omega)$ and*

$$\partial^\alpha \psi = 0 \text{ on } \operatorname{supp} u \text{ for all } \alpha \in \mathbb{N}_0 \text{ with } |\alpha| \le k. \tag{9.6}$$

Then $u(\psi) = 0$.

Proof. Let us first show that we may assume that u is compactly supported. In order to do that we assume that $(\chi_n)_{n \in \mathbb{N}}$ is a partition of unity as in Theorem 1.11. Then, for any $\varphi \in \mathcal{D}(\Omega)$, $u(\varphi)$ equals the sum over $\chi_n u(\varphi)$, for which only finitely many are possibly nonzero. As $\operatorname{supp} \chi_n u \subset \operatorname{supp} u$, it suffices to show the statement for "$\chi_n u$" instead of "u".

Instead, we will assume that u is compactly supported. Let $F = \operatorname{supp} u$. Let $\mathfrak{C} > 0$ be such that

$$|u(\varphi)| \le \mathfrak{C}\|\varphi\|_{C^k} \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Let $\varepsilon > 0$ be such that $[F]_{3\varepsilon} \subset \Omega$. By Lemma 9.3 there exists a $\eta_\varepsilon \in \mathcal{D}(\Omega)$ with (9.4) and (9.5). Then $u = \eta_\varepsilon u$ and thus

$$|u(\varphi)| \le \mathfrak{C}\|\eta_\varepsilon \varphi\|_{C^k} \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Let $\psi \in \mathcal{D}(\Omega)$ be such that (9.6). By the above inequality it suffices to show $\|\eta_\varepsilon \psi\|_{C^k} \to 0$ as $\varepsilon \downarrow 0$. By Leibniz' rule 1.14, for $x \in [F]_{3\varepsilon}$ and $\alpha \in \mathbb{N}_0^d, |\alpha| \le k$,

Let $M = \sum_{\beta \in \mathbb{N}_0^d : |\beta| \le k+1} \|\partial^\beta \psi\|_\infty$ and $C > 0$ be as in (9.5). Then by Leibniz' rule 1.14 we obtain for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$ and $x \in [F]_{3\varepsilon}$

$$\begin{aligned} |\partial^\alpha (\eta_\varepsilon \psi)(x)| &\le \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le \alpha}} \binom{\alpha}{\beta} |\partial^\beta \eta_\varepsilon(x)| |\partial^{\alpha-\beta} \psi(x)| \\ &\le \sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le \alpha}} \binom{\alpha}{\beta} M \varepsilon^{-|\beta|} C \varepsilon^{k+1-|\alpha-\beta|} \le MC \Big(\sum_{\substack{\beta \in \mathbb{N}_0^d \\ \beta \le \alpha}} \binom{\alpha}{\beta} \Big) \varepsilon. \end{aligned}$$

Therefore $\|\eta_\varepsilon \psi\|_{C^k} \xrightarrow{\varepsilon \downarrow 0} 0$ and thus $u(\psi) = 0$. □

Theorem 9.5. *If u is a distribution supported by $\{x\}$, then there exist a $k \in \mathbb{N}_0$ and $c_\alpha \in \mathbb{F}$ for $\alpha \in \mathbb{N}_0^d, |\alpha| \le k$ such that*

$$u = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| \le k} c_\alpha \partial^\alpha \delta_x.$$

Moreover, $c_\alpha = \langle \iota^{-1}(u), \boldsymbol{x}^\alpha \rangle$, where $\boldsymbol{x} : x \mapsto x$ (and thus $\boldsymbol{x}^\alpha : x \mapsto x^\alpha$) and with $\iota : \mathcal{E}'(\Omega) \to \mathcal{D}(\Omega)$ as in Theorem 5.10.

Proof. By taking a translation of the distribution, we may as well assume that $x = 0$. Let $\varepsilon > 0$ be such that $B(0, \varepsilon) \subset \Omega$. Let $k \in \mathbb{N}_0$ be the order of u. By Taylor's formula (see Theorem B.7) $\varphi = P + \psi$ on $B(0, \varepsilon)$, for a polynomial P of order k given by

$$P(x) = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \frac{1}{\alpha!} \partial^\alpha \varphi(0) x^\alpha \qquad (x \in \mathbb{R}^d),$$

and ψ satisfying $\partial^\alpha \psi(0) = 0$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$. Let χ be a testfunction that equals 1 on $B(0, \frac{\varepsilon}{2})$ and has support within $B(0, \varepsilon)$. Then $u(\varphi) = u(\chi\varphi) = u(P\chi)$ by the previous theorem. And thus,

$$u(\varphi) = u(P\chi) = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \frac{1}{\alpha!} \partial^\alpha \varphi(0) u(\boldsymbol{x}^\alpha \chi).$$

□

Theorem 9.6. *Let $k \in \mathbb{N}_0$. Let $u \in \mathcal{D}'(\Omega)$ be of order k. Then there exists exactly one linear extension of u, $v : C_c^k(\Omega) \to \mathbb{F}$ which is continuous with respect to $\|\cdot\|_{C^k}$, i.e., $v = u$ on $\mathcal{D}(\Omega)$ and there exists a $C > 0$ such that*

$$|v(f)| \le C \|f\|_{C^k} \qquad (f \in C_c^k(\Omega)). \tag{9.7}$$

Proof. Let $C > 0$ be such that

$$|u(\varphi)| \le C\|\varphi\|_{C^k} \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Let ψ be a mollifier. By Theorem 7.15 (b) follows that $\partial^\alpha f * \psi_\varepsilon \to \partial^\alpha f$ uniformly for all $f \in C_c^k(\Omega)$ and $\alpha \in \mathbb{N}_0^d, |\alpha| \le k$. Therefore

$$\|f * \psi_\varepsilon - f\|_{C^k} \xrightarrow{\varepsilon \downarrow 0} 0 \qquad (f \in C_c^k(\Omega)).$$

Therefore, $(u(f * \psi_{n^{-1}}))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$ for all $\psi \in C_c^k(\Omega)$. We define $v : C_c^k(\Omega) \to \mathbb{F}$ by

$$v(f) = \lim_{n\to\infty} u(f * \psi_{n^{-1}}) \qquad (f \in C_c^k(\Omega)).$$

Then v is linear, $v = u$ on $\mathcal{D}(\Omega)$ and for $f \in C_c^k(\Omega)$ we have for all $n \in \mathbb{N}$

$$|v(f)| \le |v(f - f * \psi_{n^{-1}})| + |u(f * \psi_{n^{-1}})| \le |v(f) - u(f * \psi_{n^{-1}})| + C\|f * \psi_{n^{-1}}\|_{C^k},$$

so by taking the limit $n \to \infty$, we obtain (9.7). The uniqueness follows by the continuity. □

Definition 9.7. For $k \in \mathbb{N}_0 \cup \{\infty\}$ we define $C_0^k(\Omega)$ to be the space of functions $f \in C^k(\Omega)$ with $\partial^\alpha f \in C_0(\Omega)$ for all $\alpha \in \mathbb{N}_0^d, |\alpha| \le k$ ($C_0(\Omega)$ is defined in Definition 2.29). For $k \in \mathbb{N}_0$ the space $C_0^k(\Omega)$ is equipped with the norm $\|\cdot\|_{C^k}$ and $C_0^\infty(\Omega)$ is equipped with the seminorms $\|\cdot\|_{C^k}$ for $k \in \mathbb{N}_0$.

For $k \in \mathbb{N}_0$, $C_0^k(\Omega)$ is a Banach space and $C_0^\infty(\Omega)$ is a Fréchet space, and $\mathcal{D}(\Omega)$ is sequentially dense in $C_0^k(\Omega)$ for each $k \in \mathbb{N}_0 \cup \{\infty\}$ (see Exercise 9.B).

Exercise 9.B. Let $k \in \mathbb{N}_0$. Prove:

(a) $C_0^k(\Omega)$ is a Banach space,

(b) $C_0^\infty(\Omega)$ a Fréchet space,

(c) $\mathcal{D}(\Omega)$ is dense in $C_0^k(\Omega)$ and in $C_0^\infty(\Omega)$,

(d) For each linear $v : C_c^k(\Omega) \to \mathbb{F}$ that is continuous with respect to $\|\cdot\|_{C^k}$ there exists exactly one continuous extension of v, $w : C_0^k(\Omega) \to \mathbb{F}$.

Corollary 9.8. *Let $k \in \mathbb{N}_0$. Let $u \in \mathcal{D}'(\Omega)$ be of order k. Then there exists exactly one continuous linear extension of u, $v : C_0^k(\Omega) \to \mathbb{F}$.*

Proof. This follows by Theorem 9.6 and the fact that $C_c^k(\Omega)$ is dense in $C_0^k(\Omega)$ (see also Exercise 9.B). □

We are ready to give the proof of Theorem 2.28:

Theorem 2.28. *A distribution u is of order 0 if and only if $u = u_\mu$ for a $\mu \in \mathcal{M}(\Omega, \mathbb{F})$.*

Proof. The "if" statement is trivial. Let $u \in \mathcal{D}'(\Omega)$ be of order 0. By Corollary 9.8 u extends to a continuous linear $C_0(\Omega, \mathbb{F})$, which by the Riesz representation theorem (Theorem 2.30) is given by a $\mu \in \mathcal{M}(\Omega, \mathbb{F})$. It follows that $u = u_\mu$. □

9.9 (The distributional structure of $\mathcal{M}(\Omega, \mathbb{F})$)**.** Observe that if $\mu \in \mathcal{M}(\Omega, \mathbb{F})$ and μ has compact support, in the sense that $\operatorname{supp} u_\mu$ is compact, then by Theorem 6.1 there exists a continuous function $f \in C(\Omega)$ such that for $\alpha = (2, \dots, 2)$, $u_\mu = \partial u_f$.

Moreover, by Theorem 6.3, for any $\mu \in \mathcal{M}(\Omega, \mathbb{F})$ there exist continuous functions $g_\alpha \in C(\Omega)$ for $\alpha \in \mathbb{N}_0^d$ with $\alpha \leq (2, \dots, 2)$ such that $u_\mu = \sum_{\alpha \in \mathbb{N}_0^d : \alpha \leq (2,\dots,2)} \partial^\alpha u_{g_\alpha}$.

For $d = 1$ this means that any Radon measure equals (as a distribution) the sum of $u_{g_0} + \partial u_{g_1} + \partial^2 u_{g_2}$, for some continuous functions g_0, g_1 and g_2.

Question **9.1.** Like for the structure theorems, can any distribution u of order k be written as $\sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq k} \partial^\alpha u_{\mu_\alpha}$ for some $\mu_\alpha \in \mathcal{M}(\Omega, \mathbb{F})$?

10 Convolutions of distributions

In this section we consider convolution as an operation between distributions. Like for locally integrable functions, one cannot expect to be able to define the convolution between any two distributions; consider for example the function $\mathbb{1}$, of which convolution with itself does not exist (in the sense of Definition 7.3). But when one considers two distributions of which one has compact support, one can define a convolution between them as we will see. Observe that the convolution between two locally integrable functions of which at least one has compact support exists in the sense of Definition 7.3.

In this section we consider only $\Omega = \mathbb{R}^d$ and write '$\mathcal{E}$' and '$\mathcal{D}$' instead of '$\mathcal{E}(\mathbb{R}^d)$' and '$\mathcal{D}(\mathbb{R}^d)$'. First, we start by characterising the operation of convoluting with a distribution as a sequentially continuous map $\mathcal{D} \to \mathcal{E}$ that commutes with translation.

Definition 10.1. We say that a linear function $A : \mathcal{D} \to \mathcal{E}$ or $A : \mathcal{E} \to \mathcal{E}$ *commutes with translations* if it commutes with the translation operators, i.e., if $\mathcal{T}_x A = A \mathcal{T}_x$ for all $x \in \mathbb{R}^d$.

Theorem 10.2.

(a) *Let $A : \mathcal{D} \to \mathcal{E}$ be linear. Then A is sequentially continuous and commutes with translations if and only if there exists a $u \in \mathcal{D}'$ such that $A\varphi = u * \varphi$ for all $\varphi \in \mathcal{D}$.*

(b) *Let $A : \mathcal{E} \to \mathcal{E}$ be linear. Then A is sequentially continuous and commutes with translations if and only if there exists a $u \in \mathcal{E}'$ such that $A\varphi = u * \varphi$ for all $\varphi \in \mathcal{D}$.*

For both (a) *and* (b)*, if A is sequentially continuous and commutes with translations, then there exists exactly one such u such that $A\varphi = u * \varphi$ for all $\varphi \in \mathcal{D}$.*

Proof. For both (a) and (b) the "if" statement, follows from the fact that convolutions commutes with translations as we have seen in Lemma 8.2 and from the fact that convolution as an operation is sequentially continuous Theorem 8.7. Therefore we assume A to be sequentially continuous and to commute with translations.

(a) Define $u : \mathcal{D} \to \mathbb{F}$ by $u(\varphi) = A\check{\varphi}(0)$ for $\varphi \in \mathcal{D}$. u is linear and sequentially continuous, therefore it is a distribution (Theorem 4.14). Then for every $x \in \mathbb{R}^d$

$$A\varphi(x) = \mathcal{T}_{-x}A\varphi(0) = A[\mathcal{T}_{-x}\varphi](0) = u([\mathcal{T}_{-x}\varphi]^{\check{}}) = u(\mathcal{T}_x\check{\varphi}) = u * \varphi(x). \tag{10.1}$$

(b) As in (a), define $u : \mathcal{E} \to \mathbb{F}$ by $u(\varphi) = A\check{\varphi}(0)$ for $\varphi \in \mathcal{E}$. Then u linear is sequentially continuous, hence continuous as $\mathcal{E}$ is a metric space, which implies $u \in \mathcal{E}'$. That $A\varphi = u * \varphi$ follows by (10.1). □

10.3. Let $v \in \mathcal{E}'$ and $u \in \mathcal{D}'$. As $u*\varphi \in \mathcal{E}$ for all $\varphi \in \mathcal{D}$, we can compose the maps $\mathcal{D} \to \mathcal{E}$, $\varphi \mapsto u*\varphi$ and $\mathcal{E} \to \mathcal{E}$, $\psi \mapsto v*\psi$. By Lemma 8.2 and Theorem 8.7 this composition forms a sequentially continuous linear map $\mathcal{D} \to \mathcal{E}$ that commutes with translations. Therefore, by Theorem 10.2 there exists a $w \in \mathcal{D}'$ such that

$$w * \varphi = u * (v * \varphi) \qquad (\varphi \in \mathcal{D}).$$

We could take this w as our definition of $u*v$; instead we define $u*v$ via another formula and show in Theorem 10.6 that it equals w.

10.4. Remember (7.1) in Corollary 7.8, which tells us that for integrable f, g and a testfunction φ we have (by viewing $f * g$ as a distribution and thus $\langle f, \varphi\rangle = \int f\varphi$)

$$\langle g * f, \varphi\rangle = \langle f * g, \varphi\rangle = \langle g, \check{f} * \varphi\rangle.$$

Moreover, if $(u, \psi) \in \mathcal{D}' \times \mathcal{D}$ or $(u, \psi) \in \mathcal{E}' \times \mathcal{E}$ and $\varphi \in \mathcal{D}$, then by the associativity property (Theorem 8.9)

$$\langle u * \psi, \varphi\rangle = (u * \psi) * \check{\varphi}(0) = u * (\psi * \check{\varphi})(0) = \langle u, \check{\psi} * \varphi\rangle.$$

This identity motivates the definition of the convolution between a distribution and a compactly supported distribution. First we make the following observation. For $v \in \mathcal{E}'$ we know by Theorem 8.7 that $\mathcal{D} \to \mathcal{D}$, $\varphi \mapsto \check{v}*\varphi$ is a sequentially continuous map. Hence, if $v \in \mathcal{E}'$ and $u \in \mathcal{D}'$, then $\mathcal{D} \to \mathbb{F}$, $\varphi \mapsto u(\check{v} * \varphi)$ is a distribution as it is sequentially continuous and linear (see Theorem 4.14).

Definition 10.5. For $u \in \mathcal{D}'$ and $v \in \mathcal{E}'$ we define $u * v$ to be the distribution given by

$$u * v(\varphi) = u(\check{v} * \varphi) \qquad (\varphi \in \mathcal{D}).$$

Moreover, we define $v * u$ to be the distribution

$$v * u(\varphi) = v(\check{u} * \varphi) \qquad (\varphi \in \mathcal{D}).$$

Theorem 10.6. *Let $u \in \mathcal{D}'$, $v \in \mathcal{E}'$ and $\varphi \in \mathcal{D}$. Then*

$$(u * v) * \varphi = u * (v * \varphi) = v * (u * \varphi) = (v * u) * \varphi. \tag{10.2}$$

Consequently, $u * v = v * u$.

Proof. The first equality in (10.2), and similarly the last one by interchanging the roles of u and v, follows from the observation that for all $x \in \mathbb{R}^d$

$$(u * v) * \varphi(x) = u * v(\mathcal{T}_x \check{\varphi}) = u(\check{v} * \mathcal{T}_x \check{\varphi}) = u(\mathcal{T}_x (v * \varphi)\check{}) = u * (v * \varphi)(x).$$

We are therefore left to prove

$$u * (v * \varphi) = v * (u * \varphi). \tag{10.3}$$

By Theorem 8.9 (a) and by the commutativity of convolution on functions (see 7.4), for $\psi \in \mathcal{D}$ we have (10.2) with "ψ" instead of "v":

$$u * (\psi * \varphi) = \psi * (u * \varphi).$$

By the fact that $\mathcal{D}$ is sequentially dense in $\mathcal{E}'$ (Theorem 5.10) and by using the sequential continuity of Theorem 8.7 we obtain (10.3) and thus (10.2). □

If $u, v \in \mathcal{E}'$, then the map $\mathcal{E} \to \mathbb{F}$, $\psi \mapsto u(\check{v} * \psi)$ is an element of $\mathcal{E}'$ as it is sequentially continuous and linear (that it is continuous follows similarly to the proof of Theorem 4.14).

Definition 10.7. For $u, v \in \mathcal{E}'$ we define $u * v \in \mathcal{E}'$ by the formula

$$u * v(\psi) = u(\check{v} * \psi) \qquad (\psi \in \mathcal{E}).$$

Lemma 10.8. *Let ι be the embedding $\mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ as in Theorem 5.10. Then*

$$\iota(u * v) = \iota(u) * v = u * \iota(v) \qquad (u, v \in \mathcal{E}').$$

Proof. The proof is rather straightforward and left to the reader (see Exercise 10.A). □

Exercise 10.A. Verify the statement in 10.7.

10.9 (Convention)**.** As we have the one-to-one correspondence of elements of $\mathcal{E}'$ and compactly supported distributions, the identity Lemma 10.8 and that $\operatorname{supp} v = \operatorname{supp}(v|_{\mathcal{D}})$ for all $v \in \mathcal{E}'$ (Definition 5.12), for $v \in \mathcal{E}'$ we will often write "v" instead of "$\iota(v)$". Similarly, for $u, v \in \mathcal{D}'$ with v compactly supported we will write $u * v$ for $u * \iota^{-1}(v)$, etc.

Lemma 10.10. *Let $(u, v) \in (\mathcal{E}' \times \mathcal{D}') \cup (\mathcal{D}' \times \mathcal{E}')$. Then*

$$\begin{aligned} \delta_0 * u &= u, \\ \delta_y * u &= \mathcal{T}_y u \qquad (y \in \mathbb{R}^d), \\ \mathcal{R}(u * v) &= \mathcal{R}(u) * \mathcal{R}(v), \\ \mathcal{T}_y(u * v) &= (\mathcal{T}_y u) * v = u * (\mathcal{T}_y v) \qquad (y \in \mathbb{R}^d), \\ \partial^\alpha (u * v) &= (\partial^\alpha u) * v = u * (\partial^\alpha v) \qquad (\alpha \in \mathbb{N}_0^d). \end{aligned}$$

Proof. The proof is left for the reader. □

Theorem 10.11. *For $u \in \mathcal{D}'$ and $v \in \mathcal{E}'$*

$$\operatorname{supp} u * v \subset \operatorname{supp} u + \operatorname{supp} v.$$

Proof. Let $x \in \operatorname{supp} u * v$. For all $\varepsilon > 0$ there exists a $\varphi \in \mathcal{D}$ supported in $B(x, \varepsilon)$ such that $u * v(\varphi) \neq 0$, i.e., $u(\check{v} * \varphi) \neq 0$. Therefore $\operatorname{supp} u \cap (\operatorname{supp} \check{v} * \varphi) \neq \emptyset$. Let y be in this intersection. By Theorem 8.6 we know that there exists a $z \in \operatorname{supp} v$ and $w \in \operatorname{supp} \varphi$ such that $y = -z + w$. Then $w = y + z \in \operatorname{supp} u + \operatorname{supp} v$ and $|x - w| < \varepsilon$. As we can find such w for each ε and $\operatorname{supp} u + \operatorname{supp} v$ is closed, we conclude that $x \in \operatorname{supp} u + \operatorname{supp} v$. □

Remark 10.12. One can also define the convolution of two distributions, where instead of assuming that one of the two has compact support the map $\Sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\Sigma(x, y) = x + y$ is proper on $\operatorname{supp} u \times \operatorname{supp} v$, meaning that $\Sigma^{-1}(K) \cap \operatorname{supp} u \times \operatorname{supp} v$ is a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$ for all compact sets $K \subset \mathbb{R}^d$. The details can be found for example in [DK10, Section 11] .

11 Fundamental solutions of partial differential operators

In this section we consider partial differential operators and corresponding fundamental solutions.

Definition 11.1. We call a map $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ a *linear partial differential operator with constant coefficients* if there exist an $m \in \mathbb{N}$ and $c_\alpha \in \mathbb{F}$ for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ such that

$$P = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \leq m}} c_\alpha \partial^\alpha.$$

Often, the following notation is also used. When we take $p : \mathbb{R}^d \to \mathbb{F}$ the polynomial

$$p(x) = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \leq m}} c_\alpha x^\alpha,$$

then it is common to write "$p(\partial)$" for "P", so that one interpret $p(\partial)$ as the formal polynomial (i.e., finite formal powerseries) evaluated at ∂. One also uses "D" instead of "∂" in literature, so that one writes "$p(\mathrm{D})$" for "P". We will not use "D" in this context because we will use this in the context of Fourier multipliers in Section 19.

A distribution E is called a *fundamental solution* to P if $PE = \delta$, where δ is the Dirac measure at zero.

Theorem 11.2. *Let P be a linear partial differential operator with constant coefficients and E a fundamental solution to P. For all $v \in \mathcal{E}'(\mathbb{R}^d)$ we have*

$$P(E * v) = v = E * (Pv).$$

Proof. This follows by the fact that $\partial^\alpha(E * v) = (\partial^\alpha E) * v = E * (\partial^\alpha v)$. □

11.3. Observe that if E is a fundamental solution to a linear partial differential operator with constant coefficients P, and if $u \in \mathcal{D}'(\mathbb{R}^d)$ satisfies $Pu = 0$, then $E + u$ is also a fundamental solution to P.

11.4. Let P, E be as in Theorem 11.2 and $v \in \mathcal{E}'(\mathbb{R}^d)$. One says that $u = E * v$ is a *solution to the partial differential equation*

$$Pu = v. \tag{11.1}$$

Hence, by Theorem 11.2 one can derive solutions of partial differential equations of the form (11.1) when one knowns a fundamental solution to P.

Definition 11.5 (Laplacian)**.** We write Δ for the linear partial differential operator

$$\Delta = \sum_{i=1}^d \partial_i^2,$$

and call it the *Laplacian*.

Example 11.6 (Fundamental solution to Δ)**.** Let E be the function on $\mathbb{R}^d$ (for $d \geq 2$) defined by $E(0) = 0$ and

$$E(x) = \begin{cases} \frac{1}{(2-d)V_d}|x|^{2-d} & d \neq 2, \\ \frac{1}{2\pi} \log|x| & d = 2, \end{cases} \tag{11.2}$$

where V_d is the $d-1$ dimensional volume of the sphere $\{x \in \mathbb{R}^d : |x| = 1\}$ (observe that $2\pi = V_2$). Then E is a fundamental solution to Δ (see Exercise 11.A).

Exercise 11.A. (a) For $i \in \{1, \dots, d\}$ let v_i be the function on $\mathbb{R}^d$ defined by $v_i(0) = 0$ and

$$v_i(x) = \frac{x_i}{|x|^d} \qquad (x \in \mathbb{R}^d \setminus \{0\}).$$

Prove that v_i is locally integrable on $\mathbb{R}^d$ and that in $\mathcal{D}'$

$$\sum_{i=1}^d \partial_i v_i = V_d \delta,$$

where V_d is the $d-1$ dimensional volume of the sphere $\{x \in \mathbb{R}^d : |x| = 1\}$. (Hint: Observe that $\langle \partial_i v_i, \varphi \rangle = -\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} v_i \partial_i \varphi$ and apply integration by parts (see Theorem C.1).)

(b) Prove that E as in (11.2) is locally integrable on $\mathbb{R}^d$ and that E is a fundamental solution to Δ, i.e., $\Delta E = \delta$ (first you might want to prove that $\partial_i E = cv_i$ for some $c \in \mathbb{R}$).

11.7. With E being a fundamental solution to Δ as defined in (11.2), we conclude that for $v \in \mathcal{E}'(\mathbb{R}^d)$ we have a solution to the *Poisson equation*

$$\Delta u = v,$$

given by $u = E * v \in \mathcal{D}'(\mathbb{R}^d)$.

Definition 11.8. A function $f \in C^2(\Omega)$ is called *harmonic*, or an *harmonic function* if $\Delta f = 0$. A distribution $u \in \mathcal{D}'(\Omega)$ is called *harmonic* if $\Delta u = 0$.

Exercise **11.B.** For $\mathbb{F} = \mathbb{C}$ and $d \geq 2$, check that for all $k \in \mathbb{N}_0$ the polynomial $x \mapsto (x_1 + \mathrm{i}x_2)^k$ is harmonic.

Observe that if Ω is connected and $f \in C^2(\mathbb{R})$ (i.e., $d = 1$), then $\Delta f = f'' = 0$ if and only if $f(x) = a + bx$ for some $a, b \in \mathbb{F}$.

As is mentioned in 11.3, if u is a harmonic distribution, then $E + u$ is a fundamental solution to Δ. Wehl's theorem, see Theorem 11.14, states that each harmonic distribution is actually (represented by) a harmonic function in $C^\infty(\mathbb{R}^d)$. We prove this theorem by proving a more general result, Theorem 11.12, which is about singular supports; this support indicates "where a distribution is smooth".

Definition 11.9 (Singular support)**.** Let $u \in \mathcal{D}'(\Omega)$. If $U \subset \Omega$ is open we say that u is *smooth on* U if there exists an $f \in C^\infty(U)$ with $u(\varphi) = \int_U f\varphi$ for all $\varphi \in \mathcal{D}(U)$. Let $\mathcal{U}$ be the collection of all open subsets of Ω on which u is smooth. By Theorem 1.11 u is smooth on $\bigcup \mathcal{U}$, as $u = \sum_{n\in\mathbb{N}} \chi_n u$ for some partition of unity $(\chi_n)_{n\in\mathbb{N}}$ subordinate to $\mathcal{U}$, so that $\chi_n u$ is (represented by) a smooth function with compact support. The complement of $\bigcup \mathcal{U}$ is defined to be the *singular support* of u,

$$\operatorname{sing\,supp} u.$$

Observe that if $U \subset \Omega$ is open and u vanishes on U, then u is smooth on U. Consequently

$$\operatorname{sing\,supp} u \subset \operatorname{supp} u \qquad (u \in \mathcal{D}'),$$

If $U \subset \Omega$ is open and $u, v \in \mathcal{D}'$ are both smooth on U, then $u + v$ is smooth on U. Consequently

$$\operatorname{sing\,supp}(u + v) \subset \operatorname{sing\,supp} u \cup \operatorname{sing\,supp} v \qquad (u, v \in \mathcal{D}'),$$

Let $\chi \in C_c^\infty(\Omega, [0, 1])$ and $U \subset \Omega$ be open. Then u is smooth on U if and only if χu and $(1 - \chi)u$ are smooth on U. Consequently

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp}(\chi u) \cup \operatorname{sing\,supp}((1 - \chi)u).$$

The above extends in the following sense to a partition of unity $(\chi_n)_{n\in\mathbb{N}}$ as in Theorem 1.11, due to the fact that the sets $\{x \in \Omega : \chi_n(x) > 0\}$ form a locally finite cover;

$$\operatorname{sing\,supp} u = \bigcup_{n\in\mathbb{N}} \operatorname{sing\,supp}(\chi_n u). \tag{11.3}$$

The singular support satisfies the same rule as the support does for convolutions:

Lemma 11.10. *Let $u \in \mathcal{D}'(\mathbb{R}^d)$ and $v \in \mathcal{E}'(\mathbb{R}^d)$. Then*

$$\operatorname{sing\,supp} u * v \;\subset\; \operatorname{sing\,supp} u + \operatorname{sing\,supp} v. \tag{11.4}$$

Proof. Let us write A for $\operatorname{sing\,supp} u$ and B for $\operatorname{sing\,supp} v$. Let $\delta > 0$. By Lemma 9.3 there exists a $\chi \in C^\infty$ such that χ is equal to 1 on $[A]_{\frac{\delta}{2}}$ and 0 outside $[A]_\delta$. Then $u_2 := (1-\chi)u$ is (represented by) a smooth function and so $u = u_1 + u_2$ for $u_1 = \chi u$, and $\operatorname{supp} u_1 \subset [A]_\delta$. Similarly, we can write $v = v_1 + v_2$, where $\operatorname{supp} v_1 \subset [B]_\delta$ and v_2 is (represented by) a smooth function. Then

$$u * v = u_1 * v_1 + u_1 * v_2 + u_2 * v_1 + u_2 * v_2.$$

The last three terms are smooth (by Theorem 8.4) and the support of $u_1 * v_1$ is included in $[A]_\delta + [B]_\delta$ (Theorem 10.11), which in turn is included in $[A+B]_{2\delta}$. Therefore

$$\operatorname{sing\,supp} u * v \subset [A+B]_{2\delta}.$$

Now observe that B is compact as it is a subset of the support of v. As δ is chosen arbitrarily and the set $A+B$ is closed (see Lemma 7.11), we have $\bigcap_{\delta>0}[A+B]_{2\delta} = A+B$ and conclude (11.4). □

Definition 11.11. Let P be a linear partial differential operator with constant coefficients. A distribution E is called a *parametrix* of P if there exists a $\psi \in \mathcal{E}(\mathbb{R}^d)$ such that $PE = \delta + \psi$.

Observe that any fundamental solution to P is a parametrix of P.

Theorem 11.12. *Let P be a linear partial differential operator with constant coefficients. Suppose E is a parametrix of P with $\operatorname{sing\,supp} E = \{0\}$. Then for all open $\Omega \subset \mathbb{R}^d$*

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Pu \qquad (u \in \mathcal{D}'(\Omega)). \tag{11.5}$$

Proof. Similarly to 8.5 we have $\operatorname{sing\,supp} Pu \subset \operatorname{sing\,supp} u$, which basically means that 'Pu is smooth where u is'.

By (11.3) we may assume that u has compact support, so that we may as well assume that $u \in \mathcal{D}'(\Omega)$. Let $\psi \in \mathcal{E}(\mathbb{R}^d)$ be such that $PE = \delta + \psi$. Then

$$E * (Pu) = (PE) * u = (\delta + \psi) * u = u + \psi * u.$$

Therefore $\operatorname{sing\,supp} u = \operatorname{sing\,supp} E * (Pu)$ as $\psi * u \in \mathcal{E}(\mathbb{R}^d)$ (Theorem 8.4). Therefore, by Lemma 11.10

$$\operatorname{sing\,supp} u \subset \operatorname{sing\,supp} E + \operatorname{sing\,supp} Pu = \operatorname{sing\,supp} Pu,$$

as $\operatorname{sing\,supp} E = \{0\}$. □

11.13. Let P and E are as in Theorem 11.12. This theorem tells us that a solution u to $Pu = v$ for a $v \in \mathcal{E}'(\mathbb{R}^d)$ is smooth where v is, in the sense that if U is open and v is smooth on U, then u is smooth on U. Therefore, in particular we obtain Weyl's theorem as a consequence.

Theorem 11.14 (Weyl's Theorem)**.** *Every harmonic distribution is (represented by) a smooth harmonic function. Moreover, if $u \in \mathcal{D}'$ is such that $\Delta u = \psi$ for a $\psi \in \mathcal{E}$, then u is a smooth function.*

Example 11.15. For $t > 0$ we define the function $h_t : \mathbb{R}^d \to \mathbb{R}$ by

$$h_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{1}{4t}|x|^2} \qquad (x \in \mathbb{R}^d). \tag{11.6}$$

Then (see Exercise 11.C)

$$\frac{\partial}{\partial t} h_t(x) = \Delta h_t(x) \qquad ((t,x) \in (0,\infty) \times \mathbb{R}^d). \tag{11.7}$$

Exercise **11.C.** Show that (11.7) is satisfied for h_t as in (11.6).

11.16. Observe that

$$\int_{\mathbb{R}^d} h_t(x)\ \mathrm{d}x = \left(\int_{\mathbb{R}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{1}{4t}s^2}\ \mathrm{d}s \right)^d = 1,$$

which follows by the fact that

$$\int_{\mathbb{R}} e^{-x^2}\ \mathrm{d}x = \sqrt{\pi}.$$

The latter identity can be proved using polar coordinates:

$$\begin{aligned} \left(\int_{\mathbb{R}} e^{-x^2}\ \mathrm{d}x \right)^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x^2+y^2)}\ \mathrm{d}x\ \mathrm{d}y = 2\pi \int_0^\infty r e^{-r^2}\ \mathrm{d}r \\ &= 2\pi \int_0^\infty \frac{1}{2} e^{-s}\ \mathrm{d}s = \pi. \end{aligned}$$

From this we can show that

$$\langle h_t, \varphi \rangle \xrightarrow{t \downarrow 0} \varphi(0) \qquad (\varphi \in C_{\mathrm{b}}(\mathbb{R}^d)). \tag{11.8}$$

Indeed,

$$\langle h_t, \varphi\rangle - \varphi(0) = \int_{\mathbb{R}^d} h_t(x)(\varphi(x) - \varphi(0))\, \mathrm{d}x.$$

By a substitution $y = \frac{x}{\sqrt{t}}$ we have

$$\int_{\mathbb{R}^d} h_t(x)(\varphi(x) - \varphi(0))\, \mathrm{d}x = \int_{\mathbb{R}^d} h_1(y)(\varphi(\sqrt{t}y) - \varphi(0))\, \mathrm{d}y.$$

So that by the Lebesgue dominated convergence theorem we indeed obtain (11.8).

Exercise **11.D.** Calculate the limit in $\mathcal{D}'(\mathbb{R}^d)$ of $\frac{\partial}{\partial t} h_t$ as $t \downarrow 0$.

In the theory of partial differential equations one is often looking for a function on $(0,\infty) \times \mathbb{R}^d$ where the first variable represents the "time variable". As one distinguishes the "time variable" from the "space variables", it makes sense to introduce the following notation.

Definition 11.17. Let $\mathbb{R}^\dagger = \mathbb{R}$. Let $\Omega \subset \mathbb{R}^\dagger \times \mathbb{R}^d$ be open. For $\alpha \in \mathbb{N}_0^d$, we write ∂^α for the operation $\mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ given by

$$\partial^\alpha \varphi(t,x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_1}} \varphi(t,x) \qquad (\varphi \in \mathcal{D}(\Omega), (t,x) \in \mathbb{R}^\dagger \times \mathbb{R}^d),$$

in other words, ∂^α is written for the operation $\partial^{(0,\alpha)}$ when we view Ω as a subset of $\mathbb{R}^{1+d}$. Moreover, we write $\partial_\dagger$ for the operation $\mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ given by

$$\partial_\dagger \varphi(t,x) = \frac{\partial}{\partial t} \varphi(t,x) \qquad (\varphi \in \mathcal{D}(\Omega), (t,x) \in \mathbb{R}^\dagger \times \mathbb{R}^d),$$

i.e., $\partial_\dagger$ is the operation $\partial^{(1,0)}$ when we view Ω as a subset of $\mathbb{R}^{1+d}$.

Remark 11.18. In the literature it is rather common to write "∂_t" instead of "$\partial_\dagger$". We avoid this as, on the one hand, we have already defined ∂_i for $i \in \{1,\dots,d\}$, on the other hand we prefer not to attach a meaning to "t" other than a variable which can be equal to 1, 2, etc.

Definition 11.19 (Heat operator)**.** The *heat operator* is the linear partial differential operator $\partial_\dagger - \Delta$.

Example 11.20. Let h_t for $t > 0$ be as in Example 11.15. The function $f : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ defined by $f(t,x) = h_t(x)$ for $(t,x) \in (0,\infty)\times\mathbb{R}^d$ solves the *heat equation* on $(0,\infty)\times\mathbb{R}^d$:

$$\partial_\dagger f = \Delta f.$$

Example 11.21. Define $E : \mathbb{R}^\dagger \times \mathbb{R}^d \to \mathbb{R}$ by

$$E(t,x) = \begin{cases} h_t(x) & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ 0 & (t,x) \in (-\infty, 0] \times \mathbb{R}^d. \end{cases}$$

Then (see Exercise 11.E) E is a fundamental solution to the heat operator $\partial_\dagger - \Delta$ (one also says, E is a fundamental solution to the heat equation).

Definition 11.22. The *gamma function* is the function $\Gamma : (0, \infty) \to (0, \infty)$ given by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, \mathrm{d}t \qquad (s \in (0, \infty)).$$

It is sometimes also defined on the complex plane for those numbers for which the real part is strictly positive. By partial integration it follows that $\Gamma(s + 1) = s\Gamma(s)$. As $\Gamma(1) = 1$, it follows that $\Gamma(n) = (n - 1)!$ for $n \in \mathbb{N}$. Moreover, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Exercise 11.E. Let E be as in Example 11.21.

(a) Calculate $\int_0^\infty h_t(x) \, \mathrm{d}t$ for $x \neq 0$ (in terms of the gamma function).

(b) Show that $\lim_{t \downarrow 0} \int_{\mathbb{R}^d} h_t(x)\varphi(t, x) \, \mathrm{d}x = \varphi(0)$ for any $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$.

(c) Show that E is locally integrable and conclude that the order of $(\partial_\dagger - \Delta)E$ is at most 2.

(d) Calculate $\operatorname{sing\,supp} E$.

(e) Show that $\operatorname{supp}(\partial_\dagger - \Delta)E \subset \{0\}$.

(f) Show that E is a fundamental solution to $\partial_\dagger - \Delta$ (Hint: Observe that $\langle (\partial_\dagger - \Delta)E, \varphi \rangle = \lim_{s \downarrow 0} \lim_{T \uparrow \infty} - \int_s^T \int_{\mathbb{R}^d} h_t(x)(\partial_\dagger + \Delta)\varphi(t, x) \, \mathrm{d}x \, \mathrm{d}t$ and apply integration by parts.)

(g) Conclude that if $v \in \mathcal{E}'(\mathbb{R}^{d+1})$ is smooth on an open set U, then so is a solution u of $(\partial_\dagger - \Delta)u = v$.

Remark 11.23. In [DK10, Section 12] one finds references for the proof of the statement that every linear partial differential operator with constant coefficients, of which at least one coefficient is nonzero, has a fundamental solution.

12 Sobolev spaces

In this section we consider Sobolev spaces as subspaces of $\mathcal{D}'$. These spaces are subsets of L^p for which not only the function itself, but also its derivatives (in the distributional sense) up to a certain order are all included in L^p.

Definition 12.1. Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. A $g \in L^1_{\mathrm{loc}}(\Omega)$ is called the α-th *weak partial derivative* of f if $u_g = \partial^\alpha u_f$, i.e., if

$$\int g\varphi = \int f \cdot (-1)^{|\alpha|} \partial^\alpha \varphi \qquad (\varphi \in \mathcal{D}(\Omega)).$$

12.2. Let $f, g, h \in L^1_{\mathrm{loc}}(\Omega)$ and $\alpha, \beta \in \mathbb{N}_0^d$.

(1) By Lemma 2.9 f has at most one α-th weak partial derivative.

(2) If g is an α-th weak partial derivative of f, then we write $\partial^\alpha f = g$.

(3) If $g = \partial^\alpha f$ and $h = \partial^\beta g$, then $h = \partial^{\alpha+\beta} f$.

(4) Following our convention 4.2 that we may write "f" instead of "u_f", for a distribution $u \in \mathcal{D}'(\Omega)$ we write "$\partial^\alpha u \in L^1_{\text{loc}}$" if there exists a $g \in L^1_{\text{loc}}$ such that $\partial^\alpha u = u_g$.

Definition 12.3. Let $p \in [1, \infty]$ and $k \in \mathbb{N}_0$. We define the *Sobolev space* of *order* k and *integrability* p, denoted $W^{k,p}(\Omega)$, by

$$W^{k,p}(\Omega) = \{u \in \mathcal{D}'(\Omega) : \partial^\beta u \in L^p(\Omega) \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| \le k\}.$$

Observe:

(1) $\partial^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ for all $u \in W^{k,p}(\Omega)$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$.

(2) If U is an open subset of Ω and $u \in W^{k,p}(\Omega)$, then $u|_U \in W^{k,p}(U)$.

Exercise **12.A.** Consider $\Omega = (0, 2)$, $f, g \in L^1_{\text{loc}}(\Omega)$ given by

$$f(x) = \begin{cases} x & x \in (0,1], \\ 1 & x \in (1,2), \end{cases} \qquad g(x) = \begin{cases} x & x \in (0,1], \\ 2 & x \in (1,2). \end{cases}$$

(a) Show that f has a weak derivative that is in L^p, so that $f \in W^{1,p}(\Omega)$ for all $p \in [1, \infty]$.

(b) Show that g has no weak derivative, but calculate ∂u_g.

(c) Give an example of an element $h \in W^{1,p}(0, 2)$ such that the function g defined on $\mathbb{R}$ by $g(x) = h(x)$ for $x \in (0, 2)$ and $g(x) = 0$ for other x, is not in $W^{1,p}(\mathbb{R})$.

Definition 12.4. We equip the Sobolev space $W^{k,p}(\Omega)$ for $p \in [1, \infty]$ with the norm

$$\|u\|_{W^{k,p}} = \max_{\beta \in \mathbb{N}_0^d, |\beta| \le k} \|\partial^\beta u\|_{L^p}.$$

Exercise **12.B.** Verify that $\|\cdot\|_{W^{k,p}}$ indeed defines a norm on $W^{k,p}(\Omega)$.

Let us recall the definition of equivalent norms:

Definition 12.5. Let $\mathfrak{X}$ be a normed space and $\|\cdot\|_1, \|\cdot\|_2 : \mathfrak{X} \to [0, \infty)$ be norms on $\mathfrak{X}$. They are said to be *equivalent* if they define the same topology.

Two norms $\|\cdot\|_1, \|\cdot\|_2$ are equivalent if and only if (see for example [Con90, Proposition III.1.5]) there exist $c, C > 0$ such that

$$c\|f\|_1 \le \|f\|_2 \le C\|f\|_1 \qquad (f \in \mathfrak{X}).$$

12.6. For $p \in [1, \infty)$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$ let $|x|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$. In 1.4, we mentioned that $|\cdot|_1$ is equivalent to $|\cdot|_\infty$ (where $|x|_\infty = \max_{i=1}^d |x_i|$ for $x \in \mathbb{R}^n$). Similarly, $|\cdot|_p$ and $|\cdot|_\infty$ are equivalent, as

$$|x|_\infty \le |x|_p \le n^{\frac{1}{p}} |x|_\infty \qquad (x \in \mathbb{R}^d).$$

Therefore, for each $q \in [1,\infty)$ there exists a $C > 0$ such that

$$\|u\|_{W^{k,p}} \le \Big(\sum_{\beta \in \mathbb{N}_0^d, |\beta| \le k} \|\partial^\beta u\|_{L^p}^q \Big)^{\frac{1}{q}} \le C \|u\|_{W^{k,p}},$$

i.e., the norm $u \mapsto (\sum_{\beta \in \mathbb{N}_0^d, |\beta| \le k} \|\partial^\beta u\|_{L^p}^q)^{\frac{1}{q}}$ is equivalent to $\| \cdot \|_{W^{k,p}}$ (in other books one might find the norm on $W^{k,p}$ to be defined by this norm with either $q = 1$ or $q = p$).

Observe that $W^{0,p}(\Omega) = L^p(\Omega)$, so that the Sobolev space of 0-th order is a Banach space. This extends to any order:

Theorem 12.7. *For all $p \in [1,\infty]$ and $k \in \mathbb{N}_0$, $W^{k,p}(\Omega)$ is a Banach space.*

Proof. Suppose that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{k,p}(\Omega)$. Then $(\partial^\alpha u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$. As $L^p(\Omega)$ is a Banach space, there exist $u^{(\alpha)} \in L^p(\Omega)$ such that $\partial^\alpha u_n \to u^{(\alpha)}$ in L^p for all such α.

Let us write u for $u^{(0)}$. We are finished by showing that $\partial^\alpha u = u^{(\alpha)}$ in $\mathcal{D}'(\Omega)$ for all such $\alpha \in \mathbb{N}_0^d$, as this implies $u_n \to u$ in $W^{k,p}(\Omega)$. This follows by testing against a testfunction φ, using; if $f_n \to f$ in L^p, then $\int f_n \varphi \to \int f \varphi$ (which follows by Hölder's inequality):

$$\langle \partial^\alpha u, \varphi \rangle = \int u \cdot (-1)^{|\alpha|} \partial^\alpha \varphi = \lim_{n\to\infty} \int u_n \cdot (-1)^{|\alpha|} \partial^\alpha \varphi = \lim_{n\to\infty} \int \partial^\alpha u_n \cdot \varphi = \langle u^{(\alpha)}, \varphi \rangle.$$

As this holds for all $\varphi \in \mathcal{D}(\Omega)$, we have $\partial^\alpha u = u^{(\alpha)}$ (by Lemma 2.9). □

Let us consider the continuity of the operator ∂^α on the C^m space and embeddings of such spaces and then show the analogue statements when we instead of C^m spaces consider Sobolev spaces.

Lemma 12.8. *Let $k, m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$. If $k \le m$, then for all compact sets $K \subset \mathbb{R}^d$,*

$$\|\partial^\alpha f\|_{C^{k-|\alpha|},K} \le \|f\|_{C^m,K} \qquad (f \in C^m),$$

and

$$\|\partial^\alpha f\|_{C^{k-|\alpha|}} \le \|f\|_{C^m} \qquad (f \in C_{\mathrm{b}}^m).$$

In particular, $\partial^\alpha : C^m \to C^{m-|\alpha|}$ is continuous for all $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d, |\alpha| \le m$, and

$$C^m \hookrightarrow C^k, \qquad C_{\mathrm{b}}^m \hookrightarrow C_{\mathrm{b}}^k \qquad (k \le m).$$

Proof. This easily follows by the definitions of the norms $\| \cdot \|_{C^m,K}$ and $\| \cdot \|_{C^m}$, we leave it to the reader to check this. □

Lemma 12.9. *Let $p \in [1,\infty]$. Let $k, m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$. If $k \le m$, then*

$$\|\partial^\alpha u\|_{W^{k-|\alpha|,p}} \le \|u\|_{W^{m,p}} \qquad (u \in W^{m,p}).$$

In particular, $\partial^\alpha : W^{m,p} \to W^{m-|\alpha|,p}$ is continuous for all $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d, |\alpha| \le m$ and

$$W^{m,p} \hookrightarrow W^{k,p} \qquad (k \le m).$$

Proof. The proof is again rather straightforward and left to the reader. □

Similar to Proposition 5.3 one has:

Lemma 12.10. *For all $k \in \mathbb{N}_0$ there exists a $C > 0$ such that for all $p \in [1,\infty]$ and $r, q \in [1,\infty]$ with $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$,*

$$\|uv\|_{W^{k,p}} \le C\|u\|_{W^{k,r}}\|v\|_{W^{k,q}} \qquad (u \in W^{k,r}(\Omega), v \in W^{k,q}(\Omega)).$$

Consequently, the function $W^{k,r}(\Omega) \times W^{k,q}(\Omega) \to W^{k,p}(\Omega)$ given by $(u,v) \mapsto uv$ is continuous.

Proof. See the proof of Proposition 5.3 and use additionally that $\|fg\|_{L^p} \le \|f\|_{L^r}\|g\|_{L^q}$ for $f \in L^p(\Omega)$ and $g \in L^\infty(\Omega)$, which follows by Hölder's inequality. □

Theorem 12.11. *Let $p \in [1,\infty)$. Then $\mathcal{D}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. Let χ and χ_R for $R > 0$ be as in Lemma 8.16. By Lemma 12.10 $\chi_R|_\Omega u \in W^{k,p}(\Omega)$ for all $R > 0$. By Lemma 8.16 $\lim_{R\to\infty} \chi_R|_\Omega u = u$ in $W^{k,p}(\Omega)$. Therefore it is sufficient to show that for all compactly supported $u \in W^{k,p}(\Omega)$ there exist $u_\varepsilon \in \mathcal{D}(\Omega)$ for $\varepsilon > 0$ such that $u_\varepsilon \xrightarrow{\varepsilon\downarrow 0} u$ in $W^{k,p}(\Omega)$. This follows similarly as in Theorem 8.17 by using (8.5). We leave the details to the reader. □

Definition 12.12. Let $k \in \mathbb{N}_0$ and $p \in [1,\infty]$.

(a) We write $W_0^{k,p}(\Omega)$ for the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

(b) We write $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

(c) We define $\langle\cdot,\cdot\rangle_{H^k} : H^k(\Omega) \times H^k(\Omega) \to \mathbb{F}$ by

$$\langle u, v\rangle_{H^k} = \sum_{\alpha\in\mathbb{N}_0^d : |\alpha|\le k} \langle\partial^\alpha u, \partial^\alpha v\rangle_{L^2} \qquad (u, v \in H^k(\Omega)),$$

and $\|\cdot\|_{H^k} : H^k(\Omega) \to [0,\infty)$ by

$$\|u\|_{H^k} = \sqrt{\langle u, u\rangle_{H^k}} \qquad (u \in H^k(\Omega)).$$

Remark 12.13. One interprets $W_0^{k,p}(\Omega)$ as the subspace of $W^{k,p}(\Omega)$ of elements that vanish at the boundary of Ω, in symbols, $u = 0$ on $\partial\Omega$.

Similarly to Theorem 12.7, in which we showed that $W^{k,p}$ is a Banach space by using that L^p is a Banach space, one can show that H^k is a Hilbert space because L^2 is:

Theorem 12.14. *Let $k \in \mathbb{N}_0$. $\langle \cdot, \cdot \rangle_{H^k}$ is an inner product on $H^k(\Omega)$, so that $H^k(\Omega)$ (and $H_0^k(\Omega)$) equipped with this inner product is a Hilbert space.*

Proof. We leave it for the reader to check that $\langle \cdot, \cdot \rangle_{H^k}$ defines an inner product. The rest follows from Theorem 12.7 and the fact that $\| \cdot \|_{H^k}$ is equivalent to $\| \cdot \|_{W^{2,k}}$ (see 12.6). $\square$

There is a lot of theory on Sobolev spaces, which we will not treat here. Sobolev spaces play a central role in the theory of partial differential equations. In the following section we consider an application to elliptic partial differential operators. One classical reference for PDE theory, which contains a whole section on Sobolev spaces is [Eva98] (see Section 5). There are various estimates that are useful, of which we present one important example, the Poincaré inequality.

Definition 12.15 ($\mathcal{L}^p(\Omega, \mathbb{R}^d)$ and $L^p(\Omega, \mathbb{R}^d)$)**.** $\mathcal{L}^p(\Omega, \mathbb{R}^d)$ is the space of (Lebesgue) measurable $f : \Omega \to \mathbb{R}^d$ such that

$$\|f\|_{L^p(\Omega,\mathbb{R}^d)} := \left(\int |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

$L^p(\Omega, \mathbb{R}^d)$ is the space of all equivalence classes in $\mathcal{L}^p(\Omega, \mathbb{R}^d)$. We mostly write "$\| \cdot \|_{L^p}$" instead of "$\| \cdot \|_{L^p(\Omega,\mathbb{R}^d)}$". $L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$ is the space of functions that are locally in $L^p(\Omega, \mathbb{R}^d)$, i.e., $f \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$ if and only if $f \mathbb{1}_K \in L^p(\Omega, \mathbb{R}^d)$ for all compact sets $K \subset \Omega$.

Let $f : \Omega \to \mathbb{R}^d$ be measurable and $f_1, \dots, f_d : \Omega \to \mathbb{R}^d$ be its coordinates, i.e., $f(x) = (f_1(x), \dots, f_d(x))$. Then f_i is measurable for each $i \in \{1, \dots, d\}$ and $f \in L^p(\Omega, \mathbb{R}^d)$ if and only if $f_i \in L^p(\Omega)$ for all $i \in \{1, \dots, d\}$. Moreover,

$$\|f\|_{L^p} = \left(\int \left(|f_1(x)|^2 + \cdots + |f_d(x)|^2 \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}}.$$

On the other hand, $\| \cdot \|_{L^p}$ is equivalent to (see 12.6)

$$f \mapsto \left(\int |f_1(x)|^p + \cdots + |f_d(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

and to $f \mapsto \|f_1\|_{L^p} + \cdots + \|f_d\|_{L^p}$.

Definition 12.16. An open set $\Omega \subset \mathbb{R}^d$ is said to be *of finite width* if there exist $a, b \in \mathbb{R}$, $a < b$ a $y \in \mathbb{R}^d$ with $|y| = 1$ and

$$\{x \cdot y : x \in \Omega\} \subset [a, b], \tag{12.1}$$

where $\cdot$ is the inner product on $\mathbb{R}^d$. The Ω is said to be of *width* w if

$$w = \inf\Big\{b - a \ : \ a, b \in \mathbb{R}, a < b, y \in \mathbb{R}^d, |y| = 1, \{x \cdot y : x \in \Omega\} \subset [a, b]\Big\}.$$

Theorem 12.17 (The Poincaré inequality)**.** *Let* $w > 0$*. Let* Ω *be a nonempty open subset of* $\mathbb{R}^d$ *of finite width* w*. For all* $p \in [1, \infty)$

$$\|u\|_{L^p} \leq \frac{w}{p^{\frac{1}{p}}} \|\nabla u\|_{L^p} \qquad (u \in W_0^{1,p}(\Omega)). \tag{12.2}$$

Proof. Let $u \in W_0^{1,p}(\Omega)$. We extend u to $\mathbb{R}^d$ by defining it to be equal to 0 outside Ω.

Let $y \in \mathbb{R}^d$, $|y| = 1$ and $a, b \in \mathbb{R}$, $a < b$ with $b = a + w$ be such that (12.1). In the first step we show the inequality in case $y = e_1$. In the second step we show how we can reduce the general case to the case where $y = e_1$.

Step 1 Let $x \in \mathbb{R}^d$, $x' = (x_2, \ldots, x_d)$. Then by Hölder's inequality,

$$\begin{aligned} |u(x_1, x')| &= |u(x_1, x') - u(a, x')| \\ &\leq \int_a^{x_1} |\partial_1 u(s, x')| \ \mathrm{d}s \\ &\leq (x_1 - a)^{\frac{p-1}{p}} \|\partial_1 u(\cdot, x')\|_{L^p(\mathbb{R})}. \end{aligned}$$

Therefore (using Fubini's theorem)

$$\begin{aligned} \int |u|^p &= \int_{\mathbb{R}^{d-1}} \int_a^b |u(x_1, x')|^p \ \mathrm{d}x_1 \ \mathrm{d}x' \\ &\leq \int_{\mathbb{R}^{d-1}} \int_a^b (x_1 - a)^{p-1} \ \mathrm{d}x_1 \|\partial_1 u(\cdot, x')\|^p_{L^p(\mathbb{R})} \ \mathrm{d}x' \\ &= \frac{(b-a)^p}{p} \|\partial_1 u\|_{L^p(\mathbb{R}^d)} \leq \frac{w^p}{p} \|\nabla u\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Step 2 Let $v_1, \ldots, v_d$ be an orthonormal basis of $\mathbb{R}^d$ with $v_1 = y$. Then the matrix which columns equal $v_1, \ldots, v_d$; $R = [v_1 \ v_2 \ \cdots \ v_d]$ is such that $Re_1 = v_1 = y$ and, by Pythagoras' theorem,

$$|Rx| = |x| \qquad (x \in \mathbb{R}^d).$$

Then, writing also R for the linear bijection $v \mapsto Rv$, $R^{-1}\Omega \subset [a, b]e_1 + (\mathbb{R}e_1)^\perp$ (where $(\mathbb{R}e_1)^\perp$ is the orthogonal complement of $\mathbb{R}e_1$) and $\|u\|_{L^p} = \|u \circ R\|_{L^p}$, $\nabla(u \circ R) = R[(\nabla u) \circ R]$ and

$$\|\nabla u\|_{L^p} = \|R[(\nabla u) \circ R]\|_{L^p}.$$

Therefore the Poincaré inequality follows from Step 1. $\square$

Corollary 12.18. *Let Ω be a nonempty open subset of $\mathbb{R}^d$ of finite width. Then the norms $W_0^{1,p} \to [0,\infty)$,*

$$u \mapsto \|u\|_{W^{1,p}}, \qquad u \mapsto \|\nabla u\|_{L^p},$$

are equivalent.

Exercise **12.C.** Prove the following statement. *Let Ω be a bounded open subset of $\mathbb{R}^d$. Let $p \in [1,\infty)$. For each $r \in [1,p]$ there exists a $C > 0$ such that*

$$\|u\|_{L^r} \le C\|\nabla u\|_{L^p} \qquad (u \in W_0^{1,p}(\Omega)).$$

Exercise **12.D.** In this exercise we show that for open sets that do not have finite width, an inequality as the Poincaré inequality (12.2) does not hold (for any $w > 0$): Suppose $\Omega \subset \mathbb{R}^d$ is an open set that contains each ball $B(x_n, n)$ for $n \in \mathbb{N}$ and some sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$. Show that for each $m \in \mathbb{N}$ there exists a $u_m \in W_0^{1,p}(\Omega)$ with $\|\nabla u_m\|_{L^p} = 1$ and $\|u_m\|_{L^p} \ge m$.

13 Solutions to elliptic PDEs in Sobolev spaces

In this section we consider the existence of solutions to elliptic partial differential equations equations. The notion of solution will be defined in the language of Sobolev spaces.

In this section the scalar field is the real numbers, i.e., $\mathbb{F} = \mathbb{R}$ and Ω is an open subset of $\mathbb{R}^d$.

Definition 13.1. We call a map $P : \mathcal{D}(\Omega) \to \mathcal{D}'(\Omega)$ a *linear partial differential operator with variable coefficients* if there exist an $m \in \mathbb{N}$ and $u_\alpha \in \mathcal{D}'(\Omega)$ for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$ such that

$$P = \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le m}} u_\alpha \partial^\alpha,$$

i.e.,

$$P\varphi = \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le m}} (\partial^\alpha \varphi) u_\alpha \qquad (\varphi \in \mathcal{D}(\Omega)).$$

m is called the *order* of P.

Definition 13.2. A linear partial differential operator P of order 2 is called a *second order linear partial differential operator*. Let P be such an operator. Then there exist $\mathfrak{a}_{ij}, \mathfrak{b}_i, \mathfrak{c} \in \mathcal{D}'(\Omega)$ for $i, j \in \{1, \dots, d\}$ such that

$$P = -\sum_{i,j=1}^d \mathfrak{a}_{ij}\partial_i\partial_j + \sum_{i=1}^d \mathfrak{b}_i\partial_i + \mathfrak{c}.$$

If $\mathfrak{a}_{ij} \in L^1_{\mathrm{loc}}(\Omega)$ for all $i, j \in \{1, \dots, d\}$, then P is called *elliptic* if there exists a $\theta > 0$ such that

$$\sum_{i,j=1}^{d} \mathfrak{a}_{ij}(x) y_i y_j \geq \theta |y|^2 \qquad (\text{almost all } x \in \Omega, y \in \mathbb{R}^d). \tag{13.1}$$

Let $\mathfrak{a} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{d\times d})$ be the matrix valued function such that $(\mathfrak{a})_{ij} = \mathfrak{a}_{ij}$. Then P is elliptic if and only if $\mathfrak{a} - \theta I$ is positive definite for some $\theta > 0$, where I is the identity matrix.

Observe that $-\Delta$ is an elliptic operator.

13.3. As we allow the coefficients u_α to be distributions, the domain of the operator P is $\mathcal{D}(\Omega)$ and not $\mathcal{D}'(\Omega)$ as in Definition 11.1. But observe the following.

- If $u_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ for each $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$, then $P\varphi \in L^1_{\mathrm{loc}}(\Omega)$ for all $\varphi \in \mathcal{D}(\Omega)$ and
$$P\varphi(x) = \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\leq m}} u_\alpha(x) \partial^\alpha \varphi(x) \qquad (\varphi \in \mathcal{D}(\Omega), x \in \Omega).$$
- If $u_\alpha \in \mathcal{E}(\Omega)$ for each $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$, then $P\varphi \in \mathcal{D}(\Omega)$ for all $\varphi \in \mathcal{D}(\Omega)$ and P extends to an operator $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$.

13.4 (Assumptions)**.** In this section we consider the following setting. Let

$$\mathfrak{a}_{ij}, \mathfrak{b}_i, \mathfrak{c} \in L^\infty(\Omega), \qquad (i, j \in \{1, \dots, d\}).$$

Let $L : \mathcal{D}(\Omega) \to \mathcal{D}'(\Omega)$ be defined by

$$L\varphi = -\sum_{i,j=1}^{d} \partial_i(\mathfrak{a}_{ij}\partial_j\varphi) + \sum_{i=1}^{d} \mathfrak{b}_i \partial_i \varphi + \mathfrak{c}\varphi \qquad (\varphi \in \mathcal{D}(\Omega)). \tag{13.2}$$

Observe that with $\tilde{\mathfrak{b}}_i = \mathfrak{b}_i - \sum_{j=1}^d \partial_j \mathfrak{a}_{ij}$ we have

$$L\varphi = -\sum_{i,j=1}^{d} \mathfrak{a}_{ij}\partial_{ij}\varphi + \sum_{i=1}^{d} \tilde{\mathfrak{b}}_i \partial_i \varphi + \mathfrak{c}\varphi \qquad (\varphi \in \mathcal{D}(\Omega)), \tag{13.3}$$

from which we see that L is a second order linear partial differential operator with variable coefficients. Let $\mathfrak{a} \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ be the matrix valued function given by $(\mathfrak{a})_{ij} = \mathfrak{a}_{ij}$ and $\mathfrak{b} \in L^\infty(\Omega, \mathbb{R}^d)$ the vector valued function given by $(\mathfrak{b})_i = \mathfrak{b}_i$. Then $\mathfrak{a}\nabla\varphi$ is an element of $L^\infty(\Omega, \mathbb{R}^d)$. By writing $\nabla \cdot \mathfrak{f} = \sum_{i=1}^d \partial_i \mathfrak{f}$ for any $\mathfrak{f} \in L^\infty(\Omega, \mathbb{R}^d)$, L can be written as

$$L\varphi = -\nabla \cdot (\mathfrak{a}\nabla\varphi) + \mathfrak{b} \cdot \nabla\varphi + \mathfrak{c}\varphi \qquad (\varphi \in \mathcal{D}(\Omega)).$$

$\nabla \cdot \mathfrak{f}$ is also called the *divergence* of $\mathfrak{f}$.

We consider the following problem. Fix a function $f : \Omega \to \mathbb{R}$. We want to find a distribution u such that $Lu = f$ and u "equals zero on $\partial\Omega$". For a distribution (or an element of L^1_{loc}) "$u = 0$ on $\partial\Omega$" does not make sense. Instead, we consider the problem of finding a u in a Sobolev space $W_0^{k,p}$ such that $Lu = f$ (remember Remark 12.13). We formalise this in Definition 13.5 with the help of the bilinear form associated with L.

Observe that

$$\begin{aligned}\langle L\varphi, \psi\rangle &= -\sum_{i,j=1}^{d} \langle \partial_i(\mathfrak{a}_{ij}\partial_j\varphi), \psi\rangle + \sum_{i=1}^{d}\langle \mathfrak{b}_i\partial_i\varphi, \psi\rangle + \langle \mathfrak{c}\varphi, \psi\rangle \\ &= \int_\Omega \left(\sum_{i,j=1}^{d} \mathfrak{a}_{ij}(\partial_i\varphi)(\partial_j\psi) + \sum_{i=1}^{d} \mathfrak{b}_i(\partial_i\varphi)\psi + \mathfrak{c}\varphi\psi \right) \qquad (\varphi, \psi \in \mathcal{D}(\Omega)).\end{aligned}$$

As the $\mathfrak{a}_{ij}, \mathfrak{b}_i$ and $\mathfrak{c}$ are bounded, we have

$$|\langle L\varphi, \psi\rangle| \le \left(\sum_{i,j=1}^{d} \|\mathfrak{a}_{ij}\|_{L^\infty} + \sum_{i=1}^{d} \|\mathfrak{b}_i\|_{L^\infty} + \|\mathfrak{c}\|_{L^\infty} \right) \|\varphi\|_{H^1}\|\psi\|_{H^1} \quad (\varphi, \psi \in \mathcal{D}(\Omega)). \tag{13.4}$$

Therefore, as $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ by definition, the bilinear form $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R}$, $(\varphi, \psi) \mapsto \langle L\varphi, \psi\rangle$ extends to a bilinear form $H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$.

Definition 13.5. Let L be as in 13.4.

(a) The *bilinear form associated with* L is the function $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$B(u, v) = \int_\Omega \sum_{i,j=1}^{d} \mathfrak{a}_{ij}(\partial_i u)(\partial_i v) + \sum_{i=1}^{d} \mathfrak{b}_i(\partial_i u)v + \mathfrak{c}uv.$$

(b) If $f \in L^2(\Omega)$, $u \in H_0^1(\Omega)$ and

$$B(u, v) = \langle f, v\rangle_{L^2} \qquad (v \in H_0^1(\Omega)),$$

then we call u a *weak solution* to the *Dirichlet boundary problem*

$$\begin{cases} Lu = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.5}$$

Exercise 13.A. Let L be as in 13.4 and suppose that $\mathfrak{a}_{ij} = \mathfrak{a}_{ji}$ and $\mathfrak{b}_i = 0$ for all $i, j \in \{1, \ldots, d\}$. Show that B is symmetric; $B(u, v) = B(v, u)$ for all $u, v \in H_0^1(\Omega)$.

We will use tools from functional analysis to prove that under certain conditions there exists a weak solution to the Dirichlet boundary problem (13.5). Let us first recall the Riesz-Fréchet theorem, for a proof see for example [Rud91, Theorem 12.5].

Theorem 13.6 (Riesz–Fréchet)**.** *Let H be a Hilbert space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle$. If $A : H \to \mathbb{R}$ is a bounded linear functional, then there exists exactly one $a \in H$ such that*

$$Ax = \langle a, x \rangle \qquad (x \in H).$$

Theorem 13.7 (Lax–Milgram)**.** *Let H be a Hilbert space over $\mathbb{R}$, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $B : H \times H \to \mathbb{R}$ be a bilinear map. Suppose there exist $c, C > 0$ such that*

$$|B(u,v)| \leq C\|u\|\|v\| \qquad (u, v \in H), \tag{13.6}$$

$$c\|u\|^2 \leq B(u,u) \qquad (u \in H). \tag{13.7}$$

(a) *There exists a linear homeomorphism $A : H \to H$ such that*

$$B(u,v) = \langle Au, v \rangle \qquad (v \in H).$$

(b) *Let $g : H \to \mathbb{R}$ be a bounded linear functional. Then there exists exactly one $u \in H$ such that*

$$B(u,v) = g(v) \qquad (v \in H).$$

Proof. Observe that (b) follows from (a) and the Riesz-Fréchet theorem, Theorem 13.6. Let us prove (a).

As for $u \in H$ the map $v \mapsto B(u,v)$ is a bounded linear functional, the Riesz-Fréchet theorem implies that there exists an element in H, for which we write Au, such that

$$B(u,v) = \langle Au, v \rangle \qquad (v \in H).$$

Then A defines a linear map $H \to H$. By the Inverse Mapping Theorem, see for example [Con90, Theorem III.12.5], it is sufficient to show that A is bounded and bijective.

We have

$$\|Au\|^2 = \langle Au, Au \rangle = B(u, Au) \leq C\|u\|\|Au\| \qquad (u \in H).$$

Therefore $\|Au\| \leq C\|u\|$ for $u \in H$, so that A is bounded.

By (13.7) we have

$$c\|u\|^2 \leq B(u,u) = \langle Au, u \rangle \leq \|Au\|\|u\| \qquad (u \in H),$$

and thus

$$c\|u\| \leq \|Au\| \qquad (u \in H),$$

from which follows that A is injective and its range, AH, is closed in H: If $(u_n)_{n \in \mathbb{N}}$ is a sequence in H such that Au_n converges, then it follows that $(u_n)_{n \in \mathbb{N}}$ is Cauchy and therefore has a limit u in H. By the boundedness of A follows that $Au_n \to Au$.

Now, let us prove that AH, the range of A, equals H. As $A(H)$ is closed we have $AH + (AH)^\perp = H$ (where $(AH)^\perp$ is the orthogonal complement of AH), so it is sufficient to show that $(AH)^\perp = \{0\}$. Let $w \in (AH)^\perp$. Then $0 = \langle Aw, w\rangle = B(w,w) \geq c\|w\|^2$. So $w = 0$. $\square$

Exercise 13.B. Let H and B be as in Theorem 13.7. Suppose furthermore that B is symmetric, in the sense that $B(u,v) = B(v,u)$ for all $u, v \in H$. Why does statement (b) directly follow from the Riesz–Fréchet theorem?

Let us verify the assumptions of the Lax-Milgram theorem for the bilinear form associated with L.

Theorem 13.8. *Let L be as in 13.4 and B be the bilinear form associated with L. Suppose L is elliptic and Ω is of finite width. There exist a $\gamma \geq 0$ and $c, C > 0$ such that*

$$|B(u,v)| \leq C\|u\|_{H^1}\|v\|_{H^1} \qquad (u,v \in H_0^1(\Omega)), \tag{13.8}$$

$$c\|u\|_{H^1}^2 \leq B(u,u) + \gamma\|u\|_{L^2}^2 \qquad (u \in H_0^1(\Omega)). \tag{13.9}$$

Proof. (13.8) follows from the following estimate, see also (13.4),

$$|B(u,v)| \leq \left(\sum_{i,j=1}^d \|\mathfrak{a}_{ij}\|_{L^\infty} + \sum_{i=1}^d \|\mathfrak{b}_i\|_{L^\infty} + \|\mathfrak{c}\|_{L^\infty}\right)\|u\|_{H^1}\|v\|_{H^1} \qquad (u,v \in H_0^1(\Omega)).$$

On the other hand, for $\theta > 0$ as in (13.1) we have for $u \in H_0^1(\Omega)$

$$\begin{aligned}
\theta \sum_{i=1}^d \int_\Omega |\partial_i u|^2 &\leq \int_\Omega \sum_{i,j=1}^d \mathfrak{a}_{ij}(\partial_i u)(\partial_j u) \\
&= B(u,u) - \int_\Omega \sum_{i=1}^d \mathfrak{b}_i u \partial_i u - \int_\Omega \mathfrak{c} u^2 \\
&\leq B(u,u) + \sum_{i=1}^d \|\mathfrak{b}_i\|_{L^\infty} \int_\Omega |\partial_i u||u| + \|\mathfrak{c}\|_{L^\infty} \int_\Omega u^2.
\end{aligned}$$

As $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ for any $a, b \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\int_\Omega |\partial_i u||u| \leq \varepsilon \int_\Omega |\partial_i u|^2 + \frac{1}{4\varepsilon}\int_\Omega |u|^2 \qquad (u \in H_0^1(\Omega)).$$

Let $M = \sum_{i=1}^d \|\mathfrak{b}_i\|_{L^\infty}$ and take ε small enough such that $\varepsilon M \leq \frac{\theta}{2}$. Then

$$\sum_{i=1}^d \|\mathfrak{b}_i\|_{L^\infty} \int_\Omega |\partial_i u||u| \leq \frac{\theta}{2}\sum_{i=1}^d \int |\partial_i u|^2 + \frac{M}{4\varepsilon}\int_\Omega u^2.$$

By the Poincaré inequality (see Theorem 12.17) there exists a $\beta > 0$ such that

$$\beta \|u\|_{H^1}^2 \leq \sum_{i=1}^d \int_\Omega |\partial_i u|^2 \qquad (u \in H_0^1(\Omega)).$$

Thus

$$\beta \frac{\theta}{2} \|u\|_{H^1}^2 \leq \left(\theta - \frac{\theta}{2}\right) \sum_{i=1}^d \int_\Omega |\partial_i u|^2 \leq B(u,u) + \left(\|\mathfrak{c}\|_{L^\infty} + \frac{M}{4\varepsilon}\right) \int_\Omega u^2.$$

□

Now we can prove that under certain conditions (13.5) has a weak solution.

Theorem 13.9. *Let L be as in 13.4. Suppose L is elliptic and Ω is of finite width. There exists a $\gamma \geq 0$ such that for all $\beta \geq \gamma$ and $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of the Dirichlet boundary problem*

$$\begin{cases} Lu + \beta u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.10}$$

Proof. Let $\gamma \geq 0$ be as in Theorem 13.8. Let $\beta \geq \gamma$. We apply the Lax-Milgram theorem to B_β, the bilinear operator corresponding to the elliptic operator L_β given by $L_\beta u = Lu + \beta u$:

$$B_\beta(u,v) = B(u,v) + \beta \langle u, v \rangle_{L^2} \qquad (u, v \in H_0^1(\Omega)).$$

Observe that for $f \in L^2(\Omega)$ the map $g : H_0^1(\Omega) \to \mathbb{R}$ given by $g(v) = \langle f, v \rangle_{L^2}$ is bounded and linear, because $\|v\|_{L^2}^2 \leq \|v\|_{H^1}^2$. So by the Lax-Milgram theorem there exists exactly one $u \in H_0^1(\Omega)$ such that $B_\beta(u,v) = \langle f, v \rangle_{L^2}$ for all $v \in H_0^1(\Omega)$, which means that u is a weak solution to (13.10). □

For more theory on weak solutions of elliptic Dirichlet boundary problems, we refer the reader to [Eva98, Section 6.2] . Moreover, one can show that the solutions have a certain regularity that depends on the regularity of the coefficients $\mathfrak{a}_{i,j}, \mathfrak{b}_i, \mathfrak{c}$, see [Eva98, Section 6.3] .

Exercise **13.C.** Show that one can choose $\gamma = 0$ in Theorem 13.8 and Theorem 13.9 in case $\mathfrak{b}_i = 0$ for all i and $\mathfrak{c} = 0$.

14 The Schwartz space

In this section we introduce the Schwartz space, which is the space of smooth functions that are of rapid decay. This space is suitable for the Fourier transformation, as the

Fourier transformation maps the Schwartz space onto itself. We will turn to that later and first discuss here the topological properties of the Schwartz space. In Section 15 we consider its dual, the space of tempered distributions. As our underlying space we consider $\mathbb{R}^d$ (only). For this reason we can leave out the part "$(\mathbb{R}^d)$" in the notation of function spaces or spaces of distributions.

Definition 14.1. We say that a function $f : \mathbb{R}^d \to \mathbb{F}$ is of *rapid decay* if

$$\lim_{|x|\to\infty} P(x)f(x) = 0,$$

for all polynomials P, where $\lim_{|x|\to\infty} g(x) = a$ means that for all $\varepsilon > 0$ there exists an $R > 0$ such that for all $x \in \mathbb{R}^d$ with $|x| > R$, $|g(x) - a| < \varepsilon$.

Observe that f is of rapid decay if and only if $\lim_{|x|\to\infty} x^\alpha f(x) = 0$ for all $\alpha \in \mathbb{N}_0^d$. As $x_j \le (1 + |x|^2)$ for all $j \in \{1, \dots, d\}$ and $x \in \mathbb{R}^d$ it follows that for each polynomial P there exists a $C > 0$ and $k \in \mathbb{N}$ such that

$$|P(x)| \le C(1 + |x|)^k \qquad (x \in \mathbb{R}^d).$$

Therefore, f is of rapid decay if and only if $\lim_{|x|\to\infty}(1 + |x|)^k f(x) = 0$ for all $k \in \mathbb{N}_0$.

See Exercise 14.A for equivalent descriptions of rapid decay for continuous functions.

Exercise **14.A.** Prove the following statement (Hint: First prove: $x_j \le (1 + |x|^2)$ for all $j \in \{1, \dots, d\}$ and $x \in \mathbb{R}^d$):
Let $f : \mathbb{R}^d \to \mathbb{F}$ be continuous. The following statements are equivalent:

(a) *f is of rapid decay.*

(b) *$x \mapsto P(x)f(x)$ is bounded for all polynomials P.*

(c) *For all $k \in \mathbb{N}_0$ there exists an $M < \infty$ such that*

$$|f(x)| \le M(1 + |x|)^{-k} \qquad (x \in \mathbb{R}^d).$$

(d)

$$\|(1 + |\cdot|)^k f\|_\infty = \sup_{x\in\mathbb{R}^d} (1 + |x|)^k |f(x)| < \infty \qquad (k \in \mathbb{N}_0).$$

Definition 14.2. A smooth function φ is called a *Schwartz function* if the function and all its derivatives are of rapid decay: if $\partial^\alpha\varphi$ is of rapid decay for all $\alpha \in \mathbb{N}_0^d$.

We write $\mathcal{S}$ (or $\mathcal{S}(\mathbb{R}^d)$) for the space of Schwartz functions and call it the *Schwartz space.* For $k \in \mathbb{N}_0$ we define $\|\cdot\|_{k,\mathcal{S}} : \mathcal{S} \to [0, \infty)$ by

$$\|\varphi\|_{k,\mathcal{S}} := \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le k}} \|(1 + |\cdot|)^k \partial^\alpha\varphi\|_{L^\infty} \qquad (\varphi \in \mathcal{S}). \tag{14.1}$$

$\|\cdot\|_{k,\mathcal{S}}$ is a norm for all $k \in \mathbb{N}_0$. The space $\mathcal{S}$ is equipped with the topology generated by the seminorms $\|\cdot\|_{k,\mathcal{S}}$.

Each testfunction is a Schwartz function. Gaussian functions are examples of Schwartz functions that are not compactly supported:

Definition 14.3. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called a *Gaussian function* if there exist $a, b \in \mathbb{R}$, $a > 0$, $y \in \mathbb{R}^d$ such that

$$f(x) = be^{-a|x-y|^2}.$$

Each such function f is smooth and is a Schwartz function: Let $\alpha \in \mathbb{N}_0^d$ and $k = |\alpha|$. Then

$$|\partial^\alpha f(x)| \le |b|(2a)^k |x-y|^k e^{-a|x-y|^2} \qquad (x \in \mathbb{R}^d),$$

and thus, using that $(1+|x+y|)^k \le (1+|y|)^k(1+|x|)^k$,

$$\|f\|_{k,\mathcal{S}} \le |b|(2a(1+|y|))^k \sup_{x\in\mathbb{R}^d} (1+|x|)^{2k} e^{-a|x|^2} < \infty. \tag{14.2}$$

14.4 (Equivalent norms)**.** In the literature one finds different definitions of norms or seminorms on $\mathcal{S}$, which all generate the same topology. For example, for $k \in \mathbb{N}_0$ the function $|||\cdot|||_{k,\mathcal{S}} : \mathcal{S} \to [0,\infty)$ defined by

$$|||\varphi|||_{k,\mathcal{S}} = \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le k}} \|(1+|\cdot|^2)^{\frac{k}{2}} \partial^\alpha \varphi(x)\|_{L^\infty},$$

is a norm that is equivalent to $\|\cdot\|_{k,\mathcal{S}}$, which can be seen by the estimate

$$1+|x|^2 \le (1+|x|)^2 \le 2(1+|x|^2) \qquad (x \in \mathbb{R}^d). \tag{14.3}$$

On the other hand, the topology on $\mathcal{S}$ generated by the seminorms $\|\cdot\|_{k,\mathcal{S}}$ with $k \in \mathbb{N}_0$ is equal to the topology generated by the seminorms $|||\cdot|||_{k,\alpha}$ with $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$, which are defined by

$$|||\cdot|||_{k,\alpha} = \|(1+|\cdot|^2)^k \partial^\alpha \varphi\|_{L^\infty} \qquad (\varphi \in \mathcal{S}).$$

Exercise **14.B.** Verify that the seminorms $|||\cdot|||_{k,\alpha}$ with $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$ generate the same topology on $\mathcal{S}$ as the norms $\|\cdot\|_{k,\mathcal{S}}$ with $k \in \mathbb{N}_0$.

Exercise **14.C.** Give an example of function φ in $C^\infty(\mathbb{R})$ for which φ is of rapid decay but its derivative φ' is not.

The operations of reflection, translation and derivation are operations $\mathcal{S} \to \mathcal{S}$. On the other hand, a Schwartz function multiplied by a smooth function may not be a Schwartz function; take $\varphi \in \mathcal{S}$, $\psi \in \mathcal{S}$ given by $\varphi(x) = e^{-|x|^2}$ and $\psi(x) = e^{2|x|^2}$ for $x \in \mathbb{R}^d$, then $\varphi\psi(x) = e^{|x|^2}$ which is clearly not of rapid decay and therefore not a Schwartz function. Multiplication with a C_b^∞ function is an operation $\mathcal{S} \to \mathcal{S}$, but we may allow for more smooth functions:

Definition 14.5. Let $\Omega \subset \mathbb{R}^d$ be open. A function $f : \Omega \to \mathbb{F}$ is said to be *of at most polynomial growth* if either Ω is bounded or there exists a polynomial P such that

$$\lim_{|x|\to\infty} \frac{f(x)}{P(x)} = 0,$$

or equivalently (as in Definition 14.1), there exists a $k \in \mathbb{N}_0$ such that

$$\lim_{|x|\to\infty} \frac{f(x)}{(1+|x|)^k} = 0.$$

Exercise 14.D. Prove the following statement:
Let $f : \mathbb{R}^d \to \mathbb{F}$ be continuous. The following statements are equivalent:

(a) *f is of at most polynomial growth.*

(b) *There exists a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $|f(x)| \leq p(|x|)$ for all $x \in \mathbb{R}^d$.*

(c) *There exist a $C > 0$ and a $k \in \mathbb{N}_0$ such that*

$$|f(x)| \leq C(1+|x|)^k \qquad (x \in \Omega).$$

(d) *There exists a $k \in \mathbb{N}_0$ such that*

$$\|(1+|\cdot|)^{-k} f\|_{L^\infty} < \infty.$$

Definition 14.6. We write $C_\mathrm{p}^\infty(\Omega)$ for the set of smooth functions σ such that for all $\alpha \in \mathbb{N}_0^d$, the function $\partial^\alpha \sigma$ is of at most polynomial growth.

For $C_\mathrm{p}^\infty = C_\mathrm{p}^\infty(\mathbb{R}^d)$ and $\sigma \in C^\infty$ we have $\sigma \in C_\mathrm{p}^\infty$ if and only if for all $m \in \mathbb{N}_0$ there exists an $k \in \mathbb{N}_0$ such that

$$\mathfrak{q}_{m,k}(\sigma) := \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\leq m}} \|(1+|\cdot|^2)^{-k} \partial^\alpha \sigma\|_{L^\infty} < \infty. \tag{14.4}$$

Observe that $\mathfrak{q}_{m,0}(\sigma) = \|\sigma\|_{C^m}$ for $\sigma \in C_\mathrm{b}^\infty(\Omega)$.

Exercise 14.E. Let $m, k \in \mathbb{N}_0$. Let $C_{\mathrm{p},k}^m(\Omega)$ be the space of functions $f \in C^m(\Omega)$ such that $\mathfrak{q}_{m,k}(f) < \infty$. Show that $\mathfrak{q}_{m,k}$ is a norm on $C_{\mathrm{p},k}^m(\Omega)$ under which it is a Banach space.

Let $C_{\mathrm{p},k}^\infty(\Omega)$ be the space of functions $f \in C^\infty(\Omega)$ such that $\mathfrak{q}_{m,k}(f) < \infty$ for all $m \in \mathbb{N}_0$. Show that $C_{\mathrm{p},k}^\infty(\Omega)$ equipped with the seminorms $(\mathfrak{q}_{m,k})_{m\in\mathbb{N}}$ is a Fréchet space and

$$C_{\mathrm{p},k}^\infty(\Omega) \hookrightarrow \mathcal{E}.$$

Lemma 14.7. *For all $\sigma \in C_{\mathrm{p}}^\infty$ and $\varphi \in \mathcal{S}$ we have $\sigma\varphi \in \mathcal{S}$. Moreover, for all $m \in \mathbb{N}_0$ there exists a $C > 0$ such that*

$$\|\sigma\varphi\|_{m,\mathcal{S}} \le C\mathfrak{q}_{m,k}(\sigma)\|\varphi\|_{m+k,\mathcal{S}} \qquad (\sigma \in C_{\mathrm{p}}^\infty, k \in \mathbb{N}_0, \varphi \in \mathcal{S}), \tag{14.5}$$

in particular,

$$\|\sigma\varphi\|_{m,\mathcal{S}} \le C\|\sigma\|_{C^m}\|\varphi\|_{m,\mathcal{S}} \qquad (\sigma \in C_{\mathrm{b}}^\infty, \varphi \in \mathcal{S}), \tag{14.6}$$

$$\|\varphi\psi\|_{m,\mathcal{S}} \le C\|\varphi\|_{m,\mathcal{S}}\|\psi\|_{m,\mathcal{S}} \qquad (\varphi, \psi \in \mathcal{S}). \tag{14.7}$$

Proof. Let $k, m \in \mathbb{N}_0$. By (5.3), which relies on Leibniz' rule, there exists a $C > 0$ such that for all $\sigma \in C_{\mathrm{p}}^\infty$ and $\varphi \in \mathcal{S}$

$$\begin{aligned}\|\sigma\varphi\|_{m,\mathcal{S}} &= \max_{\substack{\alpha\in\mathbb{N}_0^d\\|\alpha|\le m}} \sup_{x\in\mathbb{R}^d} (1+|x|)^m|\partial^\alpha(\sigma\varphi)(x)| \\ &\le C\Big(\max_{\substack{\alpha\in\mathbb{N}_0^d\\|\alpha|\le m}} \sup_{x\in\mathbb{R}^d} (1+|x|)^{-k}|\partial^\alpha\sigma(x)|\Big)\Big(\max_{\substack{\beta\in\mathbb{N}_0^d\\|\beta|\le m}} \sup_{x\in\mathbb{R}^d} (1+|x|)^{m+k}|\partial^\beta\varphi)(x)|\Big).\end{aligned}$$

□

Exercise **14.F.** Let $C_{\mathrm{p},k}^\infty$ be as in Exercise 14.E. Show that $C_{\mathrm{p},k}^\infty \times \mathcal{S} \to \mathcal{S}$, $(\sigma, \varphi) \mapsto \sigma\varphi$ is continuous.

14.8. The reflection operator $\mathcal{R}$ (see 2.12), the translation operator $\mathcal{T}_y$, the derivation operator ∂^α, multiplication with a C_{p}^∞ function and the operation of composing with a linear bijection form continuous maps $\mathcal{S} \to \mathcal{S}$.

14.9 (Notation)**.** For $\lambda \in \mathbb{R} \setminus \{0\}$ and a function $f : \Omega \to \mathbb{F}$ we write $l_\lambda f$ for the function $\frac{1}{\lambda}\Omega \to \mathbb{F}$ given by

$$l_\lambda f(x) = f(\lambda x) \qquad (x \in \tfrac{1}{\lambda}\Omega).$$

The following rather elementary estimates and convergences will be used multiple times.

Lemma 14.10.

(a) *Let $K \subset \mathbb{R}^d$ be compact and $m \in \mathbb{N}_0$. Then*

$$\|\varphi\|_{C^m,K} \le \|\varphi\|_{C^m} \le \|\varphi\|_{m,\mathcal{S}} \qquad (\varphi \in \mathcal{S}),$$
$$\|\varphi\|_{m,\mathcal{S}} \le (1 + \sup_{x\in\operatorname{supp}\varphi} |x|)^m\|\varphi\|_{C^m} \qquad (\varphi \in \mathcal{D}).$$

(b) *Let $r > 0$. If $\psi \in \mathcal{S}$ and $\psi = 0$ on $B(0, r)$, then for $k \in \mathbb{N}_0$*

$$\|\psi\|_{k,\mathcal{S}} \le \frac{\|\psi\|_{k+1,\mathcal{S}}}{1+r}.$$

(c) *Let* $\chi \in C_c^\infty(\mathbb{R}^d, [0,1])$ *be equal to* 1 *on a neighbourhood of* 0*. Then*

$$(l_\lambda \chi)\varphi \xrightarrow{\lambda \downarrow 0} \varphi \text{ in } \mathcal{S} \qquad (\varphi \in \mathcal{S}).$$

(d) *Let* $(\chi_n)_{n\in\mathbb{N}}$ *be a partition of unity with* $\sup_{N\in\mathbb{N}} \| \sum_{n=1}^N \chi_n \|_{C^k} < \infty$ *for all* $k \in \mathbb{N}_0$*. Then*

$$\sum_{n=1}^N \chi_n \varphi \xrightarrow{N\to\infty} \varphi \text{ in } \mathcal{S} \qquad (\varphi \in \mathcal{S}).$$

Proof. (a) can be easily checked. (b) follows by

$$\begin{aligned} \|\psi\|_{k,\mathcal{S}} &= \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le k}} \sup_{\substack{x\in\mathbb{R}^d \\ |x|\ge r}} (1+|x|)^k |\partial^\alpha \psi(x)| \le \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le k}} \sup_{x\in\mathbb{R}^d} \frac{(1+|x|)^{k+1}}{1+r} |\partial^\alpha \psi(x)| \\ &\le \frac{\|\psi\|_{k+1,\mathcal{S}}}{1+r}. \end{aligned}$$

(c) follows by (b) and Lemma 14.7 and using that $\|l_\lambda \chi - \mathbb{1}\|_{C^k} \le \|\chi\|_{C^k} + 1$ for all $\lambda \in (0,1)$. We leave the details and the proof of (d) to the reader, see Exercise 14.G. □

Exercise 14.G. Prove Lemma 14.10(c) and (d).

Theorem 14.11. *$\mathcal{D}$ is sequentially continuously embedded in $\mathcal{S}$ and $\mathcal{S}$ is continuously embedded in C_b^∞, so that*

$$\mathcal{D} \hookrightarrow_{\text{seq}} \mathcal{S} \hookrightarrow C_b^\infty \hookrightarrow \mathcal{E}.$$

Proof. $\mathcal{D} \hookrightarrow_{\text{seq}} \mathcal{S} \hookrightarrow C_b^\infty$ follows by Lemma 14.10 (a), $C_b^\infty \hookrightarrow \mathcal{E}$ is already observed in Definition 9.2. □

Theorem 14.12. *$\mathcal{S}$ is a separable Fréchet space and $\mathcal{D}$ is dense in $\mathcal{S}$.*

Proof. As $\mathcal{S}$ is equipped with a countable number of seminorms, it is metrizable with a translation invariant metric Theorem 3.8. If $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}$, then there exists a φ in C_b^∞ such that

$$\|\varphi_n - \varphi\|_{C^k} \to 0.$$

as $\mathcal{S}$ is continuously embedded in C_b^∞. As $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy, it is bounded; for each $k \in \mathbb{N}_0$ there exists an $M > 0$ such that $\|\varphi_n\|_{k,\mathcal{S}} < M$ for all $n \in \mathbb{N}$, and thus

$$|\partial^\alpha \varphi_n(x)| \le M(1+|x|)^{-k} \qquad (x \in \mathbb{R}^d),$$

so that $|\partial^\alpha \varphi(x)| \le M(1+|x|)^{-k}$ for all $x \in \mathbb{R}^d$ as $\partial^\alpha \varphi_n$ converges uniformly to $\partial^\alpha \varphi$. Therefore $\varphi \in \mathcal{S}$ (see also Exercise 14.A).

That $\mathcal{D}$ is dense in $\mathcal{S}$ follows from Lemma 14.10(c). The separability then follows by the fact that $\mathcal{D}$ is separable, see Theorem 8.15. □

Let us recall the integrability of the function $(1+|\cdot|)^{\alpha}$.

Lemma 14.13. *Let $\alpha \in \mathbb{R}$.*

(a) *The functions $\mathbb{R}^d \to \mathbb{R}$, $x \mapsto (1+|x|)^{-\alpha}$ and $x \mapsto (1+|x|^2)^{-\frac{\alpha}{2}}$ are integrable if and only if $\alpha > d$.*

(b) *The functions $\mathbb{Z}^d \to \mathbb{R}$, $k \mapsto (1+|k|)^{-\alpha}$ and $k \mapsto (1+|k|^2)^{-\frac{\alpha}{2}}$ are summable if and only if $\alpha > d$.*

Proof. (a) It sufficient to show that $x \mapsto (1+|x|)^{-\alpha}$ is integrable if and only if $\alpha > d$ (see (14.3)). Integrating this function on $B(0,1)$ gives a finite integral for each $\alpha \in \mathbb{R}$. It will be clear that $\alpha > 0$ is required. By changing to spherical coordinates and observing that $(2r)^{-\alpha} \le (1+r)^{-\alpha} \le r^{-\alpha}$ for $\alpha > 0$ and $r \ge 1$, we see that $(1+|x|)^{-\alpha}$ is integrable if and only if $\int_1^\infty r^{d-1-\alpha}\, \mathrm{d}r$ is finite. The latter is of course the case if and only if $\alpha > d$.

(b) It sufficient to show that $k \mapsto (1+|k|^2)^{-\frac{\alpha}{2}}$ is summable if and only if $\alpha > d$ (see (14.3)). We write $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)$ for $x \in \mathbb{R}^d$ where $\lfloor x_1 \rfloor$ is the largest integer that is smaller than or equal to x_1. Then $\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-\frac{\alpha}{2}} = \int_{\mathbb{R}^d}(1+|\lfloor x \rfloor|^2)^{-\frac{\alpha}{2}}\, \mathrm{d}x$. Note that $|x - \lfloor x \rfloor| \le \sqrt{d}$. Therefore, if $|x| \ge 2\sqrt{d}$ we have

$$\tfrac{1}{2}|x| \le |x| - \sqrt{d} \le |\lfloor x \rfloor| \le |x| + \sqrt{d} \le \tfrac{3}{2}|x|.$$

Hence, $\frac{1}{4}(1+|x|^2) \le (1+|\lfloor x \rfloor|^2) \le \frac{9}{4}(1+|x|^2)$ for those x and so the statement follows by (a). □

By Lemma 14.13 it follows that all measurable functions of rapid decay are integrable, and in L^p for any $p \in [1,\infty]$.

Lemma 14.14. *Let $p \in [1,\infty)$. For all $m \in \mathbb{N}$ such that $pm > d$ there exists a $C > 0$ such that*

$$\|\cdot\|_{L^p} \le C\|\cdot\|_{m,\mathcal{S}}.$$

As $\|\cdot\|_{L^\infty} = \|\cdot\|_{0,\mathcal{S}}$, we therefore have for all $p \in [1,\infty]$ that $\mathcal{S}$ is continuously embedded in L^p,

$$\mathcal{S} \hookrightarrow L^p \qquad (p \in [1,\infty]).$$

Moreover, $\mathcal{S}$ is dense in L^p.

Proof. Let $m \in \mathbb{N}$ be such that $pm > d$. By Lemma 14.13 $C := \|(1+|\cdot|)^{-m}\|_{L^p}$ is finite. Let $f \in \mathcal{S}$. By definition of $\|\cdot\|_{m,\mathcal{S}}$ we have

$$|f(x)| \le \|f\|_{m,\mathcal{S}}(1+|x|)^{-m} \qquad (x \in \mathbb{R}^d),$$

and thus $\|f\|_{L^p} \le C\|f\|_{m,\mathcal{S}}$.

The fact $\mathcal{S}$ is dense in L^p follows from the fact that $\mathcal{D}$ is dense in L^p, see Theorem 8.17. □

By the continuity of the partial derivation one then derives that the Schwartz space is continuously embedded in the Sobolev spaces. The Schwartz space is also dense in the Sobolev space $W^{k,p}$ when p is not infinite. This statement and its proof are postponed to Theorem 23.8.

Lemma 14.15. *Let $k \in \mathbb{N}_0$ and $p \in [1, \infty]$. Then*

$$\mathcal{S} \hookrightarrow W^{k,p}.$$

Exercise **14.H.** Prove Lemma 14.15.

15 Tempered distributions

In this section we consider tempered distributions, which form a subspace of the space of distributions. Even though not every locally integrable function is a tempered distribution, the space of tempered distributions has the benefit that we can define the Fourier transform on it, which we heavily use in the sequel.

Definition 15.1. A linear continuous map $\mathcal{S} \to \mathbb{F}$ is called a *tempered distribution.* We write $\mathcal{S}'$ (or $\mathcal{S}'(\mathbb{R}^d)$) for the space of tempered distributions. In other words, $u \in \mathcal{S}'$ if and only if u is linear and there exist a $k \in \mathbb{N}_0$ and a $C > 0$ such that

$$|u(\varphi)| \le C\|\varphi\|_{k,\mathcal{S}} \qquad (\varphi \in \mathcal{S}).$$

If u is a tempered distribution, then $u|_{\mathcal{D}}$ is a distribution by Lemma 14.10 (a). Therefore each tempered distribution corresponds to a $u \in \mathcal{D}'(\mathbb{R}^d)$ that continuously extends to a function $\mathcal{S}(\mathbb{R}^d) \to \mathbb{F}$.

$\mathcal{S}'$ is equipped with the $\sigma(\mathcal{S}', \mathcal{S})$ topology, where $\langle \cdot, \cdot \rangle : \mathcal{S}' \times \mathcal{S} \to \mathbb{F}$ is given by $\langle u, \varphi \rangle = u(\varphi)$ for $u \in \mathcal{S}', \varphi \in \mathcal{S}$ (we expect that no confusion will arise with $\langle \cdot, \cdot \rangle$ as defined in Definition 4.1).

By Lemma 14.10 (a) it follows that $u|_{\mathcal{S}}$ is a tempered distribution for all $u \in \mathcal{E}'$.

Theorem 15.2. *$\mathcal{S}' \to \mathcal{D}'$, $u \mapsto u|_{\mathcal{D}}$ is a sequentially continuous embedding and $\mathcal{E}' \to \mathcal{S}'$, $u \mapsto u|_{\mathcal{S}}$ is a continuous embedding,*

$$\mathcal{E}' \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{D}'.$$

Proof. Both maps are injective as $\mathcal{D}$ is dense in $\mathcal{S}$, and, $\mathcal{D}$ and thus $\mathcal{S}$ are dense in $\mathcal{E}$ (Theorem 5.10). The sequential continuity of the embeddings $\mathcal{S}' \to \mathcal{D}'$ and $\mathcal{E}' \to \mathcal{S}'$ is straightforward. □

As in 5.13 we show in 15.11 that the embeddings in Theorem 14.11 and Theorem 15.2 are not homeomorphisms onto their image.

We define the support and the operations we defined for distributions in a similar way. For the multiplication with a smooth function we restrict to those of at most polynomial growth. If $\psi \in C_{\mathrm{p}}^{\infty}$ and $u \in \mathcal{S}'$, then $\varphi \mapsto u(\psi\varphi)$ is again in $\mathcal{S}'$ due to Lemma 14.7.

Definition 15.3. For $u \in \mathcal{S}'$ we define the *support* of u, $\operatorname{supp} u$, to be the support of the corresponding distribution,

$$\operatorname{supp} u = \operatorname{supp} u|_{\mathcal{D}}.$$

Let $y \in \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$ and $l : \mathbb{R}^d \to \mathbb{R}^d$ linear and bijective. For a $u \in \mathcal{S}'$ we define $\check{u}, \mathcal{T}_y u, \partial^\alpha u$ and $u \circ l$ by the formulas as in Definition 2.14 but replacing "$\mathcal{D}$" everywhere by "$\mathcal{S}$".

Let $\sigma \in C_{\mathrm{p}}^{\infty}$. By Lemma 14.7 we have $\sigma\varphi \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$ and the function $\mathcal{S} \to \mathcal{S}$, $\varphi \mapsto \sigma\varphi$ is continuous. Thus for each $u \in \mathcal{S}'$ the function $\varphi \mapsto u(\sigma\varphi)$ is a tempered distribution.

For $\sigma \in C_{\mathrm{p}}^{\infty}$ and $u \in \mathcal{S}'$ we define $\sigma u \in \mathcal{S}'$ by

$$\sigma u(\varphi) = u(\sigma\varphi) \qquad (\varphi \in \mathcal{S}).$$

Again, it is straightforward to check that $\check{u}, \mathcal{T}_y u, \partial^\alpha u$ and $u \circ l$ are all in $\mathcal{S}'$ and moreover that with $\iota : \mathcal{S}' \to \mathcal{D}'$ being the embedding function of $\mathcal{S}'$ into $\mathcal{D}'$,

$$\begin{aligned}
&\iota(\check{u}) = \iota(u)\check{}, \\
&\iota(\mathcal{T}_y u) = \mathcal{T}_y \iota(u), \\
&\iota(\partial^\alpha u) = \partial^\alpha \iota(u), \\
&\iota(\sigma u) = \sigma\iota(u), \\
&\iota(u \circ l) = \iota(u) \circ l.
\end{aligned}$$

15.4. Let $y \in \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$, $\sigma \in C_{\mathrm{p}}^{\infty}$ and $l : \mathbb{R}^d \to \mathbb{R}^d$ be a linear bijection. Observe that as in 4.4 the operations $\check{}\,$, $\mathcal{T}_y, \partial^\alpha$, multiplication by σ and composition with l, i.e.,

$$\begin{array}{llll}
\mathcal{S} \to \mathcal{S}, & \varphi \mapsto \check{\varphi}, & \mathcal{S}' \to \mathcal{S}', & u \mapsto \check{u}, \\
\mathcal{S} \to \mathcal{S}, & \varphi \mapsto \mathcal{T}_y\varphi, & \mathcal{S}' \to \mathcal{S}', & u \mapsto \mathcal{T}_y u, \\
\mathcal{S} \to \mathcal{S}, & \varphi \mapsto \partial^\alpha\varphi, & \mathcal{S}' \to \mathcal{S}', & u \mapsto \partial^\alpha u, \\
\mathcal{S} \to \mathcal{S}, & \varphi \mapsto \sigma\varphi, & \mathcal{S}' \to \mathcal{S}', & u \mapsto \sigma u, \\
\mathcal{S} \to \mathcal{S}, & \varphi \mapsto \varphi \circ l, & \mathcal{S}' \to \mathcal{S}', & u \mapsto u \circ l,
\end{array}$$

are continuous. We leave it to the reader to check this (as $\mathcal{S}$ is metrizable, it is sufficient to show sequential continuity for the operations $\mathcal{S} \to \mathcal{S}$).

15.5 (Convention)**.** Following 10.9, we will identify elements of $\mathcal{S}'$ with their corresponding distributions and elements of $\mathcal{E}'$ with their corresponding tempered distribution, e.g., for $u \in \mathcal{S}'$ we will also write "u" for "$u|_{\mathcal{D}}$", and, for a $v \in \mathcal{E}'$ we will also write "v" instead of "$v|_{\mathcal{S}}$".

Let us consider more examples of tempered distributions, than only the compactly supported distributions. Each locally integrable function defines a distribution, but it might not be tempered. Consider for example the function g given by $g(x) = e^{|x|^2}$ for $x \in \mathbb{R}^d$. In Theorem 15.6 (b) we see that certain "tempered" functions define tempered distributions.

Exercise **15.A.** Verify that $x \mapsto e^{|x|^2}$ is not in $\mathcal{S}'$.

Theorem 15.6. (a) *Let $p \in [1, \infty]$. L^p is continuously embedded in $\mathcal{S}'$,*

$$L^p \hookrightarrow \mathcal{S}'.$$

(b) *Let $h : \mathbb{R}^d \to \mathbb{F}$ be Borel measurable. If there exist $k \in \mathbb{N}_0$ and $p \in [1, \infty]$ such that $(1 + |\cdot|)^{-k} h \in L^p$, then $h \in \mathcal{S}'$.*

Proof. (a) Let $g \in L^p$. Let $q \in [1, \infty]$ and $m \in \mathbb{N}$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad qm > d.$$

By Hölder's inequality and Lemma 14.14 there exists a $C > 0$ such that

$$|\langle g, \varphi \rangle| \le \int |g\varphi| \le \|g\|_{L^p} \|\varphi\|_{L^q} \le C \|g\|_{L^p} \|\varphi\|_{m,\mathcal{S}} \qquad (\varphi \in \mathcal{S}).$$

The injectivity follows from the injectivity of $L^p \to \mathcal{D}'$ and as $\mathcal{D}$ is dense in $\mathcal{S}$.

(b) follows by (a) as $(1 + |\cdot|)^k \in C_{\mathrm{p}}^\infty$ and multiplication with such a function is an operation $\mathcal{S}' \to \mathcal{S}'$. □

Corollary 15.7. *Let $k \in \mathbb{N}_0$ and $p \in [1, \infty]$. Then*

$$W^{k,p} \hookrightarrow \mathcal{S}'.$$

Proof. This follows by Theorem 15.6 because $W^{k,p} \hookrightarrow L^p$. □

Observe that Theorem 15.6 implies that every Borel measurable $h : \mathbb{R}^d \to \mathbb{F}$ of at most polynomial growth defines a tempered distribution.

Exercise **15.B.** Let $f : \mathbb{R} \to \mathbb{F}$ be given by

$$f(x) = \begin{cases} \log|x| & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f and u defined by

$$u(\varphi) := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{\varphi(x)}{x} \, \mathrm{d}x \qquad (\varphi \in \mathcal{S}(\mathbb{R})),$$

define tempered distributions. (Hint: Exercise 2.E.)

Exercise **15.C.** Show that if $f \in L^p$ for some $p \in [1, \infty]$, then there exists an $m \in \mathbb{N}_0$ such that $(1 + |\cdot|)^{-m} f \in L^1$.

Conclude the following. If $h : \mathbb{R}^d \to \mathbb{F}$ is Borel measurable and there there exist $k \in \mathbb{N}_0$ and $p \in [1, \infty]$ such that $(1 + |\cdot|)^{-k} h \in L^p$, then there exists an $m \in \mathbb{N}_0$ such that $(1 + |\cdot|)^{-m} h \in L^1$.

Exercise **15.D.** Let $C^m_{\mathrm{p},k}$ be the Banach space equipped with the norm $\mathfrak{q}_{m,k}$ as in Exercise 14.E. Prove that for all $k \in \mathbb{N}_0$ and all $m \in \mathbb{N}_0 \cup \{\infty\}$

$$C^m_{\mathrm{p},k} \hookrightarrow \mathcal{S}'.$$

Exercise **15.E.** Let $p \in [1, \infty]$ and $k \in \mathbb{N}_0$. Define $L^p_{\mathrm{p},k}$ to be the space of (equivalence classes of) Borel measurable functions $g : \mathbb{R}^d \to \mathbb{F}$ such that $\mathfrak{n}_{p,k}(g) := \|(1+|\cdot|)^{-k} g\|_{L^p} < \infty$. Show that $\mathfrak{n}_{p,k}$ is a norm such that $L^p_{\mathrm{p},k}$ equipped with this norm is a Banach space and

$$L^p_{\mathrm{p},k} \hookrightarrow \mathcal{S}'.$$

Conclude by Exercise 15.C that there exists an $m \in \mathbb{N}_0$ such that

$$L^p_{\mathrm{p},k} \hookrightarrow L^1_{\mathrm{p},m}.$$

For Radon measures there exist analogous statements to Theorem 15.6:

Theorem 15.8. (a) *$\mathcal{M}$ is continuously embedded in $\mathcal{S}'$,*

$$\mathcal{M} \hookrightarrow \mathcal{S}'.$$

(b) *$\mu : \mathrm{Borel}(\mathbb{R}^d) \to [0, \infty]$ is a measure and there exists a $k \in \mathbb{N}_0$ such that*

$$\int (1 + |\cdot|)^{-k} \, \mathrm{d}\mu < \infty,$$

then μ defines a tempered distribution.

Proof. See Exercise 15.F. □

Exercise **15.F.** Prove Theorem 15.8.

Exercise **15.G.** (a) Prove that for any $u \in \mathcal{S}'(\mathbb{R})$, the following are equivalent

(1) $\partial u = 0$,

(2) there exists a $c \in \mathbb{F}$ such that $u = c\mathbb{1}$.

(Hint: Let $\psi \in \mathcal{S}$ with $\int \psi = 1$. Prove that for all $\varphi \in \mathcal{S}$, $\varphi - (\int \varphi)\psi$ has a primitive in $\mathcal{S}$.)

(b) $\diamond$ Prove that for all $u \in \mathcal{S}'(\mathbb{R})$ there exists a $v \in \mathcal{S}'(\mathbb{R})$ such that $u = \partial v$.

Exercise **15.H.** Show that $\sum_{n=1}^{\infty} n\delta_n \in \mathcal{S}'$.

Example 15.9. There exist Borel measurable functions $g : \mathbb{R}^d \to \mathbb{F}$ which define tempered distributions but are not as in Theorem 15.6 (b), that is, for which for all $k \in \mathbb{N}_0$ and $p \in [1, \infty]$,

$$\|(1 + |\cdot|)^{-k} h\|_{L^p} = \infty.$$

Consider for example $d = 1$ and $h = g'$ where g defined by

$$g(x) = \sin(e^{x^2}).$$

Then $g \in L^\infty$ and thus g and its derivative h are in $\mathcal{S}'$, but $h(x) = 2xe^{x^2}\cos(e^{x^2})$ for all $x \in \mathbb{R}$ so that h is not of the form as in Theorem 15.6 (b).

Theorem 15.10. (a) *Let $\mathcal{U} \subset \mathcal{S}'$ and assume*

$$\sup_{u \in \mathcal{U}} |u(\varphi)| < \infty \qquad (\varphi \in \mathcal{S}).$$

Then, there exist $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|u(\varphi)| \le C\|\varphi\|_{m,\mathcal{S}} \qquad (\varphi \in \mathcal{S}, u \in \mathcal{U}).$$

(b) *The space $\mathcal{S}'$ is weak* sequentially complete.*

(c) *The pairing map $\mathcal{S}' \times \mathcal{S} \to \mathbb{F}$, $(u, \varphi) \mapsto u(\varphi) = \langle u, \varphi \rangle$ is sequentially continuous.*

Consequently (by Lemma 14.7), the product map $\mathcal{S}' \times \mathcal{S} \to \mathcal{S}'$, $(u, \varphi) \mapsto \varphi u$ is sequentially continuous.

Proof. Both statements (a) and (b) follow by the arguments as in the proofs of Theorem 4.24 and Theorem 4.26 with "$\mathcal{D}(\Omega)$" and "$\mathcal{D}_K(\Omega)$" both replaced by "$\mathcal{S}$". (c) follows from (a) similarly as the proof of Proposition 4.25. □

15.11. We show that (a) the relative topology of $\mathcal{D}$ as a subspace of $\mathcal{S}$ is different from the topology on $\mathcal{D}$; (b) the relative topology of $\mathcal{S}$ as a subspace of $\mathcal{E}$ is different from the topology on $\mathcal{E}$; (c) the relative topology of $\mathcal{S}'$ as a subset of $\mathcal{D}'$ is not equal to the topology of $\mathcal{S}'$ and (d) the relative topology of $\mathcal{E}'$ as a subset of $\mathcal{S}'$ is not equal to the topology of $\mathcal{E}'$.

We consider $d = 1$ for convenience.

(a) Let f_n be the Gaussian function given by $f_n(x) = e^{-n(1+x^2)}$ for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. By (14.2) $\lim_{n\to\infty} \|f_n\|_{k,\mathcal{S}} = 0$ for all $k \in \mathbb{N}_0$. Let $\phi \in \mathcal{D}$ be nonzero and define $\psi_n = f_n \mathcal{T}_n \phi$. Observe that $\psi_n \in \mathcal{D}$. By Lemma 14.7 $\|\psi_n\|_{k,\mathcal{S}} \le \|\phi\|_{C^k} \|f_n\|_{k,\mathcal{S}} \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}_0$. Hence $\psi_n \to 0$ in $\mathcal{S}$. However, by Theorem 4.11 $(\psi_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{D}$ as there is no compact set which contains the support of ψ_n for all $n \in \mathbb{N}$.

(b) Let $\phi_n = \frac{1}{n}\mathcal{T}_n\phi$ for $n \in \mathbb{N}$, where $\phi \in C_c^\infty(\mathbb{R}^d, [0,1])$ and $\phi(0) = 1$. Then $\phi_n \to 0$ in $\mathcal{E}$, but $(\phi_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{S}$ as

$$\sup_{x\in\mathbb{R}^d} (1 + |x|)|\phi_n(x)| \geq 1 \qquad (n \in \mathbb{N}_0).$$

(c) $e^{n^2}\delta_n \to 0$ in $\mathcal{D}'$ but not in $\mathcal{S}'$ (and not in $\mathcal{E}'$), indeed, for $\varphi \in \mathcal{S}$ the Gaussian function $\varphi(x) = e^{-x^2}$ for $x \in \mathbb{R}$, we have $e^{n^2}\delta_n(\varphi) = 1$ for all $n \in \mathbb{N}$.

(d) $\delta_n \to 0$ in $\mathcal{S}'$ but not in $\mathcal{E}'$.

15.12. Observe that by Lemma 14.7 the following holds. If $\sigma_n \in C_{\mathrm{p}}^\infty$ for all $n \in \mathbb{N}$ and $\sigma \in C_{\mathrm{p}}^\infty$ and for all $m \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $\mathfrak{q}_{m,k}(\sigma_n - \sigma) \xrightarrow{n\to\infty} 0$, then

$$\sigma_n\varphi \xrightarrow{n\to\infty} \sigma\varphi \text{ in } \mathcal{S} \qquad (\varphi \in \mathcal{S}),$$
$$\sigma_n u \xrightarrow{n\to\infty} \sigma u \text{ in } \mathcal{S}' \qquad (u \in \mathcal{S}').$$

16 The Fourier transformation

In this section we consider $\mathbb{R}^d$ to be the space on which our functions and distributions are defined, we therefore leave out the notation "$(\mathbb{R}^d)$" in the considered function spaces or spaces of distributions.

Definition 16.1 (Fourier transform of a function)**.** Let $f : \mathbb{R}^d \to \mathbb{F}$ be an integrable function. The *Fourier transform* of f, $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi\mathrm{i}\langle x,\xi\rangle} f(x) \, \mathrm{d}x, \tag{16.1}$$

where $\langle x, \xi\rangle$ is the inner product on $\mathbb{R}^d$ (the notation $\langle\cdot,\cdot\rangle$ is of course also used as the pairing between distributions, but we trust that there will be no confusing arising).

In case g is another integrable function that equals f almost everywhere, then $\widehat{f} = \widehat{g}$. This enables us to define the Fourier transform of an element of L^1 as the Fourier transform of one of its representatives and we will use the formula (16.1) also for $f \in L^1$.

Example 16.2. Let $a, b \in \mathbb{R}, a < b$. The Fourier transform of the indicator function $\mathbb{1}_{[a,b]}$ is given by

$$\widehat{\mathbb{1}_{[a,b]}}(\xi) = \begin{cases} \frac{e^{-2\pi\mathrm{i}a\xi} - e^{-2\pi\mathrm{i}b\xi}}{2\pi\mathrm{i}\xi} & \xi \in \mathbb{R}\setminus\{0\}, \\ b - a & \xi = 0. \end{cases}$$

Observe that $\widehat{\mathbb{1}_{[a,b]}}$ is continuous.

Example 16.3. The Fourier transform of the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \max(1 - |x|, 0) \qquad (x \in \mathbb{R}),$$

is given by

$$\widehat{f}(\xi) = \begin{cases} \frac{(1-\cos 2\pi\xi)}{2\pi^2\xi^2} & \xi \neq 0, \\ 1 & \xi = 0. \end{cases}$$

Exercise **16.A.** (a) Verify that $\widehat{\mathbb{1}_{[a,b]}}$ and $\widehat{f}$ are given by the formulas in Example 16.2 and Example 16.3.

(b) Check that they are both continuous (at 0).

(c) The function sinc (also found under the name "cardinal sinus") is defined by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

Prove that

$$\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}}(\xi) = \operatorname{sinc} \pi\xi, \qquad \widehat{f}(\xi) = \operatorname{sinc}^2 \pi\xi.$$

The Fourier transform of an integrable function is an element of $C_0(\mathbb{R}^d, \mathbb{F})$ (Definition 2.29), see Theorem 16.6. In order to prove that statement, we first introduce some auxiliary lemmas.

Lemma 16.4. *Let $f \in L^1$. Then for all $a \in \mathbb{R}^d$*

$$\lim_{x \to a} \|\mathcal{T}_a f - \mathcal{T}_x f\|_{L^1} = 0.$$

Proof. For $\varphi \in C_c$ it holds that $\lim_{x\to a} \|\mathcal{T}_a\varphi - \mathcal{T}_x\varphi\|_{L^\infty} = 0$ by uniform continuity, and therefore $\lim_{x\to a} \|\mathcal{T}_a\varphi - \mathcal{T}_x\varphi\|_{L^1} = 0$. As C_c is dense in L^1, by a 3ε argument one can finish the proof. ☐

Lemma 16.5 (Lemma of Riemann–Lebesgue)**.** *Let $g \in L^1(\mathbb{R})$. Then*

$$|\widehat{g}(a)| \leq \frac{1}{2}\|g - \mathcal{T}_{\frac{1}{2a}} g\|_{L^1} \qquad (a \in \mathbb{R}, a \neq 0).$$

Proof. Let $a \in \mathbb{R}$, $a \neq 0$. As $e^{\pi \mathrm{i}} = -1$ we have

$$\int_{\mathbb{R}} g(x) e^{-2\pi \mathrm{i} a x} \, \mathrm{d}x = \int_{\mathbb{R}} g(x - \frac{1}{2a}) e^{-2\pi \mathrm{i} a (x - \frac{1}{2a})} \, \mathrm{d}x = -\int_{\mathbb{R}} \mathcal{T}_{\frac{1}{2a}} g(x) e^{-2\pi \mathrm{i} a x} \, \mathrm{d}x.$$

Therefore

$$\int_{\mathbb{R}} g(x) e^{-2\pi \mathrm{i} a x} \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}} [g(x) - \mathcal{T}_{\frac{1}{2a}} g(x)] e^{-2\pi \mathrm{i} a x} \, \mathrm{d}x,$$

so that the desired inequality follows. ☐

Theorem 16.6. *If $f \in L^1$, then $\widehat{f} \in C_0(\mathbb{R}^d, \mathbb{C})$ and*

$$\|\widehat{f}\|_{L^\infty} \le \|f\|_{L^1}.$$

Proof. The norm estimate is straightforward. The continuity follows by Lebesgue's dominated convergence theorem, so that $f \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{C})$ for all $f \in L^1$. That $\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0$ follows as by Lemma 16.5

$$|\widehat{f}(\xi)| \le \tfrac{1}{2}\|f - \mathcal{T}_{\frac{1}{2\xi_i}e_i} f\|_{L^1} \qquad (i \in \{1, \dots, d\}, \xi \in \mathbb{R}^d),$$

and $\|f - \mathcal{T}_{\frac{1}{2\xi_i}e_i} f\|_{L^1}$ converges to zero as $|\xi_i| \to \infty$, for each $i \in \{1, \dots, d\}$, by Lemma 16.4. □

Definition 16.7 (Fourier transformation)**.** We write $\mathcal{F}$ for the linear function $L^1 \to C_0(\mathbb{R}^d, \mathbb{C})$, $f \mapsto \widehat{f}$ and call this map the *Fourier transformation.*

The Fourier transformation turns out very useful as it turns certain operations into other operations, see for example Theorem 16.9 and Theorem 16.13.

16.8 (Notation)**.** The symbol $\mathfrak{x}$ is used to denote the identity map $\mathbb{R}^d \to \mathbb{R}^d$.

By substitution rules for integration we obtain the following.

Theorem 16.9. *Let $f \in L^1$.*

(a) *For $y \in \mathbb{R}^d$*

$$\mathcal{F}(\mathcal{T}_y f) = e^{-2\pi \mathrm{i}\langle \mathfrak{x}, y\rangle} \widehat{f}, \qquad \mathcal{T}_y \widehat{f} = \mathcal{F}(e^{2\pi \mathrm{i}\langle \mathfrak{x}, y\rangle} f). \tag{16.2}$$

(b) *Let $l : \mathbb{R}^d \to \mathbb{R}^d$ be linear and bijective. Then*

$$\mathcal{F}(f \circ l) = \frac{1}{|\det l|} \widehat{f} \circ l_*,$$

where l_* is the transpose of l^{-1}, which means that $\langle l^{-1}y, \xi\rangle = \langle y, l_*(\xi)\rangle$ for all $x, \xi \in \mathbb{R}^d$.

In particular, for $\lambda \in \mathbb{R} \setminus \{0\}$ (for the notation see 14.9)

$$\mathcal{F}(l_\lambda f) = \mathcal{F}(f(\lambda \mathfrak{x})) = |\lambda|^{-d} l_{\frac{1}{\lambda}} \widehat{f}.$$

Exercise **16.B.** Verify the statements of Theorem 16.9.

Theorem 16.10. *Let $f, g \in L^1$. Then $f\widehat{g}, \widehat{f}g \in L^1$ and*

$$\int f\widehat{g} = \int \widehat{f}g. \tag{16.3}$$

Proof. The integrability follows by Theorem 16.6. The identity follows by Fubini's theorem (Exercise 16.C). $\square$

Exercise **16.C.** Check that (16.3) holds.

Definition 16.11. For $f \in L^1_{\text{loc}}(\mathbb{R})$ we say that a function $g : \mathbb{R} \to \mathbb{R}$ is an *indefinite integral* of f if

$$g(b) - g(a) = \int_a^b f \qquad (a, b \in \mathbb{R}, a < b).$$

Observe that any continuously differentiable function g is the indefinite integral of its derivative g'.

Theorem 16.12. *Let $g \in L^1(\mathbb{R})$.*

(a) *If $\mathfrak{x}g \in L^1(\mathbb{R})$, then $\widehat{g}$ is continuously differentiable and*

$$\widehat{g}' = \mathcal{F}(-2\pi \mathrm{i}\mathfrak{x}g). \tag{16.4}$$

(b) *If g is an indefinite integral of a function $h \in L^1(\mathbb{R})$, then $\widehat{h} = 2\pi\mathrm{i}\mathfrak{x}\widehat{g}$.*

In particular, if g is continuously differentiable and $g' \in L^1(\mathbb{R})$, then $\mathcal{F}(g') = 2\pi\mathrm{i}\mathfrak{x}\widehat{g}$.

Proof. (a) Let $a, b \in \mathbb{R}$, $a < b$. Then, by Theorem 16.10 (see also Example 16.2)

$$\begin{aligned}
\int_a^b \mathcal{F}(-2\pi\mathrm{i}\mathfrak{x}g) &= \int_{\mathbb{R}} \mathcal{F}(-2\pi\mathrm{i}\mathfrak{x}g)\mathbb{1}_{[a,b]} \\
&= \int_{\mathbb{R}} -2\pi\mathrm{i}xg(x)\mathcal{F}(\mathbb{1}_{[a,b]})(x) \,\mathrm{d}x \\
&= \int_{\mathbb{R}} -2\pi\mathrm{i}xg(x)\frac{e^{-2\pi\mathrm{i}bx} - e^{-2\pi\mathrm{i}ax}}{-2\pi\mathrm{i}x} \,\mathrm{d}x \\
&= \int_{\mathbb{R}} g(x)(e^{-2\pi\mathrm{i}bx} - e^{-2\pi\mathrm{i}ax}) \,\mathrm{d}x = \widehat{g}(b) - \widehat{g}(a).
\end{aligned}$$

As the Fourier transform of an integrable function is continuous, we conclude that $\widehat{g}$ is continuously differentiable with derivative given by (16.4).

(b) For $\xi \in \mathbb{R}$ we have (integration by parts)

$$\begin{aligned}
\widehat{h}(\xi) - 2\pi\mathrm{i}\xi\widehat{g}(\xi) &= \lim_{N\to\infty} \int_{-N}^N h(x)e^{-2\pi\mathrm{i}\xi x} + g(x)(-2\pi\mathrm{i}\xi)e^{-2\pi\mathrm{i}\xi x} \,\mathrm{d}x \\
&= \lim_{N\to\infty} (g(N)e^{-2\pi\mathrm{i}N\xi} - g(-N)e^{2\pi\mathrm{i}N\xi}).
\end{aligned}$$

Therefore it suffices to show that $\lim_{|x|\to\infty} g(x) = 0$. As g is the indefinite integral of h, which means for example that $g(y) = g(0) + \int_0^y h(x) \,\mathrm{d}x$, both $\lim_{y\to\infty} g(y)$ and $\lim_{y\to-\infty} g(y)$ exist. By the integrability of g, these limits need to be equal to zero. $\square$

We can so to say 'apply' Theorem 16.12 to any of the directions in $\mathbb{R}^d$, to obtain the following.

Theorem 16.13. *Let $k \in \mathbb{N}_0$ and $f \in L^1$.*

(a) *If $\mathfrak{x}^\beta f \in L^1$ for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \le k$, then $\widehat{f} \in C^k$ and*

$$\partial^\beta \widehat{f} = \mathcal{F}((-2\pi \mathrm{i}\mathfrak{x})^\beta f) \qquad (\beta \in \mathbb{N}_0^d, |\beta| \le k).$$

(b) *If $f \in C^k$ and $\partial^\beta f \in L^1$ for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \le k$, then $\mathfrak{x}^\beta \widehat{f} \in C_0$ and*

$$\mathcal{F}(\partial^\beta f) = (2\pi \mathrm{i}\mathfrak{x})^\beta \widehat{f} \qquad (\beta \in \mathbb{N}_0^d, |\beta| \le k).$$

Proof. By an induction argument it suffices to consider $k = 1$ and $\beta \in \mathbb{N}_0^d$ with $|\beta| = 1$. Let $j \in \{1, \dots, d\}$ be such that $\beta_j = 1$, i.e., $\beta = e_j$. For $j \in \{1, \dots, d\}$ let $\mathcal{F}_j$ denote the one-dimensional Fourier transformation acting on the j-th coordinate. That is

$$\mathcal{F}_j f(x) = \int_{\mathbb{R}} e^{-2\pi \mathrm{i} x_j y_j} f(x_1, \dots, y_j, \dots, x_d) \, \mathrm{d}y_j.$$

Then it follows for $i \ne j$ that $\mathcal{F}_j$ commutes with ∂_i and with multiplication by $\mathfrak{x}_j$, where $\mathfrak{x}_j = \langle \mathfrak{x}, e_j \rangle$;

$$\partial_i \mathcal{F}_j(f) = \mathcal{F}_j(\partial_i f), \quad \mathfrak{x}_i \mathcal{F}_j(f) = \mathcal{F}_j(\mathfrak{x}_i f) \qquad (i, j \in \{1, \dots, d\}, i \ne j). \tag{16.5}$$

(a) then follows as by Theorem 16.13 (a) the function $\xi_j \mapsto \mathcal{F}_j f(x_1, \dots, \xi_j, \dots, x_d)$ is continuously differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}\xi_j} \mathcal{F}_j f(x_1, \dots, \xi_j, \dots, x_d) = \mathcal{F}_j((-2\pi \mathrm{i}\mathfrak{x})^\beta f).$$

(b) follows also by the above commutation rules (16.5) and as by Theorem 16.13 (b)

$$\mathcal{F}_j(\partial_j f) = 2\pi \mathrm{i} \mathfrak{x}_j \widehat{f}.$$

□

As functions of rapid decay are integrable (see for example Lemma 14.13), we obtain the following corollary of Theorem 16.13.

Corollary 16.14. *Let $f : \mathbb{R}^d \to \mathbb{F}$ be measurable.*

(a) *If f is of rapid decay, then $\widehat{f}$ is smooth.*
(b) *If f is smooth and $\partial^\beta f \in L^1$ for all $\beta \in \mathbb{N}_0^d$, then $\widehat{f}$ is of rapid decay.*
(c) *If f is a Schwartz function, then $\widehat{f}$ is a Schwartz function.*

The Fourier transformation actually forms a bijection $\mathcal{S}(\mathbb{R}^d,\mathbb{C}) \to \mathcal{S}(\mathbb{R}^d,\mathbb{C})$, see Theorem 16.16. In the proof of that theorem we use that the Fourier transform of a Gaussian function is another Gaussian function:

Theorem 16.15. *Let $a > 0$, $y \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ be the Gaussian function $f(x) = e^{-a|x-y|^2}$, then $f \in L^1$ and*

$$\widehat{f}(\xi) = \left(\frac{\pi}{a}\right)^{\frac{d}{2}} e^{-\frac{\pi^2|\xi|^2}{a}} e^{-2\pi \mathrm{i}\langle y,\xi\rangle}. \tag{16.6}$$

Proof. We consider the specific case with $d = 1$, $a = 1$ and $y = 0$ and leave it to the reader to prove the general case (Exercise 16.D).

Let $g : \mathbb{R} \to \mathbb{R}$ be the Gaussian function given by $g(x) = e^{-x^2}$ for $x \in \mathbb{R}^d$. By Theorem 16.12

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{g}(\xi) = \mathcal{F}(-2\pi\mathrm{i}\mathfrak{x}e^{-\mathfrak{x}^2})(\xi) = \pi i\mathcal{F}\left(\partial e^{-\mathfrak{x}^2}\right)(\xi) = -2\pi^2\xi\widehat{g}(\xi) \qquad (\xi \in \mathbb{R}^d).$$

By 11.16 we have $\widehat{g}(0) = \int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$. Therefore

$$\widehat{g}(\xi) = \sqrt{\pi}e^{-\pi^2\xi^2} \qquad (\xi \in \mathbb{R}^d),$$

as $h = \widehat{g}$ is the unique solution to the ordinary differential equation

$$\begin{cases} h'(\xi) = -2\pi^2\xi h(\xi), \\ h(0) = \sqrt{\pi}. \end{cases}$$

□

Exercise 16.D. Prove Theorem 16.15.

Theorem 16.16. *The Fourier transformation $\mathcal{F}$ forms a linear homeomorphism $\mathcal{S}(\mathbb{R}^d,\mathbb{C}) \to \mathcal{S}(\mathbb{R}^d,\mathbb{C})$ with*

$$f(x) = \mathcal{F}(\widehat{f})(-x) = \int_{\mathbb{R}^d} \widehat{f}(\xi)e^{2\pi\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\xi \qquad (x \in \mathbb{R}^d). \tag{16.7}$$

Proof. We already know that $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^d,\mathbb{C})$ into $\mathcal{S}(\mathbb{R}^d,\mathbb{C})$, see Corollary 16.14. Let us write "$\mathcal{S}$" for "$\mathcal{S}(\mathbb{R}^d,\mathbb{C})$" here (or differently said, assume $\mathbb{F} = \mathbb{C}$ for this proof). Let us first prove that $\mathcal{F}$ forms a bijection $\mathcal{S} \to \mathcal{S}$ by proving (16.7). Let $f \in \mathcal{S}$ and $x \in \mathbb{R}^d$. Let $g = \mathcal{T}_{-x}f$. Then $\widehat{g} = \widehat{f}e^{2\pi\mathrm{i}\langle x,\mathfrak{x}\rangle}$ by Theorem 16.9. Therefore, it is sufficient to show (16.7) for $x = 0$, which means it is sufficient to show

$$f(0) = \int_{\mathbb{R}^d} \widehat{f}(\xi)\,\mathrm{d}\xi.$$

Let h_t be as in Example 11.15, i.e.,

$$h_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{1}{4t}|x|^2} \qquad (x \in \mathbb{R}^d).$$

Let $g_t = \widehat{h_t}$. By Theorem 16.15 we have (take $a = t\pi^2 t$)

$$g_t(x) = e^{-4\pi^2 t|x|^2} \qquad (x \in \mathbb{R}^d),$$

and $\widehat{g_t} = h_t$ so that $\widehat{\widehat{h_t}} = h_t$ and (16.7) holds with $f = h_t$ for any $t > 0$. By Theorem 16.10

$$\int_{\mathbb{R}^d} f(x)h_t(x) \, \mathrm{d}x = \int_{\mathbb{R}^d} \widehat{f}(\xi)g_t(\xi) \, \mathrm{d}\xi.$$

As f is continuous and bounded, the left-hand side converges to $f(0)$ as $t \downarrow 0$ by (11.8). As $\widehat{f}$ is an element of $\mathcal{S}$ it is integrable, therefore by Lebesgue's dominated convergence theorem we have that the right-hand side converges to $\int_{\mathbb{R}^d} \widehat{f}(\xi) \, \mathrm{d}\xi$ as $t \downarrow 0$, because $g_t(\xi) \uparrow 1$ as $t \downarrow 0$ for all $\xi \in \mathbb{R}^d$. This proves (16.7).

Now let us prove that $\mathcal{F}$ defines a homeomorphism $F : \mathcal{S} \to \mathcal{S}$. As the inverse is given by the composition of F with the reflection operator $\mathcal{R}$ defined in 2.12, i.e., $F^{-1} = \mathcal{R}F$, it is sufficient to show continuity of F. Let $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$. It is sufficient (see for example 14.4) to show that there exists an $n \in \mathbb{N}_0$ and a $C > 0$ such that

$$\|(1 + |\cdot|^2)^k \partial^\alpha \widehat{f}\|_{L^\infty} \le C\|f\|_{n,\mathcal{S}} \qquad (f \in \mathcal{S}). \tag{16.8}$$

Let us first observe the following. As L^1 is continuously embedded in $\mathcal{S}$ (Theorem 15.6 (a)), there exists an $m \in \mathbb{N}_0$ and a $\mathfrak{C} > 0$ such that

$$\|\widehat{f}\|_{L^\infty} \le \|f\|_{L^1} \le \mathfrak{C}\|f\|_{m,\mathcal{S}} \qquad (f \in \mathcal{S}). \tag{16.9}$$

By Theorem 16.13

$$(1 + |\xi|^2)^k \partial^\alpha \widehat{f}(\xi) = \mathcal{F}\Big(\Big(1 - \frac{\Delta}{4\pi^2}\Big)^k \Big((2\pi \mathrm{i}\mathfrak{x})^\alpha f\Big)\Big)(\xi) \qquad (\xi \in \mathbb{R}^d).$$

Therefore by (16.9)

$$\|(1 + |\cdot|^2)^k |\partial^\alpha \widehat{f}\|_{L^\infty} \le \mathfrak{C} \Big\|\Big(1 - \frac{\Delta}{4\pi^2}\Big)^k \Big((2\pi \mathrm{i}\mathfrak{x})^\alpha f\Big)\Big\|_{m,\mathcal{S}}$$

As multiplication with $(2\pi \mathrm{i}\mathfrak{x})^\alpha$ and the operation $\Big(1 - \frac{\Delta}{4\pi^2}\Big)^k$ are continuous as functions $\mathcal{S} \to \mathcal{S}$, see 15.4, there exists a $C > 0$ and $n \in \mathbb{N}_0$ such that (16.8). □

Actually, the previous theorem extends in the following way, in the sense that the Fourier transformation is a bijection on a larger space.

Theorem 16.17. *Suppose that* $f \in L^1(\mathbb{R}^d, \mathbb{C})$ *is such that also* $\widehat{f} \in L^1(\mathbb{R}^d, \mathbb{C})$. *Then*

$$f(x) = \mathcal{F}(\widehat{f})(-x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{2\pi \mathrm{i} \langle x, \xi \rangle} \, \mathrm{d}\xi \qquad \textit{for almost all } x \in \mathbb{R}^d. \tag{16.10}$$

Consequently, $\mathcal{F}$ *also forms a bijection*

$$\{f \in L^1(\mathbb{R}^d, \mathbb{C}) : \widehat{f} \in L^1(\mathbb{R}^d, \mathbb{C})\} \to \{f \in L^1(\mathbb{R}^d, \mathbb{C}) : \widehat{f} \in L^1(\mathbb{R}^d, \mathbb{C})\}.$$

Proof. For all $\varphi \in \mathcal{S}$ we have by Theorem 16.10 and Theorem 16.16.

$$\int_{\mathbb{R}^d} \mathcal{F}(\widehat{f})\varphi = \int_{\mathbb{R}^d} \widehat{f}\widehat{\varphi} = \int_{\mathbb{R}^d} f\mathcal{F}(\widehat{\varphi}) = \int_{\mathbb{R}^d} f\check{\varphi} = \int_{\mathbb{R}^d} \check{f}\varphi.$$

Therefore, by Lemma 2.9, we have $\mathcal{F}(\widehat{f}) = \check{f}$ almost everywhere. □

16.18. Observe that by Theorem 16.6 the set $\{f \in L^1 : \widehat{f} \in L^1\}$ is included in C_b (where C_b is viewed as subset of L^∞).

Example 16.19. Let f be as in Example 16.3. Then $|\widehat{f}(x)| \le \xi^{-2}$, so that $\widehat{f} \in L^1(\mathbb{R})$. By Theorem 16.17 $\widehat{\widehat{f}}(a) = f(-a)$ for $a \in \mathbb{R}$. By taking $a = 0$ we obtain

$$\int_{\mathbb{R}} \frac{(1 - \cos 2\pi\xi)}{2\pi^2\xi^2} \, \mathrm{d}\xi = 1,$$

and thus

$$\int_{\mathbb{R}} \frac{(1 - \cos x)}{x^2} \, \mathrm{d}x = \pi,$$

We mentioned $\widehat{f}$ in Example 16.19 but not $\widehat{\mathbb{1}_{[-1,1]}}$ as it is not in L^1. In Example 16.28 we come back to this.

Exercise **16.E.** Prove

$$\int_{\mathbb{R}} \frac{\cos(xs)}{1 + x^2} \, \mathrm{d}x = \pi e^{-|s|} \qquad (s \in \mathbb{R}).$$

Exercise **16.F.** Calculate the Fourier transform of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^{-|x|}$ for $x \in \mathbb{R}$.

16.20. For $f \in L^1(\mathbb{R}^d, \mathbb{C})$ and $\varphi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ we have by Theorem 16.10

$$\int \widehat{f}\varphi = \int f\widehat{\varphi}.$$

So that with the notation of 2.6, we have

$$u_{\widehat{f}}(\varphi) = u_f(\widehat{\varphi}).$$

As the Fourier transformation is a continuous function $\mathcal{S}(\mathbb{R}^d,\mathbb{C}) \to \mathcal{S}(\mathbb{R}^d,\mathbb{C})$ it is natural to define the Fourier transform of u to be the tempered distribution $\varphi \mapsto u(\widehat{\varphi})$. We give the definition in Definition 16.22, but first discuss the situation for $\mathbb{F} = \mathbb{R}$. As in that case, if we have a $u \in \mathcal{S}'(\mathbb{R}^d,\mathbb{R})$, that is a continuous linear $u : \mathcal{S}(\mathbb{R}^d,\mathbb{R}) \to \mathbb{R}$, then we a priori are not able to pair u with $\widehat{\varphi}$ for a $\varphi \in \mathcal{S}(\mathbb{R}^d,\mathbb{R})$ as the Fourier transform of φ might attain non-real values (take for example a Gaussian function as in Theorem 16.15 with $y \neq 0$).

Let us show how we overcome this situation. First of all, let us assume that u is represented by a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$. Observe that for any Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C})$ the functions $\Re\varphi, \Im\varphi$ are Schwartz functions, where

$$(\Re\varphi)(x) = \Re(\varphi(x)), \quad (\Im\varphi)(x) = \Im(\varphi(x)) \qquad (x \in \mathbb{R}^d),$$

and $\Re a$ and $\Im a$ are the real and imaginary part of a, for $a \in \mathbb{C}$. Let us for the moment be extra careful and write g for the locally integrable function $\mathbb{R}^d \to \mathbb{C}$ given by $g(x) = f(x)$ for $\mathbb{R}^d$. We can pair g with Schwartz functions, and have

$$\langle g, \varphi\rangle = \int g\varphi = \int f\Re\varphi + \mathrm{i}\int f\Im\varphi = \langle f, \Re\varphi\rangle + \mathrm{i}\langle f, \Im\varphi\rangle \qquad (\varphi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C})).$$

By this identity it is clear that g and therefore f corresponds to a tempered distribution in $\mathcal{S}'(\mathbb{R}^d,\mathbb{C})$. We can use the above identity to generalise this to general tempered distributions:

Definition 16.21. For any $u \in \mathcal{S}'(\mathbb{R}^d,\mathbb{R})$ we define its *complex extension* $u_{\mathbb{C}} \in \mathcal{S}'(\mathbb{R}^d,\mathbb{C})$ by

$$u_{\mathbb{C}}(\varphi) = u(\Re\varphi) + \mathrm{i}u(\Im\varphi) \qquad (\varphi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C})).$$

As it is common to identify the function $f : \mathbb{R}^d \to \mathbb{R}$ with the function $g : \mathbb{R}^d \to \mathbb{C}$ with $g(x) = f(x)$ for all $x \in \mathbb{R}^d$, we identify a $u \in \mathcal{S}'(\mathbb{R}^d,\mathbb{R})$ with $u_{\mathbb{C}}$ in the sense if "$u(\varphi)$" is written for some $\varphi \in \mathcal{S}(\mathbb{R}^d,\mathbb{C})$, then it is interpreted as "$u_{\mathbb{C}}(\varphi)$".

Definition 16.22. Let $u \in \mathcal{S}'$. We define the *Fourier transform* of u, $\widehat{u}$ by

$$\widehat{u}(\varphi) = u(\widehat{\varphi}) \qquad (\varphi \in \mathcal{S}).$$

(As mentioned in Definition 16.21 for $\mathbb{F} = \mathbb{R}$ we interpret "$u(\widehat{\varphi})$" to be "$u_{\mathbb{C}}(\widehat{\varphi})$".)

From here on we write $\mathcal{F}$ for the map $\mathcal{S}' \to \mathcal{S}'$, $u \mapsto \widehat{u}$.

Example 16.23. The function $\mathbb{1}$ represents a tempered distribution, and so does δ_0. We calculate their Fourier transforms. For $\varphi \in \mathcal{S}$ we have

$$\begin{aligned} \langle \widehat{\delta_0}, \varphi\rangle = \delta_0(\widehat{\varphi}) = \widehat{\varphi}(0) = \int \varphi = \langle \mathbb{1}, \varphi\rangle, \\ \langle \widehat{\mathbb{1}}, \varphi\rangle = \int \widehat{\varphi} = \varphi(0) = \langle \delta_0, \varphi\rangle, \end{aligned}$$

where we used the inversion formula in the second line. Hence

$$\widehat{\delta_0} = \mathbb{1}, \qquad \widehat{\mathbb{1}} = \delta_0.$$

Exercise **16.G.** Let $\mu \in \mathcal{M}(\mathbb{R}^d, \mathbb{C})$. It is customary to define the Fourier transform of μ to be the function $\widehat{u}$ given by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi \mathrm{i}\langle x,\xi\rangle} \, \mathrm{d}\mu(x) \qquad (\xi \in \mathbb{R}^d).$$

Show that this definition is consistent with the definition of the Fourier transform of u_μ, that is, of μ as a tempered distribution. Moreover, prove that $\widehat{u}$ is bounded and uniformly continuous.

The following theorem is a consequence of Theorem 16.9, Theorem 16.13 and Theorem 16.16.

Theorem 16.24. *The Fourier transformation $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}'(\mathbb{R}^d, \mathbb{C}), u \mapsto \widehat{u}$ is a linear homeomorphism. Moreover, with $\mathcal{R}$ the reflection operator defined as in 2.12*

$$\mathcal{F}^{-1} = \mathcal{F}\mathcal{R} = \mathcal{R}\mathcal{F},$$

and for $u \in \mathcal{S}'$, $\beta \in \mathbb{N}_0^d$, $y \in \mathbb{R}^d$, $l : \mathbb{R}^d \to \mathbb{R}^d$ a linear bijection and $\lambda \in \mathbb{R}$,

$$\mathcal{F}(\partial^\beta u) = (2\pi \mathrm{i}\mathfrak{x})^\beta \widehat{u}, \qquad \partial^\beta \widehat{u} = \mathcal{F}((-2\pi \mathrm{i}\mathfrak{x})^\beta u), \tag{16.11}$$

$$\mathcal{F}(\mathcal{T}_y u) = e^{-2\pi \mathrm{i}\langle \mathfrak{x},y\rangle} \widehat{u}, \qquad \mathcal{T}_y \widehat{u} = \mathcal{F}(e^{2\pi \mathrm{i}\langle \mathfrak{x},y\rangle} u), \tag{16.12}$$

$$\mathcal{F}(u \circ l) = \frac{1}{|\det l|} \widehat{u} \circ l_*, \qquad \mathcal{F}(l_\lambda u) = \frac{1}{|\lambda|^d} l_{\frac{1}{\lambda}} \widehat{u}, \tag{16.13}$$

where l_* is the transpose of l^{-1} as in Theorem 16.9 and as in 14.9 "$l_\lambda u$" is written for "$u \circ l_\lambda$".

Exercise **16.H.** Prove the following.

(a) *If $v \in \mathcal{S}'(\mathbb{R})$ and $\mathfrak{x}v = 0$, then there exists a $c \in \mathbb{F}$ such that $v = c\delta_0$. (Hint: Exercise 15.G.)*

(b) *If $v \in \mathcal{S}'(\mathbb{R})$ and $\mathfrak{x}v = \mathbb{1}$, then there exists a $c \in \mathbb{F}$ such that $v = c\delta_0 + u$, where u is as in Exercise 15.B, i.e.,*

$$u(\varphi) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x} \, \mathrm{d}x \qquad (\varphi \in \mathcal{S}(\mathbb{R})).$$

Exercise **16.I.** Determine the Fourier transform in $\mathcal{S}'(\mathbb{R}, \mathbb{C})$ of the following distributions:

(a) $\mathfrak{x}^2$.

(b) $\mathbb{1}_{[0,\infty)}$. (Hint: Exercise 8.E and Exercise 16.H, and $\mathbb{1}_{(-\infty,0]} = \mathcal{R}\mathbb{1}_{[0,\infty)}$.)

(c) $\mathbb{1}_{(-\infty,a)}$, for $a \in \mathbb{R}$.

Exercise **16.J.** (a) Give an example of a $u \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ such that $u \neq 0$ and $u = \widehat{u}$.

(b) Give an example of a $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \setminus \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ such that $u \neq 0$ and $u = \widehat{u}$. (Hint: Example 16.23.)

(c) Suppose $u \in \mathcal{S}'$ and $\widehat{u} = cu$ for some $c \in \mathbb{C}$, what can be said about c?

Exercise **16.K.** Prove that in $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$

$$(1 - \Delta)e^{-|\boldsymbol{x}|} = 2\delta_0.$$

Observe that this proves that $\frac{1}{2}e^{-|\boldsymbol{x}|}$ is a fundamental solution to $(1 - \Delta)$.

Definition 16.25. The inverse of the Fourier transformation on $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$, $\mathcal{F}^{-1}$, is called the *inverse Fourier transformation.* For an $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ we call $\mathcal{F}^{-1}(u)$ the *Fourier inverse* of u.

16.26. Let us write $\overline{f}$ for the complex conjugate of a function $f : \mathbb{R}^d \to \mathbb{C}$, i.e., $\overline{f}(x) = \Re(f(x)) - \mathrm{i}\Im(f(x))$ for $x \in \mathbb{R}^d$. Then

$$\widehat{\overline{f}} = \overline{\check{\widehat{f}}} \qquad \overline{\widehat{f}} = \overline{\widehat{\check{f}}} \qquad (f \in L^1(\mathbb{R}^d)).$$

Therefore, as a consequence of Theorem 16.10 we have

$$\langle f, \widehat{\varphi} \rangle_{L^2} = \int f\overline{\widehat{\varphi}} = \int f\widehat{\overline{\check{\varphi}}} = \int \widehat{f}\overline{\check{\varphi}} = \langle \widehat{f}, \check{\varphi} \rangle_{L^2} \qquad (f \in L^1, \varphi \in \mathcal{S}). \tag{16.14}$$

By the above observation and the Fourier inversion formula, we obtain the following identity, which is due to Parseval and Plancherel.

Theorem 16.27 (Parseval, Plancherel)**.** *$\mathcal{F}$ forms an isometric isomorphism*

$$L^2(\mathbb{R}^d, \mathbb{C}) \to L^2(\mathbb{R}^d, \mathbb{C}),$$

so that in particular

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2} \qquad (f \in L^2(\mathbb{R}^d, \mathbb{C})), \tag{16.15}$$

and that the Fourier inversion formula (16.10) *holds for any $f \in L^2(\mathbb{R}^d, \mathbb{C})$.*

Proof. We refrain from writing "$(\mathbb{R}^d, \mathbb{C})$" in this proof. As $\mathcal{S}$ is dense in L^2, see Lemma 14.14, it turns out, as we argue later, that it is sufficient to show

$$\|\widehat{\varphi}\|_{L^2} = \|\varphi\|_{L^2} \qquad (\varphi \in \mathcal{S}). \tag{16.16}$$

Let $\varphi \in \mathcal{S}$. By applying (16.14) with $f = \widehat{\varphi}$ and using that $\mathcal{F}\widehat{\varphi} = \check{\varphi}$,

$$\|\widehat{\varphi}\|_{L^2}^2 = \langle \widehat{\varphi}, \widehat{\varphi} \rangle_{L^2} = \langle \mathcal{F}(\widehat{\varphi}), \check{\varphi} \rangle_{L^2} = \langle \varphi, \varphi \rangle_{L^2} = \|\varphi\|_{L^2}^2.$$

As the Fourier transform is bijective on $\mathcal{S}'$, it is sufficient to show that the Fourier transform of any L^2 function is given by an L^2 function. Now take any $f \in L^2$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$ that converges to f in L^2:

$$\|\varphi_n - f\|_{L^2} \to 0.$$

By (16.16) it follows that $(\widehat{\varphi}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2 and thus converges in L^2 to some g. As L^2 is continuously embedded in $\mathcal{S}'$ (Theorem 15.6 (a)), it follows that $g = \widehat{f}$ in L^2. ☐

Example 16.28. As $\widehat{\mathbb{1}}_{[-\frac{1}{2},\frac{1}{2}]} = \operatorname{sinc} \pi\mathfrak{x}$ is in L^2, see Example 16.2 and Exercise 16.A, by Theorem 16.27 we have

$$\mathcal{F}(\operatorname{sinc}) = \pi\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(\pi\mathfrak{x}).$$

17 Convolution of tempered distributions

We have already seen that the Fourier transform turns certain operations into other operations, like differentiation become multiplication with polynomials. In this section the Fourier transform is used to consider convolutions, namely it turns the operation of convolution into the operation of multiplication.

Theorem 17.1. *Let $f, g \in L^1$. Then $f * g \in L^1$ and*

$$\mathcal{F}(f * g) = \widehat{f}\widehat{g}.$$

Proof. By Young's inequality, Theorem 7.7 we have $f * g \in L^1$. Therefore, by Fubini's theorem, we have for $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\mathcal{F}(f * g)(\xi) &= \int_{\mathbb{R}^d} f * g(x) e^{-2\pi\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y\, e^{-2\pi\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d} g(x-y)e^{-2\pi\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d} g(x)e^{-2\pi\mathrm{i}\langle x+y,\xi\rangle}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} f(y)\widehat{g}(\xi)e^{-2\pi\mathrm{i}\langle y,\xi\rangle}\,\mathrm{d}y = \widehat{f}(\xi)\widehat{g}(\xi),
\end{aligned}$$

where we used Theorem 16.9 (a). ☐

Exercise 17.A. Show that $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} * \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} = f$, where f is as in Example 16.3. Observe that indeed $\widehat{f} = (\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}})^2$, see also Exercise 16.A.

As a direct consequence, by Theorem 16.16 and as multiplication is a continuous operation $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ (see Lemma 14.7):

Theorem 17.2. *Let $\varphi, \psi \in \mathcal{S}$. Then*

$$\mathcal{F}(\varphi * \psi) = \widehat{\varphi}\widehat{\psi}, \qquad \mathcal{F}(\varphi\psi) = \widehat{\varphi} * \widehat{\psi}. \tag{17.1}$$

Consequently, $\varphi * \psi \in \mathcal{S}$ and the function $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$, $(f, g) \mapsto f * g$ is continuous.

Analogously to Definition 8.1 we define the convolution between elements of $\mathcal{S}'$ and $\mathcal{S}$ as follows.

Definition 17.3. Let $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. We define the *convolution* of u with φ to be the function $\mathbb{R}^d \to \mathbb{F}$ defined by

$$u * \varphi(x) = u(\mathcal{T}_x \check{\varphi}) \qquad (x \in \mathbb{R}^d).$$

The following is the analogues statement of Lemma 8.2.

Lemma 17.4. *Let (u, φ) be in $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. Then*

$$\begin{aligned}
\delta_0 * \varphi &= \varphi, \\
\delta_y * \varphi &= \mathcal{T}_y \varphi \qquad (y \in \mathbb{R}^d), \\
\mathcal{R}(u * \varphi) &= \mathcal{R}u * \mathcal{R}\varphi, \\
\mathcal{T}_y(u * \varphi) &= (\mathcal{T}_y u) * \varphi = u * (\mathcal{T}_y \varphi), \\
u(\varphi) &= u * \check{\varphi}(0).
\end{aligned}$$

Proof. The proof is left to the reader. □

Similarly to Theorem 8.4 we have that the convolution between a Schwartz function and a tempered distribution is smooth, as we will see in Theorem 17.6. However, it need not be a Schwartz function as will be clear from the following exercise.

Exercise **17.B.** Compute the convolution of the tempered distribution $\mathbb{1}$ with the Schwartz function $e^{-|\mathbf{x}|^2}$.

Let us consider the convergence of difference quotients as we did in Lemma 8.3.

Lemma 17.5. *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $j \in \{1, \dots, d\}$. Then*

$$\left(\frac{\mathcal{T}_0 - \mathcal{T}_{he_j}}{h} \right) \varphi \xrightarrow{h \to 0} \partial_j \varphi \quad \text{ in } \mathcal{S}(\mathbb{R}^d). \tag{17.2}$$

Consequently, for $u \in \mathcal{S}'(\mathbb{R}^d)$,

$$\left(\frac{\mathcal{T}_0 - \mathcal{T}_{he_j}}{h} \right) u \xrightarrow{h \to 0} \partial_j u \quad \text{ in } \mathcal{S}'(\mathbb{R}^d).$$

Proof. Let us write $\mathfrak{x}_j = \langle \mathfrak{x}, e_j\rangle$ for $j \in \{1,\dots,d\}$. Observe that for $\varphi \in \mathcal{S}$, $j \in \{1,\dots,d\}$ and $h \in \mathbb{R}\setminus\{0\}$

$$\mathcal{F}\Big(\Big(\frac{\mathcal{T}_{-he_j} - \mathcal{T}_0}{h}\Big)\varphi\Big) = \frac{e^{2\pi \mathrm{i}\langle he_j, \mathfrak{x}\rangle} - 1}{h}\widehat{\varphi}, \qquad \mathcal{F}(\partial_j \varphi) = 2\pi \mathrm{i}\mathfrak{x}_j \widehat{\varphi}.$$

Therefore (17.2) holds if and only if the following convergence holds in $\mathcal{S}$, where

$$\left(\frac{e^{2\pi \mathrm{i} h \mathfrak{x}_j} - 1}{h} - 2\pi \mathrm{i}\mathfrak{x}_j \right)\widehat{\varphi} \xrightarrow{h\to 0} 0. \tag{17.3}$$

By Lemma 14.7 (see (14.5)), the latter is the case if there exists a $k \in \mathbb{N}_0$ such that

$$\mathfrak{q}_{m,k}\left(\frac{e^{2\pi \mathrm{i} h \mathfrak{x}_j} - 1}{h} - 2\pi \mathrm{i}\mathfrak{x}_j \right) \xrightarrow{h\to 0} 0 \qquad (m \in \mathbb{N}_0). \tag{17.4}$$

Let us consider the function $\mathbb{R} \to \mathbb{C}$ given by

$$g_h(t) = \frac{e^{\mathrm{i}ht} - 1}{h} - \mathrm{i}t \qquad (t \in \mathbb{R}),$$

so that $g_h(2\pi\xi_j) = \frac{e^{2\pi \mathrm{i}h\xi_j} - 1}{h} - 2\pi\mathrm{i}\xi_j$ for all $\xi \in \mathbb{R}^d$. Now (17.4) follows for $k = 2$, by the following –which we will show–

$$\left| \frac{\mathrm{d}^n}{\mathrm{d}t^n} g_h(t) \right| \le \begin{cases} |h||t|^2 & n = 0, \\ |h||t| & n = 1, \\ |h|^{n-1} & n \ge 2 \end{cases} \qquad (h \in \mathbb{R}\setminus\{0\}, t \in \mathbb{R}).$$

For $n \ge 2$ the above inequality follows as

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} g_h(t) = \mathrm{i}^n h^{n-1} e^{\mathrm{i}ht} \qquad (n \ge 2).$$

By rewriting $g_h(t)$ as

$$g_h(t) = \frac{e^{\mathrm{i}ht} - 1}{h} - \mathrm{i}t = \mathrm{i}t\frac{\int_0^h e^{\mathrm{i}rt} - 1 \,\mathrm{d}r}{h} = (\mathrm{i}t)^2 \frac{\int_0^h \int_0^r e^{\mathrm{i}ut}\,\mathrm{d}u\,\mathrm{d}r}{h},$$

we see that the above inequality for $n = 0$ holds and also that

$$g_h'(t) = \mathrm{i}(e^{\mathrm{i}ht} - 1) = \mathrm{i}t\int_0^h e^{\mathrm{i}rt}\,\mathrm{d}r,$$

from which the inequality for $n = 1$ follows. □

Theorem 17.6. *Let $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. Then $u * \varphi$ is smooth and of at most polynomial growth, that is $u * \varphi \in C_{\mathrm{p}}^\infty$. Strictly speaking, $u * \varphi$ is represented by a function in C_{p}^∞. For all $\alpha \in \mathbb{N}_0^d$*

$$\partial^\alpha (u * \varphi) = u * (\partial^\alpha \varphi) = (\partial^\alpha u) * \varphi. \tag{17.5}$$

Moreover, if $C > 0$, $k \in \mathbb{N}_0$ and $u \in \mathcal{S}'$ are such that

$$|u(\psi)| \le C\|\psi\|_{k,\mathcal{S}} \qquad (\psi \in \mathcal{S}), \tag{17.6}$$

then (with $\mathfrak{q}_{m,k}$ as in (14.4)*)*

$$\mathfrak{q}_{m,k}(u * \varphi) \le C\|\varphi\|_{m+k,\mathcal{S}} \qquad (\varphi \in \mathcal{S}). \tag{17.7}$$

Proof. That $u * \varphi$ is smooth and that (17.5) holds follow by Lemma 17.5 as in the proof of Theorem 8.4. As (see Exercise 17.C)

$$\|\mathcal{T}_x\check{\varphi}\|_{m,\mathcal{S}} \le (1 + |x|)^m \|\varphi\|_{m,\mathcal{S}} \qquad (x \in \mathbb{R}^d, m \in \mathbb{N}_0), \tag{17.8}$$

we have for $u \in \mathcal{S}'$ such that (17.6),

$$|u * \varphi(x)| = |u(\mathcal{T}_x\check{\varphi})| \le C(1 + |x|)^k \|\varphi\|_{k,\mathcal{S}} \qquad (x \in \mathbb{R}^d).$$

By (17.5) we obtain (17.7). □

Exercise **17.C.** Prove (17.8).

Theorem 17.7. *Let $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. Then in $\mathcal{S}'$*

$$\mathcal{F}(u * \varphi) = \widehat{\varphi}\widehat{u}, \qquad \mathcal{F}(\varphi u) = \widehat{u} * \widehat{\varphi}.$$

Strictly speaking, $\mathcal{F}(\varphi u)$ is the tempered distribution represented by the function $\widehat{u} * \widehat{\varphi}$.

Proof. As $\widehat{\varphi}$ is a Schwartz function, $\widehat{\varphi}\widehat{u}$ is a tempered distribution. Recall that in 8.10 we have observed that

$$\langle u * \eta, \psi \rangle = \langle u, \check{\eta} * \psi \rangle \qquad (\eta, \psi \in \mathcal{D}).$$

Therefore, if $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{S}$ is such that $\widehat{\psi} \in \mathcal{D}$, then as $\check{\varphi} = \widehat{\widehat{\varphi}}$

$$\begin{aligned} \langle \mathcal{F}(u * \varphi), \psi \rangle &= \langle u * \varphi, \widehat{\psi} \rangle = \langle u, \check{\varphi} * \widehat{\psi} \rangle = \langle u, \widehat{\widehat{\varphi}} * \widehat{\psi} \rangle \\ &= \langle u, \mathcal{F}(\widehat{\varphi}\psi) \rangle = \langle \widehat{u}, \widehat{\varphi}\psi \rangle = \langle \widehat{\varphi}\widehat{u}, \psi \rangle. \end{aligned}$$

As $\mathcal{D}$ is dense in $\mathcal{S}$, the Fourier transformation is a homeomorphism, also $\{\psi \in \mathcal{S} : \widehat{\psi} \in \mathcal{D}\}$ is dense in $\mathcal{S}$. Therefore, using that $\varphi \mapsto u * \varphi$ is continuous as a function $\mathcal{S} \to \mathcal{S}'$, we obtain $\langle \mathcal{F}(u * \varphi), \psi \rangle = \langle \widehat{\varphi}\widehat{u}, \psi \rangle$ for all $\varphi, \psi \in \mathcal{S}$. The identity $\mathcal{F}(\varphi u) = \widehat{u} * \widehat{\varphi}$ then follows by the first (Exercise 17.D). □

Exercise **17.D.** Prove that the identity $\mathcal{F}(\varphi u) = \widehat{\varphi} * \widehat{u}$ holds for all $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$ by using that $\mathcal{F}(u * \varphi) = \widehat{\varphi}\widehat{u}$ holds for all $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$.

As we mentioned in 15.5, we identify elements of $v \in \mathcal{E}'$ with their corresponding tempered distribution with compact support, $v|_{\mathcal{S}}$. For the Fourier transform of $v|_{\mathcal{S}}$ we therefore also write $\widehat{v}$.

Lemma 17.8. *If $v \in \mathcal{E}'$, then $\widehat{v} \in C_{\mathrm{p}}^{\infty}$. Consequently, if $u \in \mathcal{S}'$ and* $\operatorname{supp} \widehat{u}$ *is compact, then $u \in C_{\mathrm{p}}^{\infty}$ (in the sense that u is represented by a function in C_{p}^{∞}).*

Moreover, if $v \in \mathcal{E}'$ and $\varphi \in \mathcal{S}$, then $v * \varphi \in \mathcal{S}$.

Proof. Let $\chi \in \mathcal{D}$ be such that $v = \chi v$ (see 5.6). Then

$$\widehat{v} = \mathcal{F}(\chi v) = \widehat{\chi} * \widehat{v}.$$

As χ is a Schwartz function, so is $\widehat{\chi}$. Therefore $\widehat{v} \in C_{\mathrm{p}}^{\infty}$ by Theorem 17.6.

For the "Moreover" part: for $v \in \mathcal{E}'$ and $\varphi \in \mathcal{S}$, then $\widehat{v} \in C_{\mathrm{p}}^{\infty}$ and therefore $\widehat{v}\widehat{\varphi}$ and $v * \varphi = \mathcal{F}^{-1}(\widehat{v}\widehat{\varphi})$ are in $\mathcal{S}$. □

17.9. Let $k \in \mathbb{N}_0$ and $K \subset \mathbb{R}^d$ be compact. Let

$$\mathfrak{C} = \sup_{x \in K} (1 + |x|)^k.$$

Then

$$\|\psi\|_{C^m, K} \le \mathfrak{C} \mathfrak{q}_{m,k}(\psi) \qquad (\psi \in \mathcal{E}, m \in \mathbb{N}_0). \tag{17.9}$$

Therefore, we can already conclude by Theorem 17.6 that for $u \in \mathcal{S}'$, the function $\mathcal{S} \to \mathcal{E}, \varphi \mapsto u * \varphi$ is continuous. Moreover, see for example Exercise 15.D also the function $\mathcal{S} \to \mathcal{S}', \varphi \mapsto u * \varphi$ is continuous. In Theorem 17.10 we prove that the convolution is continuous as a function of both the tempered distribution and the Schwartz function:

Theorem 17.10. (a) *Let $u \in \mathcal{S}'$, $\varphi \in \mathcal{S}$, $(u_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{S}'$, $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{S}$ and suppose*

$$(u_n, \varphi_n) \to (u, \varphi) \text{ in } \mathcal{S}' \times \mathcal{S}.$$

Then there exists a $k \in \mathbb{N}_0$ such that

$$\mathfrak{q}_{m,k}(u_n * \varphi_n - u * \varphi) \xrightarrow{n \to \infty} 0 \qquad (m \in \mathbb{N}_0). \tag{17.10}$$

(b) *The functions*

$$\mathcal{S}' \times \mathcal{S} \to \mathcal{S}', \qquad (u, \varphi) \mapsto u * \varphi, \tag{17.11}$$

$$\mathcal{S}' \times \mathcal{S} \to \mathcal{E}, \qquad (u, \varphi) \mapsto u * \varphi, \tag{17.12}$$

$$\mathcal{E}' \times \mathcal{S} \to \mathcal{S}, \qquad (v, \varphi) \mapsto v * \varphi, \tag{17.13}$$

are sequentially continuous.

(c) *If $(v_n)_{n \in \mathbb{N}}$ and v are in $\mathcal{E}'$ and $v_n \to v$ in $\mathcal{E}'$, then there exists a $k \in \mathbb{N}_0$ such that*

$$\mathfrak{q}_{m,k}(\widehat{v}_n - \widehat{v}) \xrightarrow{n \to \infty} 0 \qquad (m \in \mathbb{N}_0).$$

Proof. (a) Write $u_\infty = u$. As $u_n \to u$ in $\mathcal{S}'$, by Theorem 15.10 (a) there exist $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|u_n(\psi)| \le C\|\psi\|_{k,\mathcal{S}} \qquad (\psi \in \mathcal{S}, n \in \mathbb{N} \cup \{\infty\}). \tag{17.14}$$

We show (17.10) by showing by showing

$$\mathfrak{q}_{m,k}(u_n * (\varphi_n - \varphi)) \to 0, \tag{17.15}$$

$$\mathfrak{q}_{m,k}((u_n - u) * \varphi) \to 0. \tag{17.16}$$

(17.15) follows by (17.14) and Theorem 17.6. To prove (17.16) we use Lemma 4.8:

Without loss of generality we may assume K to be convex (as we can always choose a larger compact convex set). First observe, that as $u_n \to u$ in $\mathcal{S}'$, we have $\partial^\alpha(u_n - u) * \varphi(x) \to 0$ for all $x \in K$ and $\alpha \in \mathbb{N}_0^d, |\alpha| \le m$. Therefore, by Lemma 4.8, (17.16) follows when $((1 + |\cdot|^2)^{-k}\partial^\alpha(u_n - u) * \varphi)_{n\in\mathbb{N}}$ is a sequence of uniformly Lipschitz continuous functions on K for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le m$. For this, by Lemma 4.7, it is sufficient to show

$$M := \sup_{n\in\mathbb{N}} \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le m}} \max_{i=1}^{d} \|\partial_i[(1 + |\cdot|^2)^{-k}(\partial^\alpha(u_n - u) * \varphi]\|_{L^\infty} < \infty.$$

As

$$\left\|\partial_i[(1 + |\cdot|^2)^{-k}\right\|_{L^\infty} \le 2k(1 + |\cdot|^2)^{-k},$$

we have, using Theorem 17.6 and (17.14),

$$\begin{aligned}\frac{M}{2k+1} &\le \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le m+1}} \left\|[(1 + |\cdot|^2)^{-k}(\partial^\alpha(u_n - u) * \varphi]\right\|_{L^\infty} \\ &= \sup_{n\in\mathbb{N}} \mathfrak{q}_{m+1,k}((u_n - u) * \varphi) \le 2C\|\varphi\|_{m+1+k,\mathcal{S}}.\end{aligned}$$

(b) That the functions (17.11) and (17.12) are sequentially continuous follows by Exercise 15.D and the estimate (17.9) in 17.9. For the sequential continuity of the function (17.13) it is sufficient to prove (c) due to the identity $v * \varphi = \mathcal{F}^{-1}(\widehat{v}\widehat{\varphi})$ (Theorem 17.7), Lemma 14.7 and the continuity of the Fourier transform on $\mathcal{S}(\mathbb{R}^d, \mathbb{C})$ (Theorem 16.16).

(c) Let $\chi \in \mathcal{E}$ be such that $v = \chi v$ and thus $\widehat{v} = \widehat{\chi} * \widehat{v}$. By continuity of the product map $\mathcal{E}' \times \mathcal{E} \to \mathcal{E}$ (Proposition 5.3) it follows that $\chi v_n \to \chi v = v$ and $(\mathbb{1} - \chi)v_n \to 0$ in $\mathcal{E}'$. The latter implies $(\mathbb{1} - \chi)v_n \to 0$ in $\mathcal{S}'$ and thus $\mathcal{F}((\mathbb{1} - \chi)v_n) \to 0$. Therefore, we may as well assume that $v_n = \chi v_n$ for all $n \in \mathbb{N}$. As $\widehat{v}_n \to \widehat{v}$ in $\mathcal{S}'$ and $\widehat{\chi} \in \mathcal{S}$, by (a) there exists a $k \in \mathbb{N}_0$ such that $\mathfrak{q}_{m,k}(\widehat{v}_n * \widehat{\chi} - \widehat{v} * \chi) \xrightarrow{n\to\infty} 0$ for all $m \in \mathbb{N}_0$. □

As in Theorem 8.9 we have the following associativity rule.

Theorem 17.11. *If $u \in \mathcal{S}'$ and $\varphi, \psi \in \mathcal{S}$ then*

$$u * (\varphi * \psi) = (u * \varphi) * \psi. \tag{17.17}$$

Proof. By Theorem 17.7 and Theorem 17.2 we have

$$\mathcal{F}(u * (\varphi * \psi)) = \mathcal{F}(\varphi * \psi)\widehat{u} = \widehat{\varphi}\widehat{\psi}\widehat{u},$$
$$\mathcal{F}((u * \varphi) * \psi) = \widehat{\psi}\mathcal{F}(u * \varphi) = \widehat{\psi}\widehat{\varphi}\widehat{u}.$$

As the Fourier transformation is injective on $\mathcal{S}'$ (Theorem 16.24), we have (17.17). □

In Definition 10.5 we defined the convolution between distributions u and v, of which at least one has compact support, by

$$u * v(\varphi) = u(\check{v} * \varphi) \qquad (\varphi \in \mathcal{D}).$$

Suppose that $u \in \mathcal{S}'$ and $v \in \mathcal{E}'$. Then the functions $\mathcal{S} \to \mathbb{F}$ given by $\varphi \mapsto u(\check{v} * \varphi)$ and $\varphi \mapsto v(\check{u} * \varphi)$ are tempered distributions as they are linear and sequentially continuous by Theorem 17.10 (b) (because $\mathcal{S} \to \mathcal{S}$, $\varphi \mapsto \check{v} * \varphi$ and $\mathcal{S} \to \mathcal{E}$, $\varphi \mapsto \check{u} * \varphi$ are sequentially continuous). On other words, the distributions $u*v$ and $v*u$ as in Definition 10.5 extend to tempered distributions, for which we use the same notation:

Definition 17.12. For $u \in \mathcal{S}'$ and $v \in \mathcal{E}'$ we define $u * v$ to be the tempered distribution given by

$$u * v(\varphi) = u(\check{v} * \varphi) \qquad (\varphi \in \mathcal{S}).$$

Moreover, we define $v * u$ to be the tempered distribution

$$v * u(\varphi) = v(\check{u} * \varphi) \qquad (\varphi \in \mathcal{S}).$$

Theorem 17.13. *Let $u \in \mathcal{S}'$, $v \in \mathcal{E}'$ and $\varphi \in \mathcal{S}$.*

$$(u * v) * \varphi = u * (v * \varphi) = v * (u * \varphi) = (v * u) * \varphi, \tag{17.18}$$
$$u * v = v * u, \tag{17.19}$$
$$\mathcal{F}(u * v) = \widehat{v}\widehat{u}. \tag{17.20}$$

Proof. (17.18) holds in case $\varphi \in \mathcal{D}$, by Theorem 10.6. As $\mathcal{D}$ is dense in $\mathcal{S}$ (Theorem 14.12), by the continuity of the function $\mathcal{S} \to \mathcal{E}$, $\varphi \mapsto w * \varphi$ for w being either one of the tempered distributions $u * v$, u and $v * u$, and the continuity of the function $\mathcal{S} \to \mathcal{S}$, $\varphi \mapsto v * \varphi$ one then obtains (17.18) and thus (17.19).

For $\varphi \in \mathcal{S}$ we have, using $\mathcal{F}\mathcal{F} = \mathcal{R}$,

$$\begin{aligned}\langle \mathcal{F}(u * v), \varphi \rangle &= \langle u * v, \widehat{\varphi} \rangle = u * v * \check{\widehat{\varphi}}(0) = \langle u, \mathcal{R}v * \mathcal{F}(\varphi) \rangle \\ &= \langle u, \mathcal{F}(\widehat{v}) * \mathcal{F}(\varphi) \rangle = \langle u, \mathcal{F}(\widehat{v}\varphi) \rangle = \langle \widehat{u}, \widehat{v}\varphi \rangle = \langle \widehat{v}\widehat{u}, \varphi \rangle,\end{aligned}$$

so that we conclude (17.20). □

The relations in Lemma 8.2 between the reflection and translation extend naturally to the convolution operation between elements of $\mathcal{E}'$ and $\mathcal{S}'$, we summarize:

Lemma 17.14. *Let* $(u, v) \in (\mathcal{E}' \times \mathcal{S}') \cup (\mathcal{S}' \times \mathcal{E}')$. *Then*

$$\begin{aligned} \delta_0 * u &= u, \\ \delta_y * u &= \mathcal{T}_y u \qquad (y \in \mathbb{R}^d), \\ \mathcal{R}(u * v) &= \mathcal{R}(u) * \mathcal{R}(v), \\ \mathcal{T}_y(u * v) &= (\mathcal{T}_y u) * v = u * (\mathcal{T}_y v) \qquad (y \in \mathbb{R}^d), \\ \partial^\alpha(u * v) &= (\partial^\alpha u) * v = u * (\partial^\alpha v) \qquad (\alpha \in \mathbb{N}_0^d). \end{aligned}$$

Proof. The proof is left for the reader. □

The operation of convolution is like in Theorem 10.2 characterised by maps $\mathcal{S} \to \mathcal{E}$ that is sequentially continuous and commute with translations. The proof is similar to the proof of Theorem 10.2, and left for the reader.

Theorem 17.15. *Let* $A : \mathcal{S} \to \mathcal{E}$ *be linear. Then* A *is sequentially continuous and commutes with translations if and only if there exists a* $u \in \mathcal{S}'$ *such that* $A\varphi = u * \varphi$ *for all* $\varphi \in \mathcal{S}$.

If A *is sequentially continuous and commutes with translations, then there exists exactly one such* u *such that* $A\varphi = u * \varphi$ *for all* $\varphi \in \mathcal{S}$.

Lemma 17.16. *The function* $\mathcal{S}' \times \mathcal{E}' \to \mathcal{S}'$, $(u, v) \mapsto u * v$ *is sequentially continuous.*

Proof. The proof is left for the reader (Exercise 17.E). □

Exercise 17.E. Prove Lemma 17.16.

Exercise 17.F. Suppose $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{cases} \partial_\dagger u = \Delta u \text{ on } (0, \infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0, \end{cases}$$

for some $u_0 \in \mathcal{S}$. Take the Fourier transform of the space variable, that is, let $U(t, \xi) = \mathcal{F}(u(t, \mathfrak{x}))(\xi)$ for $(t, \xi) \in [0, \infty) \times \mathbb{R}^d$. Derive

$$U(t, \xi) = e^{-4\pi^2 t|\xi|^2} U(0, \xi) \qquad ((t, \xi) \in (0, \infty) \times \mathbb{R}^d).$$

Conclude that

$$u(t, x) = h_t * u_0(x) \qquad ((t, x) \in (0, \infty) \times \mathbb{R}^d),$$

where $h_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{1}{4t}|x|^2}$ for $(t, x) \in (0, \infty) \times \mathbb{R}^d$, as in Example 11.15.

18 $\diamond$ The Fourier transformation on $\mathcal{E}'$

In the previous section we have seen that the Fourier transformation forms a bijection $\mathcal{S}(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ and $\mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$. Moreover, Example 16.2 and Example 16.3 illustrate that the Fourier transform of a compactly supported distribution may not be compactly supported. In this section we show that the only compactly supported distribution whose Fourier transform is compact is the zero distribution (Theorem 18.7). This implies $\mathcal{D} \cap \mathcal{F}(\mathcal{D}) = \{0\}$ and $\mathcal{E}' \cap \mathcal{F}(\mathcal{E}') = \{0\}$. Moreover, as we will see, one can say much more about Fourier transforms of compactly supported distributions than we have seen in Lemma 17.8.

Definition 18.1. An *entire function* is a function $g : \mathbb{C}^d \to \mathbb{C}$, given by

$$g(z) = \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} c_\alpha z^\alpha \qquad (z \in \mathbb{C}^d),$$

where $(c_\alpha)_{\alpha \in \mathbb{N}_0^d}$ is a family of complex numbers satisfying

$$\sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} |c_\alpha| R^{|\alpha|} < \infty \text{ for all } R > 0.$$

Lemma 18.2. *With $\langle a, b \rangle$ denoting the inner product of a and b in $\mathbb{C}^d$, for $a, b \in \mathbb{C}^d$,*

$$e^{\langle a,b \rangle} = \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \frac{a^\alpha \overline{b}^\alpha}{\alpha!}.$$

Proof. Write $e^{\langle a, \bar{b} \rangle} = e^{a_1 \overline{b_1}} \cdots e^{a_d \overline{b_d}}$ and use the power series representation of the exponential function:

$$e^{\langle a,b \rangle} = \sum_{\alpha_1 = 0}^{\infty} \cdots \sum_{\alpha_d = 0}^{\infty} \frac{(a_1 \overline{b_1})^{\alpha_1} \cdots (a_d \overline{b_d})^{\alpha_d}}{\alpha_1! \cdots \alpha_d!}.$$

□

Lemma 18.3. *Let $v \in \mathcal{E}'$. Define $g : \mathbb{C}^d \to \mathbb{C}$ by*

$$g(z) = \langle v, e^{\langle z, \mathfrak{x} \rangle} \rangle \qquad (z \in \mathbb{C}^d).$$

Then g is an entire function.

Proof. Let $m \in \mathbb{N}_0$ be such that v is at most of order m. First observe

$$\sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} \frac{1}{\alpha!} \|\boldsymbol{\mathfrak{x}}^\alpha\|_{C^m, \operatorname{supp} v} R^{|\alpha|} < \infty \qquad (R > 0). \tag{18.1}$$

Therefore in particular

$$\left\| \sum_{k=0}^{K} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} \frac{z^\alpha \boldsymbol{\mathfrak{x}}^\alpha}{\alpha!} - \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} \frac{z^\alpha \boldsymbol{\mathfrak{x}}^\alpha}{\alpha!} \right\|_{C^m, \operatorname{supp} v} \xrightarrow{K \to \infty} 0 \qquad (z \in \mathbb{C}^d).$$

Therefore, for $z \in \mathbb{C}^d$ we have by Lemma 18.2,

$$g(z) = \left\langle v, \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} \frac{z^\alpha \boldsymbol{\mathfrak{x}}^\alpha}{\alpha!} \right\rangle = \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} \frac{\langle v, \boldsymbol{\mathfrak{x}}^\alpha \rangle}{\alpha!} z^\alpha,$$

whereas, for all $R \in [0, \infty)$,

$$\sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} \frac{|\langle v, \boldsymbol{\mathfrak{x}}^\alpha \rangle|}{\alpha!} R^{|\alpha|} < \infty,$$

by (18.1). □

Theorem 18.4 (Paley-Wiener)**.** *If $v \in \mathcal{E}'$ and*

$$g(z) = \langle v, e^{-2\pi i \langle z, \boldsymbol{\mathfrak{x}} \rangle} \rangle \qquad (z \in \mathbb{C}^d), \tag{18.2}$$

then g is entire and $g|_{\mathbb{R}^d}$ equals $\widehat{v}$ in $\mathcal{S}'$.

Proof. By Lemma 18.3 g is entire. In order to prove that $g|_{\mathbb{R}^d}$ equals $\widehat{v}$ in $\mathcal{S}'$, let $\chi \in \mathcal{D}$ be equal to 1 on a neighbourhood of $\operatorname{supp} v$, so that $v = \chi v$. It suffices to prove that g equals $\widehat{v} * \widehat{\chi}$ on $\mathbb{R}^d$. As $\check{\widehat{\chi}} = \mathcal{F}^{-1}(\chi)$, for all $x \in \mathbb{R}^d$:

$$\begin{aligned} \widehat{v} * \widehat{\chi}(x) &= \langle \widehat{v}, \mathcal{T}_x \mathcal{F}^{-1}(\chi) \rangle = \langle v, \mathcal{F} \mathcal{T}_x \mathcal{F}^{-1}(\chi) \rangle = \langle v, e^{-2\pi i \langle x, \boldsymbol{\mathfrak{x}} \rangle} \chi \rangle \\ &= \langle \chi v, e^{-2\pi i \langle x, \boldsymbol{\mathfrak{x}} \rangle} \rangle = \langle v, e^{-2\pi i \langle x, \boldsymbol{\mathfrak{x}} \rangle} \rangle = g(x). \end{aligned}$$

□

Definition 18.5. Let $\Omega \subset \mathbb{R}^d$ be open. A function $f : \Omega \to \mathbb{F}$ is called *analytic* if for each $y \in \mathbb{R}^d$ there exist $r > 0$ and $(c_\alpha)_{\alpha \in \mathbb{N}_0^d}$ in $\mathbb{F}$ such that $\sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{N}_0^d, |\alpha|=k} c_\alpha (x-y)^\alpha$ exists for all $x \in B(y, r)$ and

$$f(x) = \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} c_\alpha (x-y)^\alpha \qquad (x \in B(y, r)). \tag{18.3}$$

Every analytic function is smooth and if (18.3) holds, then $\partial^\alpha f(y) = c_\alpha \alpha!$.

If $g : \mathbb{C}^d \to \mathbb{C}$ is an entire function, then $f : \mathbb{R}^d \to \mathbb{C}$ given by $f = g|_{\mathbb{R}^d}$, i.e., $f(x) = g(x)$ for $x \in \mathbb{R}^d$, is an analytic function.

Lemma 18.6. *Let Ω be a connected and open subset of $\mathbb{R}^d$. Let $f : \Omega \to \mathbb{F}$ be an analytic function. Then either $f = 0$ or $\operatorname{supp} f = \Omega$.*

Proof. Let $U = \Omega \setminus \operatorname{supp} f$. For $y \in \Omega$, $y \in U$ if and only if there exists an $r > 0$ such that $f = 0$ on $B(y, r)$. As Ω is connected, it is sufficient to show that $\operatorname{supp} f$ is open. Let $y \in \operatorname{supp} f$ and let $r > 0$ and $(c_\alpha)_{\alpha \in \mathbb{N}_0^d}$ in $\mathbb{F}$ be such that (18.3). Then there exists an $\alpha \in \mathbb{N}_0^d$ such that $c_\alpha \neq 0$ and thus $\partial^\alpha f(y) = c_\alpha \neq 0$. As $\partial^\alpha f$ is continuous, there exists an $s > 0$ such that $\partial^\alpha f \neq 0$ on $B(y, s)$. Then f cannot be equal to 0 on $B(y, s)$. Therefore $B(y, s) \subset \operatorname{supp} f$. □

Theorem 18.7. *If $v \in \mathcal{E}'$ and $\widehat{v}$ has compact support, then $v = 0$.*

Proof. We are done if $\widehat{v} = 0$, which in the language of Theorem 18.4 is the case if $g = 0$ on $\mathbb{R}^d$. As g is entire $g|_{\mathbb{R}^d}$ is analytic and therefore this follows by Lemma 18.6. □

Actually, Theorem 18.4 is only a small part of the statements of the Paley-Wiener theorem, which characterizes the Fourier transforms of compactly supported distributions explicitly. We pose the statements here and refer to [Rud91, Theorem 7.23] for a proof.

Theorem 18.8 (Paley-Wiener)**.**

(a) *If $v \in \mathcal{E}'$, $R > 0$, $\operatorname{supp} v \subset B(0, R)$, v has order k and*

$$g(z) = \langle v, e^{-2\pi i \langle z, \mathfrak{x} \rangle} \rangle \qquad (z \in \mathbb{C}^d), \tag{18.4}$$

then g is entire, $g|_{\mathbb{R}^d} = \widehat{v}$ and there exists a $C > 0$ such that

$$|g(z)| \leq C(1 + |z|)^k e^{R|\Im z|} \qquad (z \in \mathbb{C}^d). \tag{18.5}$$

(b) *Conversely, if g is an entire function on $\mathbb{C}^d$ which satisfies (18.5) for some $k \in \mathbb{N}_0$ and $C > 0$, then there exists a $v \in \mathcal{E}'$ with support in $B(0, R)$ such that (18.2) holds.*

19 Fourier multipliers

Let us motivate the definition of Fourier multipliers by recalling some facts from the previous sections. By Theorem 16.24 we have

$$\partial^\beta u = \mathcal{F}^{-1}((2\pi i \mathfrak{x})^\beta \widehat{u}) \qquad (\beta \in \mathbb{N}_0^d, u \in \mathcal{S}').$$

Therefore, by writing

$$\mathrm{D}^\beta = \frac{1}{(2\pi \mathrm{i})^\beta}\partial^\beta \qquad (\beta \in \mathbb{N}_0^d),$$

we have

$$\mathrm{D}^\beta\, u = \mathcal{F}^{-1}(\mathfrak{x}^\beta \widehat{u}) \qquad (\beta \in \mathbb{N}_0^d, u \in \mathcal{S}').$$

Moreover, for any polynomial p, with $p(\mathrm{D})$ the polynomial p "evaluated in D" (as in Definition 11.1), so that $\mathfrak{x}^\alpha(\mathrm{D}) = \mathrm{D}^\alpha$,

$$p(\mathrm{D})u = \mathcal{F}^{-1}(p\widehat{u}) \qquad pu = \mathcal{F}^{-1}(p(\mathrm{D})\widehat{u}) \qquad (u \in \mathcal{S}').$$

Also, we have

$$\mathcal{T}_y u = \mathcal{F}^{-1}(e^{2\pi \mathrm{i}\langle \mathfrak{x}, y\rangle}\widehat{u}) \qquad (y \in \mathbb{R}^d, u \in \mathcal{S}').$$

So both operations D^β and $\mathcal{T}_y$ can be described by the composition of the Fourier inverse with the multiplication of the Fourier transform with a certain function, namely with $\mathfrak{x}^\beta$ and $e^{2\pi \mathrm{i}\langle \mathfrak{x}, y\rangle}$, respectively. In this section we consider those operators of the form $u \mapsto \mathcal{F}^{-1}(\sigma\widehat{u})$ for a certain class of functions σ such that $\sigma\widehat{u}$ is again tempered and hence in the domain of $\mathcal{F}^{-1}$. Those operators are called Fourier multipliers.

Definition 19.1 (Fourier multiplier)**.** For $\sigma \in C_\mathrm{p}^\infty$ we define $\sigma(\mathrm{D}) : \mathcal{S}' \to \mathcal{S}'$ by

$$\sigma(\mathrm{D})u = \mathcal{F}^{-1}(\sigma\widehat{u}) \qquad (u \in \mathcal{S}'),$$

and call $\sigma(\mathrm{D})$ a *Fourier multiplier.*

Lemma 19.2. *Let $\sigma \in C_\mathrm{p}^\infty$. Then $\sigma(\mathrm{D})$ is continuous as function $\mathcal{S}' \to \mathcal{S}'$ and it forms a continuous function $\mathcal{S} \to \mathcal{S}$. Moreover,*

$$\langle \sigma(\mathrm{D})u, \varphi\rangle = \langle u, \check{\sigma}(\mathrm{D})\varphi\rangle \qquad (u \in \mathcal{S}', \varphi \in \mathcal{S}).$$

Proof. As multiplication with σ is a continuous operation $\mathcal{S}' \to \mathcal{S}'$ and $\mathcal{S} \to \mathcal{S}$ (see 15.4) and because $\mathcal{F}$ is continuous as a function $\mathcal{S}' \to \mathcal{S}'$ and as a function $\mathcal{S} \to \mathcal{S}$ (Theorem 16.16 and Theorem 16.24), it follows that $\sigma(\mathrm{D})$ is continuous as a function $\mathcal{S}' \to \mathcal{S}'$ and as a function $\mathcal{S} \to \mathcal{S}$.

For $\sigma \in C_\mathrm{p}^\infty$, $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$

$$\begin{aligned}\langle \sigma(\mathrm{D})u, \varphi\rangle &= \langle \mathcal{R}(\sigma\widehat{u}), \mathcal{F}\varphi\rangle = \langle \check{\sigma}\mathcal{F}^{-1}(u), \widehat{\varphi}\rangle = \langle u, \mathcal{F}^{-1}(\check{\sigma}\widehat{\varphi})\rangle \\ &= \langle u, \check{\sigma}(\mathrm{D})\varphi\rangle.\end{aligned}$$

□

Example 19.3. By Theorem 16.24 we have $\partial^\beta = \sigma(\mathrm{D})$ for $\sigma = (2\pi \mathrm{i}\mathfrak{x})^\beta$ and $\mathcal{T}_y = \sigma(\mathrm{D})$ for $\sigma = e^{-2\pi\mathrm{i}\langle \mathfrak{x}, y\rangle}$, i.e.,

$$\partial^\beta u = (2\pi\mathrm{i}\mathfrak{x})^\beta(\mathrm{D})u, \qquad \mathcal{T}_y u = e^{-2\pi\mathrm{i}\langle\mathfrak{x},y\rangle}(\mathrm{D})u \qquad (\beta \in \mathbb{N}_0^d, y \in \mathbb{R}^d, u \in \mathcal{S}'). \tag{19.1}$$

Let $v \in \mathcal{E}'$. In Lemma 17.8 we have seen that $\widehat{v} \in C_{\mathrm{p}}^\infty$. Therefore

$$v * u = \widehat{v}(\mathrm{D})u \qquad (u \in \mathcal{S}').$$

By the commutativity of multiplication, we obtain that Fourier multipliers commute as well. Moreover, if $\mathcal{F}^{-1}(\sigma)$ is compactly supported, then the Fourier multiplier $\sigma(\mathrm{D})$ equals convolution with $\mathcal{F}^{-1}(\sigma)$:

Lemma 19.4. *Let $\sigma, \tau \in C_{\mathrm{p}}^\infty$. Then*

$$\tau(\mathrm{D})\sigma(\mathrm{D})u = (\sigma\tau)(\mathrm{D})u = \sigma(\mathrm{D})\tau(\mathrm{D})u \qquad (u \in \mathcal{S}').$$

Consequently, Fourier multipliers commute with partial differential operators with constant coefficients, with translations and with convolutions. Moreover,

$$\sigma(\mathrm{D})(l_\lambda u) = l_\lambda[(l_\lambda\sigma)(\mathrm{D})u] \qquad (u \in \mathcal{S}', \lambda > 0), \tag{19.2}$$
$$\sigma(\mathrm{D})\varphi = \mathcal{F}^{-1}(\sigma) * \varphi \qquad (\varphi \in \mathcal{S}),$$

and if $\widehat{\sigma} \in \mathcal{E}'$ or if $\sigma \in \mathcal{S}$, then

$$\sigma(\mathrm{D})u = \mathcal{F}^{-1}(\sigma\widehat{u}) = \mathcal{F}^{-1}(\sigma) * u \qquad (u \in \mathcal{S}'). \tag{19.3}$$

Proof. Most identities are trivial. The proof of (19.2) is left as an exercise, see Exercise 19.A. ☐

Exercise **19.A.** Prove (19.2).

19.5 (Bessel potentials)**.** Let $g \in \mathcal{S}'$. We consider the problem of finding a $u \in \mathcal{S}'$ such that

$$(1 - \Delta)u = g.$$

We can write $(1 - \Delta)$ as a Fourier multiplier (by, for example (19.1)), namely $(1 - \Delta) = \sigma(\mathrm{D})$, for

$$\sigma(\xi) = (1 + 4\pi^2|\xi|^2) \qquad (\xi \in \mathbb{R}^d).$$

As this function is strictly positive, we can divide by it: We define $\tau : \mathbb{R}^d \to \mathbb{R}$ by

$$\tau(\xi) = (1 + 4\pi^2|\xi|^2)^{-1} \qquad (\xi \in \mathbb{R}^d).$$

It is not too difficult to show that $\tau \in C_{\mathrm{p}}^{\infty}$. As $\tau\sigma = \mathbb{1}$, with $I : \mathcal{S}' \to \mathcal{S}'$ being the identity map $\varphi \mapsto \varphi$, we have

$$I = (\tau\sigma)(\mathrm{D}) = \tau(\mathrm{D})\sigma(\mathrm{D}) = \tau(\mathrm{D})(1-\Delta).$$

As also $(1-\Delta)\tau(\mathrm{D}) = I$, we have $\tau(\mathrm{D}) = (1-\Delta)^{-1}$ and thus $u = \tau(\mathrm{D})g$.

Let σ^s for $s \in \mathbb{R}$ be the function given by $\sigma^s(\xi) = (\sigma(\xi))^s$ for $\xi \in \mathbb{R}^d$. Then

$$(1-\Delta)^k = \sigma^k(\mathrm{D}) \qquad (k \in \mathbb{N}_0).$$

For $s \in \mathbb{R} \setminus \mathbb{N}_0$ one writes "$(1-\Delta)^s$" instead of "$\sigma^s(\mathrm{D})$", i.e.,

$$(1-\Delta)^s u := \sigma^s(\mathrm{D})u \qquad (u \in \mathcal{S}'). \tag{19.4}$$

For $s < -d$ the function σ^s is integrable and thus $\mathcal{F}^{-1}(\sigma^s)$ is a bounded continuous function. It turns out that for any $s > 0$ the tempered distribution $\mathcal{F}^{-1}(\sigma^{-s})$ is represented by a function that is smooth on $\mathbb{R}^d \setminus \{0\}$ (see for example [Gra14, Theorem 6.1.5]). Such functions (on $\mathbb{R}^d \setminus \{0\}$) are called *Bessel potentials*. For more on Bessel potentials we refer to [Eva98, Section 4.3] and [Gra14, Section 6.1.2] . In the last reference, not the function $\mathcal{F}^{-1}(\sigma^{-s})$ but the operator $(1-\Delta)^{-s}$ is called a Bessel potential. We come back to Bessel potentials in 20.9 and 20.10.

Exercise **19.B.** Show that $(1-\Delta)^s$ forms a homeomorphism $\mathcal{S} \to \mathcal{S}$ and $\mathcal{S}' \to \mathcal{S}'$ for each $s \in \mathbb{R}$.

Similarly to the definition of $(1-\Delta)^s$ we will define Δ^s for $s \in \mathbb{R}\setminus\mathbb{N}_0$. As the function $\xi \mapsto |\xi|^{2s}$ is not smooth at 0, in order to define Δ^s via Fourier multipliers, we extend the notion of a Fourier multiplier $\sigma(\mathrm{D})$ to the situation where the domain of σ is not necessarily all of $\mathbb{R}^d$.

19.6. Let $\Omega \subset \mathbb{R}^d$ be open. Let $\sigma \in C^{\infty}(\Omega)$. Suppose $u \in \mathcal{S}'$ and $[\operatorname{supp}\widehat{u}]_{3\delta} \subset \Omega$ for some $\delta > 0$. Let $\chi \in C_{\mathrm{b}}^{\infty}$ be equal to 1 on $[\operatorname{supp}\widehat{u}]_{\delta}$ and 0 outside $[\operatorname{supp}\widehat{u}]_{2\delta}$ (see Lemma 9.3). By abusing notation by writing $\chi\sigma$ for the function

$$x \mapsto \begin{cases} \chi(x)\sigma(x) & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases}$$

we have $\chi\sigma \in C_{\mathrm{p}}^{\infty}$ if $\operatorname{supp}\widehat{u}$ is compact or if $\sigma \in C_{\mathrm{p}}^{\infty}(\Omega)$. Therefore, in both cases, $(\chi\sigma)(\mathrm{D})u$ is defined and would be a good candidate for a definition of "$\sigma(\mathrm{D})u$" as, σ and $\chi\sigma$ are equal on $[\operatorname{supp}\widehat{u}]_{\delta}$.

We show that we can take this as our definition by showing that it does not depend on the choice of χ. Indeed, if $\eta \in C_{\mathrm{b}}^{\infty}$ equals 1 on $[\operatorname{supp}\widehat{u}]_{\delta}$ and 0 outside $[\operatorname{supp}\widehat{u}]_{2\delta}$, then $\chi\widehat{u} = \eta\widehat{u}$ and thus $\chi\sigma\widehat{u} = \eta\sigma\widehat{u}$, i.e., $(\chi\sigma)(\mathrm{D})u = (\eta\sigma)(\mathrm{D})u$.

Definition 19.7. Let $\Omega \subset \mathbb{R}^d$ be open, $\sigma \in C^\infty(\Omega)$. Suppose $u \in \mathcal{S}'$, $\delta > 0$ and $[\operatorname{supp} \widehat{u}]_{3\delta} \subset \Omega$. If $\sigma \in C_\mathrm{p}^\infty(\Omega)$ or $\operatorname{supp} \widehat{u}$ is compact, we define

$$\sigma(\mathrm{D})u := \mathcal{F}^{-1}(\sigma\chi\widehat{u}) = (\sigma\chi)(\mathrm{D})u,$$

where $\chi \in C_\mathrm{b}^\infty(\mathbb{R}^d, [0,1])$ equals 1 on $[\operatorname{supp} \widehat{u}]_\delta$ with $\operatorname{supp} \chi \subset [\operatorname{supp} \widehat{u}]_{2\delta}$. In this sense we obtain an operator $\sigma(\mathrm{D})$ on the set

$$\{u \in \mathcal{S}' : \text{ there exists a } \delta > 0 \text{ such that } [\operatorname{supp} \widehat{u}]_\delta \subset \Omega\}. \tag{19.5}$$

Observe that this extends Definition 16.22, as for $\Omega = \mathbb{R}^d$, one may choose $\chi = 1$ (remember also 19.6).

Definition 19.8. We define $\mathcal{S}'_\circ$ to be the space

$$\mathcal{S}'_\circ = \left\{u \in \mathcal{S}' : \operatorname{supp} \widehat{u} \subset \mathbb{R}^d \setminus \{0\}\right\}.$$

Observe that $\mathcal{S}'_\circ$ is a subset of (19.5) for $\Omega = \mathbb{R}^d \setminus \{0\}$.

Exercise 19.C. Prove

$$\begin{aligned}&\left\{\varphi \in \mathcal{S} : \partial^\alpha \widehat{\varphi}(0) = 0 \text{ for all } \alpha \in \mathbb{N}_0^d\right\}\\ &= \{\varphi \in \mathcal{S} : \langle P, \varphi\rangle = 0 \text{ for all polynomials } P\}.\end{aligned}$$

19.9 (Fractional Laplacian/Riesz potential)**.** For $s \in \mathbb{R}$ the function $\sigma^s : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ given by

$$\sigma^s(\xi) = |2\pi\xi|^{2s}, \qquad (\xi \in \mathbb{R}^d \setminus \{0\})$$

is in $C_\mathrm{p}^\infty(\mathbb{R}^d \setminus \{0\})$. Let $u \in \mathcal{S}'_\circ$. Then there exists an $\varepsilon > 0$ such that $B(0,\varepsilon) \subset \mathbb{R}^d \setminus \operatorname{supp} \widehat{u}$. Therefore there exists a $\delta > 0$ such that $[\operatorname{supp} \widehat{u}]_{3\delta} \subset \mathbb{R}^d \setminus \{0\}$ and hence $\sigma^s(\mathrm{D})u$ is defined for all $s \in \mathbb{R}$. For $k \in \mathbb{N}_0$ σ^k extends to a smooth function on $\mathbb{R}^d$ and

$$(-\Delta)^k = \sigma^k(\mathrm{D}) \text{ on } \mathcal{S}'.$$

For $s \in \mathbb{R} \setminus \mathbb{N}_0$ one writes "$(-\Delta)^s$" instead of "$\sigma^s(\mathrm{D})$", i.e.,

$$(-\Delta)^s u = \sigma^s(\mathrm{D})u \qquad (u \in \mathcal{S}'_\circ).$$

The operator $(-\Delta)^s$ is called a *fractional Laplacian* but is sometimes also called a *Riesz potential*. In probability theory for $\alpha \in (0,2)$ the operator $(-\Delta)^{\frac{\alpha}{2}}$ plays the role of the generator of an α-stable Lévy process.

Exercise 19.D. Let $d = 1$. Show that $(-\Delta)^{\frac{1}{2}} \neq \partial$.

19.10 (Pseudo differential operators). Each linear partial differential operator with constant coefficients P can be written as $p(\mathrm{D})$ for some polynomial p: Suppose $P = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\le k} c_\alpha\partial^\alpha$ for some scalars $c_\alpha \in \mathbb{F}$. Then $P = p(\mathrm{D})$ for $p = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\le k} \frac{c_\alpha}{(2\pi\mathrm{i})^\alpha}\mathfrak{x}^\alpha$.

Let $P : \mathcal{S} \to \mathcal{S}'$ be a linear partial differential operator with variable coefficients. Say $f_\alpha \in L^\infty$ for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$ are such that

$$P = \sum_{\alpha\in\mathbb{N}_0^d,|\alpha|\le k} f_\alpha\partial^\alpha.$$

Then

$$P\varphi(x) = p(x,\mathrm{D})\varphi(x) \qquad (x \in \mathbb{R}^d, \varphi \in \mathcal{S}), \tag{19.6}$$

where

$$p(x,\xi) = \sum_{\alpha\in\mathbb{N}_0^d,|\alpha|\le k} f_\alpha(x)\xi^\alpha \qquad (x,\xi \in \mathbb{R}^d).$$

Pseudo differential operators are operators of the form (19.6) for certain “nice” functions p. Fourier multipliers are examples of pseudo differential operators. For example $(-\Delta)^s$ is a pseudo differential operator.

20 Fractional Sobolev spaces

In this section we return to Sobolev spaces and describe them in terms of Fourier transforms and Fourier multipliers.

20.1 (Sobolev spaces described by Fourier transforms). Let $k \in \mathbb{N}_0$. In Theorem 12.14 we have seen that H^k (reminder: $H^k = H^k(\mathbb{R}^d)$), being the Sobolev space given by

$$H^k = W^{k,2} = \{u \in \mathcal{D}' : \partial^\beta u \in L^2(\Omega) \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| \le k\},$$

is a Hilbert space with inner product

$$\langle u, v\rangle_{H^k} = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\le k} \langle\partial^\alpha u, \partial^\alpha v\rangle_{L^2} \qquad (u, v \in H^k),$$

so that it is equipped with the norm

$$\|u\|_{H^k} = \Big(\sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\le k} \|\partial^\alpha u\|_{L^2}^2\Big)^{\frac{1}{2}} \qquad (u \in H^k).$$

This space can also be described using the Fourier transformation, as we will see in Lemma 20.3.

Lemma 20.2. *Let $\theta \in [0,1]$. The function $[0,\infty) \to [0,\infty)$, $x \mapsto x^\theta$ is subadditive, that is,*

$$(a+b)^\theta \le a^\theta + b^\theta \qquad (a,b \in [0,\infty)). \tag{20.1}$$

Proof. It suffices to show (20.1) for $a = 1$ (otherwise one can divide by a^θ). This in turn follows by

$$\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^\theta - t^\theta = \theta((1+t)^{\theta-1} - t^{\theta-1}) \le 0 \qquad (t \in [0,\infty)).$$

□

Lemma 20.3. *Let $k \in \mathbb{N}_0$.*

$$H^k = \{u \in \mathcal{S}' : (1+|\mathfrak{x}|)^k \widehat{u} \in L^2\}, \tag{20.2}$$

the norm $\|\cdot\|_{H^k}$ is equivalent to $u \mapsto \|(1+|\mathfrak{x}|)^k \widehat{u}\|_{L^2}$, which means there exists a $C > 1$ such that

$$\frac{1}{C}\|u\|_{H^k} \le \|(1+|\mathfrak{x}|)^k \widehat{u}\|_{L^2} \le C\|u\|_{H^k} \qquad (u \in H^k), \tag{20.3}$$

and

$$H^k \times H^k \to \mathbb{F}, \quad (u,v) \mapsto \langle (1+|\mathfrak{x}|)^k \widehat{u}, (1+|\mathfrak{x}|)^k \widehat{v}\rangle_{L^2} \tag{20.4}$$

is an inner product on H^k that generates the same topology as the inner product $\langle\cdot,\cdot\rangle_{H^k}$.

Proof. That (20.4) defines an inner product can be easily checked. It suffices to show (20.3). We show that for $k = 1$ there exists a $C > 0$ such that

$$\|(1+|\mathfrak{x}|)^k \widehat{u}\|_{L^2} \le C\|u\|_{H^k} \qquad (u \in H^k),$$

and leave the rest for the reader, see Exercise 20.A.

By Lemma 20.2 we have $(1+|x|)^{\frac{1}{2}} \le 1 + |x_1| + \cdots + |x_d|$ for all $x \in \mathbb{R}^d$. Therefore

$$\begin{aligned}
\|(1+|\mathfrak{x}|)^{\frac{1}{2}} \widehat{u}\|_{L^2} &\le \|\widehat{u}\|_{L^2} + \sum_{j=1}^d \|x_j \widehat{u}\|_{L^2} \\
&\le \|\widehat{u}\|_{L^2} + \sum_{j=1}^d \frac{1}{2\pi}\|\mathcal{F}(\partial_j u)\|_{L^2} \\
&= \|u\|_{L^2} + \sum_{j=1}^d \frac{1}{2\pi}\|\partial_j u\|_{L^2} \le \|u\|_{W^{1,2}}.
\end{aligned}$$

The latter is equivalent to $\|\cdot\|_{H^k}$, see 12.6. □

To prove (20.3) and thus (20.2) the Multinomial Theorem can be beneficial.

Theorem 20.4 (Multinomial Theorem)**.** *For* $x = (x_1, \dots, x_d) \in \mathbb{F}^d$ *and* $k \in \mathbb{N}$

$$(x_1 + \cdots + x_d)^k = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \binom{k}{\alpha} x^\alpha, \tag{20.5}$$

where with $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_d!$*,*

$$\binom{k}{\alpha} = \frac{k!}{\alpha!} = \frac{k!}{\alpha_1!\alpha_2! \cdots \alpha_d!}.$$

Proof. This follows by induction. For $d = 1$ the formula is trivial for all $k \in \mathbb{N}$. For $d = 2$ it is the usual binomial formula. Suppose (20.5) holds for a fixed $d \in \mathbb{N}$ and for any $k \in \mathbb{N}$. Then for $y = x_1 + \cdots + x_d$ and $z = (x_1, \dots, x_d)$ we have

$$\begin{aligned}(y + x_{d+1})^k &= \sum_{\substack{m \in \mathbb{N}_0 \\ m \le k}} \binom{k}{m} y^m x_{d+1}^{k-m} \\ &= \sum_{\substack{m \in \mathbb{N}_0 \\ m \le k}} \binom{k}{m} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \binom{m}{\alpha} z^\alpha x_{d+1}^{k-m}.\end{aligned}$$

As for $\beta = (\alpha_1, \dots, \alpha_d, k - m)$ we have $|\beta| = k$ and

$$\binom{k}{m}\binom{m}{\alpha} = \frac{k!}{(k-m)!m!} \frac{m!}{\alpha_1! \cdots \alpha_d!} = \binom{k}{\beta},$$

it follows that (20.5) is valid also for for $d + 1$. □

The description of H^k as in Lemma 20.3 extends naturally to non-integer values of k, as follows.

Definition 20.5. For $s \in \mathbb{R} \setminus \mathbb{N}_0$ we define the *fractional Sobolev space* H^s by

$$H^s = \{u \in \mathcal{S}' : (1 + |\cdot|)^s \widehat{u} \in L^2\}, \tag{20.6}$$

$\langle \cdot, \cdot \rangle_{H^s} : H^s(\Omega) \times H^s(\Omega) \to \mathbb{F}$ by

$$\langle u, v \rangle_{H^s} = \langle (1 + |\mathfrak{x}|)^s \widehat{u}, (1 + |\mathfrak{x}|)^s \widehat{v} \rangle_{L^2} \qquad (u, v \in H^s(\Omega)),$$

and $\| \cdot \|_{H^s} : H^s(\Omega) \to [0, \infty)$ by

$$\|u\|_{H^s} = \sqrt{\langle u, u \rangle_{H^s}} \qquad (u \in H^s(\Omega)).$$

Example 20.6. We have already seen that $\widehat{\delta_0} = \mathbb{1}$. As $(1+|\cdot|)^s$ is in L^2 if and only if $2s < -d$ by Lemma 14.13, it follows that $\delta_0 \in H^s$ if and only if $s < -\frac{d}{2}$.

Exercise **20.A.** Prove the following statements.

(a) *For each $k \in \mathbb{N}_0$ there exists a $C > 0$ such that for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$,*

$$\|\partial^\alpha u\|_{L^2} \leq C\|(1+|\mathfrak{x}|)^k \widehat{u}\|_{L^2} \qquad (u \in H^k).$$

Consequently, there exists a $C > 0$ such that

$$\|u\|_{H^k} \leq C\|(1+|\mathfrak{x}|)^k \widehat{u}\|_{L^2} \qquad (u \in H^k).$$

(b) *For all $s \in [0,\infty)$, the functions $u \mapsto \|(1+|\mathfrak{x}|)^s \widehat{u}\|_{L^2}$ and $u \mapsto \|(1+|\mathfrak{x}|^2)^{\frac{s}{2}} \widehat{u}\|_{L^2}$ are norms and they are equivalent.*

(c) *For $s \in \mathbb{R}$ the norm $\|\cdot\|_{H^s}$ is equivalent to $u \mapsto \|(1-\Delta)^{\frac{s}{2}} u\|_{L^2}$.*

(d) *For each $k \in \mathbb{N}_0$ there exists a $C > 0$ such that*

$$\|(1+|\mathfrak{x}|^2)^k \widehat{u}\|_{L^2} \leq C\|u\|_{H^{2k}}$$

(e) *For each $k \in \mathbb{N}_0$ there exists a $C > 0$ such that* (20.3) *holds.*

Theorem 20.7. *Let $s \in \mathbb{R}$. $\langle \cdot, \cdot \rangle_{H^s}$ is an inner product on H^s, so that H^s equipped with this inner product is a Hilbert space.*

Exercise **20.B.** Prove Theorem 20.7.

Like for the spaces of continuously differentiable functions and for Sobolev spaces, see Lemma 12.8 and Lemma 12.9, we have the following.

Lemma 20.8. *Let $s, r \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^d$. If $r \leq s$, then*

$$\|\partial^\alpha u\|_{H^{r-|\alpha|}} \leq \|u\|_{H^s} \qquad (u \in H^s).$$

In particular, $\partial^\alpha : H^s \to H^{s-|\alpha|}$ is continuous for all $s \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^d$, and

$$H^s \hookrightarrow H^r \qquad (r \leq s).$$

The Fourier multiplier D^α is thus a continuous function $H^s \to H^{s-|\alpha|}$. We consider conditions on the functions σ such that the Fourier multiplier $\sigma(\mathrm{D})$ is a continuous function between Besov spaces (in Section 25). Those spaces are generalisations of fractional Sobolev spaces and are introduced in the following section.

Exercise **20.C.** Prove Lemma 20.8.

20.9. By Exercise 20.A(b) and (c) we see that the spaces H^s can be described in terms of the Bessel potentials:

$$H^s = \{u \in \mathcal{S}' : (1-\Delta)^s u \in L^2\}.$$

Actually, H^s is a particular case of a Bessel potential space, see 20.10 for a definition.

20.10 (Bessel potential spaces)**.** We have only considered a generalisation of the Sobolev space $W^{k,p}$ for $p = 2$ but for any p one can actually define fractional Sobolev spaces, also called *Bessel potential spaces.* In [Tri83, p.88] , for example, it is shown that

$$W^{k,p} = \{f \in \mathcal{S}' : (1-\Delta)^{\frac{k}{2}} f \in L^p\}, \tag{20.7}$$

and that $\|\cdot\|_{W^{k,p}}$ is equivalent to

$$f \mapsto \|(1-\Delta)^{\frac{k}{2}} f\|_{L^p}. \tag{20.8}$$

Similarly to Definition 20.5 one defines the *fractional Sobolev space* H^s_p for $s \in \mathbb{R}$ and $p \in [1,\infty]$ by replacing "k" in (20.7) by "s":

$$H^s_p = \{u \in \mathcal{S}' : (1-\Delta)^{\frac{s}{2}} u \in L^p\}, \tag{20.9}$$

and defines a norm on H^s_p by

$$\|u\|_{H^s_p} = \|(1-\Delta)^{\frac{s}{2}} u\|_{L^p} \qquad (u \in \mathcal{S}').$$

Then $H^k_p = W^{k,p}$ and the norms $\|\cdot\|_{H^k_p}$ and $\|\cdot\|_{W^{k,p}}$ are equivalent for $k \in \mathbb{N}_0$ and by Plancherel's identity it follows that $\|\cdot\|_{H^s_2} = \|\cdot\|_{H^s}$ and thus $H^s_2 = H^s$ for $s \in \mathbb{R}$. One can also show that H^s_p is a Banach space for all $s \in \mathbb{R}$ and $p \in [1,\infty]$.

21 Besov spaces defined by Littlewood–Paley decompositions

In this section we introduce Besov spaces. The definition is more cumbersome than those of the normed spaces like $W^{k,p}$, H^k, H^s and H^s_p. First, we take a specific partition of unity of $\mathbb{R}^d$, $(\chi_n)_{n\in\mathbb{N}}$, such that in $\mathcal{S}'$

$$u = \sum_{n\in\mathbb{N}} \chi_n(\mathrm{D}) u \qquad (u \in \mathcal{S}'),$$

and then describe Besov spaces in terms of the Fourier multipliers $\chi_n(\mathrm{D})$.

21.1 (Motivation of Besov spaces by the space H^s)**.** We will see later that H^s is a special case of a Besov space. Let us motivate the definition of a general Besov space by describing H^s in a different way. For $n \in \mathbb{N}_0$ we define $j_n : \mathbb{R}^d \to \mathbb{F}$ by

$$j_n(x) = \mathbb{1}_{[n,n+1)}(|x|) \qquad (x \in \mathbb{R}^d).$$

We have $u \in H^s$ if and only if $(1+|\mathfrak{x}|)^s \widehat{u} \in L^2$. As gj_n is orthogonal to gj_k in L^2 when $k \neq n$ and $g \in L^2$, it follows that: $u \in H^s$ if and only if

$$\begin{aligned} &(1+|\mathfrak{x}|)^s \widehat{u} I_n \in L^2 \qquad (n \in \mathbb{N}_0), \\ &\Big(\|(1+|\mathfrak{x}|)^s \widehat{u} j_n\|_{L^2} \Big)_{n\in\mathbb{N}} \in \ell^2 \end{aligned}$$

and, moreover,

$$\|u\|_{H^s} = \Big\| \Big(\|(1+|\mathfrak{x}|)^s \widehat{u} j_n\|_{L^2} \Big)_{n\in\mathbb{N}} \Big\|_{\ell^2} \qquad (u \in H^s),$$

where we wrote "$\|\cdot\|_{\ell^2}$" for "$\|\cdot\|_{\ell^2(\mathbb{N}_0)}$". From this it follows that $u \in H^s$ if and only if

$$\begin{aligned} &(1+n)^s \widehat{u} j_n \in L^2 \qquad (n \in \mathbb{N}_0), \\ &\Big((1+n)^s \|\widehat{u} j_n\|_{L^2} \Big)_{n\in\mathbb{N}_0} \in \ell^2 \end{aligned}$$

and, moreover, that $\|\cdot\|_{H^s}$ is equivalent to

$$u \mapsto \Big\| (1+n)^s \Big(\|\widehat{u} j_n\|_{L^2} \Big)_{n\in\mathbb{N}_0} \Big\|_{\ell^2} \qquad (u \in H^s),$$

which follows from the fact that for $s \geq 0$ (and something similar for $s < 0$)

$$(1+n)^s j_n \leq (1+|\mathfrak{x}|)^s j_n \leq (2+n)^s j_n \leq 2^s (1+n)^s j_n \qquad (n \in \mathbb{N}_0).$$

Instead of taking intervals of length 1, we can also split the function into intervals of dyadic length: For $j \in \mathbb{N}_0$ define $h_j : \mathbb{R}^d \to \mathbb{F}$ by

$$h_j(x) = \mathbb{1}_{[2^j, 2^{j+1})}(|x|) \qquad (x \in \mathbb{R}^d),$$

and let $h_{-1} = j_0$, i.e., $h_{-1} = \mathbb{1}_{[0,1)}(|\mathfrak{x}|)$. Then $u \in H^s$ if and only if

$$\begin{aligned} &2^{js} \widehat{u} h_j \in L^2 \qquad (j \in \mathbb{N}_0 \cup \{-1\}), \\ &\Big(2^{js} \|\widehat{u} h_n\|_{L^2} \Big)_{j\in\mathbb{N}_0\cup\{-1\}} \in \ell^2, \end{aligned}$$

and, $\|\cdot\|_{H^s}$ is equivalent to

$$u \mapsto \Big\| \Big(2^{js} \|\widehat{u} h_j\|_{L^2} \Big)_{j\in\mathbb{N}_0\cup\{-1\}} \Big\|_{\ell^2} \qquad (u \in H^s). \tag{21.1}$$

As (16.3) of Theorem 16.10 extends to $f, g \in L^2$ (see Exercise 21.A), by Theorem 16.27 it follows that

$$\|\widehat{u} h_j\|_{L^2} = \|\mathcal{F}^{-1}(h_j) * u\|_{L^2}. \tag{21.2}$$

Now Besov spaces are basically spaces with a norm like (21.1) where instead of "L^2" and "ℓ^2" one has "L^p" and "ℓ^q" for some $p, q \in [1, \infty]$ and with smooth functions "φ_j"

with compact support which are similar to "h_j" in the sense that like for h_j, one has the scaling relation $\varphi_j(x) = \varphi_0(2^{-j}x)$ for all $x \in \mathbb{R}^d$. The latter is being done so that the convolution with $\mathcal{F}^{-1}(\varphi_j)$ is defined for any tempered distribution and defines a smooth function.

We will first introduce such functions φ_j.

Exercise **21.A.** Show that

$$\int f\widehat{g} = \int g\widehat{f} \qquad (f, g \in L^2).$$

21.2 (Notation $\mathbb{N}_{-1}$)**.** We write "$\mathbb{N}_{-1}$" for the set $\{-1, 0, 1, 2, \dots\}$.

Definition 21.3 (Annulus)**.** Let $a, b \in (0, \infty)$, $a < b$. We write

$$A(a, b) = \{x \in \mathbb{R}^d : a < |x| < b\}.$$

Such a set $A(a, b)$ is called an *annulus*.

Definition 21.4. A function $f : \mathbb{R}^d \to \mathbb{F}$ is called *radial* if $f(x) = f(y)$ for all $x, y \in \mathbb{R}^d$ with $|x| = |y|$.

For each radial function f there exists a function $g : [0, \infty) \to \mathbb{F}$ such that $f(x) = g(|x|)$ for $x \in \mathbb{R}^d$.

We introduce the notion of a dyadic partition of unity, which consists of one function that is supported in a ball and equals 1 on a smaller ball centered at the origin and of functions that are supported in annuli which are scaled versions of each other.

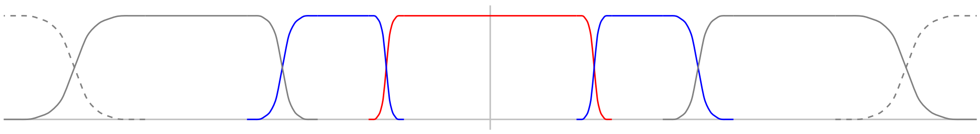

Figure 2: An example of a dyadic partition of unity.

Definition 21.5. A sequence of radial functions $(\varphi_j)_{j \in \mathbb{N}_{-1}}$ in $C_c^\infty(\mathbb{R}^d, [0, 1])$ is called a *dyadic partition of unity* if

$$\operatorname{supp} \varphi_{-1} \text{ is equal to the closure of a ball centered at the origin,}$$
$$\operatorname{supp} \varphi_0 \text{ is equal to the closure of an annulus,}$$
$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) \qquad (\xi \in \mathbb{R}^d, j \in \mathbb{N}_0),$$
$$\sum_{j \in \mathbb{N}_{-1}} \varphi_j(\xi) = 1, \qquad \frac{1}{2} \le \sum_{j \in \mathbb{N}_{-1}} \varphi_j(\xi)^2 \le 1 \qquad (\xi \in \mathbb{R}^d), \tag{21.3}$$
$$|i - j| \ge 2 \Longrightarrow \operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j = \emptyset \qquad (i, j \in \mathbb{N}_{-1}). \tag{21.4}$$

We say that a radial function φ in $C_c^\infty(\mathbb{R}^d, [0, 1])$ supported in an annulus *generates a partition of unity* if there exists a partition of unity $(\varphi_j)_{j \in \mathbb{N}_{-1}}$ with $\varphi_0 = \varphi$

Observe

(a) $\varphi_j = l_{2^{-j}}\varphi = \varphi(2^{-j}\,\cdot\,)$ for $j \in \mathbb{N}_0$.

(b) $\varphi_{-1} = 1 - \sum_{j\in\mathbb{N}_0}\varphi_j$, and

$$l_{2^{-J}}\varphi_{-1} = \varphi_{-1}(2^{-J}\,\cdot\,) = \sum_{j=-1}^{J-1}\varphi_j \qquad (J \in \mathbb{N}_0). \tag{21.5}$$

(c) $\varphi_0 = (l_{2^{-1}} - l_1)\varphi_{-1} = \varphi_{-1}(\frac{1}{2}\,\cdot\,) - \varphi_{-1}$.

Theorem 21.6. *There exists a radial function φ in $C_c^\infty(\mathbb{R}^d, [0,1])$ that generates a dyadic partition of unity.*

Proof. As the functions φ_j need to be radial, we first construct functions η_j on $[0,\infty)$ such that the functions φ_j defined by $\varphi_j(\xi) = \eta_j(|\xi|)$ form a partition of unity.

Due to the above observations, we may as well start by showing the existence of a function that is the function φ_{-1} of a partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$, as then $\varphi_0 = (l_{2^{-1}} - l_1)\varphi_{-1}$.

Let $a \in (\frac{3}{4}, 1)$. Let $\theta : [0,\infty) \to [0,1]$ be smooth on $(0,\infty)$ and equal to 1 on $[0, \frac{3}{4}]$, nonzero on $[\frac{3}{4}, a)$ and equal to 0 on $[a,\infty)$. Define $\eta = l_{\frac{1}{2}}\theta - \theta = \theta(\frac{1}{2}\,\cdot\,) - \theta$. Then $\eta = 1$ on $[a, \frac{6}{4}]$ and $\operatorname{supp}\eta \subset [0, 2a] \setminus [0, \frac{3}{4}] \subset [\frac{3}{4}, 2a]$. Define $\eta_{-1} = \theta$, $\eta_j = l_{2^{-j}}\eta = \eta(2^{-j}\,\cdot\,)$ for $j \in \mathbb{N}_0$. Then by definition

$$\sum_{j\in\mathbb{N}_{-1}} \eta_j = 1.$$

As $2 < 3 = 4 \cdot \frac{3}{4}$ we have $[\frac{3}{4}, 2] \cap 4[\frac{3}{4}, 2] = \emptyset$ and thus, as $\operatorname{supp}\eta_{-1} \subset [0,1]$ and $\operatorname{supp}\eta_j \subset 2^j[\frac{3}{4}, 2]$,

$$|i-j| \geq 2 \implies \operatorname{supp}\eta_i \cap \operatorname{supp}\eta_j = \emptyset \qquad (i, j \in \mathbb{N}_{-1}). \tag{21.6}$$

We are left to show that for $t \geq 0$,

$$\frac{1}{2} \leq \sum_{j\in\mathbb{N}_{-1}} \eta_j(t)^2. \tag{21.7}$$

Let us write $\Sigma_{odd} = \sum_{j\in 2\mathbb{N}_0 - 1}\eta_j$ and $\Sigma_{even} = \sum_{j\in 2\mathbb{N}_0}\eta_j$. As the functions η_j with $j \in 2\mathbb{N}_0 - 1$ have disjoint supports by (21.6), we have $\Sigma_{odd}^2 = \sum_{j\in 2\mathbb{N}_0-1}\eta_j^2$. Similarly, $\Sigma_{even}^2 = \sum_{j\in 2\mathbb{N}_0}\eta_j^2$. Therefore,

$$1 = (\Sigma_{odd}(t) + \Sigma_{even}(t))^2 \leq 2(\Sigma_{odd}^2(t) + \Sigma_{even}^2(t)) = 2\sum_{j\in\mathbb{N}_{-1}}\eta_j(t)^2.$$

By defining $\varphi(\xi) = \eta_0(|\xi|)$ for $\xi \in \mathbb{R}^d$, we see that φ is radial function in $C_c^\infty(\mathbb{R}^d, [0,1])$ that generates a dyadic partition of unity. □

Exercise **21.B.** (a) Let $a > 0$. Show that there exists a radial function φ with $\varphi(\xi) = 1$ for all $\xi \in A(a-\delta, a+\delta)$ for some $\delta > 0$, that generates a partition of unity. (Hint: Observe that for φ in the proof of Theorem 21.6 one has $\varphi(\xi) = 1$ for $\xi \in \mathbb{R}^d, |\xi| = 1$.)

(b) Show that there exists a radial function φ with $\varphi = 1$ on $\overline{A(\frac{3}{4}, \frac{5}{4})}$ that generates a partition of unity. (Hint: In the proof, replace "$\frac{3}{4}$" by another suitable number.)

21.7 (Notation of ordered and unordered infinite sums)**.** In the following we use the notation "$\sum_{j\in\mathbb{N}_{-1}}$", which denotes the unordered sum over a function on $\mathbb{N}_{-1}$ (with values for example in $\mathcal{S}'$). We recall the notation from the section Conventions and notation.

If $\mathbb{I}$ is a countable set and v_i is an element of a topological vector space X for all $i \in \mathbb{I}$, then we say that

$$\sum_{i\in\mathbb{I}} v_i$$

exists, if there exists a $v \in X$ such that for each bijection $q : \mathbb{N} \to \mathbb{I}$, $\sum_{n=1}^{\infty} v_{q(n)} = v$ in X, and write $\sum_{i\in\mathbb{I}} v_i$ for v.

Lemma 21.8. *Let $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ be a dyadic partition of unity. Then*

$$\sum_{j\in\mathbb{N}_{-1}} \varphi_j \psi = \psi \text{ in } \mathcal{S} \qquad (\psi \in \mathcal{S}).$$

Moreover, for all finite subsets $F \subset \mathbb{N}_{-1}$

$$\Big\| \sum_{j\in F} \varphi_j \Big\|_{C^k} \le 4\|\varphi_{-1}\|_{C^k}.$$

Proof. We invoke Lemma 14.10 (d). Observe that $\|f+g\|_{C^k} = \|f\|_{C^k} \vee \|g\|_{C^k}$ for f and g in C^k with disjoint supports.

Therefore, by (21.4), it follows that for all finite subsets $F \subset \mathbb{N}_{-1}$

$$\begin{aligned}\left\|\sum_{j\in F}\varphi_j\right\|_{C^k} &\le \Big\|\sum_{j\in 2\mathbb{N}_0-1\cap F}\varphi_j\Big\|_{C^k} + \Big\|\sum_{j\in 2\mathbb{N}_0\cap F}\varphi_j\Big\|_{C^k} \\ &\le \sup_{j\in 2\mathbb{N}_0-1}\|\varphi_j\|_{C^k} + \sup_{j\in 2\mathbb{N}_0}\|\varphi_j\|_{C^k} \le 2\sup_{j\in\mathbb{N}_{-1}}\|\varphi_j\|_{C^k} \\ &\le 2(\|\varphi_{-1}\|_{C^k} \vee \|\varphi_0\|_{C^k}) \le 4\|\varphi_{-1}\|_{C^k},\end{aligned}$$

where we used that $\|\varphi_j\|_{C^k} = \|\varphi_0(2^{-j}\,\cdot\,)\|_{C^k} \le \|\varphi_0\|_{C^k}$ and that $\varphi_0 = l_{\frac{1}{2}}\varphi_{-1} - \varphi{-1}$. From this by Lemma 14.10 (d) it follows that $\sum_{n=1}^{\infty}\varphi_{q(n)}\psi = \psi$ in $\mathcal{S}$ for any bijection $q : \mathbb{N} \to \mathbb{N}_{-1}$. ☐

By the above lemma we also have $\sum_{j\in\mathbb{N}_{-1}} \varphi_j \widehat{\psi} = \widehat{\psi}$ in $\mathcal{S}$ for all $\psi \in \mathcal{S}$. Therefore, by the continuity of $\mathcal{F}$,

$$\sum_{j\in\mathbb{N}_{-1}} \varphi_j(\mathrm{D})\psi = \sum_{j\in\mathbb{N}_{-1}} \mathcal{F}^{-1}(\varphi_j\widehat{\psi}) = \mathcal{F}^{-1}\Big(\sum_{j\in\mathbb{N}_{-1}} \varphi_j\widehat{\psi}\Big) = \psi \text{ in } \mathcal{S} \qquad (\psi \in \mathcal{S}). \tag{21.8}$$

The partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ is used to obtain the operators $\varphi_j(\mathrm{D})$. After showing some properties of Besov spaces, one can basically only work with those operators. It is therefore customary to use a shorter notation for these operators: $\Delta_j = \varphi_j(\mathrm{D})$. Those operators Δ_j for $j \in \mathbb{N}_{-1}$ are also called the *Littlewood–Paley operators*.

Lemma 21.9. *Let $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ be a dyadic partition of unity. Write $\Delta_j = \varphi_j(\mathrm{D})$. Then*

$$\psi \in \mathcal{S} \Longrightarrow \Delta_j\psi \in \mathcal{S}, \quad u \in \mathcal{S}' \Longrightarrow \Delta_j u \in C_{\mathrm{p}}^{\infty}, \quad f \in L^p \Longrightarrow \Delta_j f \in L^p \quad (j \in \mathbb{N}_{-1}),$$

and

$$\langle \Delta_j u, \psi\rangle = \langle u, \Delta_j \psi\rangle \qquad (j \in \mathbb{N}_{-1}, u \in \mathcal{S}', \psi \in \mathcal{S}) \tag{21.9}$$

$$\sum_{j\in\mathbb{N}_{-1}} \Delta_j\psi = \psi \quad \text{in } \mathcal{S} \qquad (\psi \in \mathcal{S}), \tag{21.10}$$

$$\sum_{j\in\mathbb{N}_{-1}} \Delta_j u = u \quad \text{in } \mathcal{S}' \qquad (u \in \mathcal{S}'), \tag{21.11}$$

$$\|\Delta_j f\|_{L^p} \le \|\widehat{\varphi_0}\|_{L^1}\|f\|_{L^p} \le 2\|\widehat{\varphi_{-1}}\|_{L^1}\|f\|_{L^p} \qquad (j \in \mathbb{N}_0, f \in L^p), \tag{21.12}$$

$$\Big\| \sum_{j=-1}^{J} \Delta_j f \Big\|_{L^p} \le \|\widehat{\varphi_{-1}}\|_{L^1}\|f\|_{L^p} \qquad (J \in \mathbb{N}_{-1}, f \in L^p). \tag{21.13}$$

Proof. (21.10) follows from Lemma 21.8, see (21.8). The rest we leave as an exercise (Exercise 21.C). □

Exercise 21.C. Finish the proof of Lemma 21.9.

Definition 21.10. Let X be a vector space (over the scalar field $\mathbb{F}$). We call a function $\mathfrak{n} : X \to [0,\infty]$ a *norm-like function* if

$$\mathfrak{n}(x+y) \le \mathfrak{n}(x) + \mathfrak{n}(y), \quad \mathfrak{n}(\lambda x) = |\lambda|\mathfrak{n}(x), \quad \mathfrak{n}(x) = 0 \iff x = 0 \qquad (x, y \in X, \lambda \in \mathbb{F}).$$

Like for norms, two norm-like functions $\mathfrak{n}$ and $\mathfrak{m}$ on X are called equivalent if there exists a $C > 1$ such that

$$\frac{1}{C}\mathfrak{m}(x) \le \mathfrak{n}(x) \le C\mathfrak{m}(x) \qquad (x \in X).$$

Observe that $Y = \{x \in X : \mathfrak{n}(x) < \infty\}$ is a vector space and $\mathfrak{n}$ forms a norm on Y.

21.11 (Convention)**.** Let M denote the space of measurable functions $\mathbb{R}^d \to \mathbb{F}$. For L^p spaces we have that a measurable function (or better said, an equivalence class) $f \in \mathbb{M}$ is in L^p if and only if $\mathcal{N}_{L^p}(f) < \infty$, where $\mathcal{N}_{L^p} : M \to [0, \infty]$ is the norm-like function given by

$$\mathcal{N}_{L^p}(f) = \left(\int |f|^p \right)^{\frac{1}{p}} \qquad (f \in M).$$

Of course, on L^p, $\mathcal{N}_{L^p}$ is equal to the norm $\| \cdot \|_{L^p}$. It may be convenient to work with such norm-like functions, which are allowed to take the value ∞. In general, we will not distinguish anymore between the norm-like function $\mathcal{N}_{L^p}$ and the norm $\| \cdot \|_{L^p}$. Moreover, for $q \in [1, \infty]$ and a countable set $\mathbb{I}$ and $a \in \mathbb{F}^{\mathbb{I}}$ we will also write $\|a\|_{\ell^q(\mathbb{I})}$ for $\mathfrak{l}_q(a)$, where $\mathfrak{l}_q : \mathbb{F}^{\mathbb{I}} \to [0, \infty]$ is the norm-like function given by

$$\mathfrak{l}_q(a) = \begin{cases} \left(\sum_{i \in \mathbb{I}} |a(i)|^q\right)^{1q} & q < \infty, \\ \sup_{i \in \mathbb{I}} |a(i)| & q = \infty, \end{cases} \qquad (a \in \mathbb{F}^{\mathbb{I}}).$$

Mostly, we write $\| \cdot \|_{\ell^q}$ instead of $\| \cdot \|_{\ell^q(\mathbb{I})}$.

Definition 21.12 (Besov Space)**.** Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. Let φ be a function that generates a dyadic partition of unity $(\varphi_j)_{j \in \mathbb{N}_{-1}}$. We define the function $\| \cdot \|_{B^s_{p,q}[\varphi]} : \mathcal{S}' \to [0, \infty]$ by

$$\|u\|_{B^s_{p,q}[\varphi]} := \left\| \left(2^{js} \|\varphi_j(\mathrm{D})u\|_{L^p} \right)_{j \in \mathbb{N}_{-1}} \right\|_{\ell^q} \qquad (u \in \mathcal{S}'). \tag{21.14}$$

Here we wrote "$\| \cdot \|_{\ell^q}$" as an abbreviation for "$\| \cdot \|_{\ell^q(\mathbb{N}_{-1})}$". Then $\| \cdot \|_{B^s_{p,q}[\varphi]}$ is a norm-like function (observe that if $\Delta_j u = 0$ for all $j \in \mathbb{N}_{-1}$, then $u = 0$ by Lemma 21.9). We define the *nonhomogeneous Besov space generated by* φ, $B^s_{p,q}[\varphi]$, to be the space of all tempered distributions u such that $\|u\|_{B^s_{p,q}[\varphi]} < \infty$. We write $\Delta_j = \varphi_j(\mathrm{D})$. Δ_j is called a *Littlewood–Paley operator* and for $u \in \mathcal{S}'$ one also calls $\Delta_j u$ the *Littlewood–Paley block.* With this notation,

$$\|u\|_{B^s_{p,q}[\varphi]} = \left\| \left(2^{js} \|\Delta_j u\|_{L^p} \right)_{j \in \mathbb{N}_{-1}} \right\|_{\ell^q} \qquad (u \in \mathcal{S}').$$

We will drop the notation "$[\varphi]$" later, as the space does not depend on φ; this follows from Theorem 21.18, see Corollary 21.20. First we consider some properties of tempered distributions $u \in \mathcal{S}'$ such that $\operatorname{supp} \widehat{u}$ is a subset of an annulus, or of a ball, which will be used to prove Theorem 21.18.

Exercise **21.D.** Let $p, q \in [1, \infty]$, $s, t \in \mathbb{R}$. Show that $B^t_{p,q}[\varphi] \subset B^s_{p,q}[\varphi]$ for $s < t$ and $B^s_{p,\infty}[\varphi] \subset B^{s-\varepsilon}_{p,q}[\varphi]$ for $\varepsilon > 0$.

Exercise **21.E.** Let $p \in [1, \infty]$. Show that for $f \in L^p$ and $\lambda > 0$, with the convention that $\frac{d}{\infty} = 0$,

$$\|l_\lambda f\|_{L^p} = \lambda^{-\frac{d}{p}} \|f\|_{L^p}, \tag{21.15}$$

where $l_\lambda f(x) = f(\lambda x)$ (as in 14.9).

Definition 21.13. For $\lambda \in \mathbb{R} \setminus \{0\}$ we define l_λ^* to be the operation $\lambda^{-d} l_{\frac{1}{\lambda}}$.

Observe the following facts.

(a) l_λ^* is the adjoint of l_λ as an operator on L^2, i.e.,

$$\langle l_\lambda f, g\rangle_{L^2} = \langle f, l_\lambda^* g\rangle_{L^2} \qquad (f, g \in L^2),$$

and (Definition 2.14 (e) and Definition 15.3)

$$\langle l_\lambda u, \psi\rangle = \langle u, l_\lambda^* \psi\rangle, \quad \langle l_\lambda^* u, \psi\rangle = \langle u, l_\lambda \psi\rangle \qquad (u \in \mathcal{S}', \psi \in \mathcal{S}).$$

In particular, if $d = 1$, $\delta_\lambda = l_\lambda^* \delta_1$.

(b) By Theorem 16.24 we know that for a distribution $u \in \mathcal{S}'$,

$$\mathcal{F}(l_\lambda u) = l_\lambda^* \widehat{u} \qquad \mathcal{F}(l_\lambda^* u) = l_\lambda \widehat{u}. \tag{21.16}$$

(c) Furthermore, by (21.15)

$$\|l_\lambda^* f\|_{L^1} = \|f\|_{L^1} \qquad (\lambda > 0, f \in L^1). \tag{21.17}$$

The following lemma will be used both for the proof of Theorem 21.18 but also gives us Bernstein inequalities, see Theorem 21.15.

Lemma 21.14. *Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball centered at the origin in $\mathbb{R}^d$. Let $\chi \in C_c^\infty$ be equal to 1 on a neighbourhood of $\mathcal{B}$. Let $\phi \in C_c^\infty$ be supported in an annulus and be equal to 1 on a neighbourhood of $\mathcal{A}$. Let $\lambda > 0$.*

(a) *If u is a tempered distribution with $\operatorname{supp} \widehat{u} \subset \lambda\mathcal{B}$, then for all $\alpha \in \mathbb{N}_0^d$*

$$\partial^\alpha u = \lambda^{|\alpha|+d} (l_\lambda h_\alpha) * u,$$

where $h_\alpha \in \mathcal{S}$ is given by $h_\alpha = \partial^\alpha \mathcal{F}^{-1}(\chi)$.

(b) *If u is a tempered distribution with $\operatorname{supp} \widehat{u} \subset \lambda\mathcal{A}$ and $k \in \mathbb{N}_0$, then*

$$u = \lambda^{-k} \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = k} \lambda^d (l_\lambda g_\alpha) * \partial^\alpha u,$$

where $g_\alpha \in \mathcal{S}$ for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$ is given by

$$g_\alpha = \binom{k}{\alpha} \mathcal{F}^{-1}\left((-2\pi \mathrm{i} \boldsymbol{x})^\alpha |2\pi \boldsymbol{x}|^{-2k} \phi \right).$$

Proof. First observe that $l_{\frac{1}{\lambda}} \chi$ equals 1 on a neighbourhood of $\lambda\mathcal{B}$ and $l_{\frac{1}{\lambda}} \phi$ equals 1 on a neighbourhood of $\lambda\mathcal{A}$. As

$$\partial^\alpha \mathcal{F}^{-1}(l_{\frac{1}{\lambda}} \chi) = \partial^\alpha l_{\frac{1}{\lambda}}^* \mathcal{F}^{-1}(\chi) = \lambda^{|\alpha|} l_{\frac{1}{\lambda}}^* \partial^\alpha \mathcal{F}^{-1}(\chi),$$
$$\mathcal{F}^{-1}\left((-2\pi \mathrm{i} \boldsymbol{x})^\alpha |2\pi \boldsymbol{x}|^{-2k} l_{\frac{1}{\lambda}} \phi \right) = \lambda^{|\alpha|-2k} l_{\frac{1}{\lambda}}^* \mathcal{F}^{-1}\left((-2\pi \mathrm{i} \boldsymbol{x})^\alpha |2\pi \boldsymbol{x}|^{-2k} \phi \right),$$

it is sufficient to prove the statements for $\lambda = 1$.

(a) follows from the fact that $\widehat{u} = \chi \widehat{u}$ (see Definition 5.1).

For (b), as $\widehat{u}$ is supported on an annulus, we can divide (and multiply) by $|2\pi \mathfrak{x}|^{2k}$. By the multinormial theorem (see Theorem 20.4, take $x_j = |2\pi\xi_j|^2 = (-2\pi i \xi_j)(2\pi i \xi_j)$):

$$|2\pi\xi|^{2k} = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = k} \binom{k}{\alpha} (-2\pi i \xi)^\alpha (2\pi i \xi)^\alpha \qquad (\xi \in \mathbb{R}^d). \tag{21.18}$$

Therefore by Lemma 19.4 and because $(2\pi i \mathfrak{x})^\alpha(\mathrm{D}) = \partial^\alpha$),

$$\begin{aligned} u = \mathcal{F}^{-1}(\phi \widehat{u}) &= \left(\frac{\sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = k} \binom{k}{\alpha} (-2\pi i \mathfrak{x})^\alpha (2\pi i \mathfrak{x})^\alpha}{|2\pi \mathfrak{x}|^{2k}} \phi \right) (\mathrm{D}) u \\ &= \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = k} \binom{k}{\alpha} \left(\frac{(-2\pi i \mathfrak{x})^\alpha}{|2\pi \mathfrak{x}|^{2k}} \phi \right) (\mathrm{D}) \, \partial^\alpha u, \end{aligned}$$

which proves (b). □

The next theorem follows by the previous lemma and Young's inequality.

Theorem 21.15 (Bernstein's inequalities)**.** *Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball around the origin in $\mathbb{R}^d$. For all $k \in \mathbb{N}$ there exists a $C > 0$ such that such that for all $p, q \in [1, \infty]$ with $q \ge p$ and any $u \in \mathcal{S}'$ we have for all $\lambda > 0$*

$$\operatorname{supp} \widehat{u} \subset \lambda \mathcal{B} \Longrightarrow \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \|\partial^\alpha u\|_{L^q} \le C \lambda^{k + d(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p}, \tag{21.19}$$

$$\operatorname{supp} \widehat{u} \subset \lambda \mathcal{A} \Longrightarrow \frac{1}{C} \lambda^k \|u\|_{L^p} \le \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \|\partial^\alpha u\|_{L^p} \le C \lambda^k \|u\|_{L^p}. \tag{21.20}$$

Proof. The lower bound in (21.20) follows by Lemma 21.14 and Young's inequality,

$$\begin{aligned} \|u\|_{L^p} &\le \lambda^{-k} \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = k} \|\lambda^d (l_\lambda g_\alpha)\|_{L^1} \|\partial^\alpha u\|_{L^p} \\ &\le \lambda^{-k} \Big(\sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = k} \|g_\alpha\|_{L^1} \Big) \Big(\max_{\alpha \in \mathbb{N}_0^d : |\alpha| = k} \|\partial^\alpha u\|_{L^p} \Big), \end{aligned}$$

because $\|\lambda^d (l_\lambda g_\alpha)\|_{L^1} = \|l^*_{\frac{1}{\lambda}} g_\alpha\|_{L^1} = \|g_\alpha\|_{L^1}$ and $g_\alpha \in \mathcal{S} \subset L^1$ for all $\alpha \in \mathbb{N}_0^d$.

The upper bound in (21.20) follows from (21.19), so we are left to prove the latter. Let $r \in [1, \infty]$ be such that $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$. By Young's inequality, Lemma 21.14 and an easy calculation (see Exercise 21.E)

$$\|\partial^\alpha u\|_{L^q} \le \lambda^{k+d} \|l_\lambda h_\alpha\|_{L^r} \|u\|_{L^p} \le \lambda^{k + d - \frac{d}{r}} \|h_\alpha\|_{L^r} \|u\|_{L^p}$$

so that (21.19) follows because $d - \frac{d}{r} = d(1 + \frac{1}{p} - \frac{1}{q} - 1) = d(\frac{1}{p} - \frac{1}{q})$ and because $\|h_\alpha\|_{L^r} \le \|h_\alpha\|_{L^1} + \|h_\alpha\|_{L^\infty}$, see Corollary A.12. □

Exercise **21.F.** Let $s \in \mathbb{R}$. Show that there exists a $C > 0$ such that for all $p, q \in [1, \infty]$, $q \ge p$, any $u \in \mathcal{S}'$ and $\lambda > 0$

$$\operatorname{supp} \widehat{u} \subset \lambda\mathcal{A} \Longrightarrow \|(-\Delta)^{\frac{s}{2}} u\|_{L^q} \le C\lambda^{s+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p} \qquad (u \in \mathcal{S}', \lambda > 0).$$

(Hint: Adapt Lemma 21.14 (a) and follow the proof of Theorem 21.15.)

21.16 (Towards Theorem 21.18)**.** After one more preparation, we turn to a technical theorem, Theorem 21.18, so we want to prepare the reader a little bit. This theorem implies that a Besov space does not depend on the dyadic partition of unity, but it tells us more than that. Indeed, in order to calculate $\|u\|_{B^s_{p,q}[\varphi]}$ one needs to first calculate the L^p norm of all Littlewood–Paley blocks $\Delta_j u$. Theorem 21.18 considers sequences of tempered distributions $(u_j)_{j \in \mathbb{N}_{-1}}$ with

$$\operatorname{supp} \widehat{u}_{-1} \subset \mathcal{B}, \quad \operatorname{supp} \widehat{u}_j \subset 2^j \mathcal{A} \text{ for } j \in \mathbb{N}_0, \tag{21.21}$$

$$\left\| \left(2^{js} \|u_j\|_{L^p} \right)_{j \in \mathbb{N}_{-1}} \right\|_{\ell^q} < \infty. \tag{21.22}$$

First, observe that if $u \in \mathcal{S}'$, then (21.21) is satisfied for $u_j = \Delta_j u$. And if, moreover, $u \in B^s_{p,q}$, then also (21.22) is satisfied with $u_j = \Delta_j u$ and $\|(2^{js}\|u_j\|_{L^p})_{j \in \mathbb{N}_{-1}}\|_{\ell^q} = \|u\|_{B^s_{p,q}}$. Also observe that $u_j \in C^\infty_{\mathrm{p}}$ for all $j \in \mathbb{N}_{-1}$ due to (21.21), see Lemma 17.8.

Theorem 21.18 tells us on the one hand that if (21.21) and (21.22) are satisfied, then $\sum_{j \in \mathbb{N}_{-1}} u_j$ exists in $\mathcal{S}'$, and, on the other hand, that for $u = \sum_{j \in \mathbb{N}_{-1}} u_j$ the Besov norm $\|u\|_{B^s_{p,q}[\varphi]}$ is bounded from above by $\|(2^{js}\|u_j\|_{L^p})_{j \in \mathbb{N}_{-1}}\|_{\ell^q}$.

This theorem is useful as it can be used to estimate $\|u\|_{B^s_{p,q}}[\varphi]$ in case calculating $\Delta_j u$ might be more effort. A very simple example is the following. Suppose $u \in B^s_{p,q}$ and $A \subsetneq \mathbb{N}_{-1}$. Then Theorem 21.18 tells us that $v := \sum_{j \in \mathbb{N}_{-1} \cap A} \Delta_j$ exists in $\mathcal{S}'$, and moreover, it gives an estimate on the Besov norm $\|\Delta_j v\|_{B^s_{p,q}[\varphi]}$ without calculating $\Delta_j v$ for all $j \in \mathbb{N}_{-1}$.

The Young's inequality will be helpful multiple times, not only for convolutions of functions on $\mathbb{R}^d$ (as in Theorem 7.7), but also on $\mathbb{Z}$. Both $\mathbb{R}^d$ and $\mathbb{Z}$ are groups, and one can formulate the Young's inequality more generally for locally compact groups that are equipped with a so-called Haar measure. More on such spaces can be found in books on "abstract harmonic analysis". We formulate the Young's inequality for functions on $\mathbb{Z}$ separately:

Theorem 21.17 (Young's inequality for functions on $\mathbb{Z}$)**.**
Let $p, q, r \in [1, \infty]$ be such that

$$\tfrac{1}{p} + \tfrac{1}{q} \ge 1 + \tfrac{1}{r}.$$

For $f \in \ell^p(\mathbb{Z})$, $g \in \ell^q(\mathbb{Z})$ we have $f * g \in \ell^r(\mathbb{Z})$ and

$$\|f * g\|_{\ell^r} \le \|f\|_{\ell^p} \|g\|_{\ell^q}.$$

Proof. Let $s \in [1,\infty]$, $s \geq r$, and $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{s}$. Then $\|\cdot\|_{\ell^r} \leq \|\cdot\|_{\ell^s}$, see Lemma A.8. Therefore, we may as well assume $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$.

Now the proof follows in a similar fashion as the proof of Theorem 7.7, but with applying Hölder's inequality (Theorem A.4) to the sequence spaces $\ell^p(\mathbb{Z})$ (observe $\ell^p(\mathbb{Z})$ equals $L^p(\mu)$ where μ is the counting measure on $\mathbb{Z}$). □

Now we are ready to prove the rather technical theorem with a long statement. Let us mention beforehand that (b) and (c) are similar to (a).

Theorem 21.18. *Let $s \in \mathbb{R}$ and $p,q \in [1,\infty]$. Let $\mathcal{B}$ be a ball centered at the origin and $\mathcal{A}$ be an annulus. Let φ be a function that generates a partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$.*

(a) *There exist $C>0$ and $m \in \mathbb{N}_0$ such that for all sequences of tempered distributions $(u_j)_{j\in\mathbb{N}_{-1}}$ with*

$$\operatorname{supp}\widehat{u}_{-1} \subset \mathcal{B}, \quad \operatorname{supp}\widehat{u}_j \subset 2^j\mathcal{A} \text{ for } j \in \mathbb{N}_0, \qquad \left\| \left(2^{js}\|u_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} < \infty,$$

one has that

$$\begin{cases} u := \sum_{j\in\mathbb{N}_{-1}} u_j \text{ exists in } \mathcal{S}', \\ |\langle u,\psi\rangle| \leq C \left\| (2^{js}\|u_j\|_{L^p})_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \|\psi\|_{m,\mathcal{S}} \qquad (\psi \in \mathcal{S}), \end{cases} \tag{21.23}$$

and

$$\|u\|_{B^s_{p,q}[\varphi]} \leq C \left\| \left(2^{js}\|u_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q}. \tag{21.24}$$

(Here one may take $C = 2\|\widehat{\varphi_{-1}}\|_{L^1}$ for (21.24).)

(b) *If $s>0$, then there exist $C>0$ and $m\in\mathbb{N}_0$ such that for all sequences of tempered distributions $(u_j)_{j\in\mathbb{N}_{-1}}$ with*

$$\operatorname{supp}\widehat{u}_j \subset 2^j\mathcal{B} \text{ for all } j \in \mathbb{N}_{-1}, \qquad \left\| \left(2^{js}\|u_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} < \infty, \tag{21.25}$$

(21.23) *and* (21.24) *hold. (Here there exists an $N \in \mathbb{N}$ such that we may take $C = 2\|\widehat{\varphi_{-1}}\|_{L^1}\frac{2^{Ns}}{2^s-1}$ for* (21.24).*)*

(c) *If $s=0$ and $q=1$, then there exist $C>0$ and $m \in \mathbb{N}_0$ such that for all sequences of tempered distributions $(u_j)_{j\in\mathbb{N}_{-1}}$ with* (21.25), (21.23) *holds and*

$$\|u\|_{B^0_{p,\infty}[\varphi]} = \sup_{j\in\mathbb{N}_{-1}} \|\varphi_j(\mathrm{D})u\|_{L^p} \leq \left\| (\|u_j\|_{L^p})_{j\in\mathbb{N}_{-1}} \right\|_{\ell^1}. \tag{21.26}$$

Proof. First observe that if $(u_j)_{j\in\mathbb{N}_{-1}}$ is a sequence of tempered distributions, then $\sum_{j\in\mathbb{N}_{-1}} u_j$ exists in $\mathcal{S}'$ if there exists a $u \in \mathcal{S}'$ such that for all bijections $q:\mathbb{N}\to\mathbb{N}_{-1}$

the sum $\sum_{j=1}^{J} u_{q(j)}$ converges to u in $\mathcal{S}'$ as $J \to \infty$. As $\mathcal{S}'$ is weak* sequentially complete (Theorem 15.10 (b)), it is sufficient to show $\sum_{j=-1}^{\infty} |\langle u_j, \psi\rangle| < \infty$ for all $\psi \in \mathcal{S}$. In order to also show that (21.23) holds, it is sufficient to prove

$$\sum_{j=-1}^{\infty} |\langle u_j, \psi\rangle| \le C \left\| \left(2^{js}\|u_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \|\psi\|_{m,\mathcal{S}} \qquad (\psi \in \mathcal{S}). \tag{21.27}$$

Without loss of generality, as we may take a larger ball and annulus, we may assume $\operatorname{supp}\varphi_{-1} \subset \mathcal{B}$ and $\operatorname{supp}\varphi_0 \subset \mathcal{A}$. We write $\Delta_j = \varphi_j(\mathrm{D})$ for $j \in \mathbb{N}_{-1}$.

In the rest of the proof we let $r \in [1, \infty]$ be such that

$$\frac{1}{p} + \frac{1}{r} = 1.$$

Let $n \in \mathbb{N}$ and a $C_1 > 0$ be such that (see Lemma 14.14)

$$\|\psi\|_{L^r} \le C_1 \|\psi\|_{n,\mathcal{S}} \qquad (\psi \in \mathcal{S}). \tag{21.28}$$

Let $C_2 > 0$ be such that (for example $C_2 = 2\|\widehat{\varphi_{-1}}\|_{L^1}$ see (21.12) and (21.13) with $J = -1$)

$$\|\Delta_j f\|_{L^p} \le C_2 \|f\|_{L^p} \qquad (f \in L^p). \tag{21.29}$$

Finally, let $N \in \mathbb{N}$ be such that

$$2^j\mathcal{A} \cap \mathcal{A} = 2^j\mathcal{A} \cap \mathcal{B} = \emptyset \qquad (j \ge N). \tag{21.30}$$

We start with (b) and (c) as these are the easier cases.

Proof of (c) Suppose $(u_j)_{j\in\mathbb{N}_{-1}}$ is as in (c). By Hölder's inequality

$$\sum_{j=-1}^{\infty} |\langle u_j, \psi\rangle| \le \sum_{j=-1}^{\infty} \|u_j\|_{L^p}\|\psi\|_{L^r} \le C_1 \left\|(\|u_j\|_{L^p})_{j\in\mathbb{N}_{-1}}\right\|_{\ell^1} \|\psi\|_{n,\mathcal{S}},$$

so that we obtain (21.27) with $q = 1$. We obtain (21.26) by using

$$\|\Delta_j u\|_{L^p} \le \sum_{i\in\mathbb{N}_{-1}} \|\Delta_j u_i\|_{L^p}.$$

Proof of (b) Suppose that $s > 0$ and $(u_j)_{j\in\mathbb{N}_{-1}}$ is as in (b). By Hölder's inequality we obtain

$$|\langle u_j, \psi\rangle| \le \|u_j\|_{L^p}\|\psi\|_{L^r} \le C_1 2^{-js} \left\| \left(2^{js}\|u_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \|\psi\|_{n,\mathcal{S}}.$$

As $s > 0$, we obtain (21.27) with $C = C_1 \sum_{j\in\mathbb{N}_{-1}} 2^{-js}$.

By (21.30) we have $\Delta_j u_i = 0$ for all $j \ge i + N$ and so by (21.29)

$$2^{js}\|\Delta_j u\|_{L^p} \le \sum_{i\in\mathbb{N}_{-1}: i>j-N} 2^{(j-i)s}2^{is}\|\Delta_j u_i\|_{L^p}$$
$$\le C_2 \sum_{i\in\mathbb{N}_{-1}: i>j-N} 2^{(j-i)s}2^{is}\|u_i\|_{L^p} = C_2(a*b)(j),$$

where $a(k) = 2^{ks}\mathbb{1}_{(-\infty,N)}(k)$ for $k \in \mathbb{Z}$ and $b(k) = 2^{ks}\|u_k\|_{L^p}$ for $k \in \mathbb{N}_{-1}$ and $b(k) = 0$ for $k \in \mathbb{Z} \setminus \mathbb{N}_{-1}$. By Young's inequality (Theorem 21.17), we obtain the desired bound as

$$\|a\|_{\ell^1} = \sum_{k\in\mathbb{Z}:k<N} 2^{ks} = \sum_{k\in\mathbb{N}_0} 2^{(N-1-k)s} = \frac{2^{(N-1)s}}{1-2^{-s}} = \frac{2^{Ns}}{2^s-1}.$$

Proof of (a) Suppose $(u_j)_{j\in\mathbb{N}_{-1}}$ is as in (a).

• Let us first prove (21.27). Let $k \in \mathbb{N}_0$ be such that $k > -s$. Let $\psi \in \mathcal{S}$. For all $j \in \mathbb{N}_0$ we have by Lemma 21.14 (b)

$$u_j = 2^{-jk} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} 2^{jd}(l_{2^j}g_\alpha) * \partial^\alpha u_j.$$

Therefore,

$$\langle u_j, \psi\rangle = 2^{-jk}(-1)^k \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} \langle u_j, 2^{jd}(l_{2^j}\check{g}_\alpha) * \partial^\alpha\psi\rangle.$$

By Hölder's and by Young's inequality, and then by (21.28) and (21.17),

$$|\langle u_j, \psi\rangle| \le 2^{-jk}\|u_j\|_{L^p} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} \|2^{jd}(l_{2^j}\check{g}_\alpha)\|_{L^1}\|\partial^\alpha\psi\|_{L^r}$$
$$\le C_1 2^{-jk}\|u_j\|_{L^p} \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} \|g_\alpha\|_{L^1}\|\psi\|_{n+k,\mathcal{S}}.$$

Therefore with

$$C_3 = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k} \|g_\alpha\|_{L^1},$$

we have for all $j \in \mathbb{N}_0$

$$|\langle u_j, \psi\rangle| \le C_1C_3 2^{-jk}\|u_j\|_{L^p}\|\psi\|_{n+k,\mathcal{S}}$$
$$\le C_1C_3 2^{-j(k+s)} \left\| \left(2^{js}\|u_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \|\psi\|_{n+k,\mathcal{S}}.$$

We may assume that the above also holds for $j = -1$, as by a direct application of Hölder's inequality we have

$$|\langle u_{-1}, \psi\rangle| \le \|u_{-1}\|_{L^p}\|\psi\|_{L^r} \le C_1\|u_{-1}\|_{L^p}\|\psi\|_{n,\mathcal{S}}$$
$$\le C_1 2^s \left\| \left(2^{js}\|u_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \|\psi\|_{n+k,\mathcal{S}}.$$

As $k+s>0$, there exists a $C>0$ such that (21.27) holds with $m=n+k$.

• We prove (21.24). By (21.30) $\Delta_j u_i = 0$ for all $i,j\in\mathbb{N}_{-1}$ with $|i-j|\geq N$. As $2^{(j-i)s}\leq 2^{|s|N}$ for $i,j\in\mathbb{N}_{-1}$ with $|i-j|\leq N$, with (21.29), for all $j\in\mathbb{N}_{-1}$

$$2^{js}\|\Delta_j u\|_{L^p} \leq \sum_{i=(j-N)\vee -1}^{j+N} 2^{js}\|\Delta_j u_i\|_{L^p} \leq C_2 \sum_{i=(j-N)\vee -1}^{j+N} 2^{is}\|u_i\|_{L^p} = C_2(a*b)_j,$$

where $a(k)=\mathbb{1}_{[-N,N)}(k)$ for $k\in\mathbb{Z}$ and $b(k)=2^{ks}\|u_k\|_{L^p}$ for $k\in\mathbb{N}_{-1}$ and $b(k)=0$ for $k\in\mathbb{Z}\setminus\mathbb{N}_{-1}$. Therefore by Young's inequality (Theorem 21.17),

$$\left\|(2^{js}\|\Delta_j u\|_{L^p})_{j\in\mathbb{N}_{-1}}\right\|_{\ell^q(\mathbb{N}_{-1})} \leq C_2\|a*b\|_{\ell^q(\mathbb{Z})} \leq C_2\|a\|_{\ell^1(\mathbb{Z})}\|b\|_{\ell^q(\mathbb{Z})}.$$

As $\|b\|_{\ell^q(\mathbb{Z})} = \left\|(2^{js}\|u_j\|_{L^p})_{j\in\mathbb{N}_{-1}}\right\|_{\ell^q(\mathbb{N}_{-1})}$ and $\|a\|_{\ell^1(\mathbb{Z})}=2N+1$, this finished the proof for (a). □

Remark 21.19. So again, Theorem 21.18 gives us an upper bound of the Besov norm of a tempered distribution u if u can be separated into smooth pieces u_j for $j\in\mathbb{N}_{-1}$ with certain properties which are similar to those that the $\Delta_j u$ satisfy, although less restrictive. Indeed, for $u=0$ one may take u_{-1} and u_0 to be nonzero, in L^p, such that $u_0=-u_{-1}$ and $\operatorname{supp}\widehat{u_{-1}}=\operatorname{supp}\widehat{u_0}\subset\mathcal{B}\cap\mathcal{A}$. In this case we see that the inequality (21.24) cannot be reversed as the left-hand side of (21.24) is zero and the right-hand side is not.

Corollary 21.20. *Let $s\in\mathbb{R}$ and $p,q\in[1,\infty]$. Let φ and ψ generate dyadic partitions of unity. Then $B^s_{p,q}[\varphi]=B^s_{p,q}[\psi]$ and there exists a $C>1$ such that*

$$\frac{1}{C}\|u\|_{B^s_{p,q}[\psi]} \leq \|u\|_{B^s_{p,q}[\varphi]} \leq C\|u\|_{B^s_{p,q}[\psi]} \qquad (u\in\mathcal{S}'). \tag{21.31}$$

In particular, $\|\cdot\|_{B^s_{p,q}[\varphi]}$ and $\|\cdot\|_{B^s_{p,q}[\psi]}$ are equivalent norms on $B^s_{p,q}[\varphi]$.

Proof. Let $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ and $(\psi_j)_{j\in\mathbb{N}_{-1}}$ be the dyadic partitions of unity generated by φ and ψ, respectively. There exist a ball $\mathcal{B}$ and an annulus $\mathcal{A}$ such that $\operatorname{supp}\varphi_{-1},\operatorname{supp}\psi_{-1}\subset\mathcal{B}$ and $\operatorname{supp}\varphi_0,\operatorname{supp}\psi_0\subset\mathcal{A}$. By Theorem 21.18 (a) there exists a $\mathfrak{C}>0$ such that

$$\|u\|_{B^s_{p,q}[\varphi]} \leq \mathfrak{C}\|u\|_{B^s_{p,q}[\psi]} \qquad (u\in\mathcal{S}').$$

By interchanging the roles of φ and ψ we obtain (21.31). □

Definition 21.21. Let $s\in\mathbb{R}$ and $p,q\in[1,\infty]$. We define the *nonhomogeneous Besov space* $B^s_{p,q}$ to be equal to $B^s_{p,q}[\varphi]$, where φ is a function that generates a dyadic partition of unity. Often we write $\|\cdot\|_{B^s_{p,q}}$ for $\|\cdot\|_{B^s_{p,q}[\varphi]}$ and without mentioning the partition of unity, write Δ_j for the Littlewood–Paley operators corresponding to a partition of unity.

Basically all statements that follow about the function $\|\cdot\|_{B^s_{p,q}}$ are about estimations with respect to a scalar times another function. For such statements, the choice of partition is of course irrelevant.

Exercise **21.G.** Let $p \in [1,\infty]$.

(a) Let $u \in L^p$ and suppose that $\operatorname{supp} \widehat{u}$ is compact. Show that for all $s \in \mathbb{R}$ and $q \in [1,\infty]$, $u \in B^s_{p,q}$.

(b) Let $u \in B^s_{p,q}$ for some $q \in [1,\infty]$ and $s \in \mathbb{R}$. Suppose that $\operatorname{supp} \widehat{u}$ is compact. Show that $u \in L^p$.

Example 21.22. Remember that in Example 20.6 we found that $\delta_0 \in H^s$ if and only if $s < -\frac{d}{2}$.

Now let us consider in which Besov space the Dirac delta, δ_0, lies. Using that $\widehat{\delta_0} = \mathbb{1}$, one has $\Delta_j \delta_0 = \mathcal{F}^{-1}(\varphi_j)$ for all $j \in \mathbb{N}_{-1}$. Let $p, q \in [1,\infty]$. Then, by (21.16) and Exercise 21.E, with the convention that $\frac{d}{\infty} = 0$,

$$\begin{aligned} \|\Delta_j \delta_0\|_{L^p} &= \|\mathcal{F}^{-1}(\varphi_j)\|_{L^p} = 2^{-j(\frac{d}{p}-d)} \|\mathcal{F}^{-1}(\varphi_0)\|_{L^p} \qquad (j \in \mathbb{N}_0), \\ \|\Delta_{-1}\delta_0\|_{L^p} &= \|\widehat{\varphi_{-1}}\|_{L^p}. \end{aligned}$$

Therefore,

$$\delta_0 \in B^s_{p,q} \iff s < -d(1-\tfrac{1}{p}) \qquad (q < \infty).$$

So that in particular, $\delta_0 \in B^{-d(1-\frac{1}{p})}_{p,\infty}$ and $\delta_0 \in B^{-d(1-\frac{1}{p})-\varepsilon}_{p,q}$ for all $\varepsilon > 0$.

Observe that for any $q \in [1,\infty]$ we have $\delta_0 u \in B^s_{2,q}$ if and only if $s < -\frac{d}{2}$. As the reader may already expect from the motivation in 21.1, one has $B^s_{2,2} = H^s$. The proof of this will be given in Theorem 23.2.

Exercise **21.H.** Show that for $\varepsilon > 0$ the function $z \mapsto \delta_z$ is continuous on $B^{-\varepsilon}_{1,\infty}$ but that it is not continuous on $B^0_{1,\infty}$. Hint: First prove: *for any $f \in L^1$ and $z \in \mathbb{R}^d \setminus \{0\}$,*

$$\lim_{a \to \infty} \|\mathcal{T}_{az} f - f\|_{L^1} = 2\|f\|_{L^1}.$$

This statement can be proved by approximating f by $f\mathbb{1}_K$ for a 'large' compact set K.

Similarly to Lemma 12.8, Lemma 12.9 and Lemma 20.8, we have the following.

Theorem 21.23. *For all $\alpha \in \mathbb{N}_0^d$ there exists a $C > 0$ such that for all $s, t \in \mathbb{R}$, $p_1, p_2, q_1, q_2 \in [1,\infty]$, with*

$$p_2 \geq p_1, \qquad q_2 \geq q_1, \qquad t \leq s - d(\tfrac{1}{p_1} - \tfrac{1}{p_2}), \tag{21.32}$$

one has

$$\|\partial^\alpha u\|_{B^{t-|\alpha|}_{p_2,q_2}} \leq C \|u\|_{B^s_{p_1,q_1}} \qquad (u \in \mathcal{S}'). \tag{21.33}$$

In particular, $\partial^\alpha : B^s_{p,q} \to B^{s-|\alpha|}_{p,q}$ is continuous for all $s \in \mathbb{R}$, $p, q \in [1,\infty]$ and $\alpha \in \mathbb{N}_0^d$, and for all $s, t \in \mathbb{R}$, $p_1, p_2, q_1, q_2 \in [1,\infty]$, $p_2 \geq p_1, q_2 \geq q_1$,

$$B^s_{p_1,q_1} \hookrightarrow B^t_{p_2,q_2} \qquad (t \leq s - d(\tfrac{1}{p_1} - \tfrac{1}{p_2})).$$

Proof. This follows by Bernstein's inequality, Theorem 21.15: By taking $\lambda = 2^j$ and using that $\Delta_j \partial^\alpha = \partial^\alpha \Delta_j$ it implies that there exists a $C > 0$ such that

$$\|\Delta_j \partial^\alpha u\|_{L^{p_2}} \le C 2^{j(|\alpha| + d(\frac{1}{p_1} - \frac{1}{p_2}))} \|\Delta_j u\|_{L^{p_1}} \qquad (u \in \mathcal{S}').$$

Therefore

$$\|\partial^\alpha u\|_{B^t_{p_2,q_2}} \le C \|u\|_{B^{t+|\alpha|+d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_1,q_2}} \qquad (u \in \mathcal{S}').$$

By monotonicity of the norm $\|\cdot\|_{\ell^q}$ in q (see Lemma A.8) and by monotonicity of the norm $\|\cdot\|_{B^s_{p,q}}$ in s (see Exercise 21.D) we obtain (21.33). ☐

Exercise **21.I.** Let $t \in \mathbb{R}$, $p, q \in [1, \infty]$. Show that for $u \in \mathcal{S}'_\circ$ (see Definition 19.8)

$$\|(-\Delta)^{\frac{s}{2}} u\|_{B^{t-s}_{p,q}} \le \|u\|_{B^t_{p,q}}.$$

(Hint: Exercise 21.F.)

Exercise **21.J.** (a) Let $\psi \in \mathcal{S}$. Show that $\psi \in B^0_{p,\infty}$ for all $p \in [1, \infty]$.

(b) (Optional, will be proven also in the next section.) Show that $\psi \in B^s_{p,q}$ for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$. (Hint: First consider $q = \infty$ and use (a) and Bernstein's inequalities.)

Exercise **21.K.** Show that the distribution $|\mathfrak{x}|^2$ is a tempered distribution which is not in $B^s_{p,q}$ for any $s \in \mathbb{R}$, $p, q \in [1, \infty]$.

21.24 (Some formal language as explanation)**.**
Let us formally describe what Theorem 21.23 means, but first give names to the parameters of Besov spaces. We refer to the parameters in "$B^s_{p,q}$" by calling "s" the *regularity parameter* and calling "p" and "q" the *first and second integration parameter*. In the literature one also says that Besov spaces are examples of function spaces with mixed regularity.

First, let us describe what it means for an element to be in a Besov space in this formal language. A tempered distribution u is in $B^s_{p,q}$ for some s, p and q, if $\Delta_j u$ is in L^p for all j and $\|\Delta_j u\|_{L^p}$ cannot grow too rapidly. Take for example $q = \infty$. By writing N for the Besov norm of u, $N = \|u\|_{B^s_{p,q}}$, we see that $2^{js}\|\Delta_j u\|_{L^p} < N$ and thus $\|\Delta_j u\|_{L^p} \le 2^{-js} N$ for all j. Now we see that if s is strictly positive, then $\|\Delta_j u\|_{L^p}$ needs to decrease fast enough as j increases and if s is negative, then one only allows a not too fast growth of $\|\Delta_j u\|_{L^p}$.

Now Theorem 21.23 tells us on the one hand by taking a derivative, the regularity parameter decreases by the number of (directional) derivatives taken. On the other hand, it tells us that for an element in a Besov space we can increase the first integration parameter at the cost of decreasing the regularity parameter. Also we can increase the second integration parameter without changing the first and the regularity parameter. The following lemma states that one can also decrease the second parameter by paying the littlest amount of regularity.

Lemma 21.25. *For all $q_1, q_2 \in [1,\infty]$ and $\varepsilon > 0$ there exists a $C > 0$ such that for all $s \in \mathbb{R}$ and $p \in [1,\infty]$*

$$\|u\|_{B^{s-\varepsilon}_{p,q_2}} \le C\|u\|_{B^s_{p,q_1}} \qquad (u \in \mathcal{S}'). \tag{21.34}$$

That is,

$$B^s_{p,q_1} \hookrightarrow B^{s-\varepsilon}_{p,q_2}.$$

Proof. If $q_1 \le q_2$, then this follows directly from Theorem 21.23 (even in case $\varepsilon = 0$). Therefore we assume $q_1 > q_2$. The case $q_1 = \infty$ has already been treated in Exercise 21.D. Let $u \in B^{s-\varepsilon}_{p,q_2}$ and $a_j := \|\Delta_j u\|_{L^p}$. Observe that

$$\frac{q_2}{q_1} + \frac{q_1 - q_2}{q_1} = 1.$$

Therefore, by Hölder's inequality, letting $r = \frac{q_1 q_2}{q_1 - q_2}$,

$$\begin{aligned}
\|u\|_{B^{s-\varepsilon}_{p,q_2}} &= \|(2^{j(s-\varepsilon)}a_j)_{j\in\mathbb{N}_{-1}}\|_{\ell^{q_2}} = \Big(\sum_{j\in\mathbb{N}_{-1}} 2^{-j\varepsilon q_2}(2^{js}a_j)^{q_2}\Big)^{\frac{1}{q_2}} \\
&\le \Big(\sum_{j\in\mathbb{N}_{-1}} 2^{-j\varepsilon\frac{q_1q_2}{q_1-q_2}}\Big)^{\frac{q_1-q_2}{q_1q_2}} \Big(\sum_{j\in\mathbb{N}_{-1}} (2^{js}a_j)^{q_1}\Big)^{\frac{1}{q_1}} \\
&= \|(2^{-j\varepsilon})_{j\in\mathbb{N}_{-1}}\|_{L^r}\|(2^{js}a_j)_{j\in\mathbb{N}_{-1}}\|_{\ell^{q_1}}.
\end{aligned}$$

So that with $C = \|(2^{-j\varepsilon})_{j\in\mathbb{N}_{-1}}\|_{L^r}$ we obtain (21.34). □

Theorem 21.23 and Lemma 21.25 imply for all $s \in \mathbb{R}$, $q_1, q_2 \in [1,\infty]$ with $q_2 \ge q_1$ and $\varepsilon > 0$

$$B^s_{\infty,q_1} \subset B^s_{\infty,q_2}, \qquad B^s_{\infty,q_2} \subset B^{s-\varepsilon}_{\infty,q_1}.$$

The following example illustrates (at least for $d = 1$) that those inclusions are strict, i.e.,

$$B^s_{\infty,q_1} \subsetneq B^s_{\infty,q_2}, \qquad \text{for } q_2 > q_1 \quad B^s_{\infty,q_2} \subsetneq B^{s-\varepsilon}_{\infty,q_1}.$$

We extend the idea behind the following example in the proof of Theorem 22.5 to show that all inclusions that one obtains from Theorem 21.23 and Lemma 21.25 are strict. Example 21.26 will be considered again in the next section.

Example 21.26. Let $d = 1$. Let $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ be a dyadic partition of unity such that $\varphi_0 = 1$ on a neighbourhood V of 1 (so that $\varphi_i = 1$ on $\pm 2^j$ if $i = j$ and 0 otherwise). For $n \in \mathbb{N}$, define $v_n, u_n \in \mathcal{S}'$ either by

$$v_n = \tfrac{1}{2}(\delta_{2^n} + \delta_{-2^n}) \qquad u_n = \cos(2\pi 2^n \mathfrak{x}),$$

or by

$$v_n = \tfrac{1}{2\mathrm{i}}(\delta_{2^n} - \delta_{-2^n}) \qquad u_n = \sin(2\pi 2^n \mathfrak{x}).$$

Observe that for $n \in \mathbb{N}$,

- $\widehat{v_n} = u_n$ and $\widehat{u_n} = v_n$,
- $u_n \in L^\infty$ and $\|u_n\|_{L^\infty} = 1$,
- $\varphi_j v_n = v_n$ and $\Delta_j u_n = u_n$ if $j = n$ and 0 otherwise for all $j \in \mathbb{N}_{-1}$.

Let $q \in [1, \infty]$, $s \in \mathbb{R}$. Define $a, b : \mathbb{N} \to \mathbb{R}$ by ($\frac{1}{\infty} = 0$)

$$a(n) = n^{-\frac{1}{q}}, \qquad b(n) = 2^{-ns} a(n) \qquad (n \in \mathbb{N}),$$

so that

$$a \in \ell^r \iff r \geq q, \qquad \|(2^{ns} b(n))_{n\in\mathbb{N}}\|_{\ell^r} = \|a\|_{\ell^r} \qquad (r \in [1, \infty]).$$

Because $\|(2^{ns}\|b(n)u_n\|_{L^\infty})_{n\in\mathbb{N}}\|_{\ell^q} = \|a\|_{\ell^q} < \infty$, by Theorem 21.18 (a),

$$u := \sum_{n\in\mathbb{N}} b(n) u_n \text{ exists in } \mathcal{S}',$$

Moreover, for $r \in [1, \infty]$ and $t \in \mathbb{R}$

$$\|u\|_{B^t_{\infty,r}} = \|(2^{n(t-s)} a(n))_{n\in\mathbb{N}}\|_{\ell^r}$$

which is finite if and only if $t = s$ and $r \geq q$ or $t < s$. Therefore

$$u \in B^t_{\infty,r} \iff \begin{cases} t = s, r \geq q, \\ t < s. \end{cases}$$

The previous example can be used to explain the usage of the word "frequencies", a term that is used regularly in the literature:

21.27 (Formal language in terms of frequencies)**.**
Let us consider the formal language in 21.24 with regard to Example 21.26. We can say that the frequency of u_n is equal (or proportional to) 2^n. Observe that $\Delta_n u$ is a multiple of u_n, which is either of the functions $\cos(2\pi 2^n \mathfrak{x})$ and $\sin(2\pi 2^n \mathfrak{x})$ for $n \in \mathbb{N}$. It is rather natural to call "2^n" the frequency of such a u_n. Formally, for general tempered distributions u, one often says: "$\Delta_j u$ captures the behaviour of u at those frequencies of order 2^j". Moreover, for example in [Saw18], the support of $\widehat{u}$ is also called the "frequency support of u".

With this language, the decomposition of u in terms of its Littlewood–Paley blocks $\Delta_j u$, also called the *Littlewood–Paley decomposition*, is a decomposition of u into frequency levels (or blocks of frequencies) of different (dyadic) order.

Now with this "frequency" language, one could say that $u \in B^s_{p,q}$ if the behaviour of u at the frequencies of order 2^j does not increase too fast ($s < 0$) or decreases fast enough ($s > 0$).

In Theorem 21.32 we will show that Besov spaces are Banach spaces. Moreover, we prove another property, namely: Every bounded sequence in a Besov space has a subsequence that converges in $\mathcal{S}'$ to an element of that Besov space whose Besov norm is bounded by the lim inf of the norms of the subsequence. Probably due to this limiting inequality, this property is in the literature sometimes called the “Fatou property”.

In order to prove this property for Besov spaces, we first prove it for the spaces L^p and $\mathcal{M}$. For that, let us recall two facts from Functional Analysis. For a proof of the following theorem, see for example [Con90, Theorems III.5.5 and III.5.7]. (For the definitions of $\mathcal{M}$ and C_0 see Definition 2.23 and Definition 2.29.)

Theorem 21.28. (a) *Let $p \in (1,\infty]$ and $q \in [1,\infty)$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Then L^p is isometrically isomorphic to $(L^q)'$, the dual of L^q. In particular, with $\langle v, f\rangle = \int vf$ for $v \in L^p, f \in L^q$,*

$$\|v\|_{L^p} = \sup\{|\langle v, f\rangle| : f \in L^q, \|f\|_{L^q} \le 1\} \qquad (v \in L^p).$$

(b) *$\mathcal{M}$ is isometrically isomorphic to C_0', the dual of C_0. In particular, with $\langle \mu, f\rangle = \int f \,\mathrm{d}\mu$ for $\mu \in \mathcal{M}, f \in C_0$,*

$$\|\mu\|_{\mathcal{M}} = \sup\{|\langle \mu, f\rangle| : f \in C_0, \|f\|_{C_0} \le 1\} \qquad (\mu \in \mathcal{M}).$$

The other fact we recall from Functional Analysis is the separable version of Alaoglu's theorem. For a proof combine [Con90, Theorem III.3.1 and III.5.1] or [Rud91, Theorem 3.15 and 3.16].

Theorem 21.29 (Alaoglu's theorem, separable version)**.**
Let $\mathfrak{X}$ be a separable normed space. Then the closed unit ball in $\mathfrak{X}'$, $\{f \in \mathfrak{X}' : \|f\|_{\mathfrak{X}'} \le 1\}$ is weak-sequentially compact.*

Lemma 21.30. *Let $p \in (1,\infty]$. Let $\mathfrak{X}$ be either the Banach space L^p or $\mathcal{M}$. If $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{X}$ that is bounded with respect to the norm on $\mathfrak{X}$, then it has a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ that converges in $\mathcal{S}'$ to an element u of $\mathfrak{X}$ with*

$$\|u\|_{\mathfrak{X}} \le \liminf_{m\to\infty} \|u_{n_m}\|_{\mathfrak{X}}. \tag{21.35}$$

Proof. Let $\mathfrak{Y}$ be either $\mathfrak{Y} = L^q$ or $\mathfrak{Y} = C_0$ (see Theorem 21.28), so that $\mathfrak{X}$ is isometrically isomorphic to $\mathfrak{Y}'$, the dual of $\mathfrak{Y}$. Then there is a pairing $\langle\cdot,\cdot\rangle$ between $\mathfrak{X}$ and $\mathfrak{Y}$ with

$$\|u\|_{\mathfrak{X}} = \sup\{|\langle u, f\rangle : f \in \mathfrak{Y}, \|f\|_{\mathfrak{Y}} \le 1\} \qquad (u \in \mathfrak{X}).$$

Alaoglu's theorem ensures the existence of a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ and an element $u \in \mathfrak{Y}'$ such that

$$\lim_{m\to\infty} \langle u_{n_m}, f\rangle = \langle u, f\rangle \qquad (f \in \mathfrak{Y}).$$

For $f \in \mathfrak{Y}$ we have

$$|\langle u, f\rangle| = \liminf_{m\to\infty} |\langle u_{n_m}, f\rangle| \le \liminf_{m\to\infty} \|u_{n_m}\|_{\mathfrak{X}} \|f\|_{\mathfrak{Y}},$$

so that $\|u\|_{\mathfrak{X}} \le \liminf_{m\to\infty} \|u_{n_m}\|_{\mathfrak{X}}$.

The continuous embedding $\mathcal{S} \hookrightarrow \mathfrak{Y}$ entails that u may be viewed as an element of $\mathcal{S}'$ and that $\lim_{m\to\infty} u_{n_m} = u$ in $\mathcal{S}'$. □

Exercise **21.L.** Show that the statement in Lemma 21.30 for $p = 1$ does not hold.

With an additional assumption on the supports of the Fourier transforms, we do derive the same conclusion as in Lemma 21.30 even for $p = 1$.

Lemma 21.31. *Let $p \in [1,\infty]$. Let $K \subset \mathbb{R}^d$ be compact. If $(u_n)_{n\in\mathbb{N}}$ is a sequence in L^p that is bounded in the L^p norm and such that $\operatorname{supp} \widehat{u}_n \subset K$ for all $n \in \mathbb{N}$, then it has a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ that converges in $\mathcal{S}'$ to an element u of L^p, $\operatorname{supp} \widehat{u} \subset K$, and*

$$\|u\|_{L^p} \le \liminf_{m\to\infty} \|u_{n_m}\|_{L^p}. \tag{21.36}$$

Proof. If $p > 1$, then this immediately follows from Lemma 21.30. If $p = 1$ and $(u_n)_{n\in\mathbb{N}}$ is bounded in L^1, then it is bounded in $\mathcal{M}$ as (see Lemma 2.26)

$$\|f\|_{L^1} = \|f\|_{\mathcal{M}} \qquad (f \in L^1), \tag{21.37}$$

(where we identified f with the distribution f which corresponds to the Radon measure $f\lambda$ with λ the Lebesgue measure). By Lemma 21.30 it has a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ that converges in $\mathcal{S}'$ to an element u of $\mathcal{M}$ such that (21.35) holds. As $\widehat{u}_n \to \widehat{u}$, $\operatorname{supp} \widehat{u} \subset K$. This implies $u \in C_{\mathrm{p}}^\infty$, and therefore, by (21.37) we have $u \in L^1$ and (21.36) for $p = 1$. □

Theorem 21.32. *Let $s \in \mathbb{R}$ and $p, q \in [1,\infty]$. $B^s_{p,q}$ is a Banach space that is continuously embedded in $\mathcal{S}'$. Moreover, if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $B^s_{p,q}$ that is bounded in the $B^s_{p,q}$ norm, then it has a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ that converges in $\mathcal{S}'$ to an element u, which is also in $B^s_{p,q}$ and*

$$\|u\|_{B^s_{p,q}} \le \liminf_{m\to\infty} \|u_{n_m}\|_{B^s_{p,q}}.$$

Proof. That $B^s_{p,q}$ is continuously embedded in $\mathcal{S}'$ follows from (21.23) in Theorem 21.18. We will prove that $B^s_{p,q}$ is complete after proving the “Moreover” statement.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence that is bounded in $B^s_{p,q}$. Without loss of generality we may assume that $\|u_n\|_{B^s_{p,q}} \le 1$ for all $n \in \mathbb{N}$. Then

$$\|\Delta_j u_n\|_{L^p} \le 2^{-sj} \qquad (n \in \mathbb{N}, j \in \mathbb{N}_{-1}).$$

By applying Lemma 21.31 to $(\Delta_j u_n)_{n\in\mathbb{N}}$ for each j, and applying Cantor's diagonal argument, we find a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ and $v_j \in \mathcal{S}'$ for all $j \in \mathbb{N}_{-1}$ such that

$$\Delta_j u_{n_m} \xrightarrow{m\to\infty} v_j \text{ in } \mathcal{S}', \qquad \|v_j\|_{L^p} \le \liminf_{m\to\infty} \|\Delta_j u_{n_m}\|_{L^p} \le 2^{-sj} \qquad (j \in \mathbb{N}_{-1}).$$

As the support of the Fourier transform of $\Delta_j u_n$ is in the annulus $2^j\mathcal{A}$ (or ball $\mathcal{B}$), so is the support of $\widehat{v}_j$ for $j \in \mathbb{N}_0$ (for $j = -1$).

As $\|(2^{sj}\|v_j\|_{L^p})_{j\in\mathbb{N}_{-1}}\|_{\ell^\infty} \le 1$, Theorem 21.18 (a) implies that $v := \sum_{j\in\mathbb{N}_{-1}} v_j$ exists in $\mathcal{S}'$. Observe that $v_j = \Delta_j v$ for all $j \in \mathbb{N}_{-1}$: As Δ_j is continuous as a map $\mathcal{S}' \to \mathcal{S}'$ for $j \in \mathbb{N}_{-1}$, we have by letting $\Delta_{-2} u = 0$ for all $u \in \mathcal{S}'$,

$$\Delta_j u_{n_m} = \Delta_j \sum_{i=j-1}^{j+1} \Delta_i u_{n_m} \to \Delta_j \sum_{i=j-1}^{j+1} v_j = \Delta_j v \qquad (j \in \mathbb{N}_{-1}).$$

Therefore

$$\begin{aligned}
\|v\|_{B^s_{p,q}} &= \left\| \left(2^{js}\|v_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \\
&\le \left\| \left(2^{js} \liminf_{m\to\infty} \|\Delta_j u_{n_m}\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \\
&\le \liminf_{m\to\infty} \left\| \left(2^{js}\|\Delta_j u_{n_m}\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} = \liminf_{m\to\infty} \|u_{n_m}\|_{B^s_{p,q}}.
\end{aligned}$$

To prove that $B^s_{p,q}$ is complete, we assume that the sequence $(u_n)_{n\in\mathbb{N}}$ as above is also Cauchy. It suffices to show that with $u = v$ as above, $u_n \to u$ in $B^s_{p,q}$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ be such that $m, k \ge N$ implies $\|u_k - u_m\|_{B^s_{p,q}} < \varepsilon$. Let $k \ge N$. Apply the above limiting argument to the sequence $(u_n - u_k)_{n\in\mathbb{N}}$, so that for some sequence $(n_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$

$$\|u - u_k\|_{B^s_{p,q}} \le \liminf_{m\to\infty} \|u_{n_m} - u_k\|_{B^s_{p,q}} < \varepsilon.$$

Then $u_n \to u$ in $B^s_{p,q}$. □

21.1 Comments ...

For our purposes we did not need this, but one can be a bit more specific about the dependence of the C in the Bernstein inequalities on k as in Theorem 21.15 (or differently said, in the statement we can interchange the "for all $k \in \mathbb{N}_0$" and "there exists a $C > 0$"), as follows. For a proof see Exercise 25.A.

Theorem 21.33 (Bernstein inequalities)**.** *Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball around the origin in $\mathbb{R}^d$. There exists a $C > 0$ such that for all $k \in \mathbb{N}$ and $p, q \in [1, \infty]$ with*

$q \geq p$ and any $u \in \mathcal{S}'$ we have for all $\lambda > 0$

$$\operatorname{supp} \widehat{u} \subset \lambda \mathcal{B} \Longrightarrow \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}, \tag{21.38}$$

$$\operatorname{supp} \widehat{u} \subset \lambda \mathcal{A} \Longrightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \max_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} \|\partial^\alpha u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}. \tag{21.39}$$

22 ⋄ Diversity of Besov spaces and inclusions

In Example 21.26 we have seen that the inclusions that one obtains from the embeddings in Theorem 21.23 and Lemma 21.25 are strict in case $p_1 = p_2 = \infty$. In this section we show this for general p_1 and p_2 in $[1, \infty]$ and consider under which conditions Besov spaces are equal or included in each other.

We recall the Inverse Mapping Theorem, which is a consequence of the Open Mapping Theorem (see for example [Con90, Theorem III.12.1 and III.12.5] and [Rud91, 2.11 and 2.12]).

Theorem 22.1 (Inverse Mapping Theorem)**.** *Let X and Y be Banach spaces. If $A : X \to Y$ is a continuous linear bijection, then A is a homeomorphism.*

Theorem 22.2. *Let Z be a topological vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norm-like functions on Z (Definition 21.10). Let*

$$X_i = \{x \in Z : \|x\|_i < \infty\} \qquad (i \in \{1, 2\}).$$

Suppose that $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are Banach spaces, that are both continuously embedded in Z,

$$(X_1, \|\cdot\|_1) \hookrightarrow Z, \qquad (X_2, \|\cdot\|_2) \hookrightarrow Z. \tag{22.1}$$

If $X_1 \subset X_2$, then there exists a $C > 0$ such that $\|\cdot\|_2 \leq C\|\cdot\|_1$. Consequently, if the sets X_1 and X_2 are equal, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent; $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are homeomorphic.

Proof. Suppose $X_1 \subset X_2$. Let us write $X = X_1$. Define $\|\cdot\|_3 : Z \to [0, \infty]$ by $\|\cdot\|_3 = \|\cdot\|_1 + \|\cdot\|_2$. Then $\|\cdot\|_3$ is a norm-like function and $X = \{x \in Z : \|x\|_3 < \infty\}$. By definition, the identity map $(X, \|\cdot\|_3) \to (X, \|\cdot\|_1)$ is continuous (and linear and bijective). We are done if we show that $(X, \|\cdot\|_3)$ is a Banach space, as then, by Theorem 22.1, it follows that also the identity map $(X, \|\cdot\|_1) \to (X, \|\cdot\|_3)$ is continuous, i.e., there exists a $C > 0$ such that $\|\cdot\|_3 \leq C\|\cdot\|_1$ and thus $\|\cdot\|_2 \leq (C-1)\|\cdot\|_1$. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $(X, \|\cdot\|_3)$. Then it converges in $(X, \|\cdot\|_1)$ to some limit x and in $(X_2, \|\cdot\|_2)$ to some limit y. By (22.1) it follows that $x = y$. Therefore $\|x_n - x\|_3 \to 0$ and so $(X, \|\cdot\|_3)$ is indeed a Banach space. □

Lemma 22.3. *Let $q \in [1,\infty]$. Let $a : \mathbb{N} \to \mathbb{R}$ be given by ($\frac{1}{\infty} = 0$)*

$$a(n) = \Big(\frac{1}{(n+1)(\log(n+1))^2}\Big)^{\frac{1}{q}} \qquad (n \in \mathbb{N}).$$

Then $a \in \ell^r$ if and only if $r \geq q$. Let $f : [2,\infty) \to \mathbb{R}$ be given by $f(x) = (x(\log x)^2)^{-1}$ ($x \in [2,\infty)$). Then, for $\alpha \in (0,\infty)$, $\int_2^\infty f^\alpha < \infty$ if and only if $\alpha \geq 1$.

Proof. It is sufficient to show the statement for the function f as $a(n)^q \leq f$ on $[n, n+1]$ and $f \leq a(n)^q$ on $[n+1, n+2]$ for all $n \in \mathbb{N}$ with $n \geq 2$. That f^α is integrable for $\alpha > 1$ will be clear. That f itself is integrable, follows as it is the derivative of $(\log \mathfrak{x})^{-1}$. Let $\alpha \in (0,1)$ and choose $\varepsilon > 0$ such that $\alpha(1+2\varepsilon) < 1$. As $x^{-\varepsilon}\log x \to 0$ as $x \to \infty$, there exists a $C > 0$ such that $\log x \leq Cx^\varepsilon$ on $[2,\infty)$, therefore

$$\Big(\frac{1}{x(\log x)^2}\Big)^\alpha \geq \Big(\frac{1}{Cx^{1+2\varepsilon}}\Big)^\alpha \geq \frac{1}{C^\alpha x^{\alpha(1+2\varepsilon)}} \qquad (x \in [2,\infty)).$$

So that f^α is not integrable. □

Remark 22.4. Observe that one needs the square on the logarithm, in the sense that $(x\log x)^{-1}$ is not integrable.

Theorem 22.5. *Let $s, s_1, s_2 \in \mathbb{R}$, $p, p_1, p_2, q, q_1, q_2 \in [1,\infty]$ and $\varepsilon > 0$.*

(a) *There exists a $u \in B^s_{p,q}$ such that for all $m, r \in [1,\infty]$ and $t \in \mathbb{R}$,*

$$u \in B^t_{m,r} \iff \begin{cases} t \leq s - d(\frac{1}{p} - \frac{1}{m}), r \geq q, \\ t < s - d(\frac{1}{p} - \frac{1}{m}), r < q. \end{cases} \tag{22.2}$$

(b) $B^{s_1}_{p_1,q_1} = B^{s_2}_{p_2,q_2} \iff p_1 = p_2, q_1 = q_2, s_1 = s_2$.

(c)

$$p_1 < p_2 \implies B^s_{p_1,q_1} \subsetneq B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,q_2}, \qquad q_2 > q_1 \implies B^s_{p,q_2} \subsetneq B^{s-\varepsilon}_{p,q_1}.$$

(d)

$$B^{s_1}_{p_1,q_1} \subset B^{s_2}_{p_2,q_2} \iff p_1 \leq p_2 \text{ and } \begin{cases} s_2 \leq s_1 - d(\frac{1}{p_1} - \frac{1}{p_2}), & q_1 \leq q_2, \\ s_2 < s_1 - d(\frac{1}{p_1} - \frac{1}{p_2}), & q_1 > q_2. \end{cases}$$

Proof. We consider a similar setting as in Example 21.26 (although, without restricting d to be equal to 1. Let $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ be a dyadic partition of unity such that $\varphi := \varphi_0 = 1$ on the annulus $A(1-2\varepsilon, 1+2\varepsilon)$ and $\varphi_{-1} = 1$ on the ball $B(0, \frac{1}{2} + 2\varepsilon)$ for some $\varepsilon > 0$ (such a dyadic partition of unity exists, see the proof of Theorem 21.6). Let $\psi \in C_c^\infty$ be such that $\operatorname{supp}\psi \subset A(1-\varepsilon, 1+\varepsilon)$. Observe that $\widehat{\psi} \in L^p$ for all $p \in [1,\infty]$ as $\widehat{\psi}$ is a Schwartz function. For $n \in \mathbb{Z}$ define

$$v_n = l_{2^n}^* \psi = 2^{-n} l_{2^{-n}} \psi, \qquad u_n = \widehat{v}_n = l_{2^n} \widehat{\psi}.$$

Then,

- $u_n \in L^m$ and $\|u_n\|_{L^m} = 2^{-\frac{nd}{m}} \|\widehat{\psi}\|_{L^m}$ for all $n \in \mathbb{Z}$, $m \in [1,\infty]$ (Exercise 21.E).
- For $n \in \mathbb{N}$ and $j \in \mathbb{N}_{-1}$, $\varphi_j v_n = v_n$ and $\Delta_j u_n$ is u_n if $j = n$ and 0 if $j \neq n$.
- $\operatorname{supp} \widehat{u}_{-n} \subset 2^{-n} A(1-\varepsilon, 1+\varepsilon) \subset B(0, \frac{1}{2} + \varepsilon)$ for all $n \in \mathbb{N}$.
- For $n \in \mathbb{N}$ and $j \in \mathbb{N}_{-1}$, $\Delta_j u_{-n} = u_{-n}$ if $j = -1$ and 0 if $j \in \mathbb{N}_0$.

(a) Let us construct the u such that (22.2). Let $p, q, m, r \in [1,\infty]$, $s, t \in \mathbb{R}$. Let a be as in Lemma 22.3, so that $a \in \ell^r$ if and only if $r \geq q$. Define $b : \mathbb{N} \to \mathbb{R}$ by

$$b(n) = 2^{-ns + \frac{nd}{p}} a(n) \qquad (n \in \mathbb{N}).$$

Then $u := \sum_{n\in\mathbb{N}} b(n) u_n$ exists in $\mathcal{S}'$ (Theorem 21.18 (a)) and

$$\|u\|_{B^t_{m,r}[\varphi]} = \|(2^{nt} \|b(n) u_n\|_{L^m})_{n\in\mathbb{N}}\|_{\ell^r} = \|(2^{n(t-s+d(\frac{1}{p}-\frac{1}{m}))} a(n))_{n\in\mathbb{N}}\|_{\ell^r} \|\widehat{\psi}\|_{L^m}.$$

Therefore, (22.2) holds.

(b) Then, for all $m, r \in [1,\infty]$ and $t \in \mathbb{R}$

$$\|u_{-n}\|_{B^t_{m,r}} = \|u_{-n}\|_{L^m} = 2^{\frac{nd}{m}} \|\widehat{\psi}\|_{L^m}.$$

By Theorem 22.2 $\|\cdot\|_{B^{s_1}_{p_1,q_1}}$ and $\|\cdot\|_{B^{s_2}_{p_2,q_2}}$ cannot be equivalent if $p_1 \neq p_2$. Moreover, if $p_1 < p_2$, then by the above we see that for each $C > 0$ there exists an $n \in \mathbb{N}$ such that $\|u_{-n}\|_{B^{s_1}_{p_1,q_1}} = 2^{\frac{nd}{p_1}} > C 2^{\frac{nd}{p_2}} = C\|u_{-n}\|_{B^{s_2}_{p_2,q_2}}$, so that there does not exist a $C > 0$ such that $\|\cdot\|_{B^{s_1}_{p_1,q_1}} \leq C\|\cdot\|_{B^{s_2}_{p_2,q_2}}$, hence $B^{s_2}_{p_2,q_2} \not\subset B^{s_1}_{p_1,q_1}$:

$$p_1 < p_2 \implies B^{s_2}_{p_2,q_2} \not\subset B^{s_1}_{p_1,q_1}. \tag{22.3}$$

Let $u \in B^s_{p,q}$ be as in (a). If $t > s$, then $u \notin B^t_{p,r}$ for any $r \in [1,\infty]$:

$$s_1 \neq s_2 \implies B^{s_1}_{p,q_1} \neq B^{s_2}_{p,q_2}.$$

And if $r < q$, then $u \notin B^s_{p,r}$:

$$q_1 \neq q_2 \implies B^s_{p,q_1} \neq B^s_{p,q_2}.$$

By the above three implications we obtain (b).

(c) follows by (b) and Theorem 21.23 and Lemma 21.25.

(d) The $\Longleftarrow$ we have already seen in Theorem 21.23. Suppose

$$B^{s_1}_{p_1,q_1} \subset B^{s_2}_{p_2,q_2}. \tag{22.4}$$

By (a) with $p = p_1, q = q_1$ and $s = s_1$ this implies either one of the following cases

$$\begin{cases} s_2 \le s_1 - d(\frac{1}{p_1} - \frac{1}{p_2}), & q_1 \le q_2, \\ s_2 < s_1 - d(\frac{1}{p_1} - \frac{1}{p_2}), & q_1 > q_2. \end{cases} \tag{22.5}$$

(22.3) implies $p_1 \le p_2$. □

23 Embeddings of Besov spaces and Sobolev spaces

In this section we consider embeddings between Besov and Sobolev spaces. First we prove that $B^s_{2,2}$ is equal to the fractional Sobolev spaces H^s (Definition 20.5), as the motivation of Besov spaces 21.1 already suggested. Then we compare Besov spaces with Sobolev spaces and consider in which of these the spaces the testfunctions are dense.

For the proof of $B^s_{2,2} = H^s$ we use the following lemma.

Lemma 23.1. *Let $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ be a dyadic partition of unity. For all $s \in \mathbb{R}$ there exist $c, C > 0$ such that*

$$c\left(1+|\xi|^2\right)^s \le \sum_{j\in\mathbb{N}_{-1}} 2^{2sj}\varphi_j(\xi)^2 \le C\left(1+|\xi|^2\right)^s \qquad (\xi \in \mathbb{R}^d). \tag{23.1}$$

Proof. Let $a, b > 0$, $a < b$ be such that $\operatorname{supp}\varphi_0 \subset A(a,b)$. Because $\operatorname{supp}\varphi_0 \subset A(a,b)$, we have $\operatorname{supp}\varphi_{-1} \subset B(0,\frac{b}{2}) = 2^{-1}B(0,b)$ (as for example $\varphi_{-1}+\varphi_0 = l_{\frac{1}{2}}\varphi_{-1}$ and $\operatorname{supp}\varphi_{-1}+\varphi_0 \subset B(0,b)$). Without loss of generality, we may assume $a < 1$ and $b > 1$. Using that $1+2^{2j}b^2 \le b^2(1+2^{2j})$ and that $1 \le 4\cdot 2^{2j}$ for all $j \in \mathbb{N}_{-1}$, we obtain

$$\begin{aligned} &\xi \in \operatorname{supp}\varphi_j \Longrightarrow \xi \in 2^jB(0,b) \Longrightarrow 1+|\xi|^2 \le 5b^22^{2j} &&(j \in \mathbb{N}_{-1}), \\ &\xi \in \operatorname{supp}\varphi_j \Longrightarrow \xi \in 2^jA(a,b) \Longrightarrow 1+|\xi|^2 \ge 2^{2j}a^2 &&(j \in \mathbb{N}_0), \\ &\xi \in \operatorname{supp}\varphi_{-1} \Longrightarrow 1+|\xi|^2 \ge 1 \ge 2^{-2}a^2. \end{aligned}$$

Therefore

$$\xi \in \operatorname{supp}\varphi_j \Longrightarrow \frac{1+|\xi|^2}{5b^22^{2j}} \le 1 \le \frac{1+|\xi|^2}{a^22^{2j}} \qquad (j \in \mathbb{N}_0).$$

Let $A_j = \{\xi \in \mathbb{R}^d : \varphi_j^2(\xi) \ge \frac{1}{4}\}$ for $j \in \mathbb{N}_{-1}$. Observe that by definition of the dyadic partition of unity, see (21.3) and (21.4), it follows that

$$\bigcup_{j\in\mathbb{N}_{-1}} A_j = \mathbb{R}^d.$$

Let $t \geq 0$. Then

$$\xi \in A_j \Longrightarrow \frac{1}{4}\left(\frac{1+|\xi|^2}{5b^2 2^{2j}}\right)^t \leq \varphi_j^2(\xi) \quad \text{and} \quad \frac{1}{4}\left(\frac{1+|\xi|^2}{a^2 2^{2j}}\right)^{-t} \leq \varphi_j^2(\xi),$$

$$\xi \in \operatorname{supp}\varphi_j \Longrightarrow \varphi_j^2(\xi) \leq \left(\frac{1+|\xi|^2}{a^2 2^{2j}}\right)^t \quad \text{and} \quad \varphi_j^2(\xi) \leq \left(\frac{1+|\xi|^2}{5b^2 2^{2j}}\right)^{-t}.$$

Then, because

$$|i-j| \geq 2 \Longrightarrow A_i \cap A_j = \operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset \qquad (i, j \in \mathbb{N}_{-1}),$$

we have

$$\frac{1}{4(5b^2)^t}\left(1+|\xi|^2\right)^t \leq \sum_{j\in\mathbb{N}_{-1}} 2^{2tj}\varphi_j(\xi)^2 \leq a^{-2t}\left(1+|\xi|^2\right)^t \qquad (\xi \in \mathbb{R}^d),$$

$$\frac{a^{2t}}{4}\left(1+|\xi|^2\right)^{-t} \leq \sum_{j\in\mathbb{N}_{-1}} 2^{-2tj}\varphi_j(\xi)^2 \leq (5b^2)^t\left(1+|\xi|^2\right)^{-t} \qquad (\xi \in \mathbb{R}^d).$$

□

Theorem 23.2. *For all $s \in \mathbb{R}$ we have*

$$B^s_{2,2} = H^s,$$

with equivalent norms.

Proof. By the Plancherel formula (Theorem 16.27),

$$\begin{aligned}\|u\|^2_{B^s_{2,2}} &= \sum_{j\in\mathbb{N}_{-1}} 2^{2sj}\|\varphi_j(\mathrm{D})u\|^2_{L^2} = \sum_{j\in\mathbb{N}_{-1}} 2^{2sj}\|\varphi_j\widehat{u}\|^2_{L^2} \\ &= \int_{\mathbb{R}^d}\sum_{j\in\mathbb{N}_{-1}} 2^{2sj}|\varphi_j(\xi)|^2|\widehat{u}(\xi)|^2\,\mathrm{d}\xi.\end{aligned}$$

The rest follows from Lemma 23.1. □

23.3. In particular, Theorem 23.2 implies $L^2 = B^0_{2,2}$. However, there do not exist $s \in \mathbb{R}$, $p, q \in [1,\infty]$ such that $L^1 = B^s_{p,q}$, see Exercise 23.A.

Exercise **23.A.** Show that there do not exist $s \in \mathbb{R}$, $p, q \in [1,\infty]$ such that $L^1 = B^s_{p,q}$. Hint: Use the property of Theorem 21.32 and Exercise 21.L and observe that by Theorem 22.2 the sets L^1 and $B^s_{p,q}$ are equal if and only if the norm-like functions $\|\cdot\|_{L^1}$ and $\|\cdot\|_{B^s_{p,q}}$ are equivalent.

Theorem 23.4. *Let $k \in \mathbb{N}_0$ and $p, q \in [1,\infty]$. Then*

$$\begin{aligned}&B^k_{p,1} \hookrightarrow W^{k,p} \hookrightarrow B^k_{p,\infty}, \\ &B^t_{p,q} \hookrightarrow W^{k,p} \hookrightarrow B^s_{p,q} \qquad (s, t \in \mathbb{R}, s < k < t).\end{aligned}$$

More specifically:

(a) *For all $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$ with $s < k$ there exists a $C > 0$ such that*

$$\|u\|_{B^s_{p,q}} \le C\|u\|_{W^{k,p}} \qquad (u \in \mathcal{S}', p, q \in [1,\infty]) \tag{23.2}$$

$$\|u\|_{B^k_{p,\infty}} \le C\|u\|_{W^{k,p}} \qquad (u \in \mathcal{S}', p \in [1,\infty]). \tag{23.3}$$

(b) *For all $k \in \mathbb{N}_0$ and $t \in \mathbb{R}$ with $t > k$ there exists a $C > 0$ such that*

$$\|u\|_{W^{k,p}} \le C\|u\|_{B^t_{p,q}} \qquad (u \in \mathcal{S}', p, q \in [1,\infty]) \tag{23.4}$$

$$\|u\|_{W^{k,p}} \le C\|u\|_{B^k_{p,1}} \qquad (u \in \mathcal{S}', p \in [1,\infty]). \tag{23.5}$$

Proof. Proof of (a). Let $k \in \mathbb{N}_0$. By Bernstein's inequality (Theorem 21.15) there exists a $C_1 > 0$ such that

$$\|\Delta_j u\|_{L^p} \le C_1 2^{-kj} \max_{\beta \in \mathbb{N}_0^d : |\beta| = k} \|\partial^\beta \Delta_j u\|_{L^p} \qquad (u \in \mathcal{S}', j \in \mathbb{N}_0, p \in [1,\infty]).$$

By Lemma 21.9 (in particular (21.12) and (21.13)) there exists a $C_2 > 0$ such that

$$\|\Delta_j u\|_{L^p} \le C_2\|u\|_{L^p} \qquad (u \in \mathcal{S}', j \in \mathbb{N}_{-1}, p \in [1,\infty]).$$

Let $s \in \mathbb{R}$ and $q \in [1,\infty]$ be such that either $s < k$ or $s = k$ and $q = \infty$. Then

$$M = \|2^{(s-k)j})_{j \in \mathbb{N}_{-1}}\|_{\ell^q} < \infty.$$

and thus, with $C' = C_1 C_2 M$,

$$\|u\|_{B^s_{p,q}} \le C'\Big(\|u\|_{L^p} \vee \max_{\beta \in \mathbb{N}_0^d : |\beta| = k} \|\partial^\beta u\|_{L^p}\Big) \le C'\|u\|_{W^{k,p}} \qquad (u \in \mathcal{S}', p \in [1,\infty]).$$

This implies (23.3). If $s < k$, then $M \le \|2^{(s-k)j})_{j \in \mathbb{N}_{-1}}\|_{\ell^1}$ (Lemma A.8) so that (23.2) holds for all $q \in [1,\infty]$ with $C = C_1 C_2 \|2^{(s-k)j})_{j \in \mathbb{N}_{-1}}\|_{\ell^1}$.

Proof of (b). First, observe the following. Let $r \in [1,\infty]$ being such that $1 = \frac{1}{r} + \frac{1}{q}$. Then by Hölder's inequality (Corollary A.9),

$$\|u\|_{L^p} \le \sum_{j=-1}^{\infty} 2^{-aj} 2^{aj} \|\Delta_j u\|_{L^p} \le \|(2^{-aj})_{j \in \mathbb{N}_{-1}}\|_{\ell^r} \|u\|_{B^a_{p,q}} \qquad (u \in \mathcal{S}', a \in \mathbb{R}, p \in [1,\infty]).$$

By Theorem 21.23 there exists a $C_3 > 0$ such that

$$\|\partial^\alpha u\|_{B^{t-k}_{p,q}} \le C_3\|u\|_{B^t_{p,q}} \qquad (u \in \mathcal{S}', \alpha \in \mathbb{N}_0^d, |\alpha| \le k, p, q \in [1,\infty]).$$

Let $t \in \mathbb{R}$ and $q \in [1,\infty]$ be such that either $t > k$ or $t = k$ and $q = 1$ (and thus $r = \infty$). Then

$$N = \|(2^{-(t-k)j})_{j \in \mathbb{N}_{-1}}\|_{\ell^r} < \infty,$$

and thus for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$,

$$\|\partial^\alpha u\|_{L^p} \le N\|\partial^\alpha u\|_{B^{t-k}_{p,q}} \le C_3 N\|u\|_{B^t_{p,q}} \qquad (u \in \mathcal{S}', p \in [1,\infty]).$$

This implies (23.5). If $t > k$, then $N \le \|(2^{-(t-k)j})_{j \in \mathbb{N}_{-1}}\|_{\ell^1}$ so that (23.4) holds for all $p, q \in [1,\infty]$ with $C = C_3\|(2^{-(t-k)j})_{j \in \mathbb{N}_{-1}}\|_{\ell^1}$. $\square$

For $p = \infty$ we do not only have that $B^0_{\infty,1}$ is embedded into the bounded functions, but also into the continuous bounded functions. Moreover, we have the following statement.

Theorem 23.5. *For all $k \in \mathbb{N}_0$*

$$B^k_{\infty,1} \hookrightarrow C^k_{\mathrm{b}} \hookrightarrow B^k_{\infty,\infty}.$$

Proof. That $C^k_{\mathrm{b}} \hookrightarrow B^k_{\infty,\infty}$ follows by the fact that $W^{k,\infty} \hookrightarrow B^k_{\infty,\infty}$ and $C^k_{\mathrm{b}} \hookrightarrow W^{k,\infty}$. By Theorem 23.4 it is sufficient to show the inclusions $B^k_{\infty,1} \subset C^k_{\mathrm{b}}$. As for $u \in B^k_{\infty,1}$ one has $\partial^\alpha u \in B^0_{\infty,1}$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$ (by Theorem 21.23), it is sufficient to show $B^0_{\infty,1} \subset C^0_{\mathrm{b}}$. We leave the proof of this for the reader (see Exercise 23.B). □

Exercise 23.B. Prove that any element of $B^0_{\infty,1}$ is (represented by) a continuous function.

Corollary 23.6. *Let $s \in \mathbb{R}$ and $p, q \in [1,\infty]$. Then*

$$\mathcal{D} \hookrightarrow_{\mathrm{seq}} \mathcal{S} \hookrightarrow B^s_{p,q}.$$

Proof. By Theorem 14.11 $\mathcal{D} \hookrightarrow_{\mathrm{seq}} \mathcal{S}$. Let $k \in \mathbb{N}_0$ be such that $k > s$. By Lemma 14.15 and Theorem 23.4

$$\mathcal{S} \hookrightarrow W^{k,p} \hookrightarrow B^s_{p,q}.$$

□

In Theorem 23.8 we will show that $\mathcal{D}$ is also dense in $B^s_{p,q}$ in case p and q are both finite. For this we will use the following lemma.

Lemma 23.7. *Let $s \in \mathbb{R}$ and $p, q \in [1,\infty]$. Suppose $q < \infty$. Then $\sum_{j=-1}^J \Delta_j u \to u$ in $B^s_{p,q}$ as $J \to \infty$ for all $u \in B^s_{p,q}$.*

Exercise 23.C. Prove Lemma 23.7.

Theorem 23.8. *Let $s \in \mathbb{R}$ and $p, q \in [1,\infty]$.*

(a) *If $q < \infty$ then $C^\infty_{\mathrm{b}} \cap B^s_{p,q}$ is dense in $B^s_{p,q}$.*
(b) *If $p < \infty$ and $q < \infty$, then $\mathcal{D}$ is dense in $B^s_{p,q}$.*

Proof. By (b) it is sufficient to prove (a) for $p = \infty$. But this follows from Lemma 23.7 as $\Delta_j u \in L^\infty$ for all $j \in \mathbb{N}_{-1}$ and $u \in B^s_{p,q}$.

Let $p, q < \infty$. By Lemma 23.7 it is sufficient to show that for all $u \in B^s_{p,q}$ such that $\operatorname{supp} \widehat{u}$ is compact, there exist $u_R \in \mathcal{D}$ for $R > 0$ such that $u_R \xrightarrow{R\to\infty} u$ in $B^s_{p,q}$.

Let $u \in B^s_{p,q}$ be such that $\operatorname{supp} \widehat{u}$ is compact. Then $u \in C^\infty_{\mathrm{p}}$ (Lemma 17.8) and $\partial^\alpha u \in L^p$ for all $\alpha \in \mathbb{N}_0^d$ by Bernstein's inequality (Theorem 21.15) and because $u = \sum_{j=-1}^J \Delta_j u$ for some $J \in \mathbb{N}_{-1}$ and $\Delta_j u \in L^p$ for all $j \in \mathbb{N}_{-1}$. Consequently, $u \in W^{k,p}$ for all $k \in \mathbb{N}_0$.

Let χ and χ_R for $R > 0$ be as in Lemma 8.16. Then $\chi_R u \in \mathcal{D}$ and

$$\|\partial^\alpha(\chi_R - \mathbb{1}) \cdot \partial^\beta u\|_{L^p} \xrightarrow{R\to\infty} 0 \qquad (\alpha, \beta \in \mathbb{N}_0^d). \tag{23.6}$$

Let $k \in \mathbb{N}_0$, $k > s$. As $W^{k,p} \hookrightarrow B^s_{p,q}$ (by Theorem 23.4) it is sufficient to show that $\|(\chi_R - \mathbb{1})u\|_{W^{k,p}} \xrightarrow{R\to\infty} 0$. But this follows by (23.6) because by Leibniz' rule (see 1.14) there exists a $C > 0$ such that

$$\|(\chi_R - 1)u\|_{W^{k,p}} \le C \max_{\substack{\beta\in\mathbb{N}_0^d \\ |\beta|\le k}} \max_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le k}} \|\partial^\alpha(\chi_R - 1) \cdot \partial^\beta u\|_{L^p}.$$

□

Exercise **23.D.** Show that $\mathbb{1} \in B^s_{\infty,q}$ if and only if either $s = 0$ and $q = \infty$ or $s < 0$. Conclude: $C_\mathrm{b}^\infty \subset B^s_{\infty,q}$ *if and only if either* $s = 0$ *and* $q = \infty$ *or* $s < 0$.

23.9 ($\mathcal{S}$ is not dense in $B^s_{\infty,q}$)**.** As $\Delta_{-1}\mathbb{1} = \mathbb{1}$, we have $\mathbb{1} \in B^s_{\infty,q}$ for all $s \in \mathbb{R}$ and $q \in [1,\infty]$. For each $\varphi \in \mathcal{S}$ we have $\Delta_{-1}\varphi \in \mathcal{S}$ and thus $\|\Delta_{-1}\varphi - \Delta_{-1}\mathbb{1}\|_{L^\infty} = 1$. Therefore $\mathcal{S}$ and thus $\mathcal{D}$ are not dense in $B^s_{\infty,q}$ for any $s \in \mathbb{R}$ and $q \in [1,\infty]$.

The following lemma implies that if $\mathcal{S}$ is dense in $B^s_{p,q}$, that for all $u \in B^s_{p,q}$ one has $\lim_{j\to\infty} 2^{js}\|\Delta_j u\|_{L^p} = 0$. Observe however, that the converse is not the case: If $u \in \mathcal{S}'$ is such that (23.8), this need not to imply that u is in the closure of $\mathcal{D}$ in $B^s_{p,q}$ (even though this is claimed to be obvious in [BCD11, Remark 2.75]); indeed, for $p = \infty$ this is the case for $\mathbb{1}$, see 23.9.

Lemma 23.10. *For* $\psi \in \mathcal{S}$

$$\lim_{j\to\infty} 2^{js}\|\Delta_j\psi\|_{L^p} = 0 \qquad (s \in \mathbb{R}, p \in [1,\infty]). \tag{23.7}$$

Consequently, if $s \in \mathbb{R}, p, q \in [1,\infty]$, $u \in B^s_{p,q}$ *and* u *is the limit of testfunctions in* $B^s_{p,q}$, *in other words,* u *is in the closure of* $\mathcal{S}$ *in* $B^s_{p,q}$, *then*

$$\lim_{j\to\infty} 2^{js}\|\Delta_j u\|_{L^p} = 0. \tag{23.8}$$

Proof. (23.7) basically follows because $\mathcal{S} \subset B^r_{p,q}$ for all $r \in \mathbb{R}$ and $p, q \in [1,\infty]$. The details and the proof of the rest of the statement are left for the reader (Exercise 23.E). □

Exercise **23.E.** Complete the proof of Lemma 23.10.

In the following example we will show that the inclusion $W^{k,\infty} \subset B^k_{\infty,\infty}$ which one obtains by Theorem 23.4 is strict. Therefore, in particular $L^\infty \subsetneq B^0_{\infty,\infty}$.

Example 23.11 ($W^{k,\infty} \subsetneq B^k_{\infty,\infty}$)**.** Consider the setting of Example 21.26 with $q = \infty$. Let

$$u_n = \cos(2\pi 2^n \mathfrak{x}), \qquad w_n = \sin(2\pi 2^n \mathfrak{x}) \qquad (n \in \mathbb{N}).$$

Then $U_k := \sum_{n\in\mathbb{N}} 2^{-nk} u_n$ and $W_k := \sum_{n\in\mathbb{N}} 2^{-nk} w_n$ are tempered distributions for $k \in \mathbb{N}_0$ and

$$U_k, W_k \in B^t_{\infty,r} \iff \begin{cases} t = k, r = \infty \\ t < k \end{cases} .$$

And, as $\partial 2^{-n} w_n = 2\pi u_n$, $\partial 2^{-n} u_n = -2\pi w_n$ for all $n \in \mathbb{N}$,

$$\partial W_{k+1} = 2\pi U_k \text{ and } \partial U_{k+1} = -2\pi W_k \text{ in } \mathcal{S}' \qquad (k \in \mathbb{N}_0).$$

Let $k \in \mathbb{N}$. We will show that $L^\infty \subsetneq B^k_{\infty,\infty}$ by showing that $\partial^k U_k \notin L^\infty$ (remember $L^\infty = W^{0,\infty}$).

It is sufficient to show that $\partial^k U_k \notin L^1_{\text{loc}}$.

$\partial^k U_k$ is a multiple of W_0 or U_0. Therefore, we may as well assume $k = 0$. Write $u = U_0$ and $w = \frac{1}{2\pi} W_1$ so that $\partial w = u$ in $\mathcal{S}'$.

w is a Weierstrass function, as Hardy showed, see [Har16] (he showed that functions of the form $\sum_{n\in\mathbb{N}_0} a^n \cos(b^n \pi \mathfrak{x})$ or $\sum_{n\in\mathbb{N}_0} a^n \sin(b^n \pi \mathfrak{x})$ for $a \in (0,1)$ and $b \in (1,\infty)$ with $ab \geq 1$ are nowhere differentiable). This means that w is a continuous function that is nowhere differentiable.

By showing the following statement we conclude that u is not given by a locally integrable function and therefore not in L^∞:

Lemma 23.12. *Let $u, w \in L^1_{\text{loc}}(\mathbb{R})$ and suppose $\partial w = u$ in $\mathcal{D}'(\mathbb{R})$. Then w is almost everywhere differentiable with derivative u.*

Proof. First observe that for $a, b \in \mathbb{R}$, $a < b$ and $\varphi \in \mathcal{D}(\mathbb{R})$, $\varphi * \mathbb{1}_{[a,b]}(x) = \int_{x-b}^{x-a} \varphi$ for $x \in \mathbb{R}$ and thus

$$\partial(\varphi * \mathbb{1}_{[a,b]}) = \mathcal{T}_a \varphi - \mathcal{T}_b \varphi \text{ in } \mathcal{D}(\mathbb{R}).$$

Choose a mollifier ψ with $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}$ and let ψ_ε for $\varepsilon > 0$ be as in Definition 8.11.

Let $a, b \in \mathbb{R}$, $a < b$. By Theorem 7.15 $\psi_\varepsilon * \mathbb{1}_{[a,b]} \xrightarrow{\varepsilon \downarrow 0} \mathbb{1}_{[a,b]}$ pointwise except possibly at a and at b, whereas $0 \leq \psi_\varepsilon \mathbb{1}_{[a,b]}(x) \leq 1$ for all $x \in \mathbb{R}$ and $\varepsilon > 0$. By Lebesgue's dominated convergence theorem it follows that

$$\begin{aligned}\int_a^b u = \int u \mathbb{1}_{[a,b]} = \lim_{\varepsilon\downarrow 0} \int \overline{u} \cdot \psi_\varepsilon * \mathbb{1}_{[a,b]} = \lim_{\varepsilon\downarrow 0} \langle u, \psi_\varepsilon * \mathbb{1}_{[a,b]} \rangle = \lim_{\varepsilon\downarrow 0} \langle w', \psi_\varepsilon * \mathbb{1}_{[a,b]} \rangle \\ = -\lim_{\varepsilon\downarrow 0} \langle w, \partial(\psi_\varepsilon * \mathbb{1}_{[a,b]}) \rangle = \lim_{\varepsilon\downarrow 0} \langle w, \mathcal{T}_b \psi_\varepsilon - \mathcal{T}_a \psi_\varepsilon \rangle = w(b) - w(a).\end{aligned}$$

By Lebesgue's differentiation theorem it then follows that w is almost everywhere differentiable with $\partial w = u$ almost everywhere, because for example

$$\left| \frac{w(x+h)-w(x)}{h} - u(x) \right| \le \frac{1}{h} \int_{x-|h|}^{x+|h|} |u(s)-u(x)| \, \mathrm{d}s \qquad (h \neq 0).$$

□

Exercise **23.F.** Show that U_k as in Example 23.11 does not lie in the closure of $\mathcal{D}$ in $B^k_{\infty,\infty}$.

24 ◇ Besov spaces related to other spaces

In this section we give an overview of Besov spaces and some other spaces, and of embeddings between them.

Definition 24.1 (Hölder spaces)**.** Let Ω be an open subset of $\mathbb{R}^d$ and $k \in \mathbb{N}_0$. We write also $C^{k,0}(\Omega)$ for $C^k(\Omega)$. Let $\alpha \in (0,1]$.

- A function $f : \Omega \to \mathbb{F}$ is *α-Hölder continuous* if there exists a $C > 0$ such that
$$|f(x) - f(y)| \le C|x-y|^\alpha \qquad (x, y \in \Omega). \tag{24.1}$$
- A function is called *Lipschitz continuous* if it is 1-Hölder continuous.
- $C^{0,\alpha}(\Omega)$ is defined to be the space of α-Hölder continuous functions $\Omega \to \mathbb{F}$. The α-Hölder coefficient of a function f is given by
$$|f|_{C^{0,\alpha}(\Omega)} = \sup_{x,y\in\Omega: x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$
- $C^{k,\alpha}(\Omega)$ is defined to be the space of functions $\Omega \to \mathbb{F}$ that are k-times continuously differentiable for which their derivatives of order k are α-Hölder continuous.

We defined $C^k_{\mathrm{b}}(\Omega)$ to be the subspace of $C^k(\Omega)$ that consists of functions f for which $\|f\|_{C^k}$ is finite. Similarly we define

$$\|f\|_{C^{k,\alpha}(\Omega)} = \|f\|_{C^k(\Omega)} + \sum_{\beta\in\mathbb{N}_0^d:|\beta|=k} |\partial^\beta f|_{C^{0,\alpha}(\Omega)} \qquad (f \in C^{k,\alpha}(\Omega)), \tag{24.2}$$

$$C^{k,\alpha}_{\mathrm{b}}(\Omega) = \{f \in C^{k,\alpha}(\Omega) : \|f\|_{C^{k,\alpha}(\Omega)} < \infty\}. \tag{24.3}$$

24.2. For the rest of this section we consider $\Omega = \mathbb{R}^d$ and write "$C^{k,\alpha}$" instead of "$C^{k,\alpha}(\mathbb{R}^d)$".

Observe that $C^{0,1}$ consists of all the Lipschitz functions and that for $k \in \mathbb{N}$, $C^{k+1}_{\mathrm{b}} \subsetneq C^{k,1}_{\mathrm{b}}$.

For $s \in (0,\infty) \setminus \mathbb{N}$ it is also common in literature to write C^s for $C^{k,\alpha}$, where $k = \lfloor s \rfloor$ and $\alpha = s - \lfloor s \rfloor$.

Exercise **24.A.** Can you classify the space of α-Hölder functions with $\alpha > 1$, that is, which functions f satisfy (24.1) for $\alpha > 1$?

In Definition 12.3 we introduced the Sobolev spaces $W^{k,p}$ for $k \in \mathbb{N}_0$ and $p \in [1, \infty]$. In Definition 20.5 and 20.10 we introduced the fractional Sobolev or Bessel–potential spaces H_p^s for $s \in \mathbb{R} \setminus \mathbb{N}_0$ and $p \in [1, \infty]$. We will now consider Slobodeckij spaces, $W^{s,p}$ with $s \in (0, \infty) \setminus \mathbb{N}$ as subspaces of $W^{k,p}$ with $k = \lfloor s \rfloor$ in a similar way as C^s or $C^{k,\alpha}$ for $\alpha \in (0, 1]$ is defined to be a subspace of C^k.

Definition 24.3 (Slobodeckij spaces)**.** Let $p \in [1, \infty)$ and $s \in (0, \infty) \setminus \mathbb{N}$. Let $k \in \mathbb{Z}$ and $\alpha \in (0, 1)$ be given by

$$k = \lfloor s \rfloor, \qquad \alpha = s - k.$$

We define the norm-like function $\| \cdot \|_{W^{s,p}} : W^{k,p} \to [0, \infty]$ by

$$\|f\|_{W^{s,p}} := \|f\|_{W^{k,p}} + \sum_{\beta \in \mathbb{N}_0^d : |\beta| = k} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|^p}{|x - y|^{d + \alpha p}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}} \qquad (f \in W^{k,p}).$$

The *Slobodeckij space* $W^{s,p}$ is then defined by

$$W^{s,p} = \{ f \in W^{k,p} : \|f\|_{W^{s,p}} < \infty \}.$$

Definition 24.4 (Zygmund spaces)**.** Let $s \in (0, \infty)$. Let $k \in \mathbb{N}_0$ and $\alpha \in (0, 1]$ be given by

$$k = \lceil s - 1 \rceil, \qquad \alpha = s - k, \tag{24.4}$$

in other words, k is such that $s - k \in (0, 1]$. We define the norm-like function $\| \cdot \|_{\mathcal{C}^s}$: $C^k \to [0, \infty]$, by

$$\|f\|_{\mathcal{C}^s} = \|f\|_{C^k} + \sum_{\beta \in \mathbb{N}_0^d : |\beta| = k} \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{\|(\mathcal{T}_h - 1)^2 \partial^\beta f\|_{C^0}}{|h|^\alpha} \qquad (f \in C^k).$$

The *Zygmund space* $\mathcal{C}^s$ is then defined by

$$\mathcal{C}^s = \{ f \in C^k : \|f\|_{\mathcal{C}^s} < \infty \}.$$

Observe that

$$\begin{aligned}(\mathcal{T}_h - 1)^2 g(x) &= (\mathcal{T}_h - 1)(\mathcal{T}_h - 1) g(x) = (\mathcal{T}_h - 1) g(x - h) - (\mathcal{T}_h - 1) g(x) \\ &= g(x - 2h) - 2g(x - h) + g(x).\end{aligned}$$

Definition 24.5 (Besov–Lipschitz spaces)**.** Let $s \in (0, \infty)$. Let $k \in \mathbb{Z}$ and $\alpha \in (0, 1]$ be as in (24.4). For $p, q \in [1, \infty]$ we define the norm-like function $\| \cdot \|_{\Lambda^s_{p,q}} : W^{k,p} \to [0, \infty]$ by

$$\|f\|_{\Lambda^s_{p,q}} := \|f\|_{W^{k,p}} + \begin{cases} \sum_{\beta \in \mathbb{N}_0^d : |\beta| = k} \left(\int_{\mathbb{R}^d} \frac{\|(\mathcal{T}_h - 1)^2 \partial^\beta f\|_{L^p}^q}{|h|^{d + \alpha q}} \, \mathrm{d}h \right)^{\frac{1}{q}} & q < \infty, \\ \sum_{\beta \in \mathbb{N}_0^d : |\beta| = k} \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{\|(\mathcal{T}_h - 1)^2 \partial^\alpha f\|_{L^p}^q}{|h|^\alpha} & q = \infty, \end{cases} \qquad (f \in W^{k,p}).$$

For $p, q \in [1, \infty]$ we define the *Besov-Lipschitz space* $\Lambda^s_{p,q}$ to be the set of functions

$$\Lambda^s_{p,q} = \{f \in W^{k,p} : \|f\|_{\Lambda^s_{p,q}} < \infty\}.$$

The Triebel–Lizorkin spaces are defined as the Besov spaces, but with the "L^p" and "ℓ^q" norm interchanged:

Definition 24.6 (Triebel–Lizorkin spaces)**.** Let $(\varphi_j)_{j \in \mathbb{N}_{-1}}$ be a dyadic partition of unity. Let $s \in \mathbb{R}$. For $p \in [1, \infty)$ and $q \in [1, \infty]$ we define the norm-like function $\| \cdot \|_{F^s_{p,q}} : \mathcal{S}' \to [0, \infty]$ by

$$\|u\|_{F^s_{p,q}} := \big\| \|(2^{js} |\Delta_i u|)_{j \in \mathbb{N}_{-1}} \|_{\ell^q} \big\|_{L^p} \qquad (u \in \mathcal{S}'),$$

for example, for $q < \infty$ this means

$$\|u\|_{F^s_{p,q}} = \left[\int_{\mathbb{R}^d} \Big(\sum_{j \in \mathbb{N}_{-1}} 2^{qjs} |\Delta_i u(x)|^q \Big)^{\frac{p}{q}} \, \mathrm{d}x \right]^{\frac{1}{p}}.$$

We define the *Triebel–Lizorkin space* $F^s_{p,q}$ to be the set

$$F^s_{p,q} = \{u \in \mathcal{S}' : \|u\|_{F^s_{p,q}} < \infty\}.$$

Remark 24.7. As for Besov spaces, the norm of $F^s_{p,q}$ depends on the choice of dyadic partition, but the space itself does not. This is shown in [Tri83, Section 2.3.2] .

24.8. Let us summarize for which parameters we have either continuous embeddings or equality between spaces with equivalent norms. Here, "$A \cong B$" means that A and B are the same space with equivalent norms, i.e., $A \hookrightarrow B \hookrightarrow A$.

(a) [Tri83, p.90, (9)] $C^s_{\mathrm{b}} \cong \mathcal{C}^s$ for $s \in (0, \infty) \setminus \mathbb{N}$ (C^s is as in 24.2).

(b) [Tri83, p.90, (9)] $W^{s,p} \cong \Lambda^s_{p,p}$ for $s \in (0, \infty) \setminus \mathbb{N}$ and $p \in (1, \infty)$.

(c) [Tri83, p.88] $H^s_p \cong F^s_{p,2}$ for $s \in \mathbb{R}$ and $p \in (1, \infty)$.

(d) [Tri83, p.88] $H^k_p \cong W^{k,p}$ for $k \in \mathbb{N}$ and $p \in (1, \infty)$.

(e) [Tri83, p.89] $B^0_{p,1} \hookrightarrow L^p \hookrightarrow B^0_{p,\infty}$ for $p \in [1, \infty)$. (See also Theorem 23.4.)

(f) [Tri83, p.89] $B^0_{\infty,1} \hookrightarrow C^0_{\mathrm{b}} \hookrightarrow B^0_{\infty,\infty}$. (See also Theorem **??**.)

(g) [Tri83, p.90, p.113] $\Lambda^s_{p,q} \cong B^s_{p,q}$ for $s > 0$, $p \in [1,\infty)$ and $q \in [1,\infty]$.

(h) [Tri83, p.90, p.113] $\mathcal{C}^s \cong B^s_{\infty,\infty}$ for $s > 0$.

(i) [Tri83, p.47] $B^s_{p,\min\{p,q\}} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max\{p,q\}}$ for $s \in \mathbb{R}$, $p \in [1,\infty)$ and $q \in [1,\infty]$.

(j) [Tri83, p.60] For $s_1, s_2 \in \mathbb{R}$, $p_1, p_2, q_1, q_2 \in [1,\infty]$:
$B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \cong B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ if and only if $s_1 = s_2$ and $p_1 = p_2, q_1 = q_2$ (see Theorem 22.5).
For $p_1, p_2 < \infty$:
$F^{s_1}_{p_1,q_1}(\mathbb{R}^d) \cong F^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ if and only if $s_1 = s_2$ and $p_1 = p_2, q_1 = q_2$,
$F^{s_1}_{p_1,q_1}(\mathbb{R}^d) \cong B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ if and only if $s_1 = s_2$ and $p_1 = p_2 = q_1 = q_2$.

Observe that we can combine some of the above to obtain:

$$\begin{aligned}
&C^s_{\mathrm{b}} = \mathcal{C}^s = B^s_{\infty,\infty} \qquad (s \in (0,\infty) \setminus \mathbb{N}),\\
&W^{s,p} = \Lambda^s_{p,p} = B^s_{p,p} = F^s_{p,p} \qquad (s \in (0,\infty) \setminus \mathbb{N}, p \in (1,\infty)).\\
&H^k_p = W^{k,p} = F^k_{p,2} \qquad (k \in \mathbb{N}, p \in (1,\infty)).
\end{aligned}$$

24.9. In 23.3 we mentioned that no Besov space is equal to L^1. We can generalise this as follows: For $r \in [1,2) \cup (2,\infty)$ there are no $s \in \mathbb{R}$, $p, q \in [1,\infty]$ such that $L^r = B^s_{p,q}$.

Exercise **24.B.** Let $r \in (1,\infty)$. Show that $B^s_{p,q} = L^r$ if and only if $p = q = r = 2$ and $s = 0$. Hint: $H^0_r = L^r$ (see 20.10).

Remark 24.10. The proof of $C^\alpha = B^\alpha_{\infty,\infty}$ for $\alpha \in (0,1)$ can also be found in [MS13, Lemma 8.6] .

25 The Hörmander-Mikhlin inequalities for Fourier multipliers

The derivation operator ∂^α maps C^k into $C^{k-|\alpha|}$. It behaves similar on Besov spaces as we have seen in Theorem 21.23, namely, ∂^α maps $B^s_{p,q}$ continuously into $B^{s-|\alpha|}_{p,q}$.

In this section we consider the action of Fourier multipliers on Besov spaces, and basically show that those who behave similarly as the Fourier multiplier ∂^α, i.e., $(2\pi i\mathfrak{x})^\alpha(\mathrm{D})$, also map $B^s_{p,q}$ continuously into $B^{s-|\alpha|}_{p,q}$. Moreover, we can also use this to treat "inverse derivation operators": for example $(1-\Delta)^{-1}$ forms a continuous map from $B^s_{p,q}$ into $B^{s+1}_{p,q}$. This turns out to be very useful in order to solve (elliptic) partial differential equations, as we will see in Section 28 (it allows us to find a solution by finding a fixed point of a map that involves an inverse of the form $(\beta-\Delta)^{-1}$).

Like Theorem 21.23 it proven by using the Bernstein inequalities Theorem 21.15, which describe the action of ∂^α on L^p functions whose Fourier transforms are supported within annuli and balls, we start here by considering similar inequalities for Fourier multipliers on L^p functions whose Fourier transforms are supported within annuli or balls. Then we use this to obtain the action of Fourier multipliers on Besov spaces and apply this to the fractional Laplacian $(-\Delta)^s$ and Bessel potentials $(1-\Delta)^s$ (see 19.9 and 19.5). A summary of the main results in this section is given at the end, without the use of some of the introduced notation in this section, see Theorem 25.18.

The first theorem is a direct consequence of Young's inequality and describes the action of a Fourier multiplier of a function whose Fourier transform is integrable on both L^p and Besov spaces.

Theorem 25.1. *If $\sigma \in C_{\mathrm{p}}^\infty$ and $\widehat{\sigma} \in L^1$, then there exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$*

$$\|\sigma(\mathrm{D})u\|_{L^p} \le C\|u\|_{L^p} \qquad (u \in L^p),$$
$$\|\sigma(\mathrm{D})u\|_{B^s_{p,q}} \le C\|u\|_{B^s_{p,q}} \qquad (u \in B^s_{p,q}).$$

So $\sigma(\mathrm{D})$ forms a continuous map $L^p \to L^p$ and $B^s_{p,q} \to B^s_{p,q}$. One may take $C = \|\widehat{\sigma}\|_{L^1}$.

Proof. The first inequality follows as $\sigma(\mathrm{D})u = \mathcal{F}^{-1}(\sigma) * u$ for all $u \in \mathcal{S}'$, so that $\|\sigma(\mathrm{D})u\|_{L^p} \le \|\widehat{\sigma}\|_{L^1}\|u\|_{L^p}$. The inequality for the Besov norms follows by applying the inequality for the L^p norm to $\Delta_j u$ for all $j \in \mathbb{N}_{-1}$. □

Now we continue to consider Fourier multipliers for smooth σ whose Fourier transform may not be integrable, but which satisfy some other conditions. We first turn to the action of Fourier multipliers $\sigma(\mathrm{D})$ on L^p functions u whose Fourier transforms $\widehat{u}$ are supported in annuli. For these Fourier multipliers the function σ does not need to be defined on the whole of $\mathbb{R}^d$ (Definition 19.7).

For example if $\Omega \subset \mathbb{R}^d$ is open, $\sigma \in C^\infty(\Omega)$ and $u \in L^p$, $\operatorname{supp} \widehat{u}$ is compact. Then $\sigma(\mathrm{D})u = (\sigma\chi)(\mathrm{D})u$ for some $\chi \in C_c^\infty$. So $\phi = \sigma\chi$ is in C_c^∞. As $\phi(\mathrm{D})u = \mathcal{F}^{-1}(\phi) * u$, by Young's inequality we have $\|\phi(\mathrm{D})u\|_{L^p} \le \|\mathcal{F}^{-1}\phi\|_{L^1}\|u\|_{L^p}$. Furthermore we have $\|\mathcal{F}^{-1}\phi\|_{L^1} = \|\mathcal{R}\mathcal{F}\phi\|_{L^1} = \|\widehat{\phi}\|_{L^1}$, so that this motivates us to consider estimates of $\|\widehat{\phi}\|_{L^1}$ first:

Lemma 25.2. *Let $k = 2\lfloor 1 + \frac{d}{2} \rfloor$ and $r \in [1, \infty)$. There exists an $M > 0$ such that*

$$\|\widehat{g}\|_{L^r} \le M\|(1-\Delta)^{\frac{k}{2}} g\|_{L^1} \qquad (g \in \mathcal{S}'). \tag{25.1}$$

Moreover, for all compact $K \subset \mathbb{R}^d$ there exists a $C > 0$ such that

$$\|\widehat{\phi}\|_{L^r} \le C\|\phi\|_{C^k} \qquad (\phi \in C_c^\infty, \operatorname{supp} \phi \subset K).$$

Proof. We have already used the following inequality a couple of times, but let us recall it. Observe that $(1 + 4\pi^2|\mathfrak{x}|^2)^{-\frac{k}{2}}$ and thus the r-th power of this function are integrable (Lemma 14.13). Let $M = \|(1 + 4\pi^2|\mathfrak{x}|^2)^{-\frac{k}{2}}\|_{L^r}$. Then

$$\|f\|_{L^r} = \|(1 + 4\pi^2|\mathfrak{x}|^2)^{-\frac{k}{2}}(1 + 4\pi^2|\mathfrak{x}|^2)^{\frac{k}{2}} f\|_{L^r} \le M\|(1 + 4\pi^2|\mathfrak{x}|^2)^{\frac{k}{2}} f\|_{L^\infty} \qquad (f \in \mathcal{S}'),$$

which implies (25.1).

By the Multinomial Theorem (Theorem 20.4) there exist $c_\alpha \in \mathbb{R}$ for $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$ such that

$$(1+4\pi^2|x|^2)^{\frac{k}{2}} = \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le k}} c_\alpha (2\pi \mathrm{i} x)^\alpha \qquad (x \in \mathbb{R}^d),$$

and thus

$$(1-\Delta)^{\frac{k}{2}} = \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le k}} c_\alpha \partial^\alpha.$$

Let $\mathfrak{C} = \sum\{|c_\alpha| : \alpha \in \mathbb{N}_0^d, |\alpha| \le k\}$. Then, with $L \in [0,\infty)$ being the Lebesgue measure of K,

$$\|(1-\Delta)^{\frac{k}{2}}\phi\|_{L^1} \le \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|\le k}} |c_\alpha| \cdot \|\partial^\alpha \phi\|_{L^1} \le \mathfrak{C} L \|\phi\|_{C^k} \qquad (\phi \in C_c^\infty, \operatorname{supp}\phi \subset K).$$

□

Exercise **25.A.** Let $\mathcal{A}, \mathcal{B}, \chi$ and ϕ be as in Lemma 21.14: Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball centered at the origin in $\mathbb{R}^d$, $\chi \in C_\mathrm{c}^\infty$ be equal to 1 on a neighbourhood of $\mathcal{B}$ and $\phi \in C_\mathrm{c}^\infty$ be supported in an annulus equal to 1 on a neighbourhood of $\mathcal{A}$.

Show that there exists a a $C > 0$ such that for all $r \in [1,\infty]$

$$\|h_\alpha\|_{L^r}, \|g_\alpha\|_{L^r} \le C^{|\alpha|+1} \qquad (\alpha \in \mathbb{N}_0^d).$$

With this, prove Theorem 21.33.

Exercise **25.B.** Prove the following statement: *Let $p \in [1,\infty]$ and $k = 2\lfloor 1 + \frac{d}{2}\rfloor$. Let $u \in L^p$. Then there exists a $C > 0$ such that*

$$|\langle \widehat{u}, \psi\rangle| \le C\|\psi\|_{k,\mathcal{S}} \qquad (\psi \in \mathcal{S}).$$

As a consequence of the previous lemma we have the following, which is already mentioned in the text preceding Lemma 25.2.

Lemma 25.3. *Let $k = 2\lfloor 1 + \frac{d}{2}\rfloor$. Then there exists a $C > 0$ such that for all open $\Omega \subset \mathbb{R}^d$, $\sigma \in C_\mathrm{p}^\infty(\Omega)$, $\phi \in C_c^\infty$ such that* $\operatorname{supp}\phi \subset \Omega$ *and* $p \in [1,\infty]$

$$\|(\sigma\phi)(\mathrm{D})u\|_{L^p} \le C\|\phi\|_{C^k}\|\sigma\|_{C^k,\operatorname{supp}\phi}\|u\|_{L^p} \qquad (u \in L^p). \tag{25.2}$$

Proof. This follows by Young's inequality, Lemma 25.2 and Proposition 5.3. □

As we will consider the complement of $\{0\}$ and of closed balls numerous times in this section, it makes sense to introduce a short notation. Remember Definition 21.3.

Definition 25.4. For $a \in [0, \infty)$ we write

$$A(a, \infty) = \{x \in \mathbb{R}^d : |x| > a\}.$$

Observe that $A(0, \infty) = \mathbb{R}^d \setminus \{0\}$ and that for $\lambda > 0$,

$$\lambda A(a, \infty) = A(\lambda a, \infty), \quad \lambda A(b, c) = A(\lambda b, \lambda c) \qquad (a \in [0, \infty), b, c \in (0, \infty), b < c).$$

We use the following functions, called Mikhlin norms, to describe the action of Fourier multipliers on L^p.

Definition 25.5 (Mikhlin norm)**.** Let $m \in \mathbb{R}$, $k = 2\lfloor 1 + \frac{d}{2} \rfloor$ and $\theta \geq 0$. We define $\mathfrak{M}_{m,\theta} : C^k(A(\theta, \infty)) \to [0, \infty]$ by

$$\mathfrak{M}_{m,\theta}(\sigma) = \max_{\alpha \in \mathbb{N}_0^d : |\alpha| \leq k} \sup_{x \in A(\theta, \infty)} |x|^{|\alpha| + m} |\partial^\alpha \sigma(x)| \qquad (\sigma \in C^k(A(\theta, \infty))).$$

Even though $\mathfrak{M}_{m,\theta}$ is only a norm on the space $\{\sigma \in C^k(A(\theta, \infty)) : \mathfrak{M}_{m,\theta}(\sigma) < \infty\}$, we call $\mathfrak{M}_{m,\theta}$ a *Mikhlin norm*.

We will also write "$\mathfrak{M}_m$" instead of "$\mathfrak{M}_{m,0}$".

Let $\sigma \in C^k(A(\theta, \infty))$. Then we observe the following facts.

(a) $\mathfrak{M}_{m,\theta}(\sigma) < \infty$ if and only if there exists a $C > 0$ such that

$$|\partial^\alpha \sigma(x)| \leq C|x|^{-m-|\alpha|} \qquad (x \in A(\theta, \infty), \alpha \in \mathbb{N}_0^d, |\alpha| \leq k). \tag{25.3}$$

Moreover, if $\mathfrak{M}_{m,\theta}(\sigma) < \infty$, then (25.3) is valid for $C = \mathfrak{M}_{m,\theta}(\sigma)$.

(b)

$$\mathfrak{M}_{m,a}(\sigma) \leq \mathfrak{M}_{m,\theta}(\sigma) \qquad (a \geq \theta). \tag{25.4}$$

(c)

$$\mathfrak{M}_{m,a}(l_\lambda \sigma) = \lambda^{-m} \mathfrak{M}_{m,\lambda a}(\sigma) \qquad (\lambda > 0, a \geq \tfrac{\theta}{\lambda}). \tag{25.5}$$

(d) If $\theta > 0$ and $\mathfrak{M}_{m,a}(\sigma)$ is finite for some $a \geq \theta$, then $\mathfrak{M}_{m,b}(\sigma)$ is finite for all $b \geq \theta$ (we view $C^k(A)$ as a subset of $C^k(B)$ if $B \subset A$).

Example 25.6. (a) For all $\beta \in \mathbb{N}_0^d$ we have $\mathfrak{M}_m(\mathfrak{x}^\beta) < \infty$ if and only if $m = -|\beta|$, and thus $\mathfrak{M}_{-n}(P) < \infty$ for $n \in \mathbb{N}_0$ and P being a polynomial of the form $\sum_{\alpha \in \mathbb{N}_0^d, |\alpha| = n} c_\alpha \mathfrak{x}^\alpha$ for some $c_\alpha \in \mathbb{F}$.

(b) $\mathfrak{M}_{m,1}(\psi) < \infty$ for $m \in \mathbb{R}$ and $\psi \in \mathcal{S}$, as every derivative of ψ is of rapid decay.

Now by the previous lemma we obtain the following inequalities for Fourier multipliers acting on L^p functions whose Fourier transforms are supported in annuli.

Lemma 25.7. *Let $m \in \mathbb{R}$. Let $\mathcal{A}$ be an annulus in $\mathbb{R}^d$. There exists a $C > 0$ such that for all $p \in [1, \infty]$, $\lambda > 0$ and all $\sigma \in C^\infty(\mathbb{R}^d \setminus \{0\})$*

$$\operatorname{supp} \widehat{u} \subset \lambda \mathcal{A} \Longrightarrow \|\sigma(\mathrm{D})u\|_{L^p} \le C \mathfrak{M}_m(\sigma) \lambda^{-m} \|u\|_{L^p} \qquad (u \in L^p). \tag{25.6}$$

Moreover, if $a > 0$ is such that $\overline{\mathcal{A}} \subset A(a, \infty)$, then there exists a $C > 0$ such that for all $p \in [1, \infty]$, $\lambda > 0$ and all $\sigma \in C^\infty(\mathbb{R}^d \setminus \{0\})$

$$\operatorname{supp} \widehat{u} \subset \lambda \mathcal{A} \Longrightarrow \|\sigma(\mathrm{D})u\|_{L^p} \le C \mathfrak{M}_{m,\lambda a}(\sigma) \lambda^{-m} \|u\|_{L^p} \qquad (u \in L^p). \tag{25.7}$$

Proof. It is sufficient to prove the "Moreover" statement because $\mathfrak{M}_{m,\lambda a}(\sigma) \le \mathfrak{M}_m(\sigma)$ for all $\lambda > 0$ and $\sigma \in C^\infty(\mathbb{R}^d \setminus \{0\})$.

Step 1 Let us first argue that it is sufficient to consider $\lambda = 1$ only. Let $\lambda > 0$ and $u \in L^p$ be such that $\operatorname{supp} \widehat{u} \subset \lambda \mathcal{A}$. Then $\operatorname{supp} l_\lambda \widehat{u} \subset \mathcal{A}$ and thus $\operatorname{supp} \mathcal{F}(l_{\frac{1}{\lambda}} u) \subset \mathcal{A}$. Let us write $v = l_{\frac{1}{\lambda}} u$ so that $u = l_\lambda v$. By Lemma 19.4 (19.2)

$$\begin{aligned} \|\sigma(\mathrm{D})v\|_{L^p} &= \|l_\lambda [(l_\lambda \sigma)(\mathrm{D})u]\|_{L^p} = \lambda^{-\frac{d}{p}} \|(l_\lambda \sigma)(\mathrm{D})u\|_{L^p}, \\ \|v\|_{L^p} &= \|l_{\frac{1}{\lambda}} u\|_{L^p} \le \lambda^{\frac{d}{p}} \|u\|_{L^p}. \end{aligned}$$

Therefore, if (25.6) holds for $\lambda = 1$, we can apply it to v and then obtain (25.6) for any $\lambda > 0$ because $\mathfrak{M}_m(l_\lambda \sigma) = \lambda^{-m} \mathfrak{M}_m(\sigma)$, see (25.5).

Step 2 Let $b, c, d > 0$ be such that $a < b < c < d$ and $\mathcal{A} = A(b, c)$. As $\overline{\mathcal{A}} \subset A(a, \infty)$, we have $a < b$. Let $\phi \in C_c^\infty$ be such that $\phi = 1$ on a neighbourhood of $\mathcal{A}$ and $\operatorname{supp} \phi \subset A(a, d)$. Then $\sigma(\mathrm{D})u = (\sigma\phi)(\mathrm{D})u$ for all $u \in \mathcal{S}'$ with $\operatorname{supp} \widehat{u} \subset \mathcal{A}$ and all $\sigma \in C_{\mathrm{p}}^\infty(\mathbb{R}^d \setminus \{0\})$. By Lemma 25.3 we can conclude the existence of a $C > 0$ such that (25.6) by observing that for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \le k$ we have

$$\sup_{\xi \in \operatorname{supp} \phi} |\partial^\beta \sigma(\xi)| \le \mathfrak{M}_{m,a}(\sigma) \sup_{\xi \in A(a,d)} |\xi|^{-m-|\beta|} \qquad (\sigma \in C_{\mathrm{p}}^\infty(\mathbb{R}^d \setminus \{0\})),$$

and that $\max_{\beta \in \mathbb{N}_0^d, |\beta| \le k} \sup_{\xi \in A(a,d)} |\xi|^{-m-|\beta|} < \infty$. ☐

Exercise 25.C. Let $\mathcal{A}$ be an annulus and $k \in \mathbb{N}_0$. Show that there exists a $C > 1$ such that

$$\sup_{\xi \in \mathcal{A}} |\xi|^{-m-|\beta|} \le C^{|m|} \qquad (\beta \in \mathbb{N}_0^d, |\beta| \le k).$$

From this conclude the slightly more general statement than the one of Lemma 25.7 (similar to Theorem 21.33, see also Exercise 25.A): *Let $\mathcal{A}$ be an annulus in $\mathbb{R}^d$. There exists a $C > 0$ such that for all $m \in \mathbb{R}$, $p \in [1, \infty]$, $\lambda > 0$ and all $\sigma \in C^\infty(\mathbb{R}^d \setminus \{0\})$*

$$\operatorname{supp} \widehat{u} \subset \lambda \mathcal{A} \Longrightarrow \|\sigma(\mathrm{D})u\|_{L^p} \le C^{1+|m|} \mathfrak{M}_m(\sigma) \lambda^m \|u\|_{L^p} \qquad (u \in L^p).$$

By these inequalities we obtain the following action of Fourier multipliers acting on elements of Besov spaces whose Fourier transforms are supported away from the origin. For this we introduce the following notation:

Definition 25.8. Let $A \subset \mathbb{R}^d$. We write

$$\mathcal{S}'_A = \{u \in \mathcal{S}' : \operatorname{supp} \widehat{u} \subset A\}.$$

Theorem 25.9. *Let $m \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and $\sigma \in C^\infty_{\mathrm{p}}(\mathbb{R}^d \setminus \{0\})$*

$$\|\sigma(\mathrm{D})u\|_{B^{s+m}_{p,q}} \le C\mathfrak{M}_m(\sigma)\|u\|_{B^s_{p,q}} \qquad (u \in B^s_{p,q} \cap \mathcal{S}'_{A(\varepsilon,\infty)}). \tag{25.8}$$

In other words, $\sigma(\mathrm{D})$ forms a continuous operator

$$B^s_{p,q} \cap \mathcal{S}'_{A(\varepsilon,\infty)} \to B^{s-m}_{p,q} \cap \mathcal{S}'_{A(\varepsilon,\infty)}.$$

Proof. We can consider a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ such that for the corresponding Littlewood–Paley operators $(\Delta_j)_{j\in\mathbb{N}_{-1}}$ one has $\sum_{j\in\mathbb{N}_0} \Delta_j u = u$ for all $u \in \mathcal{S}'_{A(\varepsilon,\infty)}$.

By Lemma 25.7 (applied with $\mathcal{A}$ an annulus that contains the support of φ_0) there exists a $C > 0$ such that

$$2^{jm}\|\sigma(\mathrm{D})\Delta_j u\|_{L^p} \le C\mathfrak{M}_m(\sigma)\|\Delta_j u\|_{L^p} \qquad (j \in \mathbb{N}_0, u \in B^s_{p,q} \cap \mathcal{S}'_{A(\varepsilon,\infty)}).$$

As $\sigma(\mathrm{D})\Delta_j u = \Delta_j\sigma(\mathrm{D})u$, (25.8) follows. □

Remark 25.10 (Homogeneous Besov spaces)**.** The statement of Theorem 25.9 is rather ugly as one has to take the intersection with $\mathcal{S}'_{A(\varepsilon,\infty)}$. This requirement is done so that one only has to deal with the Littlewood–Paley blocks $\Delta_j u$ for $j \in \mathbb{N}_0$, that is, those of the form $\varphi_j(\mathrm{D})u$, where $\varphi_j = l_{2^{-j}}\varphi$. Let us define $\dot{\varphi}_j = l_{2^{-j}}\varphi$ and $\dot{\Delta}_j = \dot{\varphi}_j(\mathrm{D})$ for $j \in \mathbb{Z}$, so that $\dot{\varphi}_j = \varphi_j$ for $j \in \mathbb{N}_0$. Analogously to the definition of nonhomogeneous Besov spaces, one defines homogeneous Besov spaces $\dot{B}^s_{p,q}$ for those tempered distributions u for which $\lim_{J\to\infty}\sum_{j=-J}^\infty \dot{\Delta}_j u = u$ in $\mathcal{S}'$. We will not introduce these spaces but want mention that the inequality in (25.8) holds for all $u \in \dot{B}^s_{p,q}$.

A typical example of such a Fourier multiplier is the fractional Laplacian. For that, let us first show that the corresponding Mikhlin norm is finite.

Lemma 25.11. *Let $l \in \mathbb{R}$ and $\beta \in \mathbb{N}_0^d$. Then*

$$\mathfrak{M}_m(|\mathfrak{x}|^l) < \infty \iff m = -l, \qquad \mathfrak{M}_m(\mathfrak{x}^\beta|\mathfrak{x}|^l) < \infty \iff m = -(l + |\beta|).$$

Proof. First we observe that for $\alpha = 0$ we have

$$|\mathfrak{x}|^{|\alpha|+m} \cdot |\partial^\alpha(\mathfrak{x}^\beta|\mathfrak{x}|^l)| = |\mathfrak{x}|^m|\mathfrak{x}|^{l+|\beta|}.$$

The supremum over $\mathbb{R}^d \setminus \{0\}$ over this function is finite if and only if $m = -(l + |\beta|)$.

Let $i \in \{1, \dots, d\}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x_i}|x|^l = \frac{\mathrm{d}}{\mathrm{d}x_i}(x_1^2 + \cdots + x_d^2)^{\frac{l}{2}} = l|x|^{l-2}x_i \qquad (x \in \mathbb{R}^d), \tag{25.9}$$

and thus (on $\mathbb{R}^d \setminus \{0\}$)

$$\partial_i(\mathfrak{x}^\beta |\mathfrak{x}|^l) = \beta_i \mathfrak{x}^{\beta - e_i}|\mathfrak{x}|^l + l\mathfrak{x}^{\beta+e_i}|\mathfrak{x}|^{l-2}. \tag{25.10}$$

By induction it follows that $\partial^\alpha(\mathfrak{x}^\beta|\mathfrak{x}|^l)$ is a linear combination of functions of the form $\mathfrak{x}^\gamma|\mathfrak{x}|^a$ with $\gamma \in \mathbb{N}_0^d$, $a \in \mathbb{R}$ such that $a + |\gamma| = l + |\beta| - |\alpha|$. Therefore $|\mathfrak{x}|^{|\alpha|+m} \cdot |\partial^\alpha(\mathfrak{x}^\beta|\mathfrak{x}|^l)|$ is a bounded function for all $\alpha \in \mathbb{N}_0^d$ if $m = -(l + |\beta|)$. □

Corollary 25.12. *Let $s, t \in \mathbb{R}$, $p, q \in [1, \infty]$. For all $\varepsilon > 0$ the fractional Laplacian $(-\Delta)^s$ forms a homeomorphism*

$$B^t_{p,q} \cap \mathcal{S}'_{A(\varepsilon,\infty)} \to B^{t-s}_{p,q} \cap \mathcal{S}'_{A(\varepsilon,\infty)}.$$

Proof. The continuity follows by Theorem 25.9 and Lemma 25.11. That it is a homeomorphism follows from the fact that $(-\Delta)^{-s}$ is the inverse of $(-\Delta)^s$ on $\mathcal{S}'_{A(\varepsilon,\infty)}$. □

One of the other main examples that we will consider is the Bessel potential $(1-\Delta)^s = (1 + |\mathfrak{x}|^2)^s(\mathrm{D})$. We have seen that $\mathfrak{M}_m(|2\pi\mathfrak{x}|^s)$ is finite if and only if $m = -s$, but $\mathfrak{M}_m(1 + |\mathfrak{x}|^2)^s$ is infinite for all $s > 0$ and $m \in \mathbb{R}$ (because $(1 + |\mathfrak{x}|^2)^s$ equals 1 at 0). However, as we will see, $\mathfrak{M}_{-2s,\theta}((1 + |\mathfrak{x}|^2)^s)$ is finite for all $\theta > 0$.

Theorem 25.13. *Let $m \in \mathbb{R}$ and $\sigma \in C^\infty_{\mathrm{p}}$, $\mathfrak{M}_{m,1}(\sigma) < \infty$. Then there exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$,*

$$\|\sigma(\mathrm{D})u\|_{B^{s+m}_{p,q}} \le C\|u\|_{B^s_{p,q}} \qquad (u \in B^s_{p,q}). \tag{25.11}$$

In other words, $\sigma(\mathrm{D})$ forms a continuous operator $B^s_{p,r} \to B^{s+m}_{p,r}$.

If additionally, $\widehat{\sigma} \in L^1$ and $m > 0$, then there exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$,

$$\|(l_\mu\sigma)(\mathrm{D})u\|_{B^{s+m}_{p,q}} \le C(\mu^{-m} \vee 1)\|u\|_{B^s_{p,q}} \qquad (u \in B^s_{p,q}, \mu > 0). \tag{25.12}$$

Proof. Besov norms corresponding to different dyadic partitions of unity are equivalent, we may as well consider a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ such that $\operatorname{supp}\varphi_0 \subset A(2,\infty)$. By Lemma 25.21 (applied with $\mathcal{A} \subset A(2,\infty)$ such that $\operatorname{supp}\varphi_0 \subset \mathcal{A}$) there exists a $C_1 > 0$ such that

$$2^{jm}\|(l_\mu\sigma)(\mathrm{D})\Delta_j u\|_{L^p} \le C_1\mathfrak{M}_{m,2^j}(l_\mu\sigma)\|\Delta_j u\|_{L^p} \qquad (\mu > 0, \sigma \in C^\infty_{\mathrm{p}}, j \in \mathbb{N}_0, u \in \mathcal{S}'). \tag{25.13}$$

Let $\chi \in C_c^\infty$ be such that $\chi\varphi_{-1} = \varphi_{-1}$ and thus $\chi(\mathrm{D})\Delta_{-1} = \Delta_{-1}$ and $\sigma(\mathrm{D})\Delta_{-1} = (\sigma\chi)(\mathrm{D})\Delta_{-1}$. Then by Young's inequality

$$\|\sigma(\mathrm{D})\Delta_{-1}u\|_{L^p} \le \|\mathcal{F}^{-1}(\sigma\chi)\|_{L^1}\|\Delta_{-1}u\|_{L^p} \qquad (\sigma \in C_{\mathrm{p}}^\infty, u \in \mathcal{S}'). \tag{25.14}$$

By (25.14) and (25.13) for $\mu = 1$ we obtain (25.11), because $\mathfrak{M}_{m,2^j}(\sigma) \le \mathfrak{M}_{m,1}(\sigma)$.

Let $\sigma \in C_{\mathrm{p}}^\infty$ be such that $\widehat{\sigma} \in L^1$ and assume $m \ge 0$. Then, (see also Theorem 25.1)

$$\|(l_\mu\sigma)(\mathrm{D})\Delta_j u\|_{L^p} \le \|l_\mu^*\widehat{\sigma}\|_{L^1}\|\Delta_j u\|_{L^p} \le \|\widehat{\sigma}\|_{L^1}\|\Delta_j u\|_{L^p} \qquad (\mu > 0, j \in \mathbb{N}_{-1}, u \in \mathcal{S}'). \tag{25.15}$$

By this we may as well assume $m > 0$. Then it is sufficient to show that there exists a $C > 0$ such that

$$2^{jm}\|(l_\mu\sigma)(\mathrm{D})\Delta_j u\|_{L^p} \le C\mu^{-m}\|\Delta_j u\|_{L^p} \qquad (u \in \mathcal{S}', j \in \mathbb{N}_0, \mu > 0). \tag{25.16}$$

By the observations in Definition 25.5

$$\mathfrak{M}_{m,2^j}(l_\mu\sigma) = \mu^{-m}\mathfrak{M}_{m,\mu 2^j}(\sigma) \le \mu^{-m}\mathfrak{M}_{m,1}(\sigma) \qquad (\mu \ge 2^{-j}, j \in \mathbb{N}_0).$$

As for $\mu < 2^{-j}$ one has $2^{jm} < \mu^{-m}$, by (25.13) and (25.15), for all $\mu > 0$,

$$2^{jm}\|(l_\mu\sigma)(\mathrm{D})\Delta_j u\|_{L^p} \le \begin{cases} \mu^{-m}\mathfrak{M}_{m,1}(\sigma)\|\Delta_j u\|_{L^p} & \text{if } \mu \ge 2^{-j}, \\ \mu^{-m}\|\widehat{\sigma}\|_{L^1}\|\Delta_j u\|_{L^p} & \text{if } \mu < 2^{-j} \end{cases} \qquad (j \in \mathbb{N}_0).$$

So that (25.16) follows (with $C = \mathfrak{M}_{m,1}(\sigma) \vee \|\widehat{\sigma}\|_{L^1}$). □

One can formulate the condition $\mathfrak{M}_{m,1}(\sigma) < \infty$ for smooth σ differently:

Lemma 25.14. *Let $m \in \mathbb{R}$, $k = 2\lfloor 1 + \frac{d}{2}\rfloor$ and $\sigma \in C^\infty$. Then $\mathfrak{M}_{m,1}(\sigma) < \infty$ if and only if there exists a $C > 0$ such that*

$$|\partial^\alpha\sigma(x)| \le C(1 + |x|)^{-m-|\alpha|} \qquad (x \in \mathbb{R}^d, \alpha \in \mathbb{N}_0^d, |\alpha| \le k). \tag{25.17}$$

Proof. As

$$\frac{1}{2}(1 + |x|) \le |x| \le 1 + |x| \qquad (x \in \mathbb{R}^d, |x| \ge 1),$$

by observation (a) of Definition 25.5 we have $\mathfrak{M}_{m,1}(\sigma) < \infty$ if and only if there exists a $C > 0$ such that

$$|\partial^\alpha\sigma(x)| \le C(1 + |x|)^{-m-|\alpha|} \qquad (x \in \mathbb{R}^d, |x| \ge 1, \alpha \in \mathbb{N}_0^d, |\alpha| \le k). \tag{25.18}$$

Let us show that (25.18) is equivalent to (25.17).

As $1 \le 1 + |x| \le 2$ for all $x \in B(0,1)$, there exists an $C_1 > 0$ such that

$$1 \le C_1(1 + |x|)^{-m-|\alpha|} \qquad (x \in B(0,1), \alpha \in \mathbb{N}_0^d, |\alpha| \le k).$$

As σ is smooth, its restriction to $B(0,1)$ is bounded in C^k-norm, i.e., $C_2 = \|\sigma\|_{C^k(B(0,1))} < \infty$. Then

$$|\partial^\alpha \sigma(x)| \le C_2 \le C_1 C_2 (1+|x|)^{-m-|\alpha|} \qquad (x \in B(0,1), \alpha \in \mathbb{N}_0^d, |\alpha| \le k).$$

This shows that (25.18) is equivalent to (25.17). □

25.15. The previous lemma can be used to prove the following (Exercise 25.D): *For all $\sigma \in C_{\mathrm{p}}^\infty$ there exists a $m_0 \in \mathbb{R}$ such that $\mathfrak{M}_{m,1}(\sigma) < \infty$ for all $m \in \mathbb{R}$ with $m \le m_0$.*

Exercise **25.D.** Prove the statement in 25.15.

Lemma 25.16. *Let $a, b, l \in \mathbb{R}$, $a > 0$, $\beta \in \mathbb{N}_0^d$. Then*

$$\mathfrak{M}_{m,1}(\mathfrak{x}^\beta |\mathfrak{x}|^l (1+|\mathfrak{x}|^a)^b) < \infty \iff m \le -(|\beta| + l + ab).$$

In particular,

$$\mathfrak{M}_{-ab,1}((1+|\mathfrak{x}|^a)^b) < \infty.$$

Proof. For $x \in \mathbb{R}^d$, $|x| \ge 1$ we have $|x|^a \le 1 + |x|^a \le 2|x|^a$ and thus

$$\sup_{\substack{x \in \mathbb{R}^d \\ |x| \ge 1}} |x|^m \cdot \left| x^\beta |x|^l (1+|x|^a)^b \right| < \infty \iff m \le -(|\beta| + l + ab).$$

Let $i \in \{1, \dots, d\}$. Then, see (25.9),

$$\frac{\mathrm{d}}{\mathrm{d}x_i}(1+|x|^a)^b) = b(1+|x|^a)^{b-1} \cdot a|x|^{a-2} x_i \qquad (x \in \mathbb{R}^d).$$

Therefore, by (25.10) (on $\mathbb{R}^d \setminus \{0\}$)

$$\begin{aligned} \partial_i \mathfrak{x}^\beta |\mathfrak{x}|^l (1+|\mathfrak{x}|^a)^b = \beta_i \mathfrak{x}^{\beta - e_i} |\mathfrak{x}|^l (1+|\mathfrak{x}|^a)^b + l \mathfrak{x}^{\beta+e_i} |\mathfrak{x}|^{l-2} (1+|\mathfrak{x}|^a)^b \\ + a \mathfrak{x}^{\beta+e_i} |\mathfrak{x}|^{l+a-2} b (1+|x|^a)^{b-1}. \end{aligned}$$

By induction it follows that $\partial^\alpha (\mathfrak{x}^\beta |\mathfrak{x}|^l (1+|\mathfrak{x}|^a)^b)$ is a linear combination of functions of the form $(\mathfrak{x}^\gamma |\mathfrak{x}|^c (1+|\mathfrak{x}|^a)^d)$ with $\gamma \in \mathbb{N}_0^d$, $c, d \in \mathbb{R}$ such that $|\gamma| + c + ad = |\beta| + l + ab - |\alpha|$. Therefore $|\mathfrak{x}|^{|\alpha|-m} |\partial^\alpha (\mathfrak{x}^\beta |\mathfrak{x}|^l (1+|\mathfrak{x}|^a)^b)|$ is bounded for all $\alpha \in \mathbb{N}_0^d$ if $m \le -(|\beta| + l + ab)$. □

Corollary 25.17. *Let $s, t \in \mathbb{R}$, $p, q \in [1, \infty]$. The Bessel potential $(1-\Delta)^s$ forms a homeomorphism*

$$B_{p,q}^t \to B_{p,q}^{t-2s}.$$

Proof. This is a consequence of Theorem 25.13 and Lemma 25.16. □

We summarize the different inequalities that we have obtained, but without the introduced notations $\mathfrak{M}_{m,\theta}$ and $A(a,\infty)$ (however, incorporating the scaling properties that one obtains via for example (25.5)).

Theorem 25.18 (Hörmander–Mikhlin inequalities)**.** *Let $m \in \mathbb{R}$.*

(a) (Lemma 25.7) *Let $\mathcal{A}$ be an annulus. Let $\theta \geq 0$. Let $\sigma \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and suppose that there exists a $M > 0$ such that*

$$|\partial^\alpha \sigma(x)| \leq M|x|^{-m-|\alpha|} \qquad (x \in \mathbb{R}^d, |x| > \theta, \alpha \in \mathbb{N}_0^d, |\alpha| \leq k). \tag{25.19}$$

For all $\lambda_0 > 0$ there exists a $C > 0$ such that for all $p \in [1,\infty]$ and $\lambda > \lambda_0$

$$\operatorname{supp} \widehat{u} \subset \lambda\mathcal{A} \Longrightarrow \|(l_\mu \sigma)(\mathrm{D})u\|_{L^p} \leq C\mu^{-m}\lambda^{-m}\|u\|_{L^p} \qquad (u \in L^p).$$

(b) (Theorem 25.9) *Let $m \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1,\infty]$, and $\sigma \in C_{\mathrm{p}}^\infty(\mathbb{R}^d \setminus \{0\})$ such that* (25.19) *holds for $\theta = 0$,*

$$\|(l_\mu \sigma)(\mathrm{D})u\|_{B^{s+m}_{p,q}} \leq C\mu^{-m}\|u\|_{B^s_{p,q}} \qquad (\mu > 0, u \in B^s_{p,q}, \operatorname{supp} \widehat{u} \cap B(0,\varepsilon) = \emptyset).$$

(c) (Theorem 25.13) *Let $\sigma \in C_{\mathrm{p}}^\infty$ be such that* (25.19) *holds for some $\theta > 0$ or equivalently, such that there exists an $M > 0$ such that*

$$|\partial^\alpha \sigma(x)| \leq C(1+|x|)^{-m-|\alpha|} \qquad (x \in \mathbb{R}^d, \alpha \in \mathbb{N}_0^d, |\alpha| \leq k).$$

Then there exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1,\infty]$,

$$\|\sigma(\mathrm{D})u\|_{B^{s+m}_{p,q}} \leq C\|u\|_{B^s_{p,q}} \qquad (u \in B^s_{p,q}).$$

(d) (Theorem 25.1) *Let $\sigma \in C_{\mathrm{p}}^\infty$ be such that $\widehat{\sigma} \in L^1$. There exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1,\infty]$,*

$$\|\sigma(\mathrm{D})u\|_{B^s_{p,q}} \leq C\|u\|_{B^s_{p,q}} \qquad (u \in B^s_{p,q}).$$

(e) (Theorem 25.13) *Let $m \geq 0$ and $\sigma \in C_{\mathrm{p}}^\infty$ be such that $\widehat{\sigma} \in L^1$. Then there exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1,\infty]$,*

$$\|(l_\mu \sigma)(\mathrm{D})u\|_{B^{s+m}_{p,q}} \leq C(\mu^{-m} \vee 1)\|u\|_{B^s_{p,q}} \qquad (u \in B^s_{p,q}, \mu > 0).$$

Let us apply Theorem 25.18 (e) to the heat semigroup. We consider this semigroup later on again to find solutions to the heat equation.

Definition 25.19. For $t > 0$ let $h_t : \mathbb{R}^d \to \mathbb{R}$ be the Schwartz function (as in Example 11.15) given by

$$h_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{1}{4t}|x|^2} \qquad (x \in \mathbb{R}^d).$$

We define $H_t : \mathcal{S}' \to \mathcal{S}'$ by

$$H_t u = h_t * u \qquad (u \in \mathcal{S}'),$$

and $H_0 : \mathcal{S}' \to \mathcal{S}'$ to be the identity map, $H_0 u = u$. The collection $(H_t)_{t\in[0,\infty)}$ is called the *Heat semigroup*, in the sense that

Exercise 25.E. Show that $(H_t)_{t\in[0,\infty)}$ is a *semigroup*, i.e.,

$$H_t H_s u = H_t \circ H_s u = H_{t+s} u \qquad (s,t \in [0,\infty), u \in \mathcal{S}').$$

As the convolution of a tempered distribution with a Schwartz function is a C_{p}^∞ function, $H_t u$ is smooth for all $t > 0$. By the following theorem it follows that if u is in L^p, then $H_t u$ is in the Sobolev space $W^{k,p}$ for all $k \in \mathbb{N}_0$. Moreover, if u is in a Besov space $B^s_{p,q}$ for some $s \in \mathbb{R}$, then u is in $B^t_{p,q}$ for all $t \in \mathbb{R}$.

Theorem 25.20. *Let $s \in \mathbb{R}$, $m \geq 0$, $k \in \mathbb{N}_0$, $p, q \in [1,\infty]$.*

(a) *There exists a $C > 0$ such that*

$$\|H_t f\|_{W^{k,p}} \leq C(t^{-\frac{k}{2}} \vee 1)\|f\|_{L^p} \qquad (f \in L^p, t > 0).$$

(b) *There exists a $C > 0$ such that*

$$\|H_t u\|_{B^{s+m}_{p,q}} \leq C(t^{-\frac{m}{2}} \vee 1)\|u\|_{B^s_{p,q}} \qquad (u \in B^s_{p,q}, t > 0).$$

Exercise 25.F. Prove Theorem 25.20. (Hint for (a): Show that $\partial^\alpha h_t \in L^1$ for all $\alpha \in \mathbb{N}_0$ by showing that $y \mapsto |y|^n e^{-|y|^2}$ is integrable for all $n \in \mathbb{N}_0$; either by showing that $(1+|y|^2)^n e^{-|y|^2}$ is a Schwartz function or one can use that the gamma function Γ is finite everywhere (Definition 11.22). Hint for (b)): Example 25.6.)

25.1 Comments ...

In this section we have considered Fourier multipliers of smooth functions on $\mathbb{R}^d\setminus\{0\}$. The Mikhlin norm $\mathfrak{M}_{m,\theta}$ is however defined for C^k functions on $A(\theta,\infty)$. One can actually also consider Fourier multipliers corresponding to such $C^k(A(\theta,\infty))$ functions. Let us comment on "taking $k \in \mathbb{N}_0$ instead of $k = \infty$" and on "$\theta \geq 0$ instead of $\theta = 0$" separately.

First of all, instead of taking smooth functions one may take C^k functions basically because of Exercise 25.B: For $v \in \mathcal{S}'$ for which there exists a $C > 0$ such that $|\langle v, \varphi\rangle| \leq C\|\varphi\|_{k,\mathcal{S}}$ One can show that if $\sigma \in C^k_{\mathrm{p}} = \bigcup_{m\in\mathbb{N}_0} C^k_{\mathrm{p},m}$ (see Exercise 14.E), that is, there exists an $m \in \mathbb{N}_0$ such that $\mathfrak{q}_{k,m}(\sigma) < \infty$, then

$$\varphi \mapsto \langle v, \sigma\varphi\rangle$$

defines a tempered distribution, which we call σv. Then, by Exercise 25.B for any $u \in L^p$ we have $\sigma\hat{u} \in \mathcal{S}'$ and so we may define $\sigma(\mathrm{D})u$ to be the tempered distribution $\mathcal{F}^{-1}(\sigma\hat{u})$.

Similar as in Definition 19.7 one may extend this definition in case σ is only C_{p}^k on a neighbourhood of the support of u. The arguments in this section do not depend on the smoothness of σ, well, only in terms of the Mikhlin norm, which only requires σ to be C^k.

Consider the situation as in Lemma 25.7 but with $\sigma \in C^\infty(A(\theta,\infty))$ for some $\theta > 0$. Then $\sigma(\mathrm{D})u$ may not be defined for all $\lambda > 0$ (and $u \in L^p$ with $\operatorname{supp}\widehat{u} \subset \lambda\mathcal{A}$), as $\lambda\mathcal{A}$ needs to be a subset of $A(\theta,\infty)$. We have $\lambda\mathcal{A} \subset A(\theta,\infty)$ for $\lambda > \frac{\theta}{a}$ if $a > 0$ is such that $\overline{\mathcal{A}} \subset A(a,\infty)$. The following lemma is an extension of Lemma 25.7 which entails this in its premise.

Lemma 25.21. *Let $m \in \mathbb{R}$. Let $\mathcal{A}$ be an annulus in $\mathbb{R}^d$. Let $\theta \ge 0$ and $a > 0$ be such that $\overline{\mathcal{A}} \subset A(a,\infty)$. There exists a $C > 0$ such that for all $p \in [1,\infty]$, $\lambda > \frac{\theta}{a}$ and all $\sigma \in C^\infty(A(\theta,\infty))$*

$$\operatorname{supp}\widehat{u} \subset \lambda\mathcal{A} \Longrightarrow \|\sigma(\mathrm{D})u\|_{L^p} \le C\mathfrak{M}_{m,\lambda a}(\sigma)\lambda^{-m}\|u\|_{L^p} \qquad (u \in L^p). \tag{25.20}$$

Proof. For $\theta = 0$ this is the "Moverover" statement of Lemma 25.7. Let $\theta > 0$. Let $r > 1$ be such that $\overline{\mathcal{A}} \subset A(ra,\infty)$. Let $\chi \in C_{\mathrm{b}}^\infty(\mathbb{R}^d)$ be equal to 1 on $A(r\theta,\infty)$ and $\operatorname{supp}\chi \subset A(\theta,\infty)$. Then $\sigma(\mathrm{D})u = (\sigma\chi)(\mathrm{D})u$ for $\sigma \in C^\infty(A(\theta,\infty))$ and $u \in \mathcal{S}'$ with $\operatorname{supp}\widehat{u} \subset \overline{A}(r\theta,\infty)$, which is the case if $\operatorname{supp}\widehat{u} \subset \lambda\mathcal{A}$ for $\lambda > \frac{\theta}{a}$. As there exists a $\mathfrak{C} > 0$ such that $\mathfrak{M}_{m,\lambda a}(\chi\sigma) = \mathfrak{M}_{m,\lambda a}(\sigma)$ for $\lambda > \frac{\theta}{a}$ we obtain (25.20) by (25.7). □

Remark 25.22. We called the inequalities in Theorem 25.18 Hörmander–Mikhlin inequalities as they are closely related to what in literature are called the Hörmander–Mikhlin multiplier theorems, see [Sha66] for example, or [Mic57] (in Russian) or [H6̈0] for the work of Mikhlin and Hörmander. As unfortunately happens with names from languages with different alphabets, we also found instead of Mikhlin the names Michlin or Mihlin.

The theorems of Hörmander and Mikhlin deal with the case $m = 0$. See for example also [HvNVW16, Theorem 5.5.10] (which looks again a bit different). We decided to call the norm the Mikhlin norm as that seems to align with the literature and it seems that the Hörmander and Mikhlin statements are slightly different.

26 Products of tempered distributions

For functions $f, g : \mathbb{R}^d \to \mathbb{F}$ their product fg (or $f \cdot g$) is the function $\mathbb{R}^d \to \mathbb{F}$ defined by $fg(x) = f(x) \cdot g(x)$ for $x \in \mathbb{R}^d$. We defined the product of a smooth function with a distribution in Definition 2.14 and the product of a C_{p}^∞ function with a tempered distribution in Definition 15.3 in such a way that they extend the product of functions: if $f \in L^1_{\mathrm{loc}}$ and $\psi \in \mathcal{E}$, then the product of the testfunction with the corresponding distribution ψu_f equals the distribution $u_{\psi f}$ that corresponds to the product of the functions f and ψ (similarly this equality holds in $\mathcal{S}'$ if $f \in L^1_{\mathrm{loc}}$ is such that $u_f \in \mathcal{S}'$ and $\psi \in C_{\mathrm{p}}^\infty$).

In this section we investigate for which distributions one can make sense of their "product". It would make sense to call an operation $\times : \mathcal{S}' \times \mathcal{S}' \to \mathcal{S}'$ that extends the product $C_{\mathrm{p}}^{\infty} \times \mathcal{S}' \to \mathcal{S}'$, $(\sigma, u) \mapsto \sigma u$ a "product" if it is continuous in both variables, commutative and associative.

However, such an operation does not exist and so we may only make sense of a kind of "product" for certain pairs of (tempered) distributions.

First of all, let us observe that there does not exist an operation $\times : \mathcal{S}' \times \mathcal{S}' \to \mathcal{S}'$ that extends the product map

$$C_{\mathrm{p}}^{\infty} \times \mathcal{S}' \to \mathcal{S}', \quad (\sigma, u) \mapsto \sigma u, \tag{26.1}$$

and is continuous in both variables, as then, $\delta_0 \times \delta_0$ by some approximation argument would be the tempered distribution that is equal to the Fourier transform of $\mathbb{1} * \mathbb{1}$ (by the continuity of the product and the Fourier transform), which is not defined (or, if one wants, is infinity everywhere). On the other hand, let u be defined as in Exercise 2.E, see also Exercise 15.B:

$$u(\varphi) := \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{\varphi(x)}{x} \, \mathrm{d}x \qquad (\varphi \in \mathcal{S}(\mathbb{R})).$$

Then $\boldsymbol{x} u = \mathbb{1}$, $\mathbb{1}\delta_0 = \delta_0$ and $\boldsymbol{x}\delta_0 = 0$, and $0u = 0$. By means of these identities let us show that there does not exist an operation $\times : \mathcal{S}' \times \mathcal{S}' \to \mathcal{S}'$ that extends the product map (26.1) and is commutative and associative, that is,

$$\begin{aligned} &\psi \times u = \psi u \qquad (\psi \in C_{\mathrm{p}}^{\infty}, u \in \mathcal{S}'), \\ &u \times v = v \times u, \qquad (u \times v) \times w = u \times (u \times w) \qquad (u, v, w \in \mathcal{S}'). \end{aligned}$$

If $\times$ were such an operation, then we would have

$$\begin{aligned} \boldsymbol{x} \times (u \times \delta_0) &= (\boldsymbol{x} \times u) \times \delta_0 = (\boldsymbol{x} u) \times \delta_0 = \mathbb{1} \times \delta_0 = \mathbb{1}\delta_0 = \delta_0, \\ \boldsymbol{x} \times (u \times \delta_0) &= u \times (\boldsymbol{x} \times \delta_0) = u \times (\boldsymbol{x}\delta_0) = u \times 0 = 0u = 0. \end{aligned}$$

We use the decomposition of tempered distribution in terms of Littlewood–Paley blocks as a starting point to define a product on a class of pairs of distributions. Let Δ_j for $j \in \mathbb{N}_{-1}$ be the Littlewood–Paley operators as in Definition 21.12 (for a given dyadic partition of unity). For $u, v \in \mathcal{S}'$, by Lemma 21.9 we have

$$u = \sum_{i \in \mathbb{N}_{-1}} \Delta_i u, \quad v = \sum_{i \in \mathbb{N}_{-1}} \Delta_i v.$$

Let us write " $\cdot$ " also for the product between functions, so that $\Delta_i u \cdot \Delta_j v$ is $(\Delta_i u)\Delta_j v$ (we do not want to write $\Delta_i u \Delta_j v$ as this can be read as $\Delta_i(u\Delta_j v)$). For those tempered u and v such that

$$\sum_{i,j \in \mathbb{N}_{-1}} \Delta_i u \cdot \Delta_j v \tag{26.2}$$

exists in $\mathcal{S}'$ (for the summation notation see 21.7), one could call the tempered distribution (26.2) a "product" of u and v. A priori it is not clear whether this product agrees with fg, or better said, u_{fg} using the notation as in 2.6, if u and v are represented by locally integrable functions f and g, respectively; i.e., $u = u_f$ and $v = u_g$.

In this section we introduce a certain product on tempered distributions, and call it the Bony product. In Section 27 we consider this Bony product between elements of Besov spaces, here we it between C_{p}^∞ functions and Schwartz functions (Theorem 26.5) and between C_{p}^∞ functions and tempered distributions (Theorem 26.7) and show that in these cases the product agrees with the pointwise product and with the product as defined in Definition 15.3. Moreover, we show that the Bony product between a function in L^p and a function in L^q for p and q such that $\frac{1}{p} + \frac{1}{q} = 1$ is equal to the product of the two functions, which is an L^1 function by Hölders inequality. Finally, in Example 26.11 we given examples of locally integrable functions which represent tempered distributions for which their Bony product exists but is not equal to their product as functions (the pointwise product).

Definition 26.1. Let φ generate a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ and $\Delta_j = \varphi_j(\mathrm{D})$ for $j \in \mathbb{N}_{-1}$. Let $u, v \in \mathcal{S}'$. If

$$\sum_{i,j=-1}^{J} \Delta_i u \cdot \Delta_j v$$

converges in $\mathcal{S}'$ as $J \to \infty$, then we say that the *φ-Bony product of u and v exists* and write $u \bullet_\varphi v$ or $v \bullet_\varphi u$ for the limit and call it the *φ-Bony product of u and v*, i.e.,

$$u \bullet_\varphi v = \lim_{J\to\infty} \sum_{i,j=-1}^{J} \Delta_i u \cdot \Delta_j v.$$

If for all ψ that generate dyadic partitions of unity the ψ-Bony product of u and v exists and $u \bullet_\varphi v = u \bullet_\psi v$, then we say that the *Bony product of u and v exists* and call it the *Bony product of u and v* and write $u \bullet v$ instead of $u \bullet_\varphi v$.

The Bony product can be viewed as a bilinear operation in the following sense:

Lemma 26.2. *Let φ generate a dyadic partition of unity. Let $u, v, w \in \mathcal{S}'$ and $\lambda \in \mathbb{F}$.*

(a) *If $u \bullet_\varphi v$ and $w \bullet_\varphi v$ exist, then $(u + w) \bullet_\varphi v$ exists and*
$$(u + w) \bullet_\varphi v = u \bullet_\varphi v + w \bullet_\varphi v.$$

(b) *If $u \bullet_\varphi v$ and $u \bullet_\varphi w$ exist, then $u \bullet_\varphi (v + w)$ exists and*
$$u \bullet_\varphi (v + w) = u \bullet_\varphi v + u \bullet_\varphi w.$$

(c) *If $u \bullet_\varphi v$ exist, then $(\lambda u) \bullet_\varphi v$ and $u \bullet_\varphi (\lambda v)$ exist and*
$$\lambda(u \bullet_\varphi v) = (\lambda u) \bullet_\varphi v + u \bullet_\varphi (\lambda v).$$

Proof. The proof is straightforward and left to the reader. □

$(\diamond \diamond)$

We have already seen that $\sum_{j=-1}^{\infty} \Delta_j \varphi = \varphi$ in $\mathcal{S}$ for $\varphi \in \mathcal{S}$ and that $\sum_{j=-1}^{\infty} \Delta_j u = u$ in $\mathcal{S}'$ for $u \in \mathcal{S}'$, see Lemma 21.9. We consider some similar convergence for C_{p}^∞, that we will use for the Bony products of such functions with Schwartz functions and tempered distributions. We state this convergence in Lemma 26.4 as a consequence of the following convergence of mollifiers of C_{p}^∞ functions.

Lemma 26.3. *Let ψ be a Schwartz function with $\int \psi = 1$. Let $\psi_\varepsilon = l_\varepsilon^* \psi$ for $\varepsilon > 0$. For all $\eta \in C_{\mathrm{p}}^\infty$ and $m \in \mathbb{N}_0$ there exists a $k \in \mathbb{N}_0$ such that*

$$\mathfrak{q}_{m,k}(\psi_\varepsilon * \eta - \eta) \xrightarrow{\varepsilon \downarrow 0} 0.$$

In particular, if $n \in \mathbb{N}_0$ is such that $\mathfrak{q}_{m,n}(\eta) < \infty$, then it suffices to choose $k = n + 1$.

Proof. As $\partial^\alpha(\psi_\varepsilon * \eta) = \psi_\varepsilon * (\partial^\alpha \eta)$ and $\partial^\alpha \eta \in C_{\mathrm{p}}^\infty$ for all $\alpha \in \mathbb{N}_0^d$ and $\eta \in C_{\mathrm{p}}^\infty$, it is sufficient to prove that for all $\eta \in C_{\mathrm{p}}^\infty$ there exists a $k \in \mathbb{N}_0$ such that

$$\|(\psi_\varepsilon * \eta - \eta)(1 + |\mathfrak{x}|^2)^{-k}\|_{L^\infty} \xrightarrow{\varepsilon \downarrow 0} 0. \tag{26.3}$$

Let $n \in \mathbb{N}_0^d$ be such that $\eta(1 + |\mathfrak{x}|^2)^{-n}$ is bounded. Let $M = \|\eta(1 + |\mathfrak{x}|^2)^{-n}\|_{L^\infty}$ and $N = \|\psi(1 + |\mathfrak{x}|^2)^n\|_{L^1} \vee 1$. As

$$(1 + |x|^2)^n \le 2^n (1 + |x - y|^2)^n (1 + |y|^2)^n \qquad (x, y \in \mathbb{R}^d),$$

for $\varepsilon \in (0, 1)$ we have

$$\begin{aligned}
\|(\psi_\varepsilon * \eta)(1 + |\mathfrak{x}|^2)^{-n}\|_{L^\infty} &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \psi_\varepsilon(x - y)\eta(y)(1 + |x|^2)^{-n} \, \mathrm{d}y \\
&\le \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \psi_\varepsilon(x - y)(1 + |x - y|^2)^n \eta(y)(1 + |y|^2)^{-n} \, \mathrm{d}y \\
&= \|(\psi_\varepsilon(1 + |\mathfrak{x}|^2)^n) * (\eta(1 + |\mathfrak{x}|^2)^{-n})\|_{L^\infty} \\
&\le \|\psi_\varepsilon(1 + |\mathfrak{x}|^2)^n\|_{L^1} \|\eta(1 + |\mathfrak{x}|^2)^{-n}\|_{L^\infty} \\
&\le M \|l_\varepsilon^*(\psi(1 + |\mathfrak{x}|^2)^n)\|_{L^1} \le MN,
\end{aligned}$$

where we used that $(1 + |\mathfrak{x}|^2)^n \le (1 + |\frac{\mathfrak{x}}{\varepsilon}|^2)^n$ for all $\varepsilon \in (0, 1)$.

Let $\delta, \varepsilon \in (0, 1)$. Let $R > 0$ be such that $2MN(1 + R^2)^{-1} < \varepsilon$. Let $k = n + 1$. Then

$$\begin{aligned}
&\|(\psi_\varepsilon * \eta - \eta)(1 + |\mathfrak{x}|^2)^{-k}\|_{L^\infty} \\
&\le \sup_{x \in B(0,R)} |(\psi_\varepsilon * \eta(x) - \eta(x))(1 + |x|^2)^{-k}| \\
&\qquad + (1 + R^2)^{-1} \Big(\|(\psi_\varepsilon * \eta)(1 + |\mathfrak{x}|^2)^{-n}\|_{L^\infty} + \|\eta(1 + |\mathfrak{x}|^2)^{-n}\|_{L^\infty} \Big) \\
&\le \sup_{x \in B(0,R)} |\psi_\varepsilon * \eta(x) - \eta(x)| + \delta.
\end{aligned}$$

By Theorem 7.15 (b)

$$\sup_{x\in B(0,R)} |\psi_\varepsilon * \eta(x) - \eta(x)| \xrightarrow{\varepsilon\downarrow 0} 0,$$

so that we conclude (26.3). □

Lemma 26.4. *Let φ generate a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$. Let $\Delta_j = \varphi_j(\mathrm{D})$ for $j \in \mathbb{N}_{-1}$. Let $\eta \in C_\mathrm{p}^\infty$. For all $m \in \mathbb{N}_0$ there exists a $k \in \mathbb{N}_0$ such that*

$$\mathfrak{q}_{m,k}\Big(\sum_{j=-1}^{J} \Delta_j\eta - \eta\Big) \xrightarrow{J\to\infty} 0.$$

Proof. For $\psi = \mathcal{F}^{-1}(\varphi_{-1})$ we have $\sum_{j=-1}^{J}\Delta_j\eta = \mathcal{F}^{-1}(\sum_{j=-1}^{J}\varphi_j) * \eta = l^*_{2^{-J+1}}\psi * \eta$ (see (21.10)) and $\int\psi = \int\widehat{\varphi}_{-1} = \varphi_{-1}(0) = 1$. Therefore this follows by Lemma 26.3. □

Theorem 26.5. *For $\eta \in C_\mathrm{p}^\infty$ and $\psi \in \mathcal{S}$, $\eta \bullet \psi$ exists in $\mathcal{S}$ and*

$$\eta\psi = \eta \bullet \psi \qquad (\eta \in C_\mathrm{p}^\infty, \psi \in \mathcal{S}).$$

Proof. Let φ generate a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$. Let $\Delta_j = \varphi_j(\mathrm{D})$ for $j \in \mathbb{N}_{-1}$. It is sufficient to show $\sum_{i,j=-1}^{J}\Delta_i\eta\cdot\Delta_j\psi \to \eta\psi$ in $\mathcal{S}$ as $J\to\infty$. This follows as

$$\sum_{i,j=-1}^{J}\Delta_i\eta\cdot\Delta_j\psi - \eta\psi = \Big(\sum_{i=-1}^{J}\Delta_i\eta - \eta\Big)\sum_{i=-1}^{J}\Delta_i\psi + \eta\cdot\Big(\sum_{j=-1}^{J}\Delta_j\psi - \psi\Big). \tag{26.4}$$

The second term converges to zero in $\mathcal{S}$ by Lemma 21.9. For the first term we use Lemma 14.7. Let $m \in \mathbb{N}_0$ and let $C > 0$ be as in (14.5). Then for all $k \in \mathbb{N}_0$

$$\left\|\left(\sum_{i=-1}^{J}\Delta_i\eta - \eta\right)\sum_{i=-1}^{J}\Delta_i\psi\right\|_{m,\mathcal{S}} \le C\sup_{I\in\mathbb{N}_{-1}}\left\|\sum_{i=-1}^{I}\Delta_i\psi\right\|_{m+k,\mathcal{S}}\mathfrak{q}_{m,k}\Big(\sum_{j=-1}^{J}\Delta_j\eta - \eta\Big).$$

As $\sum_{j=-1}^{J}\Delta_j\psi \to \psi$ in $\mathcal{S}$, the supremum $\sup_{I\in\mathbb{N}_{-1}}\left\|\sum_{i=-1}^{I}\Delta_i\psi\right\|_{m+k,\mathcal{S}}$ is finite for all $k \in \mathbb{N}_0$. By Lemma 26.4 there exists a $k \in \mathbb{N}_0$ such that

$$\mathfrak{q}_{m,k}\Big(\sum_{j=-1}^{J}\Delta_j\eta - \eta\Big) \xrightarrow{J\to\infty} 0,$$

from which we conclude that the first term on the right-hand side of (26.4) also converges to zero in $\mathcal{S}$. □

The following auxiliary lemma will be used for Theorem 26.7, in which we consider the product of C_p^∞ functions with tempered distributions.

Lemma 26.6. *Let φ generate a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$. Let $\Delta_j = \varphi_j(\mathrm{D})$ for $j \in \mathbb{N}_{-1}$. For all $m \in \mathbb{N}_0$ there exists a $k \in \mathbb{N}_0$ and a $C > 0$ such that*

$$\Big\| \sum_{j\in F} \Delta_j \psi \Big\|_{m,\mathcal{S}} \le C\|\psi\|_{k,\mathcal{S}} \qquad (F \subset \mathbb{N}_{-1}, F \text{ is a finite set}, \psi \in \mathcal{S}).$$

Proof. By Lemma 14.7, by Lemma 21.8 and the continuity of the Fourier transformation in $\mathcal{S}$ there exist $C_1, C_2, C_3 > 0$ and $k, l \in \mathbb{N}_0$ such that for all finite subsets $F \subset \mathbb{N}_{-1}$ and $\psi \in \mathcal{S}$

$$\begin{aligned}\Big\| \sum_{j\in F} \Delta_j \psi \Big\|_{m,\mathcal{S}} &\le C_1 \Big\| \sum_{j\in F} \varphi_j \widehat{\psi} \Big\|_{k,\mathcal{S}} \le C_1 C_2 \Big\| \sum_{j\in F} \varphi_j \Big\|_{C^k} \|\widehat{\psi}\|_{k,\mathcal{S}} \\ &\le C_1 C_2 C_3 \|\psi\|_{l,\mathcal{S}}.\end{aligned}$$

□

Theorem 26.7. *For $\eta \in C_{\mathrm{p}}^\infty$ and $u \in \mathcal{S}'$, $\eta \bullet u$ exists in $\mathcal{S}'$ and*

$$\eta u = \eta \bullet u \text{ in } \mathcal{S}'.$$

Proof. Let φ generate a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$. Let $\Delta_j = \varphi_j(\mathrm{D})$ for $j \in \mathbb{N}_{-1}$. We have to show that

$$\Big\langle \sum_{i,j=-1}^{J} \Delta_i \eta \cdot \Delta_j u, \psi \Big\rangle \xrightarrow{J\to\infty} \langle \eta u, \psi \rangle \qquad (\psi \in \mathcal{S}). \tag{26.5}$$

Observe that $\langle \eta u, \psi \rangle = \langle u, \eta \psi \rangle$ and

$$\Big\langle \sum_{i,j=-1}^{J} \Delta_i \eta \cdot \Delta_j u, \psi \Big\rangle = \Big\langle u, \sum_{j=-1}^{J} \Delta_j \Big(\sum_{i=-1}^{J} \Delta_i \eta \cdot \psi \Big) \Big\rangle \qquad (J \in \mathbb{N}_{-1}).$$

Hence it suffices to show $\sum_{j=-1}^{J} \Delta_j (\sum_{i=-1}^{J} \Delta_i \eta \cdot \psi) \to \eta\psi$ in $\mathcal{S}$. Observe that

$$\begin{aligned}\sum_{j=-1}^{J} \Delta_j \Big(\sum_{i=-1}^{J} \Delta_i \eta \cdot \psi \Big) - \eta\psi &= \sum_{j=-1}^{J} \Delta_j (\sum_{i=-1}^{J} \Delta_i \eta \cdot \psi) - \sum_{i=-1}^{J} \Delta_i \eta \cdot \psi \\ &\quad + \sum_{i=-1}^{J} \Delta_i \eta \cdot \psi - \eta\psi. \end{aligned} \tag{26.6}$$

By Lemma 26.4 and Lemma 14.7 it follows that $\sum_{i=-1}^{J} \Delta_i \eta \cdot \psi \to \eta \cdot \psi$ in $\mathcal{S}$, and by those lemmas and Lemma 26.6 it follows that

$$\sum_{j=-1}^{J} \Delta_j \Big(\Big(\sum_{i=-1}^{J} \Delta_i \eta - \eta \Big) \cdot \psi \Big) \to 0.$$

□

Now we turn to the product of L^p functions with L^q functions, in case $\frac{1}{p} + \frac{1}{q} = 1$. As before, we first consider the convergence of the partial sums $\sum_{j=-1}^{J} \Delta_j f$ as $J \to \infty$ in L^p for $f \in L^p$. For this we adapt Theorem 7.15 (d):

Lemma 26.8. *Let $\psi \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Let $p \in [1, \infty)$. Write $\psi_\varepsilon = l_\varepsilon^* \psi$ for $\varepsilon > 0$. Then, for any $f \in L^p(\mathbb{R}^d)$*

$$\psi_\varepsilon * f \to (\int \psi) f. \tag{26.7}$$

Proof. Let $\chi \in C_c(\mathbb{R}^d)$ be equal to 1 on a neighbourhood of zero. Write $\chi_R = l_{\frac{1}{R}} \chi$ for $R > 0$. By Theorem 7.15 (d) we have

$$\left\| l_\varepsilon^*(\psi \chi_R) * f - (\int \psi \chi_R) f \right\|_{L^p} \xrightarrow{\varepsilon \downarrow 0} 0.$$

By Young's inequality we have

$$\| l_\varepsilon^*(\psi(\mathbb{1} - \chi_R)) * f \|_{L^p} \le \| \psi(\mathbb{1} - \chi_R) \|_{L^1} \| f \|_{L^p} \qquad (\varepsilon > 0).$$

As both $\| \psi(\mathbb{1} - \chi_R) \|_{L^1} \xrightarrow{R \to \infty} 0$ and $\int \psi \chi_R \xrightarrow{R \to \infty} \int \psi$, and

$$\begin{aligned}
\left\| \psi_\varepsilon * f - (\int \psi) f \right\|_{L^p} &\le \| l_\varepsilon^*(\psi(\mathbb{1} - \chi_R)) * f \|_{L^p} \\
&\quad + \left\| l_\varepsilon^*(\psi \chi_R) * f - (\int \psi \chi_R) f \right\|_{L^p} + \left\| (\int \psi \chi_R) f - (\int \psi) f \right\|_{L^p} \\
&\le \| \psi(\mathbb{1} - \chi_R) \|_{L^1} \| f \|_{L^p} \\
&\quad + \left\| l_\varepsilon^*(\psi \chi_R) * f - (\int \psi \chi_R) f \right\|_{L^p} + \left| (\int \psi \chi_R) - (\int \psi) \right| \| f \|_{L^p},
\end{aligned}$$

we conclude (26.7). □

Lemma 26.9. *Let $p \in [1, \infty)$. Let φ generate a dyadic partition of unity $(\varphi_j)_{j \in \mathbb{N}_{-1}}$. Let $\Delta_j = \varphi_j(\mathrm{D})$ for $j \in \mathbb{N}_{-1}$. Then $\sum_{j=-1}^{\infty} \Delta_j f = f$ in L^p.*

Proof. This follows from Lemma 26.8 similarly as in the proof of Lemma 26.4. □

Theorem 26.10. *Let $p, q \in [1, \infty]$ be such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $f \in L^p$ and $g \in L^q$. Then $fg \in L^1$ and $f \bullet g$ exists in L^1 (and thus in $\mathcal{S}'$) and

$$fg = f \bullet g \text{ in } L^1.$$

Or, more precisely, $u_f \bullet u_g$ exists in $\mathcal{S}'$ and $u_{fg} = u_f \bullet u_g$.

Proof. Without loss of generality we may assume $q < \infty$ (otherwise one has $p < \infty$ and we can interchange roles). We have

$$\begin{aligned}
&\left\| \sum_{i=-1}^{J} \Delta_i f \sum_{j=-1}^{J} \Delta_j g - fg \right\|_{L^1} \\
&\leq \left\| \sum_{i=-1}^{J} \Delta_i f \right\|_{L^p} \left\| \sum_{j=-1}^{J} \Delta_j g - g \right\|_{L^q} + \left\| \sum_{i=-1}^{J} \Delta_i f - f \right\|_{L^p} \|g\|_{L^q},
\end{aligned}$$

which converges to 0 as $J \to \infty$ by Lemma 26.8 and by Lemma 21.9 (in particular (21.13)). $\square$

Example 26.11. Consider $d = 1$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x_+^{-\frac{1}{2}} = \begin{cases} \frac{1}{\sqrt{x}} & x > 0, \\ 0 & x \leq 0, \end{cases} \qquad g(x) = x_-^{-\frac{1}{2}} = \begin{cases} 0 & x \geq 0, \\ \frac{1}{\sqrt{-x}} & x < 0, \end{cases} \qquad (x \in \mathbb{R}).$$

Then both f ang g are locally integrable. $fg = 0$ everywhere but, as we will show, $f \bullet g$ exists in $\mathcal{S}'$ and

$$f \bullet g = \frac{\pi}{2} \delta_0. \tag{26.8}$$

Indeed, let $\psi \in \mathcal{S}$, $\psi = \psi(-\mathfrak{x})$ and $\int \psi = 1$ and write $\psi_\varepsilon = l_\varepsilon^* \psi$. It suffices to show (see for example the proof of Lemma 26.4)

$$(\psi_\varepsilon * f) \cdot (\psi_\varepsilon * g) \xrightarrow{\varepsilon \downarrow 0} \frac{\pi}{2} \delta_0 \text{ in } \mathcal{S}'$$

For $h \in L^1_{\text{loc}}$ one has

$$\psi_\varepsilon * h = \int_{\mathbb{R}} \varepsilon^{-d} \psi(\frac{y}{\varepsilon}) h(x - y) \, \mathrm{d}y = \int_{\mathbb{R}} \psi(z) h(x - \varepsilon z) \, \mathrm{d}z = \psi * l_\varepsilon h(\frac{x}{\varepsilon}) \qquad (x \in \mathbb{R}^d).$$

Therefore, for $\varphi \in \mathcal{S}$ (the convergence follows by Lebesgue's dominated convergence theorem, as φ is bounded)

$$\begin{aligned}
\langle (\psi_\varepsilon * f) \cdot (\psi_\varepsilon * g), \varphi \rangle &= \int_{\mathbb{R}} (\psi_\varepsilon * f)(x) (\psi_\varepsilon * g)(x) \varphi(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \psi(y) f(x - \varepsilon y) \, \mathrm{d}y \Big) \Big(\int_{\mathbb{R}} \psi(y) g(x - \varepsilon z) \, \mathrm{d}z \Big) \varphi(x) \, \mathrm{d}x \\
&= \varepsilon \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \psi(y) f(\varepsilon w - \varepsilon y) \, \mathrm{d}y \Big) \Big(\int_{\mathbb{R}} \psi(y) g(\varepsilon w - \varepsilon z) \, \mathrm{d}z \Big) \varphi(\varepsilon w) \, \mathrm{d}w \\
&= \int_{\mathbb{R}} \Big(\int_{-\infty}^{w} \psi(y) \frac{1}{\sqrt{w - y}} \, \mathrm{d}y \Big) \Big(\int_{w}^{\infty} \psi(z) \frac{1}{\sqrt{z - w}} \, \mathrm{d}z \Big) \varphi(\varepsilon w) \, \mathrm{d}w \\
&\xrightarrow{\varepsilon \downarrow 0} \varphi(0) \int_{\mathbb{R}} \Big(\int_{-\infty}^{w} \psi(y) \frac{1}{\sqrt{w - y}} \, \mathrm{d}y \Big) \Big(\int_{w}^{\infty} \psi(z) \frac{1}{\sqrt{z - w}} \, \mathrm{d}z \Big) \, \mathrm{d}w \\
&= \varphi(0) \int_{\mathbb{R}} \psi(y) \int_{y}^{\infty} \psi(z) \int_{y}^{z} \frac{1}{\sqrt{w - y}} \frac{1}{\sqrt{z - w}} \, \mathrm{d}w \, \mathrm{d}z \, \mathrm{d}y.
\end{aligned}$$

Now, for all $y, z \in \mathbb{R}$, $z > y$,

$$\begin{aligned}\int_y^z \frac{1}{\sqrt{w-y}} \frac{1}{\sqrt{z-w}} \, \mathrm{d}w &= \int_0^{z-y} \frac{1}{\sqrt{w}} \frac{1}{\sqrt{z-y-w}} \, \mathrm{d}w \\ &= \int_0^1 \frac{1}{\sqrt{w}} \frac{1}{\sqrt{1-w}} \, \mathrm{d}w,\end{aligned}$$

which equals π; the latter equals the Beta function evaluated in $(\frac{1}{2}, \frac{1}{2})$, which equals $\frac{\Gamma(\frac{1}{2})^2}{\Gamma(1)} = \pi$ (see for example [AAR99, Section 1.1], and see Definition 11.22 for Γ). By using that ψ is symmetric, i.e., $\psi = \mathcal{R}\psi$, we have

$$\int_{\mathbb{R}} \psi(y) \int_y^\infty \psi(z) \, \mathrm{d}z \, \mathrm{d}y = \int_{\mathbb{R}} \psi(y) \int_{-\infty}^y \psi(z) \, \mathrm{d}z \, \mathrm{d}y$$

and as $\int_{\mathbb{R}} \psi \int_{\mathbb{R}} \psi = 1$, the above integrals equal $\frac{1}{2}$ and thus

$$\langle (\psi_\varepsilon * f) \cdot (\psi_\varepsilon * g), \varphi \rangle \xrightarrow{\varepsilon \downarrow 0} \frac{\pi}{2} \varphi(0) \qquad (\varphi \in \mathcal{S}),$$

i.e.,

$$(\psi_\varepsilon * f) \cdot (\psi_\varepsilon * g) \to \frac{\pi}{2} \delta_0.$$

We conclude (26.8).

26.1 Comments...

The statements in this section are similar to those of Johnsen [Joh95], though different tools are used for the proofs. Moreover, the statements differ slightly in the sense that in this section we have considered the Bony product in $\mathcal{S}'$, whereas [Joh95] only considers it in $\mathcal{D}'$. [Joh95, Theorem 3.8] considers a more general statement than Theorem 26.10: Instead, $f \in L^p_{\mathrm{loc}}$, $g \in L^q_{\mathrm{loc}}$ and $\frac{1}{p} + \frac{1}{q} \le 1$. It is shown that $fg = f \bullet g$ in $\mathcal{D}'$, and moreover that the product is in L^r_{loc}, with $r \in [1, \infty]$ being such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

For a more comprehensive reference on products of distributions we refer to the book by Oberguggenberger [Obe92].

Example 26.11 is [Obe92, Example 2.3] (and is also mentioned in [Joh95, Example 3.2].

27 Paraproducts and resonance products in Besov spaces

In the previous section we defined the Bony product of two tempered distributions u and v to be the limit (if it exists) of $\sum_{i,j=-1}^J \Delta_i u \cdot \Delta_j v$ as $J \to \infty$. In this section we consider the Bony product between elements of Besov spaces by considering limits of three parts of $\sum_{i,j=-1}^J \Delta_i u \cdot \Delta_j v$, namely one that considers the sum over a left-upper triangle of $\{-1, 0, 1, \dots, J\}^2$, one over a right-lower triangle and one over a thickened diagonal:

Definition 27.1. Let φ generate a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ and $\Delta_j = \varphi_j(\mathrm{D})$ for $j \in \mathbb{N}_{-1}$. For $j \in \mathbb{N}_{-1}$ we define

$$\Delta_{-3}u := 0, \qquad \Delta_{-2}u := 0, \qquad \Delta_{\le j}u := \sum_{i=-1}^{j} \Delta_i u \qquad (j \in \mathbb{N}_{-1}, u \in \mathcal{S}').$$

Let $u, v \in \mathcal{S}'$.

(a) If

$$\sum_{j=-1}^{J} \Delta_{\le j-2}u \cdot \Delta_j v = \sum_{j=1}^{J}\sum_{i=-1}^{j-2} \Delta_i u \cdot \Delta_j v$$

converges in $\mathcal{S}'$ as $J \to \infty$, then we say that the *φ-paraproduct of u with v exists* and write $u \varolessthan_\varphi v$ or $v \varogreaterthan_\varphi u$ for the limit and call it the *φ-paraproduct of u with v*, i.e.,

$$u \varolessthan_\varphi v = v \varogreaterthan_\varphi u = \sum_{j=-1}^{\infty} \Delta_{\le j-2}u \cdot \Delta_j v = \sum_{j=1}^{\infty}\sum_{i=-1}^{j-2} \Delta_i u \cdot \Delta_j v.$$

If for all ψ that generate dyadic partitions of unity the ψ-paraproduct of u and v exists and $u \varolessthan_\varphi v = u \varolessthan_\psi v$, then we say that the *paraproduct of u with v exists* and write $u \varolessthan v$ instead of $u \varolessthan_\varphi v$ and call $u \varolessthan v$ the *paraproduct product of u with v*.

(b) If

$$\sum_{j=-1}^{J} \Delta_{j-1}u \cdot \Delta_j v + \Delta_j u \cdot \Delta_j v + \Delta_j u \cdot \Delta_{j-1} v$$

converges in $\mathcal{S}'$ as $J \to \infty$, then we say that the *φ-resonance product of u and v exists* and write $u \odot_\varphi v$ for the limit and call it the *φ-resonance product of u and v*, i.e.,

$$u \odot_\varphi v = \sum_{j=-1}^{\infty} \Delta_{j-1}u \cdot \Delta_j v + \Delta_j u \cdot \Delta_j v + \Delta_j u \cdot \Delta_{j-1} v.$$

If for all ψ that generate dyadic partitions of unity the ψ-resonance product of u and v exists and $u \odot_\varphi v = u \odot_\psi v$, then we say that the *resonance product of u and v exists* and write $u \odot v$ instead of $u \odot_\varphi v$ and call $u \odot v$ the *resonance product of u with v*.

Observe that $u \odot_\varphi v = v \odot_\varphi u$.

- If both $u \varolessthan_\varphi v$ and $u \odot_\varphi v$ exist, then we write $u \mathbin{\varolessthan\!\!\!\!\!\cdot}_\varphi v = u \varolessthan_\varphi v + u \odot_\varphi v$.
- If both $u \varogreaterthan_\varphi v$ and $u \odot_\varphi v$ exist, then we write $u \mathbin{\varogreaterthan\!\!\!\!\!\cdot}_\varphi v = u \varogreaterthan_\varphi v + u \odot_\varphi v$.

- If both $u \varolessthan_\varphi v$ and $u \varogreaterthan_\varphi v$ exist, then we write $u \varolessthan\!\!\!\!\!\!\varogreaterthan_\varphi v = u \varolessthan_\varphi v + u \varogreaterthan_\varphi v$.

We refer the reader to Remark 27.18 about the different notations for paraproducts and resonance products in the literature.

Let us make the following observation, which one could interpret as bilinearity of the different products:

Lemma 27.2. *Let φ generate a dyadic partition of unity. Let $u, v, w \in \mathcal{S}'$ and $\lambda \in \mathbb{F}$.*

(a) *If $u \varolessthan_\varphi v$ and $w \varolessthan_\varphi v$ exist, then $(u + w) \varolessthan_\varphi v$ exists and*

$$(u + w) \varolessthan_\varphi v = u \varolessthan_\varphi v + w \varolessthan_\varphi v.$$

(b) *If $u \varolessthan_\varphi v$ and $u \varolessthan_\varphi w$ exist, then $u \varolessthan_\varphi (v + w)$ exists and*

$$u \varolessthan_\varphi (v + w) = u \varolessthan_\varphi v + u \varolessthan_\varphi w.$$

(c) *If $u \varolessthan_\varphi v$ exist, then $(\lambda u) \varolessthan_\varphi v$ and $u \varolessthan_\varphi (\lambda v)$ exist and*

$$\lambda(u \varolessthan_\varphi v) = (\lambda u) \varolessthan_\varphi v + u \varolessthan_\varphi (\lambda v).$$

The statements (a), (b) *and* (c) *are also valid if we replace each occurrence of "$\varolessthan_\varphi$" by "$\odot_\varphi$", or each occurrence of "$\varolessthan_\varphi$" by "$\varogreaterthan_\varphi$".*

Proof. The proof is straightforward and left to the reader. □

The existence of each of the paraproducts and the resonance product imply the existence of the Bony product:

Lemma 27.3. *Let φ generate a dyadic partition of unity. Let $u, v \in \mathcal{S}'$. If $u \varolessthan_\varphi v$, $u \odot_\varphi v$ and $u \varogreaterthan_\varphi v$ exist, then $u \bullet_\varphi v$ exists and*

$$u \bullet_\varphi v = u \varolessthan_\varphi v + u \odot_\varphi v + u \varogreaterthan_\varphi v.$$

Proof. See Exercise 27.A. □

Exercise **27.A.** Prove Lemma 27.3.

In Theorem 27.5 we consider estimates on paraproducts in Besov spaces. These rely on the Hölder inequalities and the following theorem, which essentially says: For a negative regularity index s the Besov norm is equivalent to the norm-like function which looks like the Besov norm but with "$\Delta_{\le j}$" instead of "Δ_j".

Theorem 27.4. *Let $s < 0$ and $p, q \in [1, \infty]$. Then we have for $u \in \mathcal{S}'$*

$$u \in B^s_{p,q} \iff \|(2^{js}\|\Delta_{\le j} u\|_{L^p})_{j\in\mathbb{N}_{-1}}\|_{\ell^q} < \infty.$$

Moreover,

$$(1+2^s)^{-1}\|u\|_{B^s_{p,q}} \le \|(2^{js}\|\Delta_{\le j} u\|_{L^p})_{j\in\mathbb{N}_{-1}}\|_{\ell^q} \le (1-2^s)^{-1}\|u\|_{B^s_{p,q}} \qquad (u \in \mathcal{S}'). \quad (27.1)$$

Proof. It is sufficient to prove (27.1). For the inequality on the left–hand side of (27.1):

$$2^{js}\|\Delta_j u\|_{L^p} \le 2^{js}\|\Delta_{\le j} u\|_{L^p} + 2^s 2^{(j-1)s}\|\Delta_{\le j-1} u\|_{L^p}.$$

Therefore

$$\|u\|_{B^s_{p,q}} \le (1+2^s)\|(2^{js}\|\Delta_{\le j} u\|_{L^p})_{j\in\mathbb{N}_{-1}}\|_{\ell^q}.$$

For the inequality on the right-hand side of (27.1):

$$2^{js}\|\Delta_{\le j} u\|_{L^p} \le 2^{js}\sum_{i=-1}^{j}\|\Delta_i u\|_{L^p} = \sum_{i=-1}^{j} 2^{(j-i)s}2^{is}\|\Delta_i u\|_{L^p} = (a*b)(j),$$

where $a, b : \mathbb{Z} \to \mathbb{R}$ are given for $j \in \mathbb{Z}$ by

$$a(j) = \begin{cases} 2^{js} & j \in \mathbb{N}_0, \\ 0 & j \le -1, \end{cases} \qquad b(j) = \begin{cases} 2^{js}\|\Delta_j u\|_{L^p} & j \in \mathbb{N}_{-1}, \\ 0 & j \le -2. \end{cases}$$

Hence, by Young's inequality Theorem 21.17

$$\|(2^{js}\|\Delta_{\le j} u\|_{L^p})_{j\in\mathbb{N}_{-1}}\|_{\ell^q} = \|a*b\|_{\ell^q} \le \|a\|_{\ell^1}\|b\|_{\ell^q} = \|a\|_{\ell^1}\|u\|_{B^s_{p,q}}.$$

As $s < 0$ we have $\|a\|_{\ell^1} = \sum_{j\in\mathbb{N}_0} 2^{js} = (1-2^s)^{-1}$. □

Theorem 27.5. *Let φ generate a dyadic partition of unity. Let $p, p_1, p_2, q, q_1, q_2, r \in [1, \infty]$ be such that $\frac{1}{p_1} + \frac{1}{p_2} \le 1$ and*

$$\tfrac{1}{p} = \tfrac{1}{p_1} + \tfrac{1}{p_2}, \qquad \tfrac{1}{q} = \min\{1, \tfrac{1}{q_1} + \tfrac{1}{q_2}\}, \quad (27.2)$$

(a) *For all $s \in \mathbb{R}$, $u \in L^{p_1}$, and $v \in B^s_{p_2,r}$ the paraproduct of u with v exists in $B^s_{p,r}$. Moreover, there exists a $C > 0$ such that*

$$\|u \otimes_\varphi v\|_{B^s_{p,r}} \le C\|u\|_{L^{p_1}}\|v\|_{B^s_{p_2,r}} \qquad (u \in L^{p_1}, v \in B^s_{p_2,r}). \quad (27.3)$$

(b) *For all $s < 0$, $t \in \mathbb{R}$, $u \in B^s_{p_1,q_1}$ and $v \in B^t_{p_2,q_2}$ the paraproduct of u with v exists in $B^{s+t}_{p,q}$ and*

$$\|u \otimes_\varphi v\|_{B^{s+t}_{p,q}} \le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}} \qquad (u \in B^s_{p_1,q_1}, v \in B^t_{p_2,q_2}). \quad (27.4)$$

(If the Besov norms in (27.3) *and* (27.4) *are with respect to* φ*, as in Definition 21.12,* (21.14)*, then we may take* $C = 3\|\widehat{\varphi_{-1}}\|_{L^1}$ *in* (27.3) *and* $C = 2\|\widehat{\varphi_{-1}}\|_{L^1}2^{2s}(1-2^s)^{-1}$ *in* (27.4)*.)*

Proof. Let $u, v \in \mathcal{S}'$. For $j \in \mathbb{N}_{-1}$ let $w_j = \Delta_{\le j-2}u \cdot \Delta_j v = \sum_{i=-1}^{j-2} \Delta_i u \cdot \Delta_j v$. For both (a) and (b) we apply Theorem 21.18 (a). Let us first show that there exists a ball $\mathcal{B}$ and an annulus $\mathcal{A}$ such that

$$\operatorname{supp} \widehat{w}_{-1} \subset \mathcal{B}, \quad \operatorname{supp} \widehat{w}_j \subset 2^j \mathcal{A} \qquad (j \in \mathbb{N}_0).$$

Observe that $\Delta_{\le -3}u \cdot \Delta_{-1}v = 0$ and $\Delta_{\le -2}u \cdot \Delta_0 v = 0$, and so their Fourier transform is trivially supported in any ball and annulus. Let us check that for $j \in \mathbb{N}$ the Fourier transform of $\Delta_{\le j-2}u \cdot \Delta_j v$ is supported in $2^j\mathcal{A}$ for some annulus $\mathcal{A}$.

Let $a, b > 0$, $a < b$ be such that $\operatorname{supp}\varphi \subset A(a,b)$. By the disjointness property of dyadic partitions of unity, (21.4), we may assume that a and b are such that $4A(a,b) \cap A(a,b) = \emptyset$, i.e., $4a > b$. Moreover, $\operatorname{supp}\varphi_{-1} \subset B(0, \frac{b}{2}) = 2^{-1}B(0,b)$ (as for example $\varphi_{-1} + \varphi_0 = l_{\frac{1}{2}}\varphi_{-1}$ and $\operatorname{supp}\varphi_{-1} + \varphi_0 \subset B(0,b)$). Let $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{A}$ be given by

$$\mathcal{C} = A(a,b), \qquad \mathcal{B} = B(0,b), \qquad \mathcal{A} = A(a - \tfrac{1}{4}b, \tfrac{5}{4}b) = \tfrac{1}{4}\mathcal{B} + \mathcal{C}.$$

As $\mathcal{F}(\Delta_i u \cdot \Delta_j v) = (\varphi_i \widehat{u}) * (\varphi_j \widehat{v})$, we have (see Theorem 7.10 and Lemma 7.11):

$$\operatorname{supp}\mathcal{F}(\Delta_i u \cdot \Delta_j v) \subset 2^j(2^{i-j}\mathcal{B} + \mathcal{C}) \subset 2^j(\tfrac{1}{4}\mathcal{B} + \mathcal{C}) \qquad (i \in \mathbb{N}_{-1}, j \in \mathbb{N}_0, i \le j-2),$$

and thus

$$\operatorname{supp}\widehat{w_j} \subset 2^j\mathcal{A} \qquad (j \in \mathbb{N}_0).$$

For (a), by Theorem 21.18 (a) it is then sufficient to show

$$\left\| \left(2^{js}\|w_j\|_{L^p} \right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^r} \le \|\widehat{\psi_{-1}}\|_{L^1}\|u\|_{L^{p_1}}\|v\|_{B^s_{p_2,r}}.$$

This follows by Hölder's inequality (Theorem A.4) and by (21.13):

$$\begin{aligned} \|w_j\|_{L^p} = \|\Delta_{\le j-2}u \cdot \Delta_j v\|_{L^p} &\le \|\Delta_{\le j-2}u\|_{L^{p_1}}\|\Delta_j v\|_{L^{p_2}} \\ &\le \|\widehat{\psi_{-1}}\|_{L^1}\|u\|_{L^{p_1}}\|\Delta_j v\|_{L^{p_2}} \qquad (j \in \mathbb{N}_{-1}). \end{aligned}$$

And for (b) it is sufficient to show

$$\left\| \left(2^{j(s+t)}\|w_j\|_{L^p} \right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \le \frac{2^{2s}}{1-2^s}\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}. \tag{27.5}$$

By Hölder's inequality (Corollary A.9), we get

$$\begin{aligned} \left\| \left(2^{j(s+t)}\|w_j\|_{L^p} \right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} &\le \left\| \left(2^{js}\|\Delta_{\le j-2}u\|_{L^{p_1}} 2^{jt}\|\Delta_j v\|_{L^{p_2}} \right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q} \\ &\le \left\| \left(2^{js}\|\Delta_{\le j-2}u\|_{L^{p_1}} \right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^{q_1}} \left\| \left(2^{jt}\|\Delta_j v\|_{L^{p_2}} \right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^{q_2}}. \end{aligned}$$

By Theorem 27.4

$$\left\| \left(2^{js} \|\Delta_{\le j-2} u\|_{L^{p_1}}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^{q_1}} = 2^{2s} \left\| \left(2^{js} \|\Delta_{\le j} u\|_{L^{p_1}}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^{q_1}} \le \frac{2^{2s}}{1-2^s} \|u\|_{B^s_{p_1,q_1}},$$

so that we conclude (27.5). □

27.6. Here we consider some consequences of Theorem 27.5 in combination with Theorem 21.23 and Theorem 23.4. The statements that we obtain for the " $\circledcirc$ " product are summarized in Corollary 27.7.

Let $C_1 \ge 1$ as in Theorem 21.23 be such that

$$\|u\|_{B^{\mathfrak{s}}_{\mathfrak{p},\mathfrak{q}}} \le C_1 \|u\|_{B^{\mathfrak{t}}_{\mathfrak{p},\mathfrak{q}}} \qquad (u \in \mathcal{S}', \mathfrak{p},\mathfrak{q} \in [1,\infty], \mathfrak{t},\mathfrak{s} \in \mathbb{R}, \mathfrak{t} \ge \mathfrak{s}).$$

Let $C_2 \ge 1$ as in Theorem 23.4 (b) be such that

$$\|u\|_{L^{\mathfrak{p}}} \le C_2 \|u\|_{B^{\mathfrak{t}}_{\mathfrak{p},\mathfrak{q}}} \qquad (u \in \mathcal{S}', \mathfrak{t} > 0)$$
$$\|u\|_{L^{\mathfrak{p}}} \le C_2 \|u\|_{B^{0}_{\mathfrak{p},1}} \qquad (u \in \mathcal{S}').$$

Let φ generate a dyadic partition of unity. We write $C_3 = \|\widehat{\varphi_{-1}}\|_{L^1}$ (so that (27.3) holds with $C = C_3$ for all $p, p_1, p_2, q, q_1, q_2 \in [1,\infty]$ such that (27.2) is satisfied). For $s < 0$ we define $C_s = (1-2^s)^{-1}$ (so that (27.4) holds with $C = C_s$ for all $p, p_1, p_2, q, q_1, q_2 \in [1,\infty]$ such that (27.2) is satisfied). Let $s, t \in \mathbb{R}$ and $p, p_1, p_2, q, q_1, q_2, r \in [1,\infty]$ be such that (27.2) is satisfied. Let $u \in B^s_{p_1,q_1}$ and $v \in B^t_{p_2,q_2}$.

(a) If $s < 0$ and $t \in \mathbb{R}$, then $u \otimes_\varphi v$ exists in $B^{s+t}_{p,q}$.

Observe that if $t < 0$, then the regularity of $u \otimes_\varphi v$ is worse than the regularity of each of the terms u and v and that if $t = 0$, the regularity of $u \otimes_\varphi v$ equals the regularity of u and if $t > 0$ the regularity of $u \otimes_\varphi v$ is larger than the one of u.

(b) If $s = 0$, $t \in \mathbb{R}$ and $r \in [1,\infty]$ then $u \otimes_\varphi v$ exists in $B^t_{p,r}$ if $u \in L^{p_1}$ and $v \in B^s_{p_2,r}$. Observe that if $u \in B^0_{p_1,1}$ (or differently said; if $q_1 = 1$), then $u \in L^{p_1}$ (by Theorem 23.4). Therefore,

$$\|u \otimes_\varphi v\|_{B^t_{p,r}} \le C_3 \|u\|_{L^{p_1}} \|v\|_{B^t_{p_2,r}} \le C_2 C_3 \|u\|_{B^0_{p_1,1}} \|v\|_{B^t_{p_2,r}}.$$

(c) If $s > 0$ and $t \in \mathbb{R}$, then $u \otimes_\varphi v$ exists in $B^t_{p,r}$ for all $r \ge q_2$ and

$$\|u \otimes_\varphi v\|_{B^t_{p,r}} \le C_3 \|u\|_{L^{p_1}} \|v\|_{B^t_{p_2,r}} \le C_1 C_2 C_3 \|u\|_{B^s_{p_1,q_1}} \|v\|_{B^t_{p_2,q_2}}.$$

By the above we obtain the following.

- If both s and t are in $(-\infty, 0)$, then $u \otimes_\varphi v$ and $u \circledcirc v$ and thus $u \circledast_\varphi v$ exist in $B^{s+t}_{p,q}$, and

$$\|u \circledast_\varphi v\|_{B^{s+t}_{p,q}} \le (C_s + C_t) \|u\|_{B^s_{p_1,q_1}} \|v\|_{B^t_{p_2,q_2}}.$$

- If $s<0$ and $t>0$, then $u \circledcirc v$ is in $B^{s+t}_{p,q}$ by (a) and $u \circledcirc v = v \circledcirc u$ is in $B^s_{p,r}$ for all $r \ge q_1$ by (c). Moreover, as $r \ge q_1$ implies $r \ge q$,

$$\begin{aligned}\|u \circledcirc_\varphi v\|_{B^s_{p,r}} &\le \|u \circledcirc v\|_{B^{s+t}_{p,q}} + \|v \circledcirc u\|_{B^s_{p,r}} \\ &\le (C_s + C_1C_2C_3)\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}} \qquad (r \ge q_1).\end{aligned}$$

- If $s>0$ and $t>0$, then $u \circledcirc v \in B^t_{p,q_2}$ and $v \circledcirc u \in B^s_{p,q_1}$ by (c) so that $u \circledcirc v \in B^{s\wedge t}_{p,r}$ for all $r \ge q_1 \vee q_2$. Moreover,

$$\|u \circledcirc_\varphi v\|_{B^{s\wedge t}_{p,r}} \le 2C_1C_2C_3\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}} \qquad (r \ge q_1 \vee q_2).$$

- ($\diamond\diamond\diamond$) If $s<0$ and $t=0$ and $q_2=1$ (so that $q=1$), then $u \circledcirc v$ exists in $B^s_{p,1}$ by (a) and $u \circledcirc v = v \circledcirc u$ exists in B^s_{p,q_1} by (b) and thus $u \circledcirc_\varphi v$ exists in B^s_{p,q_1}. Moreover,

$$\|u \circledcirc_\varphi v\|_{B^s_{p,q_1}} \le (C_1C_s + C_2C_3)\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,1}}$$

- ($\diamond\diamond\diamond$) If $s=0$ and $t=0$ and $q_1=q_2=1$ (so that $q=1$), then $u \circledcirc v \in B^t_{p,1}$ and $v \circledcirc u \in B^s_{p,1}$ by (b) so that $u \circledcirc v \in B^{s\wedge t}_{p,1}$. Moreover,

$$\|u \circledcirc_\varphi v\|_{B^{s\wedge t}_{p,1}} \le 2C_2C_3\|u\|_{B^s_{p_1,1}}\|v\|_{B^t_{p_2,1}}.$$

- ($\diamond\diamond\diamond$) If $s>0$ and $t=0$ and $q_1=1$ (so that $q=1$), then $u \circledcirc v$ exists in B^t_{p,q_2} by (c) and $u \circledcirc v = v \circledcirc u$ exists in $B^s_{p,q}$ by (b) and thus $u \circledcirc_\varphi v$ exists in B^0_{p,q_2}. Moreover,

$$\|u \circledcirc_\varphi v\|_{B^0_{p,q_2}} \le (C_1C_s + C_2C_3)\|u\|_{B^s_{p_1,1}}\|v\|_{B^0_{p_2,q_2}}$$

We summarize the above observations in the following corollary.

Corollary 27.7. *Let φ generate a dyadic partition of unity.*

(a) *Let $s,t \in \mathbb{R}\setminus\{0\}$. There exists a $C>0$ such that for all $p,p_1,p_2,q,q_1,q_2 \in [1,\infty]$ with $\frac{1}{p_1}+\frac{1}{p_2} \le 1$ and*

$$\tfrac{1}{p} = \tfrac{1}{p_1}+\tfrac{1}{p_2}, \qquad \tfrac{1}{q} = \min\{1, \tfrac{1}{q_1}+\tfrac{1}{q_2}\},$$

for all $u \in B^s_{p_1,q_1}$ and $v \in B^t_{p_2,q_2}$: $u \circledcirc_\varphi v$ exists in $B^{s\wedge t}_{p,r}$ for all $r \in [1,\infty]$ such that

$$r \ge \begin{cases} q & \text{if } s,t<0, \\ q_1 & \text{if } s>0, t<0, \\ q_2 & \text{if } s<0, t>0, \\ q_1 \vee q_2 & \text{if } s>0, t>0,\end{cases}$$

and for such r,

$$\|u \circledcirc_\varphi v\|_{B^{s\wedge t\wedge(s+t)}_{p,r}} \le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}} \qquad (u \in B^s_{p_1,q_1}, v \in B^t_{p_2,q_2}).$$

(b) $(\diamond\diamond\diamond)$ *Let $s,t \in \mathbb{R}$, $s \le t$ and either $s = 0$ and $t \ne 0$ or $s \ne 0$ and $t = 0$. Then there exists a $C > 0$ such that for all $p, p_1, p_2 \in [1,\infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} \le 1$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, for all $q \in [1,\infty]$, for all $u \in B^s_{p_1,q}$ and $v \in B^t_{p_2,1}$, $u \circledcirc_\varphi v$ exists in $B^s_{p,q}$ and*

$$\|u \circledcirc_\varphi v\|_{B^s_{p,q}} \le C\|u\|_{B^s_{p_1,q}}\|v\|_{B^t_{p_2,1}} \qquad (u \in B^s_{p_1,q}, v \in B^t_{p_2,1}).$$

(c) $(\diamond\diamond\diamond)$ *Then there exists a $C > 0$ such that for all $p, p_1, p_2 \in [1,\infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} \le 1$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, for all $q \in [1,\infty]$: for all $u \in B^0_{p_1,1}$ and $v \in B^t_{p_2,1}$, $u \circledcirc_\varphi v$ exists in $B^0_{p,1}$ and*

$$\|u \circledcirc_\varphi v\|_{B^0_{p,1}} \le C\|u\|_{B^0_{p_1,1}}\|v\|_{B^0_{p_2,1}} \qquad (u \in B^0_{p_1,1}, v \in B^0_{p_2,1}).$$

Example 27.8. We consider the situation as in Example 21.26: Let $d = 1$. Let $(\varphi_j)_{j\in\mathbb{N}_{-1}}$ be a dyadic partition of unity such that $\varphi_0 = 1$ on $\overline{\mathcal{A}(\frac{3}{4}, \frac{5}{4})}$ (see Exercise 21.B). For $n \in \mathbb{N}$, consider $v_n, u_n \in \mathcal{S}'$ given by

$$v_n = \tfrac{1}{2}(\delta_{2^n} + \delta_{-2^n}) \qquad u_n = \cos(2\pi 2^n \mathfrak{x}).$$

As in Example 21.26 for $s \in \mathbb{R}$ one has

$$u_s := \sum_{n\in\mathbb{N}} 2^{-2ns} u_{2n} \in B^s_{\infty,\infty}.$$

We considered here the summation over even numbers, as this simplifies the calculations that we make later.

We consider the existence of the paraproduct, resonance product and Bony product $u_s \circledcirc_\varphi u_t$, $u_s \odot_\varphi u_t$ and $u_s \bullet_\varphi u_t$ for s and t in $\mathbb{R}$.

• **Paraproducts**. Observe that $\Delta_j u_s = 2^{-2ns} u_{2n}$ if $j = 2n$ for some $n \in \mathbb{N}$ and $\Delta_j u_s = 0$ otherwise. Therefore

$$\Delta_{\le J-2} u_s \cdot \Delta_J v_t = \sum_{i=-1}^{J-2} \Delta_i u_s \cdot \Delta_J u_t = \begin{cases} w_{n,s,t} & J = 2n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$w_{n,s,t} := \sum_{i=1}^{n-1} \Delta_{2i} u_s \cdot \Delta_{2n} u_t = 2^{-2nt} \sum_{i=1}^{n-1} 2^{-2is} u_{2i} u_{2n} \qquad (n \in \mathbb{N}).$$

It follows that $u_s \circledcirc_\varphi u_t$ exists in $\mathcal{S}'$ if $\sum_{n\in\mathbb{N}} w_{n,s,t}$ exists in $\mathcal{S}'$. To show the latter we invoke Theorem 21.18 (a). By considering the trigonometric identity for the product of two cosines, or equivalently, the convolution of $v_{2i} * v_{2n}$ one finds

$$u_{2i} u_{2n} = \tfrac{1}{2}\cos(2\pi(2^{2n} + 2^{2i})\mathfrak{x}) + \tfrac{1}{2}\cos(2\pi(2^{2n} - 2^{2i})\mathfrak{x}) \qquad (i, n \in \mathbb{N}). \tag{27.6}$$

Observe that for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, n-1\}$ we have

$$\tfrac{3}{4}2^{2n} = (1 - \tfrac{1}{4})2^{2n} \le 2^{2n} - 2^{2i} \le 2^{2n} + 2^{2i} \le (1 + \tfrac{1}{4})2^{2n} = \tfrac{5}{4}2^{2n}.$$

Hence

$$\operatorname{supp} \widehat{w_{n,s,t}} = \bigcup_{i \in \{1,\ldots,n-1\}} \operatorname{supp} \widehat{u_{2i} u_{2n}} \subset 2^{2n} \overline{\mathcal{A}(\tfrac{3}{4}, \tfrac{5}{4})},$$

and thus, by our choice of dyadic partition of unity,

$$\Delta_j\Big(w_{n,s,t}\Big) = \begin{cases} w_{n,s,t} & j = 2n, \\ 0 & \text{otherwise.} \end{cases} \tag{27.7}$$

It is not difficult to see that (evaluate at the origin)

$$\Big\| w_{n,s,t} \Big\|_{L^\infty} = 2^{-2nt} \sum_{i=1}^{n-1} 2^{-2is} = \begin{cases} (n-1)2^{-2nt} & s = 0, \\ 2^{-nt} \frac{2^{-2ns} - 2^{-2s}}{2^{-2s} - 1} & s \neq 0. \end{cases}$$

Now, as for example

$$\sup_{n \in \mathbb{N}} 2^{2ns} \frac{2^{-2ns} - 2^{-2s}}{2^{-2s} - 1} = \sup_{n \in \mathbb{N}} \frac{1 - 2^{2(n-1)s}}{2^{-2s} - 1} = \frac{1}{2^{-2s} - 1} \qquad (s < 0),$$

it follows by Theorem 21.18 (a) that $u_s \otimes_\varphi u_t = \sum_{n \in \mathbb{N}} w_{n,s,t}$ exists in $B^{s+t}_{\infty,\infty}$ for $s < 0$ and $t \in \mathbb{R}$ and by (27.7)

$$\|u_s \otimes_\varphi u_t\|_{B^{s+t}_{\infty,\infty}[\varphi]} = \frac{1}{2^{-2s} - 1} = \frac{2^{2s}}{1 - 2^{2s}} \|u_s\|_{B^s_{\infty,\infty}[\varphi]} \|u_t\|_{B^t_{\infty,\infty}[\varphi]} \qquad (s < 0, t \in \mathbb{R}).$$

On the other hand, if $s > 0$, then

$$\sup_{n \in \mathbb{N}} \frac{2^{-2ns} - 2^{-2s}}{2^{-2s} - 1} = \frac{1}{2^{2s} - 1},$$

and thus $u_s \otimes_\varphi u_t$ exists in $B^t_{\infty,\infty}$ for $s > 0$ and $t \in \mathbb{R}$ and

$$\|u_s \otimes_\varphi u_t\|_{B^t_{\infty,\infty}[\varphi]} = \frac{1}{2^{2s} - 1} = \frac{1}{2^{2s} - 1} \|u_s\|_{B^s_{\infty,\infty}[\varphi]} \|u_t\|_{B^t_{\infty,\infty}[\varphi]} \qquad (s > 0, t \in \mathbb{R}).$$

If $s = 0$, then

$$\sup_{n \in \mathbb{N}} 2^{nr}(n-1) < \infty \iff r < 0,$$

so that $u_0 \otimes_\varphi u_t$ exists in $B^{r+t}_{\infty,\infty}$ for any $r < 0$, but $u_0 \otimes_\varphi u_t$ is not an element of $B^t_{\infty,\infty}$ for any $t \in \mathbb{R}$.

• **Resonance products**. By our choice of our dyadic partition of unity we have

$$\Delta_{j-1}u_s \cdot \Delta_j u_t + \Delta_j u_s \cdot \Delta_j u_t + \Delta_j u_s \cdot \Delta_{j-1} u_t$$
$$= \begin{cases} \Delta_{2n}u_s \cdot \Delta_{2n}u_t & \text{if } j = 2n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly as above, see (27.6), we have

$$\Delta_{2n}u_s \cdot \Delta_{2n}u_t = 2^{-n(t+s)} \frac{1}{2} \left(\cos(2\pi 2^{2(n+1)} \mathfrak{x}) + \mathbb{1} \right),$$

and thus

$$\sum_{i=1}^{n} \Delta_{2i}u_s \cdot \Delta_{2i}u_t = \frac{1}{2} \Big(\sum_{i=1}^{n} 2^{-i(t+s)} \Big) \mathbb{1} + \sum_{i=2}^{n+1} 2^{-(i-1)(t+s)} \frac{1}{2} u_{2i}.$$

Therefore

$$\Delta_j \Big(\sum_{i=1}^{n} \Delta_{2i}u_s \cdot \Delta_{2i}u_t \Big) = \begin{cases} \frac{1}{2} \Big(\sum_{i=1}^{n} 2^{-i(t+s)} \Big) \mathbb{1} & j = -1, \\ 2^{-(n-1)(t+s)} \frac{1}{2} u_{2n} & j = 2n \text{ for some } n \in \mathbb{N}, n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

From this we see (Exercise 27.B) that if

$$\sum_{i=1}^{n} \Delta_{2i}u_s \cdot \Delta_{2i}u_t$$

converges in $\mathcal{S}'$ as $n \to \infty$ then

$$\frac{1}{2} \Big(\sum_{i=1}^{\infty} 2^{-i(t+s)} \Big) < \infty,$$

which is only the case for $t+s > 0$. And if $t+s > 0$, then $u_s \odot_\varphi u_t = \sum_{n \in \mathbb{N}} \Delta_{2n}u_s \cdot \Delta_{2n}u_t$ exists in $B_{\infty,\infty}^{t+s}$ and

$$\|u_s \odot_\varphi u_t\|_{B_{\infty,\infty}^{s+t}[\varphi]} = 2^{-(s+t)} \frac{1}{2} \frac{2^{-(s+t)}}{1 - 2^{-(s+t)}} \vee 2^{-(t+s)} \frac{1}{2}$$

Exercise **27.B.** Show that in the situation of Example $\sum_{i=1}^{n} \Delta_{2i}u_s \cdot \Delta_{2i}u_t$ converges in $\mathcal{S}'$ as $n \to \infty$ only if

$$\frac{1}{2} \Big(\sum_{i=1}^{\infty} 2^{-i(t+s)} \Big).$$

(Hint: Show there exists a symmetric $\psi \in \mathcal{S}$ such that $\Delta_j \widehat{\psi} = 0$ for $j \in \mathbb{N}_0$ and $\langle \psi, \mathbb{1} \rangle \neq 0$ and test against φ.)

27.9 (Formal explanation of paraproducts and resonance products)**.** In 21.27 we discussed the language of frequencies corresponding to the Example 21.26 which of course closely corresponds to Example 27.8. In this language, one may say that the paraproduct $u \otimes v$ is the distribution that considers the product between the "low frequencies" of u times the "high frequencies" of v.

In Example 27.8 we have seen that the frequencies of $\Delta_{\le 2n-2} u_s \cdot \Delta_{2n} u_t$ are of order 2^{2n}, which is the same order as the frequences of $\Delta_{2n} u_t$. See for example also Figure 3 and Figure 4 for an illustration that the frequency of the product of two functions of different frequencies is close to the larger one of the two.

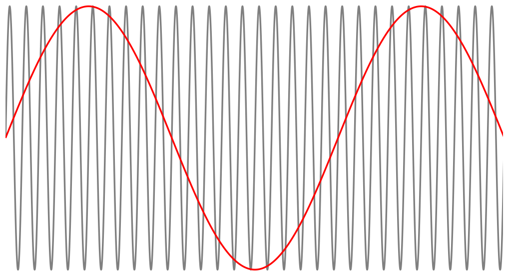

Figure 3: A function with high and one with low frequency.

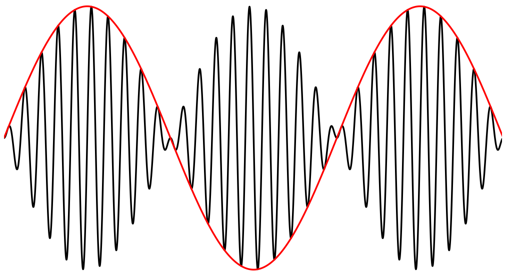

Figure 4: The product of the functions with high and low frequencies.

Observe again, as we have seen in 27.6, that the regularity of $u_s \otimes u_t$ is at most the regularity of t. One therefore could say, in the above language: "One cannot improve the regularity of u_t by multiplying the high frequencies of u_t by the low frequencies of u_s". If, however, we turn around the roles of u_s and u_t, we see that if u_s is of low regularity, say $s < 0$, then: "One can improve the regularity of u_s by multiplying the low frequences of u_s by the high frequencies of u_t". Possibly this formal language helps the reader with having some intuition about the estimates we have for paraproducts.

The term "resonance product" can be explained as follows. Contrary to the effect of the paraproduct, the order of the frequencies of $\Delta_{j-1} u_s \cdot \Delta_j u_t + \Delta_j u_s \cdot \Delta_j u_t + \Delta_j u_s \cdot \Delta_{j-1} u_t$ may range between frequencies of order 1 to frequencies of order 2^{n+1}. Indeed, in Example 27.8, we have seen that taking the product of two functions of equal frequencies may give a function of larger and a function of lower frequency: $\cos^2 \mathfrak{x} = \frac{1}{2} + \frac{1}{2} \cos 2\mathfrak{x}$ (see Figure 5). This effect relates to the word "resonance", as it relates to two 'systems' that interact on the same frequency, which may 'strengthen' the outcome.

Theorem 27.10. *Let φ generate a dyadic partition of unity. Let $p, p_1, p_2, q, q_1, q_2 \in [1,\infty]$ be such that $\frac{1}{p_1} + \frac{1}{p_2} \le 1$ and*

$$\tfrac{1}{p} = \tfrac{1}{p_1} + \tfrac{1}{p_2}, \qquad \tfrac{1}{q} = \min\{1, \tfrac{1}{q_1} + \tfrac{1}{q_2}\},$$

(a) *For all $s, t \in \mathbb{R}$ with $s + t > 0$, for $u \in B^s_{p_1,q_1}$ and $v \in B^t_{p_2,q_2}$ the resonance product*

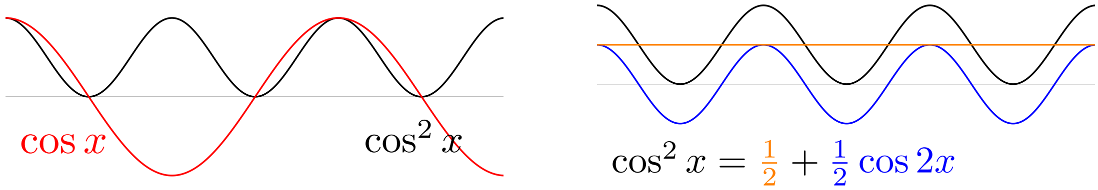

Figure 5: The sine function and its square and the decomposition of the square of the sine function in a low and high frequency function.

of u and v exists in $B^{s+t}_{p,q}$. Moreover, there exists a $C > 0$ such that

$$\|u \odot_\varphi v\|_{B^{s+t}_{p,q}} \le C\|u\|_{B^s_{p_1,q_1}} \|v\|_{B^t_{p_2,q_2}} \qquad (u \in B^s_{p_1,q_1}, v \in B^t_{p_2,q_2}), \tag{27.8}$$

(b) ($\diamond\diamond\diamond$) *For all $s, t \in \mathbb{R}$ with $s + t = 0$, for $u \in B^s_{p_1,1}$ and $v \in B^t_{p_2,\infty}$ the resonance product of u and v exists in $B^0_{p,\infty}$. Moreover, there exists a $C > 0$ such that*

$$\|u \odot_\varphi v\|_{B^0_{p,\infty}} \le C\|u\|_{B^s_{p_1,1}} \|v\|_{B^t_{p_2,\infty}} \qquad (u \in B^s_{p_1,1}, v \in B^t_{p_2,\infty}). \tag{27.9}$$

(If the Besov norms in (27.3) and (27.4) are with respect to φ, as in Definition 21.12, (21.14), then there exists an $N \in \mathbb{N}$ such that we may take $C = 2\|\widehat{\varphi_{-1}}\|_{L^1} \frac{2^{Ns}}{2^s-1}(2^s+1+2^t)$ in (27.8) and in (27.9).)

Proof. Let $u, v \in \mathcal{S}'$. For $j \in \mathbb{N}_{-1}$ let $w_j = \Delta_{j-1}u \cdot \Delta_j v + \Delta_j u \cdot \Delta_j v + \Delta_j u \cdot \Delta_{j-1} v$. For (a) we apply Theorem 21.18 (b) and for (b) we apply Theorem 21.18 (c). Let us first show that there exists a ball $\mathcal{B}$ such that

$$\operatorname{supp} \widehat{w}_j \subset 2^j \mathcal{B} \qquad (j \in \mathbb{N}_{-1}). \tag{27.10}$$

Let $\mathcal{C}$ be a ball such that $\operatorname{supp} \varphi_j \subset 2^j\mathcal{C}$ for all $j \in \mathbb{N}_{-1}$. Similarly as in the proof of Theorem 27.5, by using that $\mathcal{F}(\Delta_i u \cdot \Delta_j v) = (\varphi_i \widehat{u}) * (\varphi_j \widehat{v})$, we have

$$\operatorname{supp} \widehat{w_j} \subset \bigcup_{i \in \{0,1\}} 2^{j-i}\mathcal{C} + 2^j\mathcal{C} \subset 2^j(2\mathcal{C}) \qquad (j \in \mathbb{N}_{-1}),$$

from which we have (27.10) for $\mathcal{B} = 2\mathcal{C}$. For (a) and (b) (the latter case we have $q_1 = 1$ and $q_2 = \infty$ and thus $q = 1$), by Theorem 21.18 (b) and (c) it is sufficient to show

$$\left\| (2^{j(s+t)} \|w_j\|_{L^p})_{j \in \mathbb{N}_{-1}} \right\|_{\ell^q} \le (2^s + 1 + 2^t) \|u\|_{B^s_{p_1,q_1}} \|v\|_{B^t_{p_2,q_2}}. \tag{27.11}$$

We use $\|w_j\|_{L^p} \le \|\Delta_{j-1}u \cdot \Delta_j v\|_{L^p} + \|\Delta_j u \cdot \Delta_j v\|_{L^p} + \|\Delta_j u \cdot \Delta_{j-1} v\|_{L^p}$. By Hölder's inequality (both Theorem A.4 and Corollary A.9):

$$\begin{aligned}
\left\|(2^{j(s+t)}\|\Delta_{j-1}u\cdot\Delta_j v\|_{L^p})_{j\in\mathbb{N}_{-1}}\right\|_{\ell^q} &\le \left\|\left(2^{j(s+t)}\left\|\Delta_{j-1}u\right\|_{L^{p_1}}\|\Delta_j v\|_{L^{p_2}}\right)_{j\in\mathbb{N}_{-1}}\right\|_{\ell^q}\\
&\le 2^s\left\|\left(2^{(j-1)s}\|\Delta_{j-1}u\|_{L^{p_1}}\right)_{j\in\mathbb{N}_{-1}}\right\|_{\ell^{q_1}}\left\|\left(2^{jt}\|\Delta_j v\|_{L^{p_2}}\right)_{j\in\mathbb{N}_{-1}}\right\|_{\ell^{q_2}}\\
&= 2^s\,\|u\|_{B^s_{p_1,q_1}}\,\|v\|_{B^t_{p_2,q_2}}.
\end{aligned}$$

One obtains similar estimates with "$\Delta_{j-1}u\cdot\Delta_j v$" replaced by "$\Delta_j u\cdot\Delta_j v$" or "$\Delta_j u\cdot\Delta_{j-1}v$" by which one can conclude (27.11). □

Exercise **27.C.** Consider the situation as in Theorem 27.10: Let $p,p_1,p_2,q,q_1,q_2\in[1,\infty]$ be such that $\frac{1}{p_1}+\frac{1}{p_2}\le 1$ and

$$\tfrac{1}{p}=\tfrac{1}{p_1}+\tfrac{1}{p_2},\qquad \tfrac{1}{q}=\min\{1,\tfrac{1}{q_1}+\tfrac{1}{q_2}\}.$$

Let $s,t\in\mathbb{R}$, $u\in B^s_{p_1,q_1}$, $v\in B^t_{p_2,q_2}$. Show that if either $s+t>0$ or $s+t=0$ and $\{q_1,q_2\}=\{1,\infty\}$, then $\Delta_{J+i}u\cdot\Delta_{J+j}v\to 0$ in $\mathcal{S}'$ as $J\to\infty$ for all $i,j\in\mathbb{N}_0$.

In the following theorem we consider the product between elements of Besov spaces.

Theorem 27.11. (a) *Let $s,t\in\mathbb{R}\setminus\{0\}$, $s+t>0$. Let $\delta>0$. There exists a $C>0$ such that for all $p,p_1,p_2,q,q_1,q_2\in[1,\infty]$ such that $\frac{1}{p_1}+\frac{1}{p_2}\le 1$,*

$$\tfrac{1}{p}=\tfrac{1}{p_1}+\tfrac{1}{p_2},$$

we have for all $r\in[1,\infty]$ such that

$$r\ge\begin{cases} q_1 & \text{if } s>0,t<0,\\ q_2 & \text{if } s<0,t>0,\\ q_1\vee q_2 & \text{if } s>0,t>0,\end{cases}$$

$u\bullet v$ exists in $B^{s\wedge t}_{p,r}$ for all $u\in B^s_{p_1,q_1}$ and $v\in B^t_{p_2,q_2}$, and for such r the operation $\bullet$ forms a bilinear continuous map $B^s_{p_1,q_1}\times B^t_{p_2,q_2}\to B^{s\wedge t}_{p,r}$;

$$\|u\bullet v\|_{B^{s\wedge t}_{p,r}}\le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}\qquad (u\in B^s_{p_1,q_1},v\in B^t_{p_2,q_2})\tag{27.12}$$

and, consequently, $u\bullet v$ exists in $B^{s\wedge t-\delta}_{p,q}$ for all $u\in B^s_{p_1,q_1}$ and $v\in B^t_{p_2,q_2}$, and the operation $\bullet$ forms a bilinear continuous map $B^s_{p_1,q_1}\times B^t_{p_2,q_2}\to B^{s\wedge t-\delta}_{p,q}$;

$$\|u\bullet v\|_{B^{s\wedge t-\delta}_{p,q}}\le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}\qquad (u\in B^s_{p_1,q_1},v\in B^t_{p_2,q_2}).\tag{27.13}$$

(b) ($\diamond\diamond\diamond$) *Let $s\in(0,\infty)$. There exists a $C>0$ such that for all $p,p_1,p_2,q\in[1,\infty]$ with $\frac{1}{p_1}+\frac{1}{p_2}\le 1$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, for all $u\in B^s_{p_1,1}$ and $v\in B^0_{p_2,q}$, $u\bullet v$ exists in $B^0_{p,1}$ and*

$$\|u\bullet v\|_{B^0_{p,q}}\le C\|u\|_{B^s_{p_1,1}}\|v\|_{B^0_{p_2,q}}\qquad (u\in B^s_{p_1,1},v\in B^0_{p_2,q}).\tag{27.14}$$

(c) $(\diamond\diamond\diamond)$ *There exists a* $C > 0$ *such that for all* $p, p_1, p_2, \in [1,\infty]$ *with* $\frac{1}{p_1} + \frac{1}{p_2} \le 1$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, *for all* $u \in B^0_{p_1,\infty}$ *and* $v \in B^0_{p_2,1}$*:* $u \bullet v$ *exists in* $B^0_{p,\infty}$ *and*

$$\|u \bullet v\|_{B^0_{p,\infty}} \le C \|u\|_{B^0_{p_1,1}} \|v\|_{B^0_{p_2,1}} \qquad (u \in B^0_{p_1,1}, v \in B^0_{p_2,1}). \tag{27.15}$$

In all cases, also $v \bullet u$ *exists and is equal to* $u \bullet v$*.*

Proof. Let φ generate a dyadic partition of unity. For each of the cases, we first show the statements for "$u \bullet_\varphi v$" instead of "$u \bullet v$" and then show that the product is independent of the choice of φ. The bilinearity follows by Lemma 26.2. The continuity follows by the norm estimates, see Exercise 27.D.

(a): (27.13) follows from (27.12) by Lemma 21.25: Let r be as in the statement and q any element of $[1,\infty]$. Then for all $\delta > 0$ there exists a $C > 0$ such that

$$\|u\|_{B^{s-\delta}_{p,q}} \le \|u\|_{B^s_{p,r}} \qquad (u \in \mathcal{S}').$$

The existence of $u \bullet_\varphi v$ and the estimate (27.12) for $u \bullet_\varphi v$, i.e.,

$$\|u \bullet_\varphi v\|_{B^{s\wedge t}_{p,r}} \le C \|u\|_{B^s_{p_1,q_1}} \|v\|_{B^t_{p_2,q_2}} \qquad (u \in B^s_{p_1,q_1}, v \in B^t_{p_2,q_2}),$$

follows from Corollary 27.7 (a) and Theorem 27.10 (a).

(b): (27.14) follows from Corollary 27.7 (c) and Theorem 27.10 (a) as $\|\cdot\|_{B^0_{p,q}} \le \|\cdot\|_{B^0_{p,1}}$.

(c): (27.15) follows from Corollary 27.7 (c) and Theorem 27.10 (b) as $\|\cdot\|_{B^0_{p,\infty}} \le \|\cdot\|_{B^0_{p,1}}$.

We are left to prove that $u \bullet_\varphi v$, so to say, is independent of the choice of φ (that is, if η generates a dyadic partition of unity, then $u \bullet_\eta v = u \bullet_\varphi v$). If either u or v is in C^∞_{p}, then this follows by Theorem 26.7. We continue by using that we can approximate elements of Besov spaces whose second integration parameter is finite by C^∞_{p} functions, see Theorem 23.8 (a) or Lemma 23.7. We give the proof in the situation of (a); the proof in case of (b) and (c) is similar. Let us first assume that either $q_1 < \infty$ or $q_2 < \infty$. We may assume $q_1 < \infty$. Then there exists a sequence $(\eta_n)_{n\in\mathbb{N}}$ in $C^\infty_{\mathrm{p}} \cap B^s_{p_1,q_1}$ (take for example $\eta_n = \Delta_{\le n} u$) such that $\eta_n \to u$ in $B^s_{p_1,q_1}$. so that $\lim_{n\to\infty} \eta_n v = \lim_{n\to\infty} \eta_n \bullet_\varphi v = u \bullet_\varphi v$, so that the limit is independent of φ. Now suppose $q_1 = q_2 = \infty$. We may assume $s > 0$. Then we choose $\kappa > 0$ such that $s - \kappa > 0$ and $s + t - \kappa > 0$ and thus $u \in B^{s-\kappa}_{p_1,1}$ and $u \bullet_\varphi v \in B^{s\wedge t}_{p,r}$ for all $r \in [1,\infty]$ and

$$\|u \bullet_\varphi v\|_{B^{s\wedge t}_{p,r}} \le C \|u\|_{B^{s-\kappa}_{p_1,1}} \|v\|_{B^t_{p_2,q_2}} \qquad (u \in B^s_{p_1,q_1}, v \in B^t_{p_2,q_2}).$$

We may repeat the above limiting argument and obtain that in $u \bullet_\varphi v$ is independent of the choice of φ. $\square$

Exercise **27.D.** Let X, Y and Z be normed vector spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Suppose $F : X \times Y \to Z$ is bilinear and

$$\|F(x,y)\|_Z \le \|x\|_X \|y\|_Y \qquad (x \in X, y \in Y).$$

Show that F is continuous, i.e., if $x_n \to x$ in X and $y_n \to y$ in Y, then

$$F(x_n, y_n) \to F(x,y) \text{ in } Z.$$

Lemma 27.12. *Let $s_1, s_2, s_3 \in \mathbb{R} \setminus \{0\}$, $s_1 + s_2 > 0$ and $s_2 + s_3 > 0$. Let $p_1, p_2, p_3, q_1, q_2, q_3 \in [1,\infty]$ be such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \le 1$. For $u_1 \in B^{s_1}_{p_1,q_1}, u_2 \in B^{s_2}_{p_2,q_2}$ and $u_3 \in B^{s_3}_{p_3,q_3}$, $u_1 \bullet u_2$, $u_2 \bullet u_3$, $(u_1 \bullet u_2) \bullet u_3$ and $u_1 \bullet (u_2 \bullet u_3)$ exist in $\mathcal{S}'$ and*

$$(u_1 \bullet u_2) \bullet u_3 = u_1 \bullet (u_2 \bullet u_3). \tag{27.16}$$

Proof. By choosing $\kappa_1, \kappa_2, \kappa_3 > 0$ such that $s_1 - \kappa_1, s_2 - \kappa_2, s_3 - \kappa_3 \neq 0$, $s_1 - \kappa_1 + s_2 - \kappa_2 > 0$ and $s_2 - \kappa_2 + s_3 - \kappa_3 > 0$, we may as well assume that $q_1 = q_2 = q_3 = 1$ (as $u_1 \in B^{s_1-\kappa_1}_{p_1,q_1}, u_2 \in B^{s_2-\kappa_2}_{p_2,q_2}$ and $u_3 \in B^{s_3-\kappa_3}_{p_3,q_3}$).

Let $p, p_{1,2}, p_{2,3} \in [1,\infty]$ be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}, \qquad \frac{1}{p_{1,2}} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{p_{2,3}} = \frac{1}{p_2} + \frac{1}{p_3}.$$

By Theorem 27.11 (a) it follows that $u_1 \bullet u_2$ exists in $B^{s_1 \wedge s_2}_{p_{12},1}$, $u_2 \bullet u_3$ exists in $B^{s_2 \wedge s_3}_{p_{23},1}$ and $(u_1 \bullet u_2) \bullet u_3$ and $u_1 \bullet (u_2 \bullet u_3)$ exist in $B^{s_1 \wedge s_2 \wedge s_3}_{p,1}$. Let $\eta_n = \Delta_{\le n} u_1$ and $\zeta_n = \Delta_{\le n} u_2$ for all $n \in \mathbb{N}$. Then $\eta_n, \zeta_n \in C^\infty_{\mathrm{p}}$ and $\eta_n \to u_1$ in $B^{s_1}_{p_1,1}$ and $\zeta_n \to u_2$ in $B^{s_2}_{p_2,1}$. Moreover, by Theorem 26.7

$$(\eta_n \bullet \zeta_n) \bullet u_3 = \eta_n \zeta_n u_3 = \eta_n \bullet (\zeta_n \bullet u_3) \qquad (n \in \mathbb{N}).$$

By the continuity of the product $\bullet$ as a map

$$\begin{aligned} &B^{s_1}_{p_1,1} \times B^{s_2}_{p_2,1} \to B^{s_1 \wedge s_2}_{p_{12},1}, \\ &B^{s_2}_{p_2,1} \times B^{s_3}_{p_3,1} \to B^{s_2 \wedge s_3}_{p_{23},1}, \\ &B^{s_1 \wedge s_2}_{p_{12},1} \times B^{s_3}_{p_3,1} \to B^{s_1 \wedge s_2 \wedge s_3}_{p,1}, \\ &B^{s_1}_{p_1,1} \times B^{s_2 \wedge s_3}_{p_{23},1} \to B^{s_1 \wedge s_2 \wedge s_3}_{p,1}, \end{aligned}$$

(27.16) follows. □

Definition 27.13. For $s \in \mathbb{R}$ we define $\mathcal{C}^s = B^s_{\infty,\infty}$ and write "$\|\cdot\|_{\mathcal{C}^s}$" instead of "$\|\cdot\|_{B^s_{\infty,\infty}}$".

For those who know the notation $\mathcal{C}^t$ for $t > 0$ as in Definition 24.4 one may take the above definition only for $s \le 0$. For $t > 0$ it does not make much of a difference as the norm-like functions $\|\cdot\|_{\mathcal{C}^t}$ and $\|\cdot\|_{B^t_{\infty,\infty}}$ are equivalent, see 24.8.

By Theorem 23.2 the following statement is a consequence of Theorem 27.11.

Corollary 27.14. *Let $s > 0, t \in \mathbb{R}$ and $s + t > 0$. Let $\delta > 0$.*

(a) *If $t < 0$, $u \in H^s$ and $v \in \mathcal{C}^t$, then $u \bullet v$ exists and is an element of H^t. Moreover, the map*

$$H^s \times \mathcal{C}^t \to H^t, \quad (u, v) \mapsto u \bullet v,$$

is a bilinear continuous map and there exists a $C > 0$ such that

$$\|u \bullet v\|_{H^t} \le C \|u\|_{H^s} \|v\|_{\mathcal{C}^t} \qquad (u, v \in \mathcal{S}'). \tag{27.17}$$

(b) *If $u \in H^s$ and $v \in \mathcal{C}^t$, then $u \bullet v$ exists and is an element of $H^{s \wedge t - \delta}$. Moreover, the map*

$$H^s \times \mathcal{C}^t \to H^{s \wedge t - \delta}, \quad (u, v) \mapsto u \bullet v,$$

is a bilinear continuous map and there exists a $C > 0$ such that

$$\|u \bullet v\|_{H^{s \wedge t - \delta}} \le C \|u\|_{H^s} \|v\|_{\mathcal{C}^t} \qquad (u, v \in \mathcal{S}'). \tag{27.18}$$

Proof. As $H^s = B^s_{2,2}$ and $\mathcal{C}^t = B^t_{\infty,\infty}$ with equivalent norms, (a) follows directly by Theorem 27.11, (27.12) (with $r = 2$ and $q_1 = 2$, $q_2 = \infty$) and (b) follows similarly by (27.13) if $t \neq 0$. If $t = 0$, then (27.13) implies

$$\|u \bullet v\|_{H^{s \wedge t - \delta}} \lesssim \|u\|_{H^s} \|v\|_{\mathcal{C}^{t - \frac{\delta}{2}}} \lesssim \|u\|_{H^s} \|v\|_{\mathcal{C}^t}.$$

□

We turn now to a specific case of products between Besov spaces with $p = q = \infty$.

By Proposition 5.3 it follows that if $k, m \in \mathbb{N}_0$, then the product map $C^k_{\mathrm{b}} \times C^m_{\mathrm{b}} \to C^{k \wedge m}_{\mathrm{b}}$, $(f, g) \mapsto fg$ is a continuous bilinear map. The following statement is of a similar nature.

It is widely used in the theory of SPDEs. See for example [Hai14, Proposition 4.14] and [GIP15, Lemma 2.1 and text below] .

Corollary 27.15. *Let $s, t \in \mathbb{R} \setminus \{0\}$ and $s + t > 0$. If $u \in \mathcal{C}^s$ and $v \in \mathcal{C}^t$, then $u \bullet v$ exists and is an element of $\mathcal{C}^{s \wedge t}$. Moreover, the map*

$$\mathcal{C}^s \times \mathcal{C}^t \to \mathcal{C}^{s \wedge t}, \quad (u, v) \mapsto u \bullet v,$$

is a bilinear continuous associative symmetric map and there exists a $C > 0$ such that

$$\|u \bullet v\|_{\mathcal{C}^{s \wedge t}} \le C \|u\|_{\mathcal{C}^s} \|v\|_{\mathcal{C}^t} \qquad (u, v \in \mathcal{S}'). \tag{27.19}$$

The following corollary is a consequence of Theorem 27.5 and Theorem 27.10 and is left as an exercise:

Corollary 27.16. *Let $s \in (0,\infty)$. There exists a $C > 1$ such that for all $p,q \in [1,\infty]$*

$$\|u \bullet v\|_{B^s_{p,q}} \le C \left(\|u\|_{B^s_{p,q}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{B^s_{p,q}} \right) \qquad (u,v \in \mathcal{S}').$$

Consequently, $L^\infty \cap B^s_{p,q}$ is a Banach algebra under the norm $C(\|\cdot\|_{L^\infty} + \|\cdot\|_{B^s_{p,q}})$.

Exercise **27.E.** Prove Corollary 27.16.

Another consequence is the following:

Theorem 27.17. *Let $s,t \in \mathbb{R}$ and $s+t > 0$. Let $\delta > 0$ be such that $s \wedge t - \delta \in (0,\infty) \setminus \mathbb{N}_0$. For $u \in H^s$ and $v \in H^t$ we have $u \bullet v$ exists in $W^{s\wedge t-\delta,1}$. Moreover, the product map $H^s \times H^t \to W^{s\wedge t - \delta,1}$, $(u,v) \mapsto u \bullet v$ is continuous, i.e., there exists a $C > 0$ such that*

$$\|u \bullet v\|_{W^{s\wedge t-\delta,1}} \le C \|u\|_{H^s} \|v\|_{H^t} \qquad (u \in H^s, v \in H^t).$$

Exercise **27.F.** Prove Theorem 27.17. (Hint: Observe that it is allowed for either s or t to be equal to zero. Furthermore: Theorem 23.4.)

27.1 Comments ...

Remark 27.18 (About notation and latex)**.** In many textbooks one writes "$T_u v$" for the paraproduct instead of "$u \varolessthan v$" (for example in [BCD11]). In this sense one views T_u as an multiplying operator. Also "$\Pi(u,v)$" or "$R(u,v)$" is written for the resonance product. In the application to SPDEs in the authors of the paper [GIP15] wrote "$u \prec v$" and "$u \circ v$" for the para- and resonance product, respectively. The latter notation changed in the SPDE literature, with some authors creating new symbols, for example "<" and "=" with circles around them. In the latter case, "≤" with a circle around it is then used for the sum of the paraproduct and the resonance product, for which the authors of [GIP15] used "$\preceq$".

For the sum of the paraproduct and the resonance product we write $\mathrlap{\odot}{\olessthan}$. The following table presents the latex commands for the symbols used in these notes.

\varolessthan	$\varolessthan$
\varogreaterthan	$\varogreaterthan$
\varodot	$\varodot$
\mathrlap{\odot}{\olessthan}	$\mathrlap{\odot}{\olessthan}$
\mathrlap{\odot}{\ogreaterthan}	$\mathrlap{\odot}{\ogreaterthan}$
\mathrlap{\olessthan}{\ogreaterthan}	$\mathrlap{\olessthan}{\ogreaterthan}$
\mathrlap{\mathrlap{\odot}{\olessthan}}{\olessthan}	$\mathrlap{\mathrlap{\odot}{\olessthan}}{\olessthan}$

27.19 (The notation $\lesssim$)**.** We have seen multiple arguments in which different estimates were combined, which were for example of the form

$$\|\Phi(x)\|_Y \le C\|x\|_X \qquad (x \in X),$$

with $C > 0$, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ normed vector spaces and $\Phi : X \to Y$. Let us call such a "C" a "toleration constant" for the moment. For example in 27.6 we had toleration constants C_1, C_2, C_3 and C_s. Keeping track of those toleration constants becomes more cumbersome and is a rather uninteresting job as the number of those constants grows. Indeed, often one is only interested in the existence of such a toleration constant. For this reason the notation "$\lesssim$" has been invented. Its usage is the following. If f and g are functions on a set X with values in $[0, \infty]$, we write

$$f(x) \lesssim g(x) \qquad (x \in X),$$

to indicated the existence of a number $C > 0$ such that $f(x) \le Cg(x)$ for all $x \in X$.

For sets X, Y and functions $f, g : X \times Y \to [0, \infty]$ observe the difference between

$$f(x, y) \lesssim g(x, y) \qquad ((x, y) \in X \times Y), \tag{27.20}$$

and

$$\text{if } y \in Y, \text{ then } f(x, y) \lesssim g(x, y) \qquad (x \in X). \tag{27.21}$$

In the literature (27.20) is also rendered

$$f(x, y) \lesssim g(x, y) \qquad (x \in X), \text{ uniformly in } y \in Y,$$

also, sometimes (27.21) is rendered

$$f(x, y) \lesssim_y g(x, y) \qquad (x \in X).$$

28 Solutions to elliptic PDEs in Besov spaces

In Section 13 we have considered elliptic operators with bounded coefficients (see 13.4), i.e., second order linear partial differential operators of the form

$$P = -\sum_{i,j=1}^d \mathfrak{a}_{ij}\partial_i\partial_j + \sum_{i=1}^d \mathfrak{b}_i\partial_i + \mathfrak{c},$$

$\mathfrak{a}_{ij}, \mathfrak{b}_i, \mathfrak{c} \in L^\infty(\Omega)$ for $i, j \in \{1, \dots, d\}$ such that there exists a $\theta > 0$ such that

$$\sum_{i,j=1}^d \mathfrak{a}_{ij}(x)y_iy_j \ge \theta|y|^2 \qquad (\text{almost all } x \in \Omega, y \in \mathbb{R}^d).$$

In this section we on the one hand restrict ourselves to $\Omega = \mathbb{R}^d$ and $\mathfrak{a}_{ii} = 1$ and $\mathfrak{a}_{ij} = 0$ if $i \ne j$ for $i, j \in \{1, \dots, d\}$ but on the other hand, instead of considering $\mathfrak{b}_i$ and $\mathfrak{c}$ in L^∞, we allow $\mathfrak{b}_i$ and $\mathfrak{c}$ to be in the Besov space $B^s_{\infty,\infty}$ for some $s < 0$ (in which L^∞ is embedded, see Theorem 23.4).

Lemma 28.1. *Let $s \in \mathbb{R}$ and*

$$\mathfrak{b}_i \in B^{s+1}_{\infty,\infty}, \mathfrak{c} \in B^{s}_{\infty,\infty} \qquad (i \in \{1,\dots,d\}).$$

Let $t > 0$ be such that $s + t > 0$ and $s + 1 \le t$. Then $\mathfrak{b}_i \bullet \partial_i u$ and $\mathfrak{c} \bullet u$ exist in $H^{s-\delta}$ for all $i \in \{1,\dots,d\}$ and there exists a $C > 0$ such that

$$\Big\| \sum_{i=1}^d \mathfrak{b}_i \bullet \partial_i u + \mathfrak{c} \bullet u \Big\|_{H^{s-\delta}} \le C\Big(\|\mathfrak{c}\|_{B^{s}_{\infty,\infty}} + \sum_{i=1}^d \|\mathfrak{b}_i\|_{B^{s+1}_{\infty,\infty}} \Big) \|u\|_{H^t} \qquad (u \in \mathcal{S}').$$

Proof. This follows by Theorem 23.2, Theorem 21.23 and Theorem 27.11. Observe in particular that as $s \le t-1$, (for the $\lesssim$ notation see 27.19)

$$\begin{aligned} \|\mathfrak{b}_i \bullet \partial_i u\|_{H^{s-\delta}} &\lesssim \|\mathfrak{b}_i \bullet \partial_i u\|_{H^{(s+1)\wedge(t-1)-\delta}} \\ &\lesssim \|\mathfrak{b}_i\|_{B^{s+1}_{\infty,\infty}} \|\partial_i u\|_{H^{t-1}} \lesssim \|\mathfrak{b}_i\|_{B^{s+1}_{\infty,\infty}} \|u\|_{H^t} \qquad (i \in \{1,\dots,d\}, u \in H^t). \end{aligned}$$

□

By this lemma we can extend the elliptic operator from $\mathcal{D}$ to H^t for large enough t:

28.2 (Assumptions)**.** Let $s \in \mathbb{R}$, $t > 0$, $s + t > 0$, $s \le t$ and $\delta > 0$. Let

$$\mathfrak{b}_i \in B^{s+1}_{\infty,\infty}, \mathfrak{c} \in B^{s}_{\infty,\infty} \qquad (i \in \{1,\dots,d\}).$$

Let $L : H^t \to H^{(s-\delta)\wedge(t-2)}$ be defined by

$$Lu = -\Delta u - \sum_{i=1}^d \mathfrak{b}_i \partial_i u - \mathfrak{c} u \qquad (u \in H^t). \tag{28.1}$$

Let us describe the goal of this section for L, s and t as in 28.2. Similarly to Theorem 13.9 we want to consider conditions on L and $\mathfrak{f}$ (in some Besov space) such that for some class of real numbers β there exists a u in H^t that satisfies

$$Lu + \beta u = \mathfrak{f}.$$

This equation can be rewritten by the formula

$$(\beta - \Delta) u = \sum_{i=1}^d \mathfrak{b}_i \bullet \partial_i u + \mathfrak{c} \bullet u + \mathfrak{f},$$

which, with $(\beta - \Delta)^{-1} = (\beta + 4\pi^2 |\mathfrak{x}|^2)^{-1}(\mathrm{D})$, can be rewritten as

$$u = (\beta - \Delta)^{-1} \Big(\sum_{i=1}^d \mathfrak{b}_i \bullet \partial_i u + \mathfrak{c} \bullet u + \mathfrak{f} \Big). \tag{28.2}$$

The following strategy is often used to find a u that satisfies this formula. First one defines a function Φ by the formula $\Phi(v) = (\beta - \Delta)^{-1}(\sum_{i=1}^d \mathfrak{b}_i \bullet \partial_i v + \mathfrak{c} \bullet v + \mathfrak{f})$ (basically by "the right-hand side" of (28.2)). Then u satisfies (28.2) if and only if $\Phi(u) = u$. One also says that u is a fixed point of Φ. In order to show the existence of fixed points, we recall Banach's fixed point theorem, and leave its proof for the reader.

Definition 28.3. Let $(\mathfrak{X}, d)$ be a non-empty complete metric space. A function $\Phi : \mathfrak{X} \to \mathfrak{X}$ is called a *contraction* if there exists a $\theta \in (0,1)$ such that

$$d(\Phi(x), \Phi(y)) \leq \theta d(x,y) \qquad (x, y \in \mathfrak{X}).$$

Theorem 28.4 (Banach's fixed point theorem)**.** *Let $(\mathfrak{X}, d)$ be a non-empty complete metric space. Suppose that $\Phi : \mathfrak{X} \to \mathfrak{X}$ is a contraction. Then there exists a precisely one x_* in $\mathfrak{X}$ such that $\Phi(x_*) = x_*$. Moreover, by defining $\Phi^1 = \Phi$ and $\Phi^k = \Phi \circ \Phi^{k-1}$ for $k \in \mathbb{N}$ with $k \geq 2$, we have for each $x \in \mathfrak{X}$ that*

$$\lim_{k \to \infty} \Phi^k(x) = x_*.$$

Before we prove the existence of a fixed point, let us have a closer look at the operator $(\beta - \Delta)^{-1}$ on Besov spaces.

Lemma 28.5. *Let $m \in (0, 2]$. There exists a $C > 0$ such that for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$*

$$\|(\beta - \Delta)^{-1} u\|_{B^{s+m}_{p,q}} \leq C\beta^{-1}(\beta^{\frac{m}{2}} \vee 1)\|u\|_{B^s_{p,q}} \qquad (u \in \mathcal{S}', \beta > 0). \tag{28.3}$$

Proof. Let $\sigma : \mathbb{R}^d \to (0, \infty)$ be defined by

$$\sigma = (1 + 4\pi^2|\mathfrak{x}|^2)^{-1}.$$

By Lemma 25.16 we have $\mathfrak{M}_m(\sigma) < \infty$ for $m \in (-\infty, 2]$. Observe that

$$(\beta + 4\pi^2|\mathfrak{x}|^2)^{-1} = \beta^{-1}(1 + 4\pi^2|\sqrt{\beta}\mathfrak{x}|^2)^{-1} = \beta^{-1} l_{\beta^{-\frac{1}{2}}} \sigma.$$

Therefore

$$\|(\beta - \Delta)^{-1} u\|_{B^{s+m}_{p,q}} = \beta^{-1} \|(l_{\beta^{-\frac{1}{2}}}\sigma)(\mathrm{D})u\|_{B^{s+m}_{p,q}} \qquad (u \in \mathcal{S}', \beta > 0).$$

Hence by Theorem 25.18 (e) we obtain (28.3). □

Theorem 28.6. *Let $s \in (-2, 2)$. Let*

$$\mathfrak{b}_i \in B^{s+1}_{\infty,\infty}, \mathfrak{c} \in B^s_{\infty,\infty}, \mathfrak{f} \in H^s \qquad (i \in \{1, \dots, d\}).$$

For all $t > 0$ with $t + s > 0$, $1 + s < t < 2 + s$ there exists a $\gamma > 0$ such that for all $\beta > \gamma$, the formula

$$\Phi_{\mathfrak{f}}(v) = (\beta - \Delta)^{-1}\Big(\sum_{i=1}^d \mathfrak{b}_i \bullet \partial_i v + \mathfrak{c} \bullet v + \mathfrak{f}\Big) \qquad (v \in H^t)$$

defines a contraction $H^t \to H^t$. Consequently, for such t and γ, for each $\beta > \gamma$ there exists exactly one $u \in H^t$ such that

$$Lu + \beta u = \mathfrak{f},$$

where L is as in (28.1) *of 28.2.*

Proof. First observe that as $t > |s|$, we have $t > s$, $t > 0$ and $t + s > 0$. Let $\delta > 0$ be such that $t < 2 - \delta + s$. Then we can and do choose $m \in (0, 2)$ such that $t < s - \delta + m$. By Lemma 28.5 there exists a $C_1 > 0$ and by Lemma 28.1 there exists a $C_2 > 0$ such that for all $v \in H^t$

$$\begin{aligned}\|\Phi_{\mathfrak{f}}(v)\|_{H^{s-\delta+m}} &\le C_1 \beta^{-1}(\beta^{\frac{m}{2}} \vee 1) \Big\| \sum_{i=1}^{d} \mathfrak{b}_i \bullet \partial_i v + \mathfrak{c} \bullet v + f \Big\|_{H^{s-\delta}} \\ &\le C_1 C_2 \Big(\|\mathfrak{c}\|_{B^s_{\infty,\infty}} + \sum_{i=1}^{d} \|\mathfrak{b}_i\|_{B^{s+1}_{\infty,\infty}} \Big) \beta^{-1}(\beta^{\frac{m}{2}} \vee 1) \|v\|_{H^t} + \|f\|_{H^s}.\end{aligned}$$

As

$$\|\Phi_{\mathfrak{f}}(v)\|_{H^t} \le \|\Phi_{\mathfrak{f}}(v)\|_{H^{s-\delta+m}} \qquad (v \in H^t),$$

we see that Φ defines a function $H^t \to H^t$. As $\Phi_{\mathfrak{f}}(v - w) = \Phi_0(v - w)$, we see that $\Phi_{\mathfrak{f}}$ is a contraction if

$$C_1 C_2 \Big(\|\mathfrak{c}\|_{B^s_{\infty,\infty}} + \sum_{i=1}^{d} \|\mathfrak{b}_i\|_{B^{s+1}_{\infty,\infty}} \Big) \beta^{-1}(\beta^{\frac{m}{2}} \vee 1) < 1.$$

From this one can obtain a γ such that the statement holds, for example

$$\gamma = 1 \vee \Big(C_1 C_2 (\|\mathfrak{c}\|_{B^s_{\infty,\infty}} + \sum_{i=1}^{d} \|\mathfrak{b}_i\|_{B^{s+1}_{\infty,\infty}}) \Big)^{\frac{2}{2-m}}.$$

□

The methods used above do not restrict to second order linear partial differential operators, as the following theorem shows. The proof follows the same strategy as above and is left to the reader.

Theorem 28.7. *Let $k \in \mathbb{N}$. Let $s \in (-2k, 2k)$. Let*

$$\mathfrak{b}_\alpha \in B^{s+|\alpha|}_{\infty,\infty}, \mathfrak{f} \in B^s_{\infty,\infty} \qquad (\alpha \in \mathbb{N}_0^d, |\alpha| \le 2k - 1).$$

For all $t > 0$ with $t + s > 0$, $2k - 1 + s < t < 2k + s$ there exists a $\gamma > 0$ such that for all $\beta > \gamma$, the formula

$$\Phi_{\mathfrak{f}}(v) = (\beta - \Delta^k)^{-1} \Big(\sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le 2k-1}} \mathfrak{b}_\alpha \bullet \partial^\alpha v + \mathfrak{f} \Big) \qquad (v \in H^t)$$

defines a contraction $H^t \to H^t$. Consequently, for such t and γ, for each $\beta > \gamma$ there exists exactly one $u \in H^t$ such that

$$\beta u - \Delta^k u - \sum_{\alpha \in \mathbb{N}_0^d} \mathfrak{b}_\alpha \bullet \partial^\alpha u = \mathfrak{f}.$$

Exercise **28.A.** Prove Theorem 28.7.

29 Overview

In this section we group statements according to their type and give references to where the full statement can be found. Some formula's are given without any description of the different symbols.

29.1 Spaces of functions and distributions

$C(\Omega), C^k(\Omega), C^\infty(\Omega)$: 1.3. $\mathcal{D}(\Omega)$: 1.5 and 4.1. $\mathcal{D}'(\Omega)$: 2.1 and 2.3. L^1_{loc} : 2.5. $\mathcal{M}(\Omega$: 2.23. $C^k_{\text{b}}, C^\infty_{\text{b}}$: 9.2. $W^{k,p}$: 12.3. $\mathcal{S}$: 14.2. C^∞_{p} : 14.6. $\mathcal{S}'$: 15.1. H^s : 20.5. H^s_p : 20.10. $B^s_{p,q}[\varphi]$: 21.12. $B^s_{p,q}$: 21.21.

29.2 Characterisations

Distributions of order 0 : 2.28.
Distributions in terms of sequentially continuous maps : Theorem 4.14.
Compactly supported distributions and $\mathcal{E}'$: 5.2 and 5.6.
$\mathcal{E}'(\Omega)$ in terms of linear combinations of derivatives of continuous functions : 6.2.
$\mathcal{D}'(\Omega)$ in terms of linear combinations of derivatives of continuous functions : 6.3.
Distributions of order k : 9.6.
Convolutions : 10.2 and 17.15.
$H^k = \{u \in \mathcal{S}' : (1 + |\mathfrak{x}|)^k \widehat{u} \in L^2\}$: 20.3.
$B^s_{2,2} = H^s$: 23.2.

29.3 Convergences

- Convergences of mollifications.

Pointwise for elements in L^1_{loc}.	Theorem 7.15 (a).
Uniformly on compacts for elements in $C(U)$.	Theorem 7.15 (b).
In L^p_{loc}.	Theorem 7.15 (c).
In L^p_{loc}.	Theorem 7.15 (d).
In $\mathcal{D}, \mathcal{D}', \mathcal{E}, \mathcal{E}'$.	Theorem 8.12.
"In C^∞_{p}": $\mathfrak{q}_{m,k}(\psi_\varepsilon * \eta - \eta) \xrightarrow{\varepsilon \downarrow 0} 0$	Lemma 26.3.

- Other convergences.

$(l_\lambda \chi)\varphi \xrightarrow{\lambda \downarrow 0} \varphi$	Lemma 14.10 (c).
$\sum_{n=1}^{N} \chi_n \varphi \xrightarrow{N\to\infty} \varphi$	Lemma 14.10 (d).
$\sum_{j\in\mathbb{N}_{-1}} \varphi_j \psi = \psi$	Lemma 21.8.
$\sum_{j\in\mathbb{N}_{-1}} \Delta_j \psi = \psi$	Lemma 21.9.
$\sum_{j\in\mathbb{N}_{-1}} \Delta_j u = u$	Lemma 21.9.
$\mathfrak{q}_{m,k}\Big(\sum_{j=-1}^{J} \Delta_j \eta - \eta\Big) \xrightarrow{J\to\infty} 0$	Lemma 26.4.
$\sum_{i\in\mathbb{N}_{-1}} \Delta_i f = f$	Lemma 26.9.

29.4 Identities

- Leibniz' Rule and the Multinomial Theorem.

$\partial^\alpha(fg) = \sum_{\substack{\beta\in\mathbb{N}_0^d \\ \beta\le\alpha}} \binom{\alpha}{\beta}(\partial^\beta f)(\partial^{\alpha-\beta} g)$	1.14.
$(x_1 + \cdots + x_d)^k = \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ \lvert\alpha\rvert = k}} \binom{k}{\alpha} x^\alpha$	Theorem 20.4.

- Identities involving convolutions (and no Fourier transforms).

$\delta_0 * \varphi = \varphi$	Lemma 8.2 and Lemma 17.4
$\delta_y * \varphi = \mathcal{T}_y \varphi$	Lemma 8.2 and Lemma 17.4
$(u * \varphi)\check{} = \check{u} * \check{\varphi}$	Lemma 8.2 and Lemma 17.4
$\mathcal{T}_y(u * \varphi) = (\mathcal{T}_y u) * \varphi = u * (\mathcal{T}_y \varphi)$	Lemma 8.2 and Lemma 17.4
$u(\varphi) = u * \check{\varphi}(0)$	Lemma 8.2 and Lemma 17.4
$\partial^\alpha(u * \varphi) = u * (\partial^\alpha \varphi) = (\partial^\alpha u) * \varphi$	Theorem 8.4, Theorem 17.6.
$(u * \varphi) * \psi = u * (\varphi * \psi)$	Theorem 8.9, Theorem 17.11.
$u * v = v * u$	Theorem 10.6, Theorem 17.13.
$(u * v) * \psi = u * (v * \psi)$	Theorem 10.6, Theorem 17.13.
$\delta_0 * u = u$	Lemma 10.10, Lemma 17.14.
$\delta_y * u = \mathcal{T}_y u$	Lemma 10.10, Lemma 17.14.
$\mathcal{R}(u * v) = \mathcal{R}(u) * \mathcal{R}(v)$	Lemma 10.10, Lemma 17.14.
$\mathcal{T}_y(u * v) = (\mathcal{T}_y u) * v = u * (\mathcal{T}_y v)$	Lemma 10.10, Lemma 17.14.
$\partial^\alpha(u * v) = (\partial^\alpha u) * v = u * (\partial^\alpha v)$	Lemma 10.10, Lemma 17.14.

- Identities involving Fourier transforms.

$\int f\widehat{g} = \int \widehat{f}g$	Theorem 16.10.
$\mathcal{F}^{-1} = \mathcal{F}\mathcal{R} = \mathcal{R}\mathcal{F}$	Theorem 16.24.
$\mathcal{F}(\partial^\beta u) = (2\pi\mathrm{i}\mathfrak{x})^\beta \widehat{u}$	Theorem 16.24.
$\partial^\beta \widehat{u} = \mathcal{F}((-2\pi\mathrm{i}\mathfrak{x})^\beta u)$	Theorem 16.24.
$\mathcal{F}(\mathcal{T}_y u) = e^{-2\pi\mathrm{i}\langle\mathfrak{x},y\rangle}\widehat{u}$	Theorem 16.24.
$\mathcal{T}_y \widehat{u} = \mathcal{F}(e^{2\pi\mathrm{i}\langle\mathfrak{x},y\rangle}u)$	Theorem 16.24.
$\mathcal{F}(u \circ l) = \frac{1}{\lvert\det l\rvert}\widehat{u} \circ l_*$	Theorem 16.24.
$\mathcal{F}(l_\lambda u) = \frac{1}{\lvert\lambda\rvert^d} l_{\frac{1}{\lambda}} \widehat{u}$	Theorem 16.24.
$\mathcal{F}(f * g) = \widehat{f}\widehat{g}$	Theorem 17.1.
$\mathcal{F}(\varphi * \psi) = \widehat{\varphi}\widehat{\psi}$	Theorem 17.2.
$\mathcal{F}(u * \varphi) = \widehat{\varphi}\widehat{u}$	Theorem 17.7.
$\mathcal{F}(u * v) = \widehat{v}\widehat{u}$	Theorem 17.13.

- Identities involving norms.

$\lVert\widehat{f}\rVert_{L^2} = \lVert f\rVert_{L^2}$	Theorem 16.27.
$\lVert v\rVert_{L^p} = \sup\{\lvert\langle v, f\rangle\rvert : f \in L^q, \lVert f\rVert_{L^q} \leq 1\}$	Theorem 21.28.
$\lVert\mu\rVert_{\mathcal{M}} = \sup\{\lvert\langle \mu, f\rangle\rvert : f \in C_0, \lVert f\rVert_{C_0} \leq 1\}$	Theorem 21.28.

- Identities involving Fourier multipliers.

$\tau(\mathrm{D})\sigma(\mathrm{D})u = (\sigma\tau)(\mathrm{D})u = \sigma(\mathrm{D})\tau(\mathrm{D})u$	Lemma 19.4.
$\sigma(\mathrm{D})(l_\lambda u) = l_\lambda[(l_\lambda\sigma)(\mathrm{D})u]$	Lemma 19.4.
$\sigma(\mathrm{D})\varphi = \mathcal{F}^{-1}(\sigma) * \varphi$	Lemma 19.4.
$\sigma(\mathrm{D})u = \mathcal{F}^{-1}(\sigma\widehat{u}) = \mathcal{F}^{-1}(\sigma) * u$	Lemma 19.4.
$\langle\Delta_j u, \psi\rangle = \langle u, \Delta_j\psi\rangle$	Lemma 21.9.
$\partial^\alpha u = \lambda^{\lvert\alpha\rvert+d}(l_\lambda h_\alpha) * u$	Lemma 21.14.
$u = \lambda^{-k}\sum_{\alpha\in\mathbb{N}_0^d:\lvert\alpha\rvert=k} \lambda^d(l_\lambda g_\alpha) * \partial^\alpha u$	Lemma 21.14.

- Identities involving "products".

$(u + w) \bullet_\varphi v = u \bullet_\varphi v + w \bullet_\varphi v$	Lemma 26.2.
$u \bullet_\varphi (v + w) = u \bullet_\varphi v + u \bullet_\varphi w$	Lemma 26.2.
$\lambda(u \bullet_\varphi v) = (\lambda u) \bullet_\varphi v + u \bullet_\varphi (\lambda v)$	Lemma 26.2.
$\eta\psi = \eta \bullet \psi$	Theorem 26.5.
$\eta u = \eta \bullet u$	Theorem 26.7.
$fg = f \bullet g$	Lemma 26.10.
$(u + w) \otimes_\varphi v = u \otimes_\varphi v + w \otimes_\varphi v$	Lemma 27.2.
$u \otimes_\varphi (v + w) = u \otimes_\varphi v + u \otimes_\varphi w$	Lemma 27.2.
$\lambda(u \otimes_\varphi v) = (\lambda u) \otimes_\varphi v + u \otimes_\varphi (\lambda v)$	Lemma 27.2.
$u \bullet_\varphi v = u \otimes_\varphi v + u \odot_\varphi v + u \oslash_\varphi v$	Lemma 27.3.
$(u_1 \bullet u_2) \bullet u_3 = u_1 \bullet (u_2 \bullet u_3)$	Lemma 27.12.

29.5 Continuity of operations

Continuity of the reflection $\mathcal{R}$, translations $\mathcal{T}_y$, derivations ∂^α, multiplication with a smooth function and composition with a linear bijection: 4.4 and 15.4

- Continuity of the pairing map

$\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{F}$, $\mathcal{E}'(\Omega) \times \mathcal{E}(\Omega) \to \mathbb{F}$	4.25.
$\mathcal{S}'(\Omega) \times \mathcal{S}(\Omega) \to \mathbb{F}$.	Theorem 15.10 (c).

- Continuity of the product maps

$\mathcal{E}(\Omega) \times \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$	5.3.
$\mathcal{D}'(\Omega) \times \mathcal{E}(\Omega) \to \mathcal{D}'(\Omega)$	5.3.
$\mathcal{E}'(\Omega) \times \mathcal{E}(\Omega) \to \mathcal{E}'(\Omega)$	5.3.
$W^{k,p}(\Omega) \times W^{k,r}(\Omega) \to W^{k,q}(\Omega)$	12.10.
$C_{\mathrm{b}}^{\infty} \times \mathcal{S} \to \mathcal{S}$	14.7.
$\mathcal{S} \times \mathcal{S} \to \mathcal{S}$	14.7.

- Continuity of convolution maps

$L^p \times L^q \to L^r$.	Theorem 7.7.
$\mathcal{D}' \times \mathcal{D} \to \mathcal{E}$.	Theorem 8.7.
$\mathcal{E}' \times \mathcal{D} \to \mathcal{D}$.	Theorem 8.7.
$\mathcal{E}' \times \mathcal{E} \to \mathcal{E}$.	Theorem 8.7.
$\mathcal{S}' \times \mathcal{S} \to \mathcal{S}'$.	Theorem 17.10.
$\mathcal{S}' \times \mathcal{S} \to \mathcal{E}$.	Theorem 17.10.
$\mathcal{E}' \times \mathcal{S} \to \mathcal{S}$.	Theorem 17.10.
$\mathcal{E}' \times \mathcal{S}' \to \mathcal{S}'$.	Lemma 17.16.

- Continuity of the Fourier transformation

$L^1 \to C_0(\mathbb{R}^d,)$	Theorem 16.6.
$\mathcal{S}(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d, \mathbb{C})$	Theorem 16.16.
$\mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \to \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$	Theorem 16.24.
$L^2(\mathbb{R}^d, \mathbb{C}) \to L^2(\mathbb{R}^d, \mathbb{C})$	Theorem 16.27.

- Continuity of Fourier multipliers

$\sigma(\mathrm{D}) : \mathcal{S}' \to \mathcal{S}'$	Lemma 19.2.

29.6 Metrizability and completeness statements

$\mathcal{D}(\Omega)$ is NOT metrizable.	4.15
Countably many seminorms $\Longrightarrow$ metrizability.	Theorem 3.8
$C^m(\Omega)$ is a Fréchet space.	Theorem 4.19
$\mathcal{E}(\Omega)$ is a Fréchet space.	Theorem 4.19
$\mathcal{D}_K(\Omega)$ is a Fréchet space.	Theorem 4.19
$\mathcal{D}'(\Omega)$ is weak* complete.	Theorem 4.26
$\mathcal{E}'(\Omega)$ is weak* complete.	Theorem 4.26
$\mathcal{E}'(\Omega)$ is NOT metrizable.	5.13.
$W^{k,p}(\Omega)$ is a Banach space.	Theorem 12.7.
$H^k(\Omega)$ is a Hilbert space.	Theorem 12.14.
$\mathcal{S}$ is a Fréchet space.	Theorem 14.12.
$\mathcal{S}'$ is weak* complete.	Theorem 15.10 (b).
H^s is a Hilbert space.	Theorem 20.7.
$B^s_{p,q}$ is a Banach space.	Theorem 21.32.

29.7 Denseness and separability statements

- Denseness:

$\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{E}(\Omega)$.	Theorem 5.10
$\mathcal{E}'(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$.	Theorem 5.10
$\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$.	Lemma 8.14.
$\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.	Theorem 8.17.
$\mathcal{D}(\Omega)$ is dense in $W^{k,p}(\Omega)$.	Theorem 12.11.
$\mathcal{D}$ is dense in $\mathcal{S}$.	Theorem 14.12
$C^\infty_{\mathrm{b}} \cap B^s_{p,q}$ is dense in $B^s_{p,q}$	Theorem 23.8.
$\mathcal{D}$ is dense in $B^s_{p,q}$.	Theorem 23.8.
$\mathcal{S}$ is NOT dense in $B^s_{\infty,q}$	23.9.

- Separability:

$\mathcal{D}(\Omega), \mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ are separable.	Theorem 8.15.
$\mathcal{S}$ is separable.	Theorem 14.12.

29.8 Continuous embeddings

- Continuous embeddings

$\mathcal{D}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^p_{\mathrm{loc}}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$.	Theorem 4.28
$\mathcal{D}(\Omega) \hookrightarrow_{\mathrm{seq}} \mathcal{E}(\Omega)$.	Theorem 5.10
$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$.	Theorem 5.10
$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{E}'(\mathbb{R}^d)$.	Theorem 5.10
$C^m \hookrightarrow C^k, \qquad C^m_{\mathrm{b}} \hookrightarrow C^k_{\mathrm{b}}$.	Lemma 12.8
$W^{m,p} \hookrightarrow W^{k,p}$.	Lemma 12.9
$\mathcal{D} \hookrightarrow_{\mathrm{seq}} \mathcal{S} \hookrightarrow C^\infty_{\mathrm{b}} \hookrightarrow \mathcal{E}$.	Theorem 14.11.
$\mathcal{S} \hookrightarrow L^p$	Lemma 14.14.
$\mathcal{S} \hookrightarrow W^{k,p}$	Lemma 14.15.
$\mathcal{E}' \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{D}'$	Theorem 15.2.
$L^p \hookrightarrow \mathcal{S}'$	Theorem 15.6.
$W^{k,p} \hookrightarrow \mathcal{S}'$	Lemma 14.15.
$H^s \hookrightarrow H^r$	Lemma 20.8.
$B^s_{p_1,q_1} \hookrightarrow B^t_{p_2,q_2} \qquad \big(t \le s - d(\frac{1}{p_1} - \frac{1}{p_2})\big)$	Theorem 21.23.
$B^s_{p,q_1} \hookrightarrow B^{s-\varepsilon}_{p,q_2}$	Lemma 21.25.
$B^s_{2,2} \hookrightarrow H^s \hookrightarrow B^s_{2,2}$	Theorem 23.2.
$B^k_{p,1} \hookrightarrow W^{k,p} \hookrightarrow B^k_{p,\infty}$	Theorem 23.4.
$B^t_{p,q} \hookrightarrow W^{k,p} \hookrightarrow B^s_{p,q} \qquad (s,t \in \mathbb{R}, s < k < t)$	Theorem 23.4.
$B^k_{\infty,1} \hookrightarrow C^k_{\mathrm{b}} \hookrightarrow B^k_{\infty,\infty}$	Theorem 23.5.
$\mathcal{D} \hookrightarrow_{\mathrm{seq}} \mathcal{S} \hookrightarrow B^s_{p,q}$	Corollary 23.6.

- Embeddings which are not bijective.

$W^{k,\infty} \hookrightarrow B^k_{\infty,\infty}$	Example 23.11.

- Embeddings which are no homeomorphisms on their image.

$\mathcal{D}(\Omega) \hookrightarrow_{\mathrm{seq}} \mathcal{E}(\Omega)$	5.13
$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$	5.13
$\mathcal{D} \hookrightarrow \mathcal{S}$	15.11
$\mathcal{S} \hookrightarrow \mathcal{E}$	15.11
$\mathcal{S}' \hookrightarrow \mathcal{D}'$	15.11
$\mathcal{E}' \hookrightarrow \mathcal{S}'$	15.11

29.9 Inequalities

- Inequalities involving ‘products’ and convolutions.

$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$	Theorem 7.7.
$\|uv\|_{W^{k,p}} \le C\|u\|_{W^{k,r}}\|v\|_{W^{k,q}}$	Lemma 12.10.
$\|\sigma\varphi\|_{m,\mathcal{S}} \le C\mathfrak{q}_{m,k}(\sigma)\|\varphi\|_{m+k,\mathcal{S}}$	Lemma 14.7.
$\|f * g\|_{\ell^r} \le \|f\|_{\ell^p}\|g\|_{\ell^q}$	Theorem 21.17.

$\|\sigma\varphi\|_{m,\mathcal{S}} \le C\|\sigma\|_{C^m}\|\varphi\|_{m,\mathcal{S}}$.	Lemma 14.7.
$\|\varphi\psi\|_{m,\mathcal{S}} \le C\|\varphi\|_{m,\mathcal{S}}\|\psi\|_{m,\mathcal{S}}$.	Lemma 14.7.
$\|u \otimes_\varphi v\|_{B^s_{p,r}} \le C\|u\|_{L^{p_1}}\|v\|_{B^s_{p_2,r}}$	Theorem 27.5.
$\|u \otimes_\varphi v\|_{B^{s+t}_{p,q}} \le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}$	Theorem 27.5.
$\|u \circledast_\varphi v\|_{B^{s\wedge t\wedge(s+t)}_{p,r}} \le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}$	Corollary 27.7 (a).
$\|u \circledast_\varphi v\|_{B^s_{p,q}} \le C\|u\|_{B^s_{p_1,q}}\|v\|_{B^t_{p_2,1}}$	Corollary 27.7 (b).
$\|u \circledast_\varphi v\|_{B^0_{p,1}} \le C\|u\|_{B^0_{p_1,1}}\|v\|_{B^0_{p_2,1}}$	Corollary 27.7 (c).
$\|u \odot_\varphi v\|_{B^{s+t}_{p,q}} \le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}$	Theorem 27.10.
$\|u \odot_\varphi v\|_{B^0_{p,\infty}} \le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}$	Theorem 27.10.
$\|u \bullet v\|_{B^{s\wedge t}_{p,r}} \le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}$	Theorem 27.11 (a).
$\|u \bullet v\|_{B^{s\wedge t-\delta}_{p,q}} \le C\|u\|_{B^s_{p_1,q_1}}\|v\|_{B^t_{p_2,q_2}}$	Theorem 27.11 (a).
$\|u \bullet v\|_{B^0_{p,q}} \le C\|u\|_{B^s_{p_1,1}}\|v\|_{B^0_{p_2,q}}$	Theorem 27.11 (b).
$\|u \bullet v\|_{B^0_{p,\infty}} \le C\|u\|_{B^0_{p_1,1}}\|v\|_{B^0_{p_2,1}}$	Theorem 27.11 (c).
$\|u \bullet v\|_{H^t} \le C\|u\|_{H^s}\|v\|_{\mathcal{C}^t}$	Corollary 27.14.
$\|u \bullet v\|_{H^{s\wedge t-\delta}} \le C\|u\|_{H^s}\|v\|_{\mathcal{C}^t}$	Corollary 27.14.
$\|u \bullet v\|_{\mathcal{C}^{s\wedge t}} \le C\|u\|_{\mathcal{C}^s}\|v\|_{\mathcal{C}^t}$	Corollary 27.15.
$\|u \bullet v\|_{B^s_{p,q}} \le$ $C\left(\|u\|_{B^s_{p,q}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{B^s_{p,q}}\right)$	Corollary 27.16

- Inequalities of operations.

$\|\partial^\alpha f\|_{C^{k-\lvert\alpha\rvert},K} \le \|f\|_{C^m,K}$	Lemma 12.8.
$\|\partial^\alpha f\|_{C^{k-\lvert\alpha\rvert}} \le \|f\|_{C^m}$	Lemma 12.8.
$\|\partial^\alpha u\|_{W^{k-\lvert\alpha\rvert,p}} \le \|u\|_{W^{m,p}}$	Lemma 12.9.
$\|\partial^\alpha u\|_{H^{r-\lvert\alpha\rvert}} \le \|u\|_{H^s}$	Lemma 20.8.
$\|\Delta_j f\|_{L^p} \le \|\widehat{\varphi_0}\|_{L^1}\|f\|_{L^p}$	Lemma 21.9.
$\left\|\sum_{j=-1}^J \Delta_j f\right\|_{L^p} \le \|\widehat{\varphi_{-1}}\|_{L^1}\|f\|_{L^p}$	Lemma 21.9.
$\max_{\substack{\alpha\in\mathbb{N}_0^d\\ \lvert\alpha\rvert=k}} \|\partial^\alpha u\|_{L^q} \le C\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}$	Theorem 21.15.
$\frac{1}{C}\lambda^k\|u\|_{L^p} \le \max_{\substack{\alpha\in\mathbb{N}_0^d\\ \lvert\alpha\rvert=k}} \|\partial^\alpha u\|_{L^p} \le C\lambda^k\|u\|_{L^p}$	Theorem 21.15.
$\|\partial^\alpha u\|_{B^{t-\lvert\alpha\rvert}_{p_2,q_2}} \le C\|u\|_{B^s_{p_1,q_1}}$	Theorem 21.23.
$\|\sigma(\mathrm{D})u\|_{L^p} \le C\|u\|_{L^p}$	Theorem 25.1.
$\|\sigma(\mathrm{D})u\|_{B^s_{p,q}} \le C\|u\|_{B^s_{p,q}}$	Theorem 25.1 and Theorem 25.18 (d).
$\|(l_\mu\sigma)(\mathrm{D})u\|_{L^p} \le C\mu^{-m}\lambda^{-m}\|u\|_{L^p}$	Theorem 25.18 (a).
$\|(l_\mu\sigma)(\mathrm{D})u\|_{B^{s+m}_{p,q}} \le C\mu^{-m}\|u\|_{B^s_{p,q}}$	Theorem 25.18 (b).
$\|\sigma(\mathrm{D})u\|_{B^{s+m}_{p,q}} \le C\|u\|_{B^s_{p,q}}$	Theorem 25.18 (c).
$\|(l_\mu\sigma)(\mathrm{D})u\|_{B^{s+m}_{p,q}} \le C(\mu^{-m}\vee 1)\|u\|_{B^s_{p,q}}$	Theorem 25.18 (e).
$\left\|\sum_{j\in F}\Delta_j\psi\right\|_{m,\mathcal{S}} \le C\|\psi\|_{k,\mathcal{S}}$	Lemma 26.6.

- Inequalities related to embeddings.

$\|\varphi\|_{C^m,K} \leq \|\varphi\|_{C^m} \leq \|\varphi\|_{m,\mathcal{S}}$	Lemma 14.10.
$\|\varphi\|_{m,\mathcal{S}} \leq (1+\sup_{x\in\operatorname{supp}\varphi}\|x\|)^m \|\varphi\|_{C^m}$	Lemma 14.10.
$\|\psi\|_{k,\mathcal{S}} \leq \frac{\|\psi\|_{k+1,\mathcal{S}}}{1+r}$	Lemma 14.10.
$\|\cdot\|_{L^p} \leq C\|\cdot\|_{m,\mathcal{S}}$.	Lemma 14.14.
$\|u\|_{B^s_{p,q}[\varphi]} \leq C \left\| \left(2^{js}\|u_j\|_{L^p}\right)_{j\in\mathbb{N}_{-1}} \right\|_{\ell^q}$	Theorem 21.18.
$\frac{1}{C}\|u\|_{B^s_{p,q}[\psi]} \leq \|u\|_{B^s_{p,q}[\varphi]} \leq C\|u\|_{B^s_{p,q}[\psi]}$	Corollary 21.20.
$\|\partial^\alpha u\|_{B^{t-\|\alpha\|}_{p_2,q_2}} \leq C\|u\|_{B^s_{p_1,q_1}}$	Theorem 21.23.
$\|u\|_{B^{s-\varepsilon}_{p,q_2}} \leq C\|u\|_{B^s_{p,q_1}}$	Lemma 21.25.
$\|u\|_{\mathfrak{X}} \leq \liminf_{m\to\infty}\|u_{n_m}\|_{\mathfrak{X}}$	Lemma 21.30.
$\|u\|_{L^p} \leq \liminf_{m\to\infty}\|u_{n_m}\|_{L^p}$	Lemma 21.31.
$\|u\|_{B^s_{p,q}} \leq \liminf_{m\to\infty}\|u_{n_m}\|_{B^s_{p,q}}$	Theorem 21.32.
$c\left(1+\|\xi\|^2\right)^s \leq \sum_{j\in\mathbb{N}_{-1}} 2^{2sj}\varphi_j(\xi)^2 \leq C\left(1+\|\xi\|^2\right)^s$	Lemma 23.1.
$\|u\|_{B^s_{p,q}} \leq C\|u\|_{W^{k,p}}$	Theorem 23.4.
$\|u\|_{B^k_{p,\infty}} \leq C\|u\|_{W^{k,p}}$	Theorem 23.4.
$\|u\|_{W^{k,p}} \leq C\|u\|_{B^t_{p,q}}$	Theorem 23.4.
$\|u\|_{W^{k,p}} \leq C\|u\|_{B^k_{p,1}}$	Theorem 23.4.
$(1+2^s)^{-1}\|u\|_{B^s_{p,q}} \leq \|(2^{js}\|\Delta_{\leq j}u\|_{L^p})_{j\in\mathbb{N}_{-1}}\|_{\ell^q}$	Theorem 27.4.
$\|(2^{js}\|\Delta_{\leq j}u\|_{L^p})_{j\in\mathbb{N}_{-1}}\|_{\ell^q} \leq (1-2^s)^{-1}\|u\|_{B^s_{p,q}}$	Theorem 27.4.

- Other inequalities

Poincaré inequality	
$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p}$ (for $u \in W^{1,p}_0(\Omega)$)	Theorem 12.17.
Lemma of Riemann–Lebesgue	
$\|\widehat{g}(a)\| \leq \frac{1}{2}\|g - \mathcal{T}_{\frac{1}{2a}}g\|_{L^1}$.	Lemma 16.5.
$(a+b)^\theta \leq a^\theta + b^\theta$	Lemma 20.2.

29.10 Other statements

$\operatorname{supp} u * v \subset \operatorname{supp} u * \operatorname{supp} v$.	Theorem 7.10, Theorem 8.6, Theorem 10.11.
$v \in \mathcal{E}'$ implies $\widehat{v} \in C^\infty_{\mathrm{p}}$	Lemma 17.8.
Principle of uniform boundedness for $\mathcal{D}'$, $\mathcal{E}'$.	Theorem 4.24
$\operatorname{supp} u = \{x\} \implies u = \sum_{\alpha\in\mathbb{N}_0^d:\|\alpha\|\leq k} c_\alpha \partial^\alpha \delta_x$	Theorem 9.5
Principle of uniform boundedness for $\mathcal{S}'$.	Theorem 15.10 (a)

A Preliminaries on L^p spaces

In this section we let $(X, \mathcal{A}, \mu)$ be a measure space (see Definition A.1. For the purpose of this course, X will mostly be either be $\mathbb{R}^d$, $\mathcal{A}$ the set of Lebesgue measurable sets and μ the Lebesgue measure or X with be $\mathbb{N}$ or $\mathbb{Z}^d$ or any countable space, $\mathcal{A}$ the power set of X and μ the counting measure. As usual, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Before we turn to L^p spaces, let us recall some standard definitions from measure theory.

Definition A.1. Let X be a set. A collection $\mathcal{A}$ of subsets of X is called a *σ-algebra* on X if $X \in \mathcal{A}$ and

$$A, B \in \mathcal{A} \Longrightarrow A \setminus B \in \mathcal{A},$$
$$A_1, A_2, \dots \in \mathcal{A} \Longrightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}.$$

If $\mathcal{A}$ is a σ-algebra on X, then the pair $(X, \mathcal{A})$ is called a *measurable space.* A function f on a measurable space $(X, \mathcal{A})$ with values in $\mathbb{R}$, i.e., $f : X \to \mathbb{F}$, is called *measurable* (or $\mathcal{A}$-measurable) if $f^{-1}(U) \in \mathcal{A}$ for all open sets $U \subset \mathbb{F}$.

Let $\mathcal{A}$ be a σ-algebra on X. A function $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$ is called a *measure* if it is *countably additive*, i.e., if for any sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint elements of $\mathcal{A}$,

$$\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n).$$

A measure is called *finite* if $\mu(X) < \infty$. A *signed measure* is a countably additive function $\mu : \mathcal{A} \to \mathbb{R}$ such that $\mu(\emptyset) = 0$. A *complex measure* is a countably additive function $\mu : \mathcal{A} \to \mathbb{C}$ such that $\mu(\emptyset) = 0$.

Observe that not every measure is a signed measure, for example the Lebesgue measure is not a signed measure. But every finite measure is a signed measure.

Definition A.2. We say that a subset A of X is an *(μ-)null set*, if there exists a $B \in \mathcal{A}$ with $A \subset B$ and $\mu(B) = 0$. We write A^c for the complement of A in X, so that $A^c = X \setminus A$.

Definition A.3. Let $p \in [1, \infty)$. $\mathcal{L}^p(\mu)$ is the space of measurable functions $f : X \to \mathbb{F}$ for which

$$\int |f(x)|^p \, \mathrm{d}\mu(x) < \infty.$$

We say that two measurable functions f and g are equivalent, written $f \sim g$ if there exists a null set $A \in \mathcal{A}$ such that $f = g$ on A^c. We write $L^p(\mu)$ for the space that consists of all equivalence classes in $\mathcal{L}^p(\mu)$, in formula $L^p(\mu) = \mathcal{L}^p(\mu)/\sim$ or when we define $[f]_\sim = \{g \in \mathcal{L}^p : g \sim f\}$ for $f \in \mathcal{L}^p$, then

$$L^p(\mu) = \{[f]_\sim : f \in \mathcal{L}^p(\mu)\}.$$

We define

$$\|f\|_{L^p} := \left(\int |f(x)|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}}.$$

Similarly, we define $\mathcal{L}^\infty$ to be the space of measurable functions $f : X \to \mathbb{F}$ for which there exists a null set A such that f is bounded on A^c. We define

$$\|f\|_{\mathcal{L}^\infty} = \inf\{M > 0 : |f| \leq M \text{ a.e. } \},$$

where we abbreviated "almost everywhere" by "a.e.". Similarly as for $p \in [1, \infty)$, we define

$$L^\infty(\mu) = \{[f]_\sim : f \in \mathcal{L}^\infty(\mu)\},$$

and write for $f \in L^\infty(\mu)$ and $g \in f$ (the following is independent of the choice of g)

$$\|f\|_{L^\infty} = \|g\|_{\mathcal{L}^\infty}.$$

But from now on we 'identify' functions f with their equivalence class $[f]_\sim$, and so use also consider elements of L^p as functions.

Theorem A.4 (Hölder's inequality)**.** [BCD11, Theorem 1.1] *Let $p, q, r \in [1, \infty]$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^r(\mu)$ and

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Theorem A.5 (Generalized Hölder inequality)**.** *Let $n \in \mathbb{N}$ and $p_1, \dots, p_n, r \in [1, \infty]$. Suppose*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{r}.$$

For $i \in \{1, \dots, n\}$ let $f_i \in L^{p_i}(\mu)$. Then $f_1 \cdots f_n \in L^r$ and

$$\|f_1 \cdots f_n\|_{L^r} \leq \|f_1\|_{L^{p_1}} \cdots \|f_n\|_{L^{p_n}}.$$

Proof. Let $q \in [1, \infty]$ be such that

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}}.$$

Let $g = f_1 \cdots f_{n-1}$. If $g \in L^q$, then by the Hölder inequality, as $\frac{1}{q} + \frac{1}{p_n} = \frac{1}{r}$

$$\|gf_n\|_{L^r} \leq \|g\|_{L^q} \|f_n\|_{L^{p_n}}.$$

From this one can finish the proof by an induction argument. □

Lemma A.6. *We have $L^p(\mu) \subset L^1(\mu) + L^\infty(\mu)$ for all $p \in [1, \infty]$.*

Proof. Let $f \in L^p(\mu)$. Then $[|f| > 1]$ has finite measure. Define $f_1 := f\mathbb{1}_{[|f|\le 1]}$ and $f_2 := f\mathbb{1}_{[|f|>1]}$. Then $f_1 \in L^\infty(\mu)$ and with Hölder's inequality we have

$$\|f_2\|_{L^1} \le \|f\|_{L^p}\|\mathbb{1}_{[|f|>1]}\|_{L^q} < \infty,$$

for $q \ge 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. □

Definition A.7. Let $p \in [1, \infty]$ and $\mathbb{I}$ be a countable set. We write $\|\cdot\|_{\ell^p(\mathbb{I})}$ or $\|\cdot\|_{\ell^p}$ for the function $\mathbb{F}^{\mathbb{I}} \to [0, \infty]$ given by

$$\|x\|_{\ell^p(\mathbb{I})} = \begin{cases} \Big(\sum_{i\in\mathbb{I}} |x(i)|^p\Big)^{\frac{1}{p}} & p < \infty, \\ \sup_{i\in\mathbb{I}} |x(i)| & p = \infty, \end{cases} \qquad (x \in \mathbb{F}^{\mathbb{I}}).$$

We write $\ell^p(\mathbb{I})$ for the set of $x \in \mathbb{F}^{\mathbb{I}}$ such that $\|x\|_{\ell^p(\mathbb{I})} < \infty$. In other words, $\ell^p(\mathbb{I})$ is the space $L^p(\mu)$ in case μ is the counting measure on $\mathbb{I}$.

Lemma A.8. *Let $p, r \in [1, \infty]$ and $p \le r$. Then*

$$\|x\|_{\ell^r} \le \|x\|_{\ell^p} \qquad (x \in \mathbb{F}^{\mathbb{I}}).$$

In particular, $\ell^p \subset \ell^r$.

Proof. If $r = \infty$, this is immediate. For convenience, we may assume $\mathbb{I} = \mathbb{N}$. Let $\theta \le 1$. It suffices (take $\theta = \frac{p}{r}$) to show that

$$\Big(\sum_{i=1}^{\infty} |x(i)|\Big)^{\theta} \le \sum_{i=1}^{\infty} |x(i)|^{\theta} \qquad (x \in \mathbb{F}^{\mathbb{I}}).$$

But this follows from the subadditivity of the function $(0, \infty) \to (0, \infty)$, $t \mapsto t^a$, see Lemma 20.2. □

Corollary A.9 (Hölder's inequality for ℓ^p spaces)**.** *Let $p, q \in [1, \infty]$ and $r \in [1, \infty]$ be such that*

$$\min\{1, \frac{1}{p} + \frac{1}{q}\} = \frac{1}{r}.$$

If $f \in \ell^p$ and $g \in \ell^q$, then $fg \in \ell^r$ with

$$\|fg\|_{\ell^r} \le \|f\|_{\ell^p}\|g\|_{\ell^q}.$$

Proof. Suppose that $\frac{1}{p} + \frac{1}{q} > 1$, in the other case we can apply Hölder's inequality immediately. Then both p and q are finite, and we can find $\tilde{p}$, $\tilde{q}$ with $p \le \tilde{p} < \infty$, $q \le \tilde{q} < \infty$ such that

$$\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1,$$

Let $f \in \ell^p$ and $g \in \ell^q$. Then $f \in \ell^{\tilde{p}}$ and $g \in \ell^{\tilde{q}}$ and $\|fg\|_{\ell^1} \le \|f\|_{\ell^{\tilde{p}}}\|g\|_{\ell^{\tilde{q}}} \le \|f\|_{\ell^p}\|g\|_{\ell^q}$. □

Theorem A.10 (Log-convexity of L^p norms)**.** *Let p, r be such that $1 \leq p < r \leq \infty$. Then $L^p(\mu) \cap L^r(\mu) \subset L^q(\mu)$ for all q with $p \leq q \leq r$ and with $\theta \in [0,1]$ such that*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

we have

$$\|f\|_{L^q} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^r}^{1-\theta} \qquad (f \in L^p \cap L^r).$$

Proof. As $1 = \frac{\theta q}{p} + \frac{(1-\theta)q}{r}$, we obtain by Hölder's inequality,

$$\|f\|_{L^q}^q = \int |f|^{\theta q} |f|^{(1-\theta)q} \leq \||f|^{\theta q}\|_{L^{\frac{p}{\theta q}}} \||f|^{(1-\theta)q}\|_{L^{\frac{r}{(1-\theta)q}}} = \|f\|_{L^p}^{\theta q} \|f\|_{L^r}^{(1-\theta)q}.$$

□

Lemma A.11 (Young's inequality for products)**.** *For $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \qquad (a, b \geq 0).$$

In an other formulation; if $\theta \in [0,1]$ then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$ for all $a, b \geq 0$.

Proof. As the exponential function is convex, we have for p, q as above and $a, b \geq 0$,

$$ab = \exp\left(\frac{1}{p} \log a^p + \frac{1}{q} \log b^q\right) \leq \frac{1}{p} \exp\left(\log a^p\right) + \frac{1}{q} \exp\left(\log b^q\right) = \frac{1}{p} a^p + \frac{1}{q} b^q.$$

□

Corollary A.12. *Let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $L^1(\mu) \cap L^\infty(\mu) \subset L^p(\mu)$ and*

$$\|f\|_{L^p} \leq \frac{1}{p} \|f\|_{L^1} + \frac{1}{q} \|f\|_{L^\infty} \qquad (f \in L^1 \cap L^\infty).$$

Proof. Note that $\theta = \frac{1}{p}$ is such that $\frac{1}{p} = \frac{\theta}{1} + \frac{(1-\theta)}{\infty}$. Apply Theorem A.10, to obtain $\|f\|_{L^p} \leq \|f\|_{L^1}^{\frac{1}{p}} \|f\|_{L^\infty}^{\frac{1}{q}}$. Then apply Lemma A.11. □

A.13 (Notation)**.** Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be open. We write $\mathcal{B}(\Omega)$ for the Borel-σ-algebra on Ω. Let λ be the Lebesgue measure on the measurable space $(\Omega, \mathcal{B}^\lambda(\Omega))$, where $\mathcal{B}^\lambda(\Omega)$ is the completion of the Borel-σ-algebra on Ω, which consists of all Lebesgue measurable sets. For $p \in [1, \infty]$ we write $L^p(\Omega)$ instead of $L^p(\lambda)$.

Lemma A.14. *Let $p \in [1, \infty)$. Then $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.*

B Taylor's formula

B.1 For one dimension

Let us first recall the fundamental theorem of calculus.

Theorem B.1. [RS82, §15] *Let $g : [a, b] \to \mathbb{R}$ be continuous. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x g(y) \ \mathrm{d}y = g(x).$$

The following is a direct consequence.

Corollary B.2. *If $f \in C^1[a, b]$, then*

$$f(x) = f(a) + \int_a^x f'(y) \ \mathrm{d}y.$$

B.3. If $f \in C^2$, then we have

$$f'(y) = f'(a) + \int_a^y f''(z) \ \mathrm{d}z,$$

and thus

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(y) \ \mathrm{d}y \\ &= f(a) + \int_a^x \left(f'(a) + \int_a^y f''(z) \ \mathrm{d}z \right) \mathrm{d}y \\ &= f(a) + (x-a)f'(a) + \int_a^x \int_a^y f''(z) \ \mathrm{d}z \ \mathrm{d}y. \end{aligned}$$

This can be iterated:

For $f \in C^k[a, b]$, we have

$$f(x) = \sum_{i=0}^{k-1} \frac{(x-a)^i}{i!} \partial^i f(a) + R^k_{f,a}(x),$$

where by Fubini

$$\begin{aligned} R^k_{f,a}(x) &= \int_a^x \int_a^{y_1} \cdots \int_a^{y_{k-1}} \partial^k f(y_k) \ \mathrm{d}y_k \ \mathrm{d}y_{k-1} \cdots \mathrm{d}y_1 \\ &= \int_{[a,b]^k} \mathbb{1}_{\{y : a \le y_k \le y_{k-1} \le \cdots \le y_1 \le x\}}(y) \partial^k f(y_k) \ \mathrm{d}y. \\ &= \int_a^x \int_{y_k}^x \int_{y_{k-1}}^x \cdots \int_{y_2}^x \mathrm{d}y_1 \ \mathrm{d}y_2 \cdots \mathrm{d}y_{k-1} \partial^k f(y_k) \ \mathrm{d}y_k \end{aligned}$$

By induction one can easily see that

$$\int_{y_k}^x \int_{y_{k-1}}^x \cdots \int_{y_2}^x \mathrm{d}y_1 \ \mathrm{d}y_2 \cdots \mathrm{d}y_{k-1} = \frac{(x-y_k)^{k-1}}{(k-1)!}.$$

So we have obtained the following.

Theorem B.4. *Let $f \in C^k[a,b]$, then*

$$f(x) = \sum_{i=0}^{k-1} \frac{(x-a)^i}{i!}\partial^i f(a) + \int_a^x \frac{(x-y)^{k-1}}{(k-1)!}\partial^k f(y)\ \mathrm{d}y$$
$$= \sum_{i=0}^{k} \frac{(x-a)^i}{i!}\partial^i f(a) + \int_a^x \frac{(x-y)^{k-1}}{(k-1)!}[\partial^k f(y) - \partial^k f(a)]\ \mathrm{d}y.$$

Let

$$L = \max_{y\in[a,b]} |\partial^k f(y)|$$
$$M = \max_{y\in[a,b]} |\partial^k f(y) - \partial^k f(a)|.$$

Then

$$\left| f(x) - \sum_{i=0}^{k-1} \frac{(x-a)^i}{i!}\partial^i f(a)\right| \leq \frac{L}{k!}(x-a)^k,$$
$$\left| f(x) - \sum_{i=0}^{k} \frac{(x-a)^i}{i!}\partial^i f(a)\right| \leq \frac{M}{k!}(x-a)^k.$$

B.2 Taylor expansion in higher dimensions

Definition B.5. Let $f \in C^k(U, \mathbb{R}^p)$ for $U \subset \mathbb{R}^d$ open. Let $a \in U$. The *Taylor polynomial of order* k at the point a, written $T^k_{f,a}$, is given by

$$T^k_{f,a}(x) = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\leq k} \frac{1}{\alpha!}\partial^\alpha f(a)(x-a)^\alpha.$$

The *remainder of order* k at the point a is given by $R^k_{f,a}(x) = f(x) - T^k_{f,a}(x)$.

Lemma B.6. [DK10, Lemma 6.1] *Let $f \in C^k(U, \mathbb{R}^d)$. Then for $l \in \{0, 1, \ldots, k\}$ and $a, h \in \mathbb{R}^d$ and $t \in \mathbb{R}$ such that $a + th \in U$ we have*

$$\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}t^j} f(a+th) = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=j} \frac{h^\alpha}{\alpha!}\partial^\alpha f(a+th)$$

Theorem B.7 (Taylor's Formula)**.** [DK10, Theorem 6.2] *Let $f \in C^k(U, \mathbb{R}^p)$ for $U \subset \mathbb{R}^d$ being an open ball. Let $a \in U$. For all $l \in \{1, \ldots, k\}$ and $x \in U$*

$$f(x) = T^{l-1}_{f,a}(x) + \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=l} \frac{(x-a)^\alpha}{\alpha!}\int_0^1 \frac{(1-s)^{l-1}}{(l-1)!}\partial^\alpha f(a+s(x-a))\ \mathrm{d}s \tag{B.1}$$

$$= T^{l}_{f,a}(x) + \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=l} \frac{(x-a)^\alpha}{\alpha!}\int_0^1 \frac{(1-s)^{l-1}}{(l-1)!}\left[\partial^\alpha f(a+s(x-a)) - \partial^\alpha f(a)\right]\ \mathrm{d}s. \tag{B.2}$$

For $a, x \in U$ let us define

$$R^l_{f,a}(x) = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| = l} \frac{(x-a)^\alpha}{\alpha!} \int_0^1 \frac{(1-s)^{l-1}}{(l-1)!} \left(\partial^\alpha f(a + s(x-a)) - \partial^\alpha f(a) \right) \mathrm{d}s. \tag{B.3}$$

The map $U \times U \to \mathbb{R}$ given by $(a, x) \mapsto R^l_{f,a}(x)$ is in C^{k-l}, and for every compact $K \subset U$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|R^l_{f,a}(x)| \le \varepsilon |x-a|^l \qquad \text{for } x, a \in K \text{ and } |x-a| < \delta.$$

Moreover, for all $a \in U$ the map $R^l_{f,a} : U \to \mathbb{R}$ is in C^k and $\partial^\alpha R^l_{f,a}(a) = 0$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le l$.

Proof. Let g be the one-dimensional function given by $g(t) = f(a + t(x-a))$. Then by Theorem B.4

$$g(t) = \sum_{i=0}^{l-1} \frac{t^i}{i!} \frac{\mathrm{d}^i}{\mathrm{d}t^i} g(0) + \int_0^t \frac{(t-s)^{l-1}}{(l-1)!} \frac{\mathrm{d}^l}{\mathrm{d}s^l} g(s) \, \mathrm{d}s.$$

So that with Lemma B.6 one obtains (B.1) and (B.2). □

C Integration by parts

Theorem C.1. [Eva98, Appendix C.2, Theorem 2] *Let Ω be a bounded open set with C^1 boundary $\partial\Omega$. We write σ for the $d-1$ dimensional "surface" measure on $\partial\Omega$. For $f, g \in C(\overline{\Omega})$ which are differentiable on Ω, and $i \in \{1, \ldots, d\}$ we have*

$$\int_U f \partial_i g = -\int_U g \partial_i f + \int_{\partial U} f g \mathfrak{n}_i \, \mathrm{d}\sigma,$$

where $\mathfrak{n}(x)$ for $x \in \partial U$ is the outward pointing normal vector and $\mathfrak{n}_i$ its i-th component.

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