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From Damage to Delamination in Nonlinearly Elastic Materials at Small Strains

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Abstract

Brittle Griffith-type delamination of compounds is deduced by means of Γ-convergence from partial, isotropic damage of three-specimen-sandwich-structures by flattening the middle component to the thickness 0. The models used here allow for nonlinearly elastic materials at small strains and consider the processes to be unidirectional and rate-independent. The limit passage is performed via a double limit: first, we gain a delamination model involving the gradient of the delamination variable, which is essential to overcome the lack of a uniform coercivity arising from the passage from partial damage to delamination. Second, the delamination gradient is supressed. Noninterpenetration- and transmission-conditions along the interface are obtained.

1 Introduction

Delamination (or *debonding*) is one main reason for the macroscopic failure of compounds. Opposite, sometimes delamination is an intentional mechanism in engineering constructions designed for the efficient absorption of energy during impacts. In any case, reliable modelling of delamination is important and has recently received a considerable attention both in engineering and in mathematical communities. As many engineering contributions [All02, AC96, DBS02, Lad92] the present paper views delamination as the *damage* of interfaces. Using the ideas of continuum damage mechanics, the delamination along an interface $\Gamma_{\rm C}$ is modelled by an inner variable, the delamination variable $z : [0,T] \times \Gamma_{\rm C} \to [0,1]$, which reflects the current state of the bonding along $\Gamma_{\rm C}$, i.e. for $z(t,x) = 1$ the bonding is fully intact at $x \in \Gamma_{\rm C}$ at time $t \in [0,T]$, whereas for $z(t,x) = 0$ the bonding is completely broken. In [All02] it is suggested to understand interfaces as the limit of a thin medium, which links two constituents and which follows its own constitutive law. Such interface models have been exploited in [PS96a, PS96b] to study delamination in the framework of the adhesion models of Frémond, see e.g. [Fré88].

In the present work such a limit is rigorously performed: Starting from a sandwich-structure composed of three constituents of non-zero thickness, where the middle component is exposed to partial, isotropic damage, the delamination of two perfectly unbreakable specimen glued together with a breakable adhesive of thickness 0 is gained when flattening the thickness of the middle component to 0, see also Fig. 1. The damage models applied for this purpose where analyzed in [TM10]. The limit passage is mathematically performed via a double limit. The first limit models describe delamination with an energy functional involving the delamination gradient and they reflect transmission- and noninterpenetration conditions on the displacements u along the interface, namely

$$z[\![u]\!] = 0 \ \text{ and } \ [\![u \cdot \mathrm{n}_1]\!] \geq 0 \ \text{ a.e. on } \Gamma_{\rm C}\,, \tag{1.1}$$

where $[\![u]\!]$ is the jump of u across Γ_C and n_1 is the unit normal vector. At this point we emphasize that the noninterpenetration condition cannot be obtained from any constitutive relation in the damageable domain. Since the usage of the small strain tensor presumes infinitesimally small strains and hence excludes interpenetration in the bulk, this additional unilateral contact condition rather results from an anisotropic term in the stored energy density on the damageable domain, which involves $(\operatorname{tr} e)^-$, the negative part of the trace of the small strain tensor e.

The delamination gradient was also included in the models analyzed in [BBR08, BBR09]. Due to this term, the delamination variable can attain values between 0 and 1. This property differs from those of crack-models based on Griffith' fracture criterion [Gri21], as studied e.g. in [DMFT04, FL03, Gia05]. To overcome this discrepancy the gradient is suppressed in a second limit $\kappa \to 0$ and the delamination model discussed in [RSZ09] is obtained. In fact, Proposition 4.4 implies that z in this model only takes the values 0 or 1 for the initial datum $z_0 = 1$. Then $1 - z$ is the indicator function of the crack. Indeed, this model reflects Griffith' fracture criterion, since it expresses, that a crack expands as soon as the energy release is bigger than a critical value (the fracture toughness ϱ in (4.5)) and crack-healing is forbidden.

Both the damage and the delamination processes are considered to be *quasistatic* and hence can be analyzed using their so-called *energetic formulation*. Our general framework will solely be based on the hypothesis that the evolution is governed by a time-dependent *energy functional* $\mathcal{E}$ and a *dissipation potential* $\mathcal{R}$ being degree-1 positively homogeneous, which reflects the *rate-independence* of the process (i.e. invariance under any monotone rescaling of time). Both functionals are defined with respect to a suitable state space $\mathcal{Q}$, which is a Banach space in this work. The triple $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ is called a rate-independent system. A state $q = (u, z) \in \mathcal{U} \times \mathcal{Z} =: \mathcal{Q}$ is given by the *displacement* field u and the *inner variable* z that describes either damage or delamination. We assume that $\mathcal{R}$ involves only z, which distinguishes it as a "slow" variable while u is a "fast" variable. Within the energetic formulation of rate-independent processes one is interested in so-called *energetic solutions*, which are defined as follows:

Definition 1.1 (Energetic solution) *The process $q = (u, z) : [0, T] \to \mathcal{Q}$ is an energetic solution of the initial value problem given by $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ and the initial condition (u_0, z_0), if $q(0) = (u(0), z(0)) = (u_0, z_0)$, if $t \mapsto \partial_t \mathcal{E}(t, q(t)) \in L^1((0, T))$, if for all $t \in [0, T]$ we have $\mathcal{E}(t, q(t)) < \infty$ and if the global* stability inequality (1.2 S) *and the global* energy balance (1.2 E) *are satisfied for all $t \in [0, T]$:*

$$\text{for all } \tilde{q} \in \mathcal{Q}: \ \mathcal{E}(t, q(t)) \le \mathcal{E}(t, \tilde{q}) + \mathcal{R}(\tilde{z} - z(t)), \tag{1.2 S}$$

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{R}}(z, [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_\xi \mathcal{E}(\xi, q(\xi))\, \mathrm{d}\xi \tag{1.2 E}$$

with $\mathrm{Diss}_{\mathcal{R}}(z, [0, t]) := \sup \left\{ \sum_{j=1}^N \mathcal{R}(z(t_j) - z(t_{j-1})) \,|\, 0 = t_0 < t_1 < \ldots < t_N = t,\ N \in \mathbb{N} \right\}$.

For the limit passages we will apply the abstract result [MRS08, Theorem 3.1] for sequences of rate-independent systems, which generalizes the classical ideas of Γ-convergence to the rate-independent setting. While the classical Γ-convergence, see e.g. [DM93] ensures that minimizers of static functionals converge to minimizers of a limit functional, if the lim inf-inequality and the existence of a recovery sequence is given, these two properties are not sufficient to verify an analogous implication in the rate-independent setting. In order to guarantee that energetic solutions $q_h : [0,T] \to \mathcal{Q}$ of the approximating systems $(\mathcal{Q}, \mathcal{E}_h, \mathcal{R}_h)$ converge as $h \to 0$ to an energetic solution $q : [0,T] \to \mathcal{Q}$ of the limit system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ the properties (1.2) have to be maintained under convergence. The theorem [MRS08, Theorem 3.1], which guarantees this and which is the basis of our convergence results, is recalled in Theorem A.1 in the Appendix. In particular, the conservation of (1.2 S) can be verified by the construction of a so-called mutual recovery sequence, which must preserve the interplay of the displacements and the inner variable required by the specific form of the functionals.

In the present work the transmission condition in (1.1) makes the construction of the mutual recovery sequences extraordinarily difficult for both limit passages, since it requires a strong interaction of the displacements and the inner variables. For the first limit a reflection technique is applied to the displacements, see Section 3.2, and for the second limit a generalized Hardy's inequality is used, see Section 4.2.

Another difficulty lies in extracting the conditions (1.1) when passing from partial damage to delamination, since this entails a loss of coercivity: For the modeling of damage and delamination it is characteristic that the stored energy density links the unknowns (linearized strain tensor e, inner variable z) multiplicatively, e.g. as in $W(e,z) := z|e|^2$. Thus, the coercivity of the partial damage processes, i.e. $z \in (\varepsilon^\gamma, 1]$ with $\gamma > 0$, is lost as $\varepsilon \to 0$. Then, in general, regions with $z = 0$ isolating those with $z > 0$ from the Dirichlet boundary may occur, so that Korn's inequality does not hold. Due to this, e.g. in [BMR09] partial damage models result in a complete damage model containing no information about the displacements. Anyhow to deduce (1.1) we transform the damageable domains to a unit domain, see Fig. 1, and we use an ansatz ensuring that the limit z of a bounded sequence $(u_\varepsilon, z_\varepsilon)_{\varepsilon \in (0,\varepsilon_0]}$ is constant in the direction vanishing as $\varepsilon \to 0$, so that no isolated regions with $z > 0$ can occur.

In Section 2 the setup, tools and an existence result for the partial, isotropic damage models are introduced. In Section 3, a delamination model involving the delamination gradient is obtained as the Γ-limit of these damage models. Then, in Section 4, it is shown that the gradient delamination models Γ-converge to a model describing Griffith-type delamination, which no longer involves the (artificial) delamination gradient. Finally, in Section 5 the results are merged to a simultaneous convergence.

Remark 1.2 In [Tho10] the noninterpenetration condition from (1.1) was deduced from the term e_{11}^-, which involves only the first component of the strain tensor e, and not from the full trace $(\operatorname{tr} e)^-$, as it is done in this work in order to get closer to engineering models. Moreover, the transmission condition from (1.1) was deduced under the assumption that the damage component of states in sublevels of $\mathcal{E}$ is bounded in $W^{1,r}(\Omega_{\mathrm{D}})$ for some $r > d$, which implies the compact embedding $W^{1,r}(\Omega_{\mathrm{D}}) \Subset \mathrm{C}(\overline{\Omega_{\mathrm{D}}})$. In this work it was possible to generalize the results to $r \in (1,\infty)$. Hence, the limit passage $\varepsilon \to 0$ can be done for all $r \in (1,\infty)$ and $p \in (1,\infty)$, which satisfy a certain relation, see (3.12). Here, $W^{1,r}(\Omega_{\mathrm{D}})$ is the Sobolev space for the damage variable and $W^{1,p}(\Omega, \mathbb{R}^d)$ denotes the Sobolev space for the displacements. Relation (3.12) even admits the exponents $r = 2$ and $p = 2$ for $d = 3$. However, for technical reasons the second limit passage $\kappa \to 0$ is carried out as in [Tho10] for $p > d$. $\square$

2 The Damage Models, Assumptions and Tools

For all $\varepsilon \in (0, \varepsilon_0]$ we consider a domain $\Omega := (-L, L) \times (-H, H)^{d-1}$, which is the union of the three cuboid-type Lipschitz-domains $\Omega_-^\varepsilon := (-L, -\varepsilon) \times \Gamma_\mathrm{C}$, $\Omega_+^\varepsilon := (\varepsilon, L) \times \Gamma_\mathrm{C}$ for $L > 1$, $\Omega_\mathrm{D}^\varepsilon := (-\varepsilon, \varepsilon) \times \Gamma_\mathrm{C} \subset \mathbb{R}^d$ with the interfaces $\Gamma_\pm^\varepsilon := \{\pm\varepsilon\} \times \Gamma_\mathrm{C} \subset \mathbb{R}^{d-1}$ and $\Gamma_\mathrm{C} := (-H, H)^{d-1}$, see also Fig. 1a. We assume that the domains $\Omega_\pm^\varepsilon$ are occupied by a nonlinearly elastic material which is damage-resistive, whereas $\Omega_\mathrm{D}^\varepsilon$ refers to a material undergoing a rate-independent damage process leading to partial damage of that specimen. This damage process is assumed to be driven by slow time-dependent external loadings induced by time-dependent Dirichlet conditions on parts of the outer boundary $\Gamma_\mathrm{Dir} = \{L, -L\} \times \Gamma_\mathrm{C}$ with $\mathcal{L}^{d-1}(\Gamma_\mathrm{Dir}) > 0$. Throughout this paper $\mathcal{L}^m(A)$ denotes the m-dimensional Lebesgue-measure of the set $A \subset \mathbb{R}^m$ with $m = (d-2)$, $(d-1)$ or d.

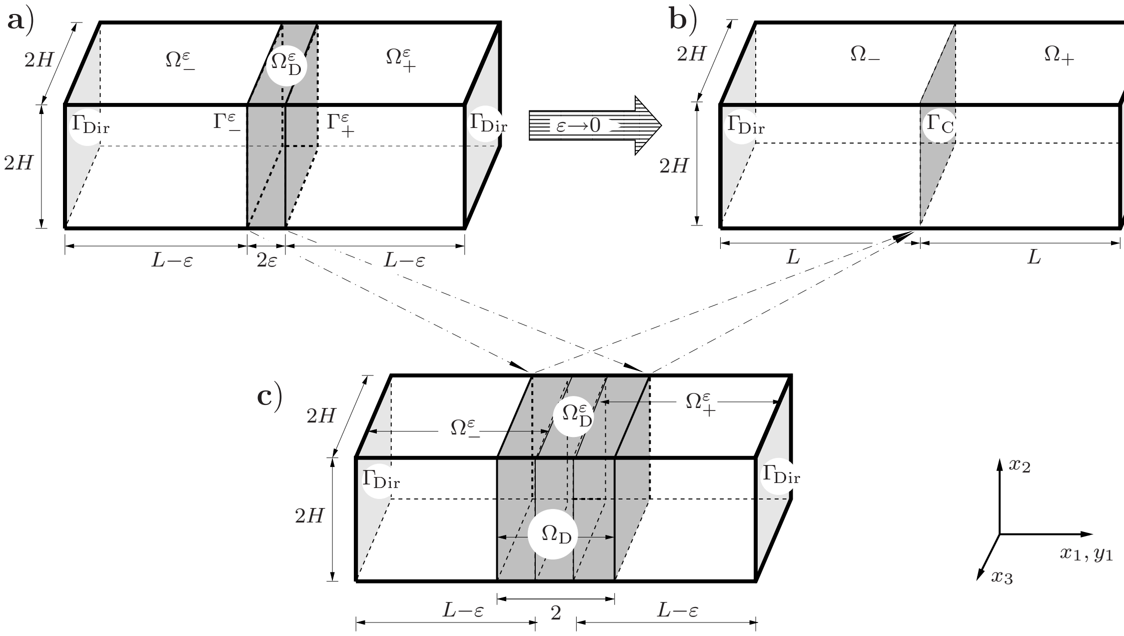

Fig.1. Geometry and notation of the cuboid-type domains and surfaces used.

a) Domain with a thin subdomain $\Omega_\mathrm{D}^\varepsilon$ undergoing possible damage. Loading is realized through Dirichlet boundary conditions prescribed on the sides Γ_Dir.

b) Domain obtained for $\varepsilon=0$ with an interface Γ_C undergoing possible delamination with a subsequent unilateral Signorini condition.

c) Setup for the analysis: the original, ε-dependent domains Ω_-^ε, Ω_+^ε and $\Omega_\mathrm{D}^\varepsilon$ are used for the displacements, whereas the auxiliary transformed damageable domain Ω_D of fixed size is used for the damage/delamination variable.

For $q = (u, z)$ the energy of the compound Ω, see Fig. 1a, is given by:

$$\widetilde{\mathcal{E}}_\varepsilon^\kappa(t, u, z) := \int_{\Omega_-^\varepsilon \cup \Omega_+^\varepsilon} W(e(u+g(t)))\,\mathrm{d}x + \int_{\Omega_\mathrm{D}^\varepsilon} \Big(W_\mathrm{D}(e(u+g(t)), z) + \tfrac{\kappa}{r\varepsilon}|\nabla z|^r + \delta_{[\varepsilon^\gamma, 1]}(z)\Big)\,\mathrm{d}x, \qquad (2.1)$$

where $r \in (1, \infty)$ and $\varepsilon, \kappa > 0$. Since we are going to perform the limit passages $\varepsilon, \kappa \to 0$, we restrict our analysis to small values $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \in (0, \kappa_0]$ for constants $0 < \varepsilon_0 \ll 1$, $0 < \kappa_0 \ll 1$. For the stored elastic energy density $W_\mathrm{D} : \mathbb{R}^{d\times d}_\mathrm{sym} \times [0, 1] \to \mathbb{R}$ of the damageable region we make a specific ansatz for all $e \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $z \in [0,1]$, namely

$$W_{\mathrm{D}}(e,z) := z\widetilde{W}(e) + \varphi(\operatorname{tr} e)\,, \tag{2.2}$$

where $\operatorname{tr}(e) = \sum_{i=1}^{d} e_{ii}$ and where $\varphi : \mathbb{R} \to [0,\infty)$ is convex and satisfies

$$\tilde{c}(a^-)^{\hat{p}} \le \varphi(a) \le c\big((a^-)^{\hat{p}-1} + 1\big)\, a^- \tag{2.3}$$

with constants $\tilde{c}$, $c > 0$ and an exponent $\hat{p} \in (1,p]$ and $a^- := \max\{0,-a\}$. Thus, φ in (2.2) only takes into account the negative part of $\operatorname{tr} e$ and hence punishes compression, which may trigger less damage than tension. More importantly, the contribution of $\varphi(\operatorname{tr} e)$ to W_{D} in (2.2) guarantees that even the totally damaged material still resists compression. As an example for (2.2) one may consider an isotropic material coupled with damage as follows

$$W_{\mathrm{D}}(e,z) := z\big(\mu_1|e|^2 + \mu_2|e|^p + \tfrac{\lambda}{2}|(\operatorname{tr} e)^+|^2\big) + \tfrac{\lambda}{2}|(\operatorname{tr} e)^-|^2\,,$$

where λ, $\mu > 0$ are the Lamé constants. Then $\hat{p} = 2$ and $\tilde{c} = c = \lambda/2$ in (2.3). The properties of W and $\widetilde{W}$ are explained in detail in Section 2.1.

In (2.1), $u : \Omega \to \mathbb{R}^d$ denotes the unknown displacement and $e(w) := \frac{1}{2}(\nabla w + \nabla w^\top)$ the linearized strain tensor for all $w : \Omega \to \mathbb{R}^d$. Thereby u satisfies homogeneous Dirichlet conditions on Γ_{Dir} and the given displacement $g(t) = g(t,\cdot) : \Omega \to \mathbb{R}^d$ with $t \in [0,T]$ incorporates the time-dependent Dirichlet condition. Its properties are specified in Subsection 2.1. Moreover, $z : [0,T] \times \Omega^\varepsilon_{\mathrm{D}} \to [0,1]$ denotes the damage variable. The functional $\widetilde{\mathcal{E}}^\kappa_\varepsilon$ allows for partial damage only, which is ensured by the indicator function $\delta_{[\varepsilon^\gamma,1]}$ of the interval $[\varepsilon^\gamma,1]$ for $\gamma>0$, i.e. $\delta_{[\varepsilon^\gamma,1]}(z)=0$ if $\varepsilon^\gamma \le z(x) \le 1$ for a.e. $x \in \Omega^\varepsilon_{\mathrm{D}}$ and $\delta_{[\varepsilon^\gamma,1]}(z) = \infty$ otherwise. However $\delta_{[\varepsilon^\gamma,1]}$ prevents total damage for each $\varepsilon \in (0,\varepsilon_0]$, but it will allow for complete delamination in the limit $\varepsilon = 0$.

We assume that the damage process is unidirectional, i.e. that healing of the material is impossible, meaning $\dot{z} \le 0$, where $\dot{z} = \partial_t z$ is the partial derivative with respect to time. The evolution of the damage variable is described by the dissipation potential

$$\widetilde{\mathcal{R}}_\varepsilon(v) := \begin{cases} \int_{\Omega^\varepsilon_{\mathrm{D}}} -\frac{\varrho}{\varepsilon} v \,\mathrm{d}x & \text{if } v \le 0 \text{ a.e. on } \Omega^\varepsilon_{\mathrm{D}}, \\ \infty & \text{otherwise,} \end{cases} \tag{2.4}$$

for a constant $\varrho > 0$ and $v = \dot{z}$.

2.1 General Assumptions and Existence Result

We now state general assumptions on the densities W, $\widetilde{W}$ and the given data, and therewith deduce the existence of energetic solutions to the model given by $\widetilde{\mathcal{E}}^\kappa_\varepsilon$ and $\widetilde{\mathcal{R}}_\varepsilon$.

We assume that the Dirichlet data satisfy

$$\left.\begin{array}{l} g \in \mathrm{C}^1([0,T], W^{1,p}(\Omega,\mathbb{R}^d)), \\ \operatorname{supp} g(t) \cap \overline{\Omega^{\varepsilon_0}_{\mathrm{D}}} = \emptyset \text{ for all } t \in [0,T] \end{array}\right\} \tag{2.5}$$

and we set $\hat{c}_g := \|g\|_{\mathrm{C}^1([0,T],W^{1,p}(\Omega,\mathbb{R}^d))}$. Note that the second assumption in (2.5) leads to $\operatorname{supp} g(t) \cap \overline{\Omega^\varepsilon_{\mathrm{D}}} = \emptyset$ even for all $\varepsilon \in (0,\varepsilon_0]$.

Furthermore we make the following hypotheses on the energy densities $W : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}$, $\widetilde{W} : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}$ of the damage-resistive and of the damageable materials:

(2.6a) **Convexity:** $W, \widetilde{W} : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}$ strictly convex.

(2.6b) **Coercivity:** $\exists\, p \in (1,\infty),\ c, \tilde{c}, \tilde{C} > 0\, \forall e,\ \hat{e} \in \mathbb{R}^{d\times d}_{\rm sym}$:
$$c|e|^p \le W(e) \le \tilde{c}(|e|^p + \tilde{C})\,, \quad c|e|^p \le \widetilde{W}(e) \le \tilde{c}(|e|^p + \tilde{C})\,.$$

(2.6c) **Continuity of the stresses:** $\exists\, c, C > 0\ \forall\, e,\ \hat{e} \in \mathbb{R}^{d\times d}_{\rm sym}$:
$$|\partial_e W(e) - \partial_e W(\hat{e})| \le C(c + |e|^{p-1} + |\hat{e}|^{p-1})\, |e - \hat{e}|\,.$$

As a direct consequence of (2.6a, b) one obtains, see [Dac00, Theorem 2.31],

(2.6d) **Continuity:** $W,\ \widetilde{W} : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}$ continuously.

Moreover, (2.6a, b) imply the following stress control for the densities

(2.6e) **Stress control:** $\exists\, c,\ C > 0\, \forall e,\ \hat{e} \in \mathbb{R}^{d\times d}_{\rm sym}$:
$$|\partial_e W(e)| \le c(|\partial_e W(e)|^{p-1} + C)\,, \quad |\partial_e \widetilde{W}(e)| \le c(|\partial_e \widetilde{W}(e)|^{p-1} + C)\,.$$

In view of (2.2) we realize that the composed density

$$\overline{W}(x,e,z) := \begin{cases} W(e) & \text{if } x \in \Omega^\varepsilon_- \cup \Omega^\varepsilon_+ \\ W_{\rm D}(e,z) & \text{if } x \in \Omega_{\rm D} \end{cases} \tag{2.7}$$

also satisfies (2.6a-e) with constants that depend on ε and

(2.6f) **Monotonicity:** $\forall \varepsilon \in (0,\varepsilon_0]\ \exists\, K > 0,\ \widetilde{K} \ge 0\ \forall e \in \mathbb{R}^{d\times d},\ \varepsilon^\gamma \le z \le \tilde{z} \le 1$:
$$\overline{W}(e,z) \le \overline{W}(e,\tilde{z}) \le K(\overline{W}(e,z) + \widetilde{K}).$$

This is a property of partial damage. Due to (2.6b) we introduce the spaces

$$\mathcal{U}_{\rm D} := \{u \in W^{1,p}(\Omega,\mathbb{R}^d)\,|\, u = 0 \text{ on } \Gamma_{\rm Dir}\}\,, \quad \mathcal{Z}_\varepsilon := W^{1,r}(\Omega^\varepsilon_{\rm D})\,, \quad \mathcal{Q}_\varepsilon := \mathcal{U}_{\rm D} \times \mathcal{Z}_\varepsilon \tag{2.8}$$

and $\widetilde{\mathcal{S}}^\kappa_\varepsilon(t) := \{q \in \mathcal{Q}_\varepsilon\,|\, \widetilde{\mathcal{E}}^\kappa_\varepsilon(t,q) < \infty,\ \widetilde{\mathcal{E}}^\kappa_\varepsilon(t,q) \le \widetilde{\mathcal{E}}^\kappa_\varepsilon(t,\tilde{q}) + \widetilde{\mathcal{R}}_\varepsilon(\tilde{z} - z) \text{ for all } \tilde{q} \in \mathcal{Q}_\varepsilon\}$ denote the stable sets at time t.

For all fixed $\varepsilon \in (0,\varepsilon_0]$, $\kappa \in (0,\kappa_0]$ the rate-independent systems $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$ thus fit to the setting studied in [TM10] so that the existence of energetic solutions is guaranteed.

Proposition 2.1 (Energetic solutions of $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$, [TM10, Theorem 3.1]) *For all $\varepsilon \in (0,\varepsilon_0]$ and $\kappa \in (0,\kappa_0]$ fixed, let the rate-independent system $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$ be defined via (2.1)-(2.5). Let $p,\ r \in (1,\infty)$. Then, for $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$ and for any initial state $q_0 \in \widetilde{\mathcal{S}}^\kappa_\varepsilon(0)$, there exists an energetic solution q of the initial-value problem $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon, q_0)$.*

2.2 The Damage Model in a Fixed State Space

First, $\kappa \in (0,\kappa_0]$ remains fixed. As $\varepsilon \to 0$ the d-dimensional domain $\Omega^\varepsilon_{\rm D}$ shrinks to the $(d{-}1)$-dimensional interface $\Gamma_{\rm C}$ between the domains $\Omega_\pm$, see Fig. 1a, b, and we want to show that $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}_\varepsilon^\kappa, \widetilde{\mathcal{R}}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ converges to a rate-independent process describing the delamination along the interface. Thus, it is necessary to reformulate the ε-problems in a fixed state space $\mathcal{Q}$. In particular, for all $\varepsilon \in (0, \varepsilon_0]$, we use damage variables that are defined on a fixed domain $\Omega_{\mathrm{D}} = (-1, 1) \times \Gamma_{\mathrm{C}}$, see Fig. 1a, c. Hence, from now on we consider $z : \Omega_{\mathrm{D}} \to [0, 1]$ and the energy functionals $\widetilde{\mathcal{E}}_\varepsilon^\kappa$ have to be adapted. This is realized with the following mapping:

$$T_\varepsilon : \Omega_{\mathrm{D}} \to \Omega_{\mathrm{D}}^\varepsilon,\, T_\varepsilon y = (\varepsilon y_1, s) = x \in \Omega_{\mathrm{D}}^\varepsilon \text{ for } y = (y_1, s) \in \Omega_{\mathrm{D}}, \tag{2.9}$$

with $s = (x_2, \ldots, x_d) \in \Gamma_{\mathrm{C}}$. For all $\varepsilon \in (0, \varepsilon_0]$ this transformation is welldefined, continuous and and invertible. Then we introduce the following transformation:

$$\Pi_\varepsilon : L^1(\Omega_{\mathrm{D}}^\varepsilon) \to L^1(\Omega_{\mathrm{D}}),\ \tilde{z} \mapsto \tilde{z} \circ T_\varepsilon\,. \tag{2.10}$$

In view of (2.9) and (2.10) we obtain that the gradient of $\tilde{z}$ transforms as follows:

$$\nabla_x \tilde{z}(x) = \nabla_y \Pi_\varepsilon \tilde{z}(y) \nabla_x y = \big(\tfrac{1}{\varepsilon}\partial_{y_1}\Pi_\varepsilon \tilde{z}(y), (\nabla_s \Pi_\varepsilon \tilde{z}(y))^\top\big)^\top =: {}_\varepsilon\nabla \Pi_\varepsilon \tilde{z}(y)\,, \tag{2.11}$$

where we used $\nabla_s := (\partial_{y_2}, \ldots, \partial_{y_d})^\top$.

We are now in a position to define a fixed state space by

$$\mathcal{U} := \{u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d) \,|\, u = 0 \text{ on } \Gamma_{\mathrm{Dir}}\}\,,\ \mathcal{Z} := L^\infty(\Omega_{\mathrm{D}})\,,\ \mathcal{Q} := \mathcal{U} \times \mathcal{Z}\,. \tag{2.12}$$

With $\mathcal{U}_{\mathrm{D}}$ as in (2.8) the state space for the approximating problems is given by

$$\mathcal{Z}_{\mathrm{D}} := W^{1,r}(\Omega_{\mathrm{D}}) \text{ with } r \in (1, \infty)\,, \quad \mathcal{Q}_{\mathrm{D}} := \mathcal{U}_{\mathrm{D}} \times \mathcal{Z}_{\mathrm{D}}\,. \tag{2.13}$$

Therewith we introduce the extended energy functionals $\mathcal{E}_\varepsilon^\kappa : [0, T]\times\mathcal{Q} \to \mathbb{R}_\infty$

$$\begin{gathered}\mathcal{E}_\varepsilon^\kappa(t, q) := \begin{cases} \Pi\mathcal{E}_\varepsilon^\kappa(t, q) & \text{if } q = (u, z) \in \mathcal{Q}_{\mathrm{D}}, \\ \infty & \text{if } q \in \mathcal{Q}\backslash\mathcal{Q}_{\mathrm{D}}, \end{cases} \qquad \text{where} \\ \Pi\mathcal{E}_\varepsilon^\kappa(t, u, z) := \int\limits_{\Omega_-^\varepsilon\cup\Omega_+^\varepsilon} W(e(u{+}g(t))\mathrm{d}x + \int\limits_{\Omega_{\mathrm{D}}^\varepsilon} W_{\mathrm{D}}(e(u), \Pi_\varepsilon^{-1}z)\mathrm{d}x + \int\limits_{\Omega_{\mathrm{D}}} \Big(\tfrac{\kappa}{r}|{}_\varepsilon\nabla z|^r + \delta_{[\varepsilon^\gamma,1]}(z)\Big)\,\mathrm{d}y.\end{gathered} \tag{2.14}$$

Here we used that $\operatorname{supp} g(t) \cap \Omega_{\mathrm{D}}^\varepsilon = 0$ for all $\varepsilon \in (0, \varepsilon_0]$ and all $t \in [0, T]$. Compared to $\widetilde{\mathcal{E}}_\varepsilon^\kappa$ in (2.1) the functional $\Pi\mathcal{E}_\varepsilon^\kappa$ allows for $z : \Omega_{\mathrm{D}} \to [0, 1]$. Therefore one has to use $\Pi_\varepsilon^{-1}z$ in in the second integral. Only the integral containing the damage gradient is transformed from $\Omega_{\mathrm{D}}^\varepsilon$ to Ω_{D}. This requires to use ${}_\varepsilon\nabla z$ from (2.11) and involves a factor ε, which cancels out $1/\varepsilon$ in (2.1). Additionally we used that $\varepsilon\delta_{[\varepsilon^\gamma,1]}(z) = \delta_{[\varepsilon^\gamma,1]}(z)$. In view of the transformations (2.9), (2.10) we note that

$$\varepsilon^\gamma \le z \le 1 \text{ a.e. on } \Omega_{\mathrm{D}} \quad \text{is equivalent to} \quad \varepsilon^\gamma \le \Pi_\varepsilon^{-1}z \le 1 \text{ a.e. on } \Omega_{\mathrm{D}}^\varepsilon\,. \tag{2.15}$$

As we now use the state space $\mathcal{Q}$ we also transform the dissipation potential (2.4) leading to the potential $\mathcal{R} : \mathcal{Z} \to [0, \infty]$ with

$$\mathcal{R}(v) := \begin{cases} \int_{\Omega_{\mathrm{D}}} -\varrho\, v(y)\,\mathrm{d}y & \text{if } v \le 0 \text{ a.e. on } \Omega_{\mathrm{D}}, \\ \infty & \text{else.} \end{cases} \tag{2.16}$$

Remark 2.2 *Since $\varrho > 0$ we find the coercivity $\mathcal{R}(v) \geq \varrho\|v\|_{L^1(\Omega_\mathrm{D})}$. Moreover, $\mathcal{R}: L^1(\Omega_\mathrm{D}) \to [0,\infty]$ is convex and both weakly and strongly lower semicontinuous. However, the lack of strong upper semicontinuity makes the theory technically difficult.*

For all $t\in[0,T]$ we now define the stable sets of the transformed problems by

$$\mathcal{S}^\kappa_\varepsilon(t) := \{q\in\mathcal{Q}\,|\,\mathcal{E}^\kappa_\varepsilon(t,q)<\infty,\ \mathcal{E}^\kappa_\varepsilon(t,q)\leq\mathcal{E}^\kappa_\varepsilon(t,\tilde q)+\mathcal{R}(\tilde z-z) \text{ for all } \tilde q\in\mathcal{Q}\}.$$

We can rewrite the rate-independent systems $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$ by the equivalent systems $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$. It remains to transfer the existence result stated in Proposition 2.1 for $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \mathcal{R}_\varepsilon)$ to $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$. For this we first show that $\partial_t\mathcal{E}^\kappa_\varepsilon(t,q)$ is well-defined for all $q\in\mathcal{Q}$ if $\mathcal{E}^\kappa_\varepsilon(t_*,q)<\infty$ for some $t_*\in[0,T]$.

Proposition 2.3 (Well-posedness of $\partial_t\mathcal{E}^\kappa_\varepsilon$) *Keep $\varepsilon\in(0,\varepsilon_0], \kappa\in(0,\kappa_0]$ fixed. Let $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ be given by (2.12), (2.14) and (2.16) so that (2.5) and (2.6) hold with $p, r\in(1,\infty)$. Then, for all $(t_q,q)\in[0,T]\times\mathcal{Q}$ with $\mathcal{E}^\kappa_\varepsilon(t_*,q)<\infty$ it is $\mathcal{E}^\kappa_\varepsilon(\cdot,q)\in\mathrm{C}^1([0,T])$ with*

$$\partial_t\mathcal{E}^\kappa_\varepsilon(t,q) = \int_{\Omega_-^{\varepsilon_0}\cup\Omega_+^{\varepsilon_0}} \partial_e W(e(u+g(t))):\partial_t e(g(t))\,\mathrm{d}x\,. \tag{2.17}$$

Proof: Because of (2.1), (2.14) and (2.10) it is $\mathcal{E}^\kappa_\varepsilon(t_*,u,z) = \widetilde{\mathcal{E}}^\kappa_\varepsilon(t_*,u,\Pi_\varepsilon^{-1}z)<\infty$. Since $\int_{\Omega_\mathrm{D}}\frac{\kappa}{r}|_\varepsilon\nabla z|^r\,\mathrm{d}y$ with $z\in\mathcal{Z}_\mathrm{D}$ does not depend on $t\in[0,T]$ we conclude that $\partial_t\mathcal{E}^\varepsilon_\kappa(t,u,z) = \partial_t\widetilde{\mathcal{E}}^\varepsilon_\kappa(t,u,\Pi_\varepsilon^{-1}z)$, which is given by formula (2.17). ■

This result is used to adapt Proposition 2.1 to the transformed functionals.

Proposition 2.4 (Energetic solutions of $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$) *For all $\varepsilon\in(0,\varepsilon_0]$, $\kappa\in(0,\kappa_0]$ fixed, let $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ be defined via (2.12), (2.14) and (2.16) such that (2.5) and (2.6) hold with $p,r\in(1,\infty)$. Then, for $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ and for any initial state $q_0\in\mathcal{S}^\kappa_\varepsilon(0)$, there exists an energetic solution $q:[0,T]\to\mathcal{Q}$ of the initial value problem $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R}, q_0)$.*

Proof: Consider $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ with the initial state $q_0=(u_0,z_0)\in\mathcal{S}^\kappa_\varepsilon(0)$. By (2.14) and (2.16) we find that $(u_0,\Pi_\varepsilon^{-1}z_0)\in\widetilde{\mathcal{S}}^\kappa_\varepsilon(0)$. Then Proposition 2.1 states the existence of an energetic solution $q=(u,z):[0,T]\to\mathcal{Q}_\varepsilon$ of $(\mathcal{Q}_\varepsilon, \widetilde{\mathcal{E}}^\kappa_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon)$ with $(u(0),z(0)) = (u_0,\Pi_\varepsilon^{-1}z_0)$. We want to show that $(u,\Pi_\varepsilon z)$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R}, q_0)$. To verify that $(u(t),\Pi_\varepsilon z(t))\in\mathcal{S}^\kappa_\varepsilon(t)$ we use that $(u(t),z(t))\in\widetilde{\mathcal{S}}^\kappa_\varepsilon(t)$. The bijectivity of $\Pi_\varepsilon\colon\mathcal{Z}_\varepsilon\to\mathcal{Z}_\mathrm{D}$ and (2.15) imply that $\widetilde{\mathcal{E}}^\kappa_\varepsilon(t,\tilde u,\Pi_\varepsilon\tilde z)<\infty$ since $\mathcal{E}^\kappa_\varepsilon(t,\tilde u,\tilde z)<\infty$. Applying Π_ε and transforming the integrals in stability condition (1.2 S) yields the stability of $(u(t),\Pi_\varepsilon z(t))$, i.e. $\mathcal{E}^\kappa_\varepsilon(t,u(t),\Pi_\varepsilon z(t))\leq\mathcal{E}^\kappa_\varepsilon(t,\tilde u,\Pi_\varepsilon\tilde z)+\mathcal{R}(\Pi_\varepsilon\tilde z-\Pi_\varepsilon z(t))$. The energy balance (1.2 E) follows directly from $\mathrm{Diss}_\mathcal{R}(\Pi_\varepsilon z,[0,t]) = \mathrm{Diss}_{\widetilde{\mathcal{R}}_\varepsilon}(z,[0,t])$ and Proposition 2.3, since $\partial_t\mathcal{E}^\kappa_\varepsilon(t,u(t),\Pi_\varepsilon z(t)) = \partial_t\widetilde{\mathcal{E}}^\kappa_\varepsilon(t,u(t),z(t))$. ■

2.3 The Topologies $\mathcal{T}$, $\mathcal{T}_T$ and a uniform Korn's Inequality

In the following we specify a suitable topology on the fixed state space $\mathcal{Q}$, which allows us to show that a subsequence of energetic solutions of $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ converges to an energetic solution of the limit system as $\varepsilon\to0$ and as $\kappa\to0$ respectively.

For the analysis we will consider sequences of systems $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})_{\varepsilon\in(0,\varepsilon_0]}$ and sequences $(t_\varepsilon, q_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]} \subset [0,T]\times\mathcal{Q}$. The notation $\varepsilon \in (0,\varepsilon_0]$ always stands for countably many indices $\varepsilon\in(0,\varepsilon_0]$ satisfying $\varepsilon\to 0$. The indications $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})_{\kappa\in(0,\kappa_0]}$ and $(q_\kappa)_{\kappa\in(0,\kappa_0]}$ have to be understood similarly.

Since $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, u_\varepsilon, z_\varepsilon) \le E$ for some $E\in[0,\infty)$ implies that $\|z_\varepsilon\|_{L^\infty(\Omega_\mathrm{D})} \le 1$, a suitable topology on $\mathcal{Z} = L^\infty(\Omega_\mathrm{D})$ is the weak*-topology of $L^\infty(\Omega_\mathrm{D})$. In view of (2.14) and (2.6b) we obtain that $\|e(u_\varepsilon + g(t_\varepsilon))\|_{L^p(\Omega^\varepsilon_-\cup\Omega^\varepsilon_+,\mathbb{R}^{d\times d})} \le E$. By the triangle inequality, assumption (2.6) and Korn's inequality on each of the domains $\Omega^\varepsilon_-\cup\Omega^\varepsilon_+$ we find a constant $\tilde{E}$ such that $\|u_\varepsilon\|_{W^{1,p}(\Omega^\varepsilon_-\cup\Omega^\varepsilon_+,\mathbb{R}^d)} \le \tilde{E}$, provided that the constants in Korn's inequality are uniformly bounded, which is ensured below. Therefore the convergence of a sequence $(u_\varepsilon, z_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ to a limit (u,z) has to be understood as follows

$$(u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{T}} (u,z) \quad \Leftrightarrow \quad \begin{cases} u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega^\nu_-\cup\Omega^\nu_+,\mathbb{R}^d) \text{ for all } \nu\in(0,\varepsilon_0], \\ z_\varepsilon \overset{*}{\rightharpoonup} z \text{ in } L^\infty(\Omega_\mathrm{D}). \end{cases} \tag{2.18}$$

With the functions $u_\varepsilon(x) = \tanh(x_1/\varepsilon)$ one can see that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega^\nu_-\cup\Omega^\nu_+)$ for all $\nu\in(0,\varepsilon_0]$ does not imply $u_\varepsilon\rightharpoonup u$ in $W^{1,p}(\Omega_-\cup\Omega_+)$.

To specify the convergence of sequences of pairs $(t_\varepsilon, q_\varepsilon)\in[0,T]\times\mathcal{Q}$ we define

$$(t_\varepsilon, q_\varepsilon) \xrightarrow{\mathcal{T}_T} (t,q) \quad \Leftrightarrow \quad t_\varepsilon \to t \quad \text{and} \quad q_\varepsilon \xrightarrow{\mathcal{T}} q. \tag{2.19}$$

As already mentioned a uniform Korn's inequality is required for the domains $\Omega^\varepsilon_-\cup\Omega^\varepsilon_+$.

Theorem 2.5 (Korn's inequality for a family of domains) *For all $0<\varepsilon\le\varepsilon_0$ let $\Omega^\varepsilon_\pm\subset\Omega_\pm$ be the Lipschitz domains depicted in Fig.1a and let $p\in(1,\infty)$. Then there is a constant $c_\mathcal{K}>0$, such that for all $0<\varepsilon\le\varepsilon_0$ and all $v\in W^{1,p}(\Omega_\pm,\mathbb{R}^d)$ with $v=0$ on Γ_Dir in the trace sense we have*

$$\|v\|_{W^{1,p}(\Omega^\varepsilon_\pm,\mathbb{R}^d)} \le c_\mathcal{K}\|e(v)\|_{L^p(\Omega^\varepsilon_\pm,\mathbb{R}^{d\times d})}\,. \tag{2.20}$$

Proof: It suffices to prove the result for Ω^ε_+ and Ω^ε_- separately. We restrict ourselves to Ω^ε_+, the proof for Ω^ε_- is analogous.

We transform $\Omega^\varepsilon_+ = (\varepsilon, L)\times\Gamma_\mathrm{C}$ into $\Omega_+ = (0,L)\times\Gamma_\mathrm{C}$ via the invertible mapping

$$\tau_\varepsilon : \Omega_+ \to \Omega^\varepsilon_+,\ (y_1, s)\mapsto(\varepsilon+\alpha(\varepsilon)y_1, s)\,,\ \text{where } \alpha(\varepsilon) = (1-\varepsilon/L)\,. \tag{2.21}$$

For $v_\varepsilon := v\circ\tau_\varepsilon \in W^{1,p}(\Omega_+,\mathbb{R}^d)$ we obtain that

$$\nabla_y v_\varepsilon(y) = \nabla_x v(\tau_\varepsilon(y))\nabla_y\tau_\varepsilon(y) \quad \text{and} \quad \nabla_x v(x) = \nabla_y v_\varepsilon(\tau_\varepsilon^{-1}(x))\,\nabla_x\tau_\varepsilon^{-1}(x)\,, \tag{2.22}$$

where $\nabla_y\tau_\varepsilon = \mathrm{diag}(\alpha(\varepsilon),1,\ldots,1)$, $y=(y_1,s)\in\Omega_+$ and $x=(x_1,s)\in\Omega^\varepsilon_+$ with $x_1 = \varepsilon+\alpha(\varepsilon)y_1$.

Using these relations and exploiting Korn's inequality on Ω_+ results in a uniform Korn's inequality for all $\varepsilon\in(0,\varepsilon_0]$:

$$\begin{aligned} \|v\|^p_{W^{1,p}(\Omega^\varepsilon_+)} &= \|v\|^p_{L^p(\Omega^\varepsilon_+)} + \|\nabla_x v\|^p_{L^p(\Omega^\varepsilon_+)} = \alpha(\varepsilon)\big(\|v_\varepsilon\|^p_{L^p(\Omega_+)} + \|\nabla_y v_\varepsilon\nabla_x\tau_\varepsilon^{-1}\|^p_{L^p(\Omega_+)}\big) \\ &\le \alpha(\varepsilon)^{-p+1}\big(\|v_\varepsilon\|^p_{L^p(\Omega_+)} + \|\nabla_y v_\varepsilon\|^p_{L^p(\Omega_+)}\big) \le \alpha(\varepsilon_0)^{-p+1}C^p_\mathcal{K}\|e(v_\varepsilon)\|^p_{L^p(\Omega_+)} \\ &\le \alpha(\varepsilon_0)^{-p}C^p_\mathcal{K}\|e(v)\|^p_{L^p(\Omega^\varepsilon_+)}\,. \end{aligned}$$

■

3 The first Γ-limit: Gradient Delamination

Our aim for this section is to show that $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})_{\varepsilon\in(0,\varepsilon_0]}$ Γ-converges to the limit system $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})$ as $\varepsilon\to 0$, see Fig. 1b, where $\mathcal{E}^\kappa:[0,T]\times\mathcal{Q}\to\mathbb{R}_\infty$ is given by

$$\mathcal{E}^\kappa(t,q):=\begin{cases}\displaystyle\int_{\Omega_-\cup\Omega_+} W(e(u{+}g(t)))\,\mathrm{d}x+\int_{\Omega_\mathrm{D}}\Big(\tfrac{\kappa}{r}|\nabla z|^r{+}\delta_{[0,1]}(z)\Big)\,\mathrm{d}y & \text{if } q{=}(u,z){\in}\mathcal{Q}_\mathrm{C},\\ \infty & \text{if } q\in\mathcal{Q}\backslash\mathcal{Q}_\mathrm{C}\,,\end{cases}\tag{3.1}$$

$$\mathcal{Z}_\mathrm{C}:=\{z\in W^{1,r}(\Omega_\mathrm{D})\,|\,\partial_{y_1}z=0\}\ \text{ with } r\in(1,\infty)\,,\tag{3.2}$$

$$\mathcal{Q}_\mathrm{C}:=\big\{q=(u,z)\in\mathcal{U}\times\mathcal{Z}_\mathrm{C}\,\big|\,T_\mathrm{C}z[\![u]\!]=0 \text{ and } [\![u\cdot\mathrm{n}_1]\!]\geq 0 \text{ a.e. on } \Gamma_\mathrm{C}\big\}\tag{3.3}$$

with $\mathcal{U}$ from (2.12). Moreover, $T_\mathrm{C}z{=}z|_{\Gamma_\mathrm{C}}$ in the trace sense, which is well-defined in $\mathcal{Z}_\mathrm{C}$, and $[\![\cdot]\!]$ denotes the jump of a function defined on $\Omega_-\cup\Omega_+$ across Γ_C in the trace sense. The constraint $T_\mathrm{C}z[\![u]\!]=0$ a.e. on Γ_C incorporates a transmission condition, namely $[\![u]\!]=0$ whenever $T_\mathrm{C}z>0$. This condition was already used in [Fré88]. Furthermore $\mathrm{n}_1:=(1,0,\dots,0)$ stands for the unit normal vector to Γ_C. Thus the condition $[\![u\cdot\mathrm{n}_1]\!]\geq 0$ a.e. on Γ_C prevents the interpenetration of the material of Ω_- and Ω_+.

If $(u,z)\in\mathcal{Q}_\mathrm{C}$ and $v\in\mathcal{Z}_\mathrm{C}$ we find that $\mathcal{E}^\kappa(t,q)$ and $\mathcal{R}(v)$ equivalently read

$$\mathcal{E}^\kappa(t,u,z)=\int_{\Omega_-\cup\Omega_+}W(e(u{+}g(t)))\,\mathrm{d}x+2\int_{\Gamma_\mathrm{C}}\Big(\tfrac{\kappa}{r}|\nabla_s T_\mathrm{C}z|^r{+}\delta_{[0,1]}(T_\mathrm{C}z)\Big)\,\mathrm{d}s\,,\tag{3.4}$$

$$\mathcal{R}(v)=\begin{cases}\displaystyle 2\int_{\Gamma_\mathrm{C}}-\varrho\,T_\mathrm{C}v(s)\,\mathrm{d}s & \text{if } T_\mathrm{C}v\leq 0\ \mathcal{L}^{d-1}\text{-a.e. on } \Gamma_\mathrm{C}\,,\\ \infty & \text{otherwise}\end{cases}\tag{3.5}$$

with $s:=(x_2,\dots,x_d)$ and $\nabla_s:=(\partial_{x_2},\dots,\partial_{x_d})$. This shows that the limit system indeed models delamination along Γ_C. For all $t{\in}[0,T]$ we introduce the stable sets

$$\mathcal{S}^\kappa(t):=\{q{=}(u,z){\in}\mathcal{Q}\,|\,\mathcal{E}^\kappa(t,q){<}\infty,\ \mathcal{E}^\kappa(t,q){\leq}\mathcal{E}^\kappa(t,\tilde q){+}\mathcal{R}(\tilde z{-}z) \text{ for all } \tilde q{=}(\tilde u,\tilde z){\in}\mathcal{Q}\}\,.$$

The convergence result, which will be proven in the next subsection, is the following:

Theorem 3.1 (Γ-convergence of the damage problems) *Let assumptions* (2.5) *and* (2.6) *be valid with* $r,p\in(1,\infty)$, *and* $\gamma\in(p{-}1,P)$ *satisfying* (3.12) *and* (3.9). *Keep* $\kappa\in(0,\kappa_0]$ *fixed. For all* $\varepsilon\in(0,\varepsilon_0]$ *let* $q_\varepsilon:[0,T]\to\mathcal{Q}$ *be an energetic solution of* $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})$ *given by* (2.12), (2.14) *and* (2.16). *If the initial values satisfy* $q_0^\varepsilon\xrightarrow{\mathcal{T}}q_0$ *and* $\mathcal{E}^\kappa_\varepsilon(0,q_0^\varepsilon)\to\mathcal{E}^\kappa(0,q_0)$, *then the damage problems* $(\mathcal{Q},\mathcal{E}^\kappa_\varepsilon,\mathcal{R})_{\varepsilon\in(0,\varepsilon_0]}$ *Γ-converge to the delamination problem* $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})$ *given by* (2.12), (3.1) *and* (2.16) *in the sense of Theorem A.1.*

Proof: Theorem 3.1 is proven by checking the assumptions (A.1)–(A.3) of Theorem A.1. The lower Γ-limit of $\mathcal{R}$, i.e. condition (A.3-C4) here follows from the weak sequential lower semicontinuity of $\mathcal{R}$ on $\mathcal{Z}$. Conditions (A.1), (A.3-C1) and (A.3-C3) are shown in Subsection 3.1 and condition (A.3-C2) is verified in Subsection 3.2.

The existence of a subsequence $(z_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ is obtained by repeating the arguments of [MM05, Theorem 3.2], using the bound (3.7b), Helly's selection principle and the fact that $\min\{\mathcal{R}(z_k - z), \mathcal{R}(z - z_k)\} \to 0$ implies $z_k \overset{*}{\rightharpoonup}$ in $L^\infty(\Omega_\mathrm{D})$. For the corresponding subsequence $(u_\varepsilon)_{\varepsilon\in(0,T]}$ the bound (3.7a) provides a further subsequence $u_{\tilde{\varepsilon}}(t) \rightharpoonup u(t)$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$ uniformly for a countable choice of indices $\nu \to 0$ and Lemma 3.9 implies that $(u(t), z(t)) \in \mathcal{S}^\kappa(t)$ for all $t \in [0, T]$. Due to the strict convexity of W by (2.6a) the functional $\mathcal{E}^\kappa(t, \cdot, z(t))$ has a unique minimizer, so that $u(t)$ is the only accumulation point, i.e. $u_\varepsilon(t) \rightharpoonup u(t)$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$ for all $\nu \in (0, \varepsilon_0]$ and all $t \in [0, T]$ even for the whole subsequence. ∎

3.1 Compactness of Sublevels, Lower Γ-limit, Conditions on $\partial_t\mathcal{E}_\varepsilon^\kappa$, $\partial_t\mathcal{E}^\kappa$

In the following we verify the conditions on the energy functionals complying with (A.1), (A.3-C1) and (A.3-C3). As a direct consequence of stability (1.2 S) one obtains that the energetic solutions of the approximating problems have an equibounded energy; to see this one may check (1.2 S) for the energetic solutions and the states $(\hat{u}, \hat{z}_\varepsilon)$ with $\hat{u} = 0$ and $\hat{z}_\varepsilon = \varepsilon^\gamma$. To ensure that the equiboundedness of the energies implies the equiboundedness of the corresponding states we establish the following a priori estimates as a consequence of the coercivity (2.6b).

Lemma 3.2 (A priori estimates uniform for $\kappa \in [0, \kappa_0]$) *Let (2.5), (2.6) hold, let $t \in [0, T]$ and keep $\kappa \in [0, \kappa_0]$ fixed. For all $\varepsilon \in (0, \varepsilon_0]$, all $\nu \in [\varepsilon, \varepsilon_0]$ and all $q = (u, z) \in \mathcal{Q}$ with $\mathcal{E}_\varepsilon^\kappa(t, q) < \infty$ it is*

$$\mathcal{E}_\varepsilon^\kappa(t, q) \geq \tfrac{2^{1-p}c}{c_\mathcal{K}^p} \|u\|^p_{W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)} + \tfrac{\kappa}{r} \|{}_\varepsilon\nabla z\|^r_{L^r(\Omega_\mathrm{D})} - C \tag{3.6}$$

with $C = c c_g^p$ and $\|{}_\varepsilon\nabla z\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \geq \|\nabla z\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \geq \|z\|^r_{W^{1,r}(\Omega_\mathrm{D})} - \mathcal{L}^d(\Omega_\mathrm{D})$ for all $\varepsilon \in (0, \varepsilon_0]$. Moreover, $\mathcal{E}_\varepsilon^\kappa(t, q) < \infty$ implies that $\|z\|_{L^\infty(\Omega_\mathrm{D})} \leq 1$.

Proof: Let $q = (u, z) \in \mathcal{Q}$ with $\mathcal{E}_\varepsilon^\kappa(t, q) < \infty$. Keep $\nu \in (0, \varepsilon_0]$ fixed. Then $\Omega_-^\nu \cup \Omega_+^\nu \subseteq \Omega_-^\varepsilon \cup \Omega_+^\varepsilon$ for all $\varepsilon \leq \nu$. From hypothesis (2.6b), (2.5) and the uniform Korn's inequality (2.20), where we exploit the Dirichlet-conditions on the Lipschitz-domains $\Omega_\pm^\nu$, we infer

$$\begin{aligned}
\mathcal{E}_\varepsilon^\kappa(t, q) &\geq \int_{\Omega_-^\nu \cup \Omega_+^\nu} W(e(u{+}g(t)))\,\mathrm{d}x + \tfrac{\kappa}{r} \|{}_\varepsilon\nabla z\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \\
&\geq c\|e(u{+}g(t))\|^p_{L^p(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^{d\times d})} + \tfrac{\kappa}{r} \|\nabla z\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \\
&\geq 2^{1-p}c\|e(u)\|^p_{L^p(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^{d\times d})} - c c_g^p + \tfrac{\kappa}{r} \|\nabla z\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \\
&\geq \tfrac{2^{1-p}c}{c_\mathcal{K}^p} \|u\|^p_{W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)} - c c_g^p + \tfrac{\kappa}{r} \|z\|^r_{W^{1,r}(\Omega_\mathrm{D})} - \tfrac{\kappa}{r} \mathcal{L}^d(\Omega_\mathrm{D}) \,,
\end{aligned}$$

where we used that $\varepsilon^{-1} > 1$ for all $0 < \varepsilon < 1$. The last statement of the lemma directly follows from $\delta_{[\varepsilon^\gamma,1]}(z(y)) = \infty$ if $z(y) \notin [\varepsilon^\gamma, 1]$ in (2.14). ∎

Proposition 3.3 (A priori estimates for energetic solutions) *Let (2.5) as well as (2.6) hold. Keep $\kappa \in (0, \kappa_0]$ fixed. For all $\varepsilon \in (0, \varepsilon_0]$ let $q_\varepsilon : [0, T] \to \mathcal{Q}$ be an energetic solution of $(\mathcal{Q}, \mathcal{E}_\varepsilon^\kappa, \mathcal{R}, q_0^\varepsilon)$. Then there are constant $\tilde{E}, C$ independent of κ and ε, such that for all $t \in [0, T]$ and for all fixed $\nu \in (0, \varepsilon_0]$ the following uniform bounds are valid*

$$\|u_\varepsilon(t)\|_{W^{1,p}(\Omega_\pm^\nu, \mathbb{R}^d)} \leq \tilde{E}\,, \qquad \|z_\varepsilon(t)\|_{L^\infty(\Omega_\mathrm{D})} \leq 1\,, \tag{3.7a}$$

$$\mathrm{Diss}_\mathcal{R}(z_\varepsilon, [0, t]) \leq C\,. \tag{3.7b}$$

Proof: For all $\varepsilon \in (0, \varepsilon_0]$ the function $q_\varepsilon : [0,T] \to \mathcal{Q}$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$. Hence, for all $t \in [0,T]$ they satisfy $\mathcal{E}^\kappa_\varepsilon(t, q_\varepsilon(t)) < \infty$, which implies that $\varepsilon^\gamma \le z_\varepsilon(t,x) \le 1$ for a.e. $x \in \Omega_\mathrm{D}$, for all $t \in [0,T]$ and all $\varepsilon \in (0, \varepsilon_0]$. Stability inequality (1.2 S) with $q_\varepsilon(t)$ and $\tilde q = (0, \varepsilon^\gamma)$ yields $\mathcal{E}^\kappa_\varepsilon(t, q_\varepsilon(t)) \le \mathcal{E}^\kappa_\varepsilon(t, \tilde q) + \mathcal{R}(\tilde z - z_\varepsilon(t)) \le E$ for all $t \in [0,T]$ by (2.5), so that $(t, q_\varepsilon(t))_{\varepsilon \in (0,T]}$ is a stable sequence and their energies are equibounded for all $t \in [0,T]$. Using estimate (3.6) finishes the proof of (3.7a).

Because of $\mathcal{E}^\kappa_\varepsilon(0, q_\varepsilon(0)) \le C$ and $\int_0^t \partial_\xi \mathcal{E}^\kappa_\varepsilon(\xi, q_\varepsilon(\xi))\,\mathrm{d}\xi \le c_g T(\hat c E + \hat C \mathcal{L}^d(\Omega))$ for all $t \in [0,T]$, which is due to stress control (2.6c), energy balance (1.2 E) yields (3.7b). ∎

With Proposition 3.4 we then ensure that the equiboundedness of sequences enables us to extract subsequences converging with respect to $\mathcal{T}$ to an element in $\mathcal{Q}_\mathrm{C}$, given by (3.3). The Items *(1.)* and *(2.a)* in Proposition 3.4 result from the coercivity inequality (3.6), which yields uniform boundedness of u_ε in $W^{1,p}(\Omega^\nu_- \cup \Omega^\nu_+, \mathbb{R}^d)$ for all fixed $\nu \in (0, \varepsilon_0]$ and hence, using Cantor's diagonal process, the convergence of a subsequence for all fixed ν. Moreover, *(2.b)* results from the uniform boundedness of the gradient term for fixed $\kappa \in (0, \kappa_0]$. Item *(2.c)* can be gained from the term $(\mathrm{tr}\, e(u_\varepsilon))^-$ included to W_D, see (2.2), using the Lebesgue-Besicovitch differentiation theorem to express $[\![u^1]\!] \in L^1(\Gamma_\mathrm{C})$ in the Lebesgue points $\hat s \in \Gamma_\mathrm{C}$ and then Gauss' theorem on balls $B_r(\hat s) \subset \Gamma_\mathrm{C}$. In this context we use the following relation for the trace mapping

$$T : \begin{cases} W^{1,p}(A) \to L^{q'}(\partial A), \\ u \mapsto u|_{\partial A}, \end{cases} \qquad \text{if} \begin{cases} p < d \quad \text{and} \quad 1 \le q' \le (d-1)p/(d-p)\,, \\ p = d \quad \text{and} \quad q' \in [1, \infty)\,, \end{cases} \tag{3.8}$$

to obtain that $\left|\int_{\partial A} \int_{-\varepsilon}^{\varepsilon} u_\varepsilon \cdot \mathrm{n}\, \mathrm{d}x_1\, \mathrm{d}a\right| \le (2\varepsilon)^{(q'-1)/q'} \mathcal{L}^{d-2}(\partial A) \|u_\varepsilon\|_{L^{q'}(I_\varepsilon \times \partial A)} \overset{!}{\to} 0$, where $A = B_r(\hat s)$ and $\|u_\varepsilon\|_{W^{1,p}(\Omega^\varepsilon_\mathrm{D}, \mathbb{R}^d)} \le C\varepsilon^{-\gamma/p}$ by $\Pi_\varepsilon^{-1} z_\varepsilon \in [\varepsilon^\gamma, 1]$. This leads to the following condition on γ:

$$\gamma < P\,, \quad \text{where} \quad P = \begin{cases} (p-1)d/(d-1) & \text{if } p < d\,, \\ p & \text{if } p \ge d\,. \end{cases} \tag{3.9}$$

Moreover, for $\gamma < p-1$ one obtains that

$$\|\nabla u_\varepsilon\|_{L^1(\Omega^\varepsilon_\mathrm{D}, \mathbb{R}^{d\times d})} \le C\varepsilon^{\frac{p-1-\gamma}{p}} \to 0\,,$$

which implies that jumps are prevented. In order to admit nontrivial displacement jumps in points where $z = 0$ we thus have to assume $\gamma > p-1$.

Item *(2.d)* is equivalent to $\int_{\Gamma_\mathrm{C}} |T_\mathrm{C} z [\![u]\!]|\,\mathrm{d}s = 0$. This is obtained by considering the traces of the approximating sequence $(u_\varepsilon)_{\varepsilon \in (0,\varepsilon_0]}$ on $\{\pm\nu\} \times \Gamma_\mathrm{C}$ and by passing to 0 first with ε, then with ν. To estimate the traces $T_\mathrm{C} \Pi_\varepsilon^{-1} z_\varepsilon$ on $\{0\} \times \Gamma_\mathrm{C}$ we use that

$$\varepsilon^\gamma \le \Pi_\varepsilon^{-1} z_\varepsilon \le 1 \;\mathcal{L}^d\text{-a.e. in } \Omega^\varepsilon_\mathrm{D} \;\Rightarrow\; \varepsilon^\gamma \le T_\mathrm{C} \Pi_\varepsilon^{-1} z_\varepsilon \le 1 \;\mathcal{H}^{d-1}\text{-a.e. on } \{0\} \times \Gamma_\mathrm{C}\,. \tag{3.10}$$

This is due to the fact that $\Omega^\varepsilon_\mathrm{D}$ can be extended to a Lipschitz domain $\tilde\Omega \supset \Omega^\varepsilon_\mathrm{D}$. Moreover, $\mathrm{C}(\tilde\Omega)$ is dense in $W^{1,r}(\tilde\Omega)$ and $\{v : \tilde\Omega \to [0,1]\}$ is a closed subset both of $\mathrm{C}(\tilde\Omega)$ and of $W^{1,r}(\tilde\Omega)$. Then (3.10) follows by density arguments.

When proving that $\int_{\Gamma_\mathrm{C}} |T_\mathrm{C} z [\![u]\!]|\,\mathrm{d}s = 0$, we have to handle terms of the form

$$\int_{\Omega^\varepsilon_\mathrm{D}} |u_\varepsilon| \left|\Pi_\varepsilon^{-1} z_\varepsilon(\varepsilon) - T_\mathrm{C} \Pi_\varepsilon^{-1} z_\varepsilon\right| \mathrm{d}s \le \|u_\varepsilon\|_{L^{q'}(\Omega^\varepsilon_\mathrm{D}, \mathbb{R}^d)} \|\partial_{x_1} \Pi_\varepsilon^{-1} z_\varepsilon\|_{L^q(\Omega^\varepsilon_\mathrm{D})}\,,$$

where $q' = q/(q-1)$. We need $q < r$ for $\Pi_\varepsilon^{-1} z_\varepsilon \in W^{1,r}(\Omega_{\mathrm{D}}^\varepsilon)$ to show that the second factor can be estimated by $c\varepsilon^\alpha \|\Pi_\varepsilon^{-1} z_\varepsilon\|_{W^{1,r}(\Omega_{\mathrm{D}}^\varepsilon)}$ with some $\alpha > 0$. Hence, we have to ensure that $u_\varepsilon \in L^{q'}(\Omega_{\mathrm{D}}^\varepsilon, \mathbb{R}^d)$ using the embedding

$$W^{1,p}(\Omega_{\mathrm{D}}^\varepsilon, \mathbb{R}^d) \to L^{q'}(\Omega_{\mathrm{D}}^\varepsilon, \mathbb{R}^d) \quad \text{if} \quad \begin{cases} p < d & \text{and} \quad 1 \le q' \le dp/(d-p)\,, \\ p = d & \text{and} \quad q' \in [1, \infty)\,, \end{cases} \tag{3.11}$$

see [Ada75, Th. 5.4]. This leads to the following admissible combinations of r and p:

$$\left.\begin{array}{lll} r \in (1,d) & \text{and} \quad p \in [rd/(rd-d+r), \infty) & \text{or} \\ r \in [d, \infty) & \text{and} \quad p \in (1, \infty)\,. & \end{array}\right\} \tag{3.12}$$

Thus, the transmission condition *(2.d)* can be verified using implication (3.10) together with the conditions (3.12) and (3.9) on r, p and γ.

Note that not every combination of $r, p < d$ is admissible. But the case $d = 3$, $r = p = 2$ is included in the first line of (3.12), since then $3 > r = p = 2 > rd/(rd-d+r) = 6/5$.

Proposition 3.4 (Properties of sequences with equibounded energies) *Let the energy functionals $\mathcal{E}_\varepsilon^\kappa$ be given by* (2.14) *such that the assumptions* (2.5) *and* (2.6) *hold. Let $\kappa \in (0, \kappa_0]$ fixed and $(t_\varepsilon)_{\varepsilon \in (0,\varepsilon_0]} \subset [0, T]$. Let $r, p \in (1, \infty)$ and $\gamma \in (p-1, P)$ such that* (3.12) *and* (3.9) *hold. Assume that $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, u_\varepsilon, z_\varepsilon) \le E$ for all $\varepsilon \in (0, \varepsilon_0]$. Then*

(1.) there is a subsequence $(u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{T}} (u, z)$ as $\varepsilon \to 0$,

(2.) the limit satisfies $(u, z) \in \mathcal{Q}_{\mathrm{C}}$, i.e.

(2.a) $u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$, $u = 0$ on Γ_{Dir} in the trace sense,

(2.b) $z \in W^{1,r}(\Omega_{\mathrm{D}})$, $0 \le z(y) \le 1$ and $\partial_{y_1} z(y) = 0$ for all $y \in \Omega_{\mathrm{D}}$,

(2.c) $[\![u \cdot \mathrm{n}_1]\!] \ge 0$ a.e. on Γ_{C},

(2.d) $T_{\mathrm{C}} z [\![u]\!] = 0$ a.e. on Γ_{C}.

Moreover, for $\gamma < p-1$ jumps are prevented, i.e. $[\![u]\!] = 0$ a.e. on Γ_{C}.

Proof: Recall $\mathcal{Q}$ from (2.12), $\mathcal{E}_\varepsilon^\kappa$ from (2.14) and $\mathcal{Q}_{\mathrm{C}}$ from (3.3).

Ad *(1.)* and *(2.a)*: From $\mathcal{E}_\varepsilon^\kappa(t_\varepsilon, q_\varepsilon) \le E$ we infer that $\varepsilon^\gamma \le z_\varepsilon \le 1$ a.e. in Ω_{D}. Since the unit ball of $L^\infty(\Omega_{\mathrm{D}})$, which is the dual space of $L^1(\Omega_{\mathrm{D}})$, is weakly* sequentially compact by the theorem of Banach-Alaoglu we find a subsequence $z_\varepsilon \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_{\mathrm{D}})$.

The equiboundedness of the energies together with coercivity estimate (3.6) yields that $\|u_\varepsilon\|_{W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)}$ are uniformly bounded for all $\varepsilon \le \nu$. For a countable set of indices ν with $\nu \to 0$ we obtain by Cantor's diagonal process that there is a subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$ as $\varepsilon \to 0$ for all ν, due to the reflexivity of $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu, \mathbb{R}^d)$. As $\nu \to 0$ we conclude that $u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ with $u = 0$ on Γ_{Dir} in the trace sense. This proves the existence of a subsequence $q_\varepsilon \xrightarrow{\mathcal{T}} q$.

Ad *(2.b)*: The equiboundedness of the energies together with (3.6) yields that $\|z_\varepsilon\|_{W^{1,r}(\Omega_{\mathrm{D}})}^r \le r(E + \mathcal{L}^d(\Omega_{\mathrm{D}}))/\kappa$ as well as $\|\partial_{y_1} z_\varepsilon\|_{L^r(\Omega_{\mathrm{D}})}^r \le \varepsilon^r r E/\kappa$. Due to the reflexivity of $W^{1,r}(\Omega_{\mathrm{D}})$ there is a subsequence $z_\varepsilon \rightharpoonup z$ in $W^{1,r}(\Omega_{\mathrm{D}})$ with $\partial_{y_1} z = 0$ a.e. in Ω_{D}. Because of the compact embedding $W^{1,r}(\Omega_{\mathrm{D}}) \Subset L^r(\Omega_{\mathrm{D}})$ and Riesz' convergence theorem there is a subsequence $[\varepsilon^\gamma, 1] \ni z_\varepsilon(y) \to z(y) \in [0, 1]$ pointwise for a.e. $y \in \Omega_{\mathrm{D}}$. Hence, $z \in [0, 1]$ a.e. in Ω_{D}.

Ad *(2.c)*: To verify that $[\![u^1]\!] \geq 0$ a.e. on Γ_{C} we use the Lebesgue-Besicovitch differentiation theorem, see [AFP05, Corollary 2.23], stating for $[\![u^1]\!] \in L^1(\Gamma_{\mathrm{C}})$ that

$$[\![u^1(\hat{s})]\!] = \lim_{r\to 0} \frac{1}{\mathcal{L}^{d-1}(B_r(\hat{s}))} \int_{B_r(\hat{s})} [\![u^1(s)]\!]\,\mathrm{d}s \quad \text{for a.a. } \hat{s} \in \Gamma_{\mathrm{C}} \tag{3.13}$$

with $B_r(\hat{s}) := \{s \in \Gamma_{\mathrm{C}}\,|\,|s-\hat{s}| < r\}$. Hence it suffices to show that

$$\int_{B_r(\hat{s})} [\![u^1(s)]\!]\,\mathrm{d}s \geq 0 \quad \text{for a.a. } \hat{s} \in \Gamma_{\mathrm{C}} \text{ and all } r \leq r(\hat{s})\,. \tag{3.14}$$

Omitting to indicate the dependence of u on s we first deduce the following relation

$$\int\limits_{B_r(\hat{s})} [\![u^1]\!]\,\mathrm{d}s = \lim_{\nu\to 0} \int\limits_{B_r(\hat{s})} \big(u^1(\nu) - u^1(-\nu)\big)\,\mathrm{d}s = \lim_{\nu\to 0}\lim_{\varepsilon\to 0} \int\limits_{B_r(\hat{s})} \big(u^1_\varepsilon(\nu) - u^1_\varepsilon(-\nu)\big)\,\mathrm{d}s \tag{3.15}$$

for the subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega^\nu_- \cup \Omega^\nu_+, \mathbb{R}^d)$ for all $\nu \in (0,\varepsilon_0]$ obtained in *(1.)* from the equiboundedness of $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, u_\varepsilon, z_\varepsilon)$. Moreover, note that the first equality results from the fact that the linear, continuous trace operators $S^\pm_\nu : W^{1,p}(\Omega_\pm) \to L^1(\Gamma_{\mathrm{C}})$, $S^\pm_\nu v{=}(v(\pm\nu, s){-}v_\pm)$, for $v_\pm$ being the trace of $v|_{\Omega_\pm} \in W^{1,p}(\Omega_\pm)$ onto Γ_{C}, satisfy the estimate $\|S^\pm_\nu\| \leq \nu^{\frac{p-1}{p}}(\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}))^{\frac{p-1}{p}}$, which follows with Hölder's inequality.

Now it remains to verify that the expression in line (3.15) is positive. Using Gauß' theorem we obtain that

$$\int\limits_{B_r(\hat{s})} \big(u^1_\varepsilon(\nu,s) - u^1_\varepsilon(-\nu,s)\big)\,\mathrm{d}s = \int\limits_{B_r(\hat{s})}\int_{-\nu}^{\nu} \operatorname{div} u_\varepsilon\,\mathrm{d}x_1\,\mathrm{d}s - \int\limits_{\partial B_r(\hat{s})}\int_{-\nu}^{\nu} u_\varepsilon\cdot \mathrm{n}\,\mathrm{d}a\,,$$

with n as the outer unit normal vector to $(-\nu,\nu)\times\partial B_r(\hat{s})$. Hence, (3.14) holds true, if

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\int_{B_r(\hat{s})}\int_{-\nu}^{\nu} \operatorname{div} u_\varepsilon\,\mathrm{d}x_1\mathrm{d}s \geq 0 \tag{3.16}$$

$$\text{and} \quad \lim_{\nu\to 0}\lim_{\varepsilon\to 0}\int_{\partial B_r(\hat{s})}\int_{-\nu}^{\nu} u_\varepsilon\cdot\mathrm{n}\,\mathrm{d}a \quad \to 0\,. \tag{3.17}$$

For (3.16) we decompose $\mathrm{div}u_\varepsilon{=}(\mathrm{div}u_\varepsilon)^+{-}(\mathrm{div}u_\varepsilon)^-$ with $(\mathrm{div}u_\varepsilon)^+{=}\max\{0, \mathrm{div}u_\varepsilon\}$ and $(\mathrm{div}u_\varepsilon)^- = \max\{0, -\mathrm{div}u_\varepsilon\}$. Showing $\lim_{\nu\to 0}\limsup_{\varepsilon\to 0}\int_{B_r(\hat{s})}\int_{-\nu}^{\nu}(\operatorname{div} u_\varepsilon)^-\,\mathrm{d}x_1\mathrm{d}s = 0$ we are done. To do so, we choose a subsequence in ε which attains the limit superior. Due to (2.2) and the coercivity inequalities (2.3) and (2.6b) for φ and W the equiboundedness of $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, u_\varepsilon, z_\varepsilon)$ yields that $\|(\mathrm{div}u_\varepsilon)^-\|_{L^{\hat{p}}(\Omega)} \leq C$ for all $\varepsilon \in (0,\varepsilon_0]$ on the domain Ω with $\hat{p} \in (1,p]$, see (2.2). Thus, we find a further subsequence $(\mathrm{div}u_\varepsilon)^- \rightharpoonup b$ in $L^{\hat{p}}(\Omega)$ and obtain

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\int_{B_r(\hat{s})}\int_{-\nu}^{\nu}(\operatorname{div} u_\varepsilon)^-\,\mathrm{d}x_1\,\mathrm{d}s = \lim_{\nu\to 0}\int_{B_r(\hat{s})}\int_{-\nu}^{\nu} b\,\mathrm{d}x_1\,\mathrm{d}s = 0\,.$$

Hence (3.16) is established.

For the proof of (3.17) we decompose the integral as follows

$$\int_{\partial B_r(\hat{s})}\int_{-\nu}^{\nu} u_\varepsilon\cdot\mathrm{n}\,\mathrm{d}x_1\,\mathrm{d}a = \int_{\partial B_r(\hat{s})}\Big(\int_{-\nu}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\nu}\Big)u_\varepsilon\cdot\mathrm{n}\,\mathrm{d}x_1\,\mathrm{d}a\,. \tag{3.18}$$

First, let $p \leq d$. Using Hölder's inequality we obtain that

$$
\begin{aligned}
\Big| \pm \int_{\partial B_r(\hat{s})} \int_{\pm\varepsilon}^{\pm\nu} u_\varepsilon \cdot \mathrm{n} \,\mathrm{d}x_1 \,\mathrm{d}a \Big| &\leq \pm \int_{\partial B_r(\hat{s})} \int_{\pm\varepsilon}^{\pm\nu} |u_\varepsilon| \,\mathrm{d}x_1 \,\mathrm{d}a \\
&\leq (\nu - \varepsilon)^{\frac{q'-1}{q'}} \mathcal{L}^{d-2}(\partial B_r(\hat{s}))^{\frac{q'-1}{q'}} \Big(\pm \int_{\pm\varepsilon}^{\pm\nu} \int_{\partial B_r(\hat{s})} |u_\varepsilon|^{q'} \,\mathrm{d}a \,\mathrm{d}x_1 \Big)^{\frac{1}{q'}} ,
\end{aligned}
\tag{3.19}
$$

which tends to 0 as $\varepsilon < \nu \to 0$ by property (3.8) for either $A = (-\nu, -\varepsilon) \times \partial B_r(\hat{s})$ or $A = (\varepsilon, \nu) \times \partial B_r(\hat{s})$.

For the integral over $I_\varepsilon = (-\varepsilon, \varepsilon)$ in (3.18) we proceed as in estimate (3.19). The equiboundedness of the energies, the assumptions (2.6b), (2.5), $\Pi_\varepsilon^{-1} z_\varepsilon \geq \varepsilon^\gamma$, Korn's inequality on Ω and property (3.8) imply the following estimate for all $\varepsilon \in (0, \varepsilon_0]$:

$$
\|u_\varepsilon\|_{L^{q'}(I_\varepsilon \times \partial B_r(\hat{s}), \mathbb{R}^d)} \leq C \|u_\varepsilon\|_{W^{1,p}(\Omega_{\mathrm{D}}^\varepsilon, \mathbb{R}^d)} \leq \varepsilon^{-\frac{\gamma}{p}} \big(\tfrac{E^{\frac{1}{p}}}{c} + \hat{c}_g \big) c_{\mathcal{K}}(\Omega) C \tag{3.20}
$$

Under the assumption that $\gamma \in (p-1, P)$ with $P = (p-1)d/(d-1)$ if $p < d$, see (3.9), we now conclude

$$
\begin{aligned}
\Big| \int_{\partial B_r(\hat{s})} \int_{-\varepsilon}^{\varepsilon} u_\varepsilon \cdot \mathrm{n} \,\mathrm{d}x_1 \,\mathrm{d}a \Big| &\leq (2\varepsilon)^{\frac{q'-1}{q'}} \mathcal{L}^{d-2}(\partial B_r(\hat{s}))^{\frac{q'-1}{q'}} \Big(\int_{\partial B_r(\hat{s})} \int_{-\varepsilon}^{\varepsilon} |u_\varepsilon|^{q'} \,\mathrm{d}x_1 \,\mathrm{d}a \Big)^{\frac{1}{q'}} \\
&\leq \varepsilon^{\frac{q'-1}{q'} - \frac{\gamma}{p}} C \ \to 0 ,
\end{aligned}
\tag{3.21}
$$

where we use that $\|u_\varepsilon\|_{L^{q'}(I_\varepsilon \times \partial B_r(\hat{s}), \mathbb{R}^d)} \leq c \|u_\varepsilon\|_{W^{1,p}(\Omega_{\mathrm{D}}^\varepsilon, \mathbb{R}^d)}$ for $q' = (d-1)p/(d-p)$. The requirement $\frac{q'-1}{q'} - \frac{\gamma}{p} > 0$ then yields $\gamma < (p-1)d/(d-1)$ as stated in condition (3.9).

Assume now that $p > d$. Then $W^{1,p}(\Omega, \mathbb{R}^d) \Subset \mathrm{C}(\overline{\Omega}, \mathbb{R}^d)$. Due to this, we can set $q' = \infty$ in the above estimates. Moreover, $q' - 1/q' = 1$, so that (3.21) implies that $\gamma < p$.

Alltogether we have verified (3.16) and (3.17), hence $[\![u \cdot \mathrm{n}_1]\!]$ a.e. on Γ_{C} by (3.14).

Ad *(2.d)*: In the following we verify $T_{\mathrm{C}} z [\![u]\!] = 0$ a.e. on Γ_{C} for the limit state (u, z).

Verifying $T_{\mathrm{C}} z [\![u]\!] = 0$ a.e. on Γ_{C} for the limit state (u, z) is equivalent to showing that $\int_{\Gamma_{\mathrm{C}}} |T_{\mathrm{C}} z [\![u]\!]| \,\mathrm{d}s = 0$. For this, we approximate u on the interface $\{0\} \times \Gamma_{\mathrm{C}}$ from the left and the right by the traces of the approximating sequence on the lines $\{\pm\nu\} \times \Gamma_{\mathrm{C}}$ and we exploit that z is constant in y_1-direction, so that $z(\pm\nu, s) = z(0, s)$ for all $s \in \Gamma_{\mathrm{C}}$ and all $\nu \in (0, \varepsilon_0]$. In particular, we use

$$
\begin{aligned}
&\int_{\Gamma_{\mathrm{C}}} \big| T_{\mathrm{C}} z \, [\![u]\!] \big| \,\mathrm{d}s = \lim_{\nu \to 0} \int_{\Gamma_{\mathrm{C}}} \big| T_{\mathrm{C}} z \, (u(\nu) - u(-\nu)) \big| \,\mathrm{d}s \\
&\leq \lim_{\nu \to 0} \Big(\sum_{\iota \in \{-,+\}} \int_{\Gamma_{\mathrm{C}}} \big| T_{\mathrm{C}} z \, (u(\iota\nu) - u_\varepsilon(\iota\varepsilon)) \big| \,\mathrm{d}s + \int_{\Gamma_{\mathrm{C}}} \big| z(\varepsilon) u(\varepsilon) - z(-\varepsilon) u(-\varepsilon)) \big| \,\mathrm{d}s \Big) \\
&\leq \lim_{\nu \to 0} \lim_{\varepsilon \to 0} \Big(\int_{\Gamma_{\mathrm{C}}} \big| \Pi_\varepsilon^{-1} z_\varepsilon(\varepsilon) u_\varepsilon(\varepsilon) - \Pi_\varepsilon^{-1} z_\varepsilon(-\varepsilon) u_\varepsilon(-\varepsilon) \big| \,\mathrm{d}s \qquad (3.22) \\
&\qquad\qquad + \sum_{\iota \in \{-,+\}} \int_{\Gamma_{\mathrm{C}}} \big| T_{\mathrm{C}} \Pi_\varepsilon^{-1} z_\varepsilon \, (u_\varepsilon(\iota\nu) - u_\varepsilon(\iota\varepsilon)) \big| \,\mathrm{d}s \Big). \qquad (3.23)
\end{aligned}
$$

In (3.23), with $\iota \in \{-,+\}$, we apply that $|T_{\mathrm{C}} \Pi_\varepsilon^{-1} z_\varepsilon| \leq 1$ a.e. on Γ_{C} by (3.10). With partial integration and Hölder's inequality we find

$$
\int_{\Gamma_{\mathrm{C}}} \big| T_{\mathrm{C}} \Pi_\varepsilon^{-1} z_\varepsilon \, (u_\varepsilon(\pm\nu) - u_\varepsilon(\pm\varepsilon)) \big| \,\mathrm{d}s \leq \|\partial_{x_1} u_\varepsilon\|_{L^1(\Omega_\pm^\varepsilon \setminus \Omega_\pm^\nu, \mathbb{R}^d)} \leq (\nu - \varepsilon)^{\frac{p-1}{p}} \|\partial_{x_1} u_\varepsilon\|_{L^p(\Omega_\pm^\varepsilon, \mathbb{R}^d)},
$$

which tends to 0 as $\varepsilon < \nu \to 0$, since the norms are uniformly bounded, as can be seen from (3.6).

When estimating the term in (3.22) we apply partial integration on $\Omega_{\mathrm{D}}^{\varepsilon}$ and we use that $\|u_\varepsilon\|_{W^{1,p}(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^d)} \le C\varepsilon^{-\gamma/p}$, due to $\Pi_\varepsilon^{-1} z_\varepsilon \in [\varepsilon^\gamma, 1]$. In particular, we obtain

$$\begin{aligned}\int_{\Gamma_{\mathrm{C}}} \left|\Pi_\varepsilon^{-1} z_\varepsilon(\varepsilon) u_\varepsilon(\varepsilon) - \Pi_\varepsilon^{-1} z_\varepsilon(-\varepsilon) u_\varepsilon(-\varepsilon)\right| \mathrm{d}s &= \int_{\Gamma_{\mathrm{C}}} \left| \int_{-\varepsilon}^{\varepsilon} \partial_{x_1}(\Pi_\varepsilon^{-1} z_\varepsilon u_\varepsilon)\,\mathrm{d}x_1 \right| \mathrm{d}s \\ \le \int_{\Gamma_{\mathrm{C}}} \left| \int_{-\varepsilon}^{\varepsilon} (\partial_{x_1}\Pi_\varepsilon^{-1} z_\varepsilon) u_\varepsilon\,\mathrm{d}x_1 \right| \mathrm{d}s &+ \int_{\Gamma_{\mathrm{C}}} \left| \int_{-\varepsilon}^{\varepsilon} \Pi_\varepsilon^{-1} z_\varepsilon \partial_{x_1} u_\varepsilon\,\mathrm{d}x_1 \right| \mathrm{d}s\,. \end{aligned} \tag{3.24}$$

For the first term in (3.24) we use again Hölder's inequality with the exponent $q = r$ for z_ε and $q' = r/(r-1)$ for u_ε. Now, we exploit the embedding $W^{1,p}(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^d) \to L^{q'}(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^d)$ for $p < d$ and $p \le q' \le dp/(d-p)$. Because of these relations we find the condition $q' = r/(r-1) \le dp/(d-p)$ which leads to $p \in [rd/(rd-d+r), d)$ in (3.12).

To estimate the second term in (3.24) we use that $\int_{\Omega_{\mathrm{D}}^{\varepsilon}} W_{\mathrm{D}}(\Pi_\varepsilon^{-1} z_\varepsilon, e(u_\varepsilon)) \le C$ due to the equiboundedness of the energies, and additionally that $\Pi_\varepsilon^{-1} z_\varepsilon^p \le \Pi_\varepsilon^{-1} z_\varepsilon$ for $\Pi_\varepsilon^{-1} z_\varepsilon \in [\varepsilon^\gamma, 1]$ and $p \in (1,\infty)$. Thus, with Hölder's inequality we obtain

$$\int_{\Gamma_{\mathrm{C}}} \left| \int_{-\varepsilon}^{\varepsilon} \Pi_\varepsilon^{-1} z_\varepsilon \partial_{x_1} u_\varepsilon\,\mathrm{d}x_1 \right| \mathrm{d}s \le \int_{\Omega_{\mathrm{D}}^{\varepsilon}} \Pi_\varepsilon^{-1} z_\varepsilon |\partial_{x_1} u_\varepsilon|\,\mathrm{d}x \le \mathcal{L}^d(\Omega_{\mathrm{D}}^{\varepsilon})^{p-1} \int_{\Omega_{\mathrm{D}}^{\varepsilon}} W_{\mathrm{D}}(\Pi_\varepsilon^{-1} z_\varepsilon, e(u_\varepsilon)) \to 0\,.$$

Hence, Item *(2.d)* is proven for $r \in (1,\infty)$ and $p \in [rd/(rd-d+r), d)$.

For $p = d$ we can apply the embedding $W^{1,p}(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^d) \to L^{q'}(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^d)$, which holds for all $q' \in [p,\infty)$ and in particular for all $q' \in [1,\infty)$. For $p > d$ we have the compact embedding $W^{1,p}(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^d) \Subset \mathrm{C}(\overline{\Omega_{\mathrm{D}}^{\varepsilon}},\mathbb{R}^d)$. Thus, in both cases the choice $q' = r/(r-1)$ in the above Hölder estimates is admissible. Note that, if $r > d$ we may use the exponent $\tilde r = d$ instead of r in the above estimates. Then the lower bound on p is $\tilde r d/(\tilde r d - d + r) = 1$. This finishes the proof of Item *(2.d)*.

Ad $[\![u]\!]$: By *(1.)* there is a subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu,\mathbb{R}^d)$ for all fixed $\nu \in (0,\varepsilon_0]$. Using partial integration we obtain for the ith component that

$$\int_{\Gamma_{\mathrm{C}}} |u_\varepsilon^i(\nu,s) - u_\varepsilon^i(-\nu,s)|\,\mathrm{d}s \le \int_{\Omega_{\mathrm{D}}^{\nu}} |\partial_{x_1} u_\varepsilon^i|\,\mathrm{d}x \le \int_{\Omega_{\mathrm{D}}^{\varepsilon}} |\nabla u_\varepsilon|\,\mathrm{d}x + \int_{\Omega_{\mathrm{D}}^{\nu}\backslash\Omega_{\mathrm{D}}^{\varepsilon}} |\nabla u_\varepsilon|\,\mathrm{d}x\,. \tag{3.25}$$

With estimate (3.20) and Hölder's inequality we find for the first term in (3.25) that

$$\|\nabla u_\varepsilon\|_{L^1(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^{d\times d})} \le \varepsilon^{\frac{p-1}{p}} \mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})^{\frac{p-1}{p}} c_{\mathcal{K}}(\Omega) E^{\frac{1}{p}} \varepsilon^{\frac{-\gamma}{p}}\,. \tag{3.26}$$

Since $\gamma < p-1$ we conclude from (3.26) that $\|\nabla u_\varepsilon\|_{L^1(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^{d\times d})} \to 0$.

Additionally the equiboundedness of the energies and the coercivity of W provide a constant $C > 0$ such that $\|\nabla u_\varepsilon\|_{L^p(\Omega_-^\varepsilon \cup \Omega_+^\varepsilon,\mathbb{R}^{d\times d})} \le C$. Thus, application of Hölder's inequality on the second term in (3.25) yields

$$\int_{\Omega_{\mathrm{D}}^{\nu}\backslash\Omega_{\mathrm{D}}^{\varepsilon}} |\nabla u_\varepsilon|\,\mathrm{d}x \le \big((\nu-\varepsilon)\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})\big)^{\frac{p-1}{p}} \|\nabla u_\varepsilon\|_{L^p((\Omega_-^\varepsilon\cup\Omega_+^\varepsilon)\backslash(\Omega_-^\nu\cup\Omega_+^\nu),\mathbb{R}^{d\times d})} \to 0\,.$$

Repeating the ideas of (3.15) we obtain $\int_{\Gamma_{\mathrm{C}}} |[\![u]\!]|\,\mathrm{d}s = 0$, if $\|\nabla u_\varepsilon\|_{L^1(\Omega_{\mathrm{D}}^{\varepsilon},\mathbb{R}^{d\times d})} \to 0$. ∎

The next lemma summarizes the properties of the limit energy $\mathcal{E}^\kappa$, which guarantee the existence of minimizers in the direct method of the calculus of variations, such as coercivity and lower semicontinuity. They yield the compactness of the sublevels of $\mathcal{E}^\kappa$.

Lemma 3.5 (Properties of the limit energy) *Let the assumptions* (2.5) *and* (2.6) *hold. Then, for all* $t\in[0,T]$ *and all* $\kappa\in(0,\kappa_0]$ *the energy functional* $\mathcal{E}^\kappa(t,\cdot):\mathcal{Q}_{\rm C}\to\mathbb{R}_\infty$ *given by* (3.1) *and* (3.3) *is coercive and weakly sequentially lower semicontinuous on* $\mathcal{Q}_{\rm C}$. *In particular,* (3.6) *holds also for* $\varepsilon=0$, *i.e.* $\Omega_-\cup\Omega_+$. *Moreover for all* $E\in\mathbb{R}$ *the sublevels* $L^\kappa_E(t):=\{q\in\mathcal{Q}\,|\,\mathcal{E}^\kappa(t,q)\le E\}$ *of the functional* $\mathcal{E}^\kappa(t,\cdot):\mathcal{Q}\to\mathbb{R}_\infty$ *are sequentially compact with respect to* $\mathcal{T}$ *from* (2.18).

Proof: Keep $\kappa\in(0,\kappa_0]$ and $t\in[0,T]$ fixed. If $(q_j)_{j\in\mathbb{N}}\subset\mathcal{Q}\backslash\mathcal{Q}_{\rm C}$, then $\mathcal{E}^\kappa(t,q_j)=\infty$ for all $j\in\mathbb{N}$. Thus, for $\|u_j\|_{W^{1,p}(\Omega_-^\nu\cup\Omega_+^\nu,\mathbb{R}^d)}\to\infty$ for some $\nu\in(0,\varepsilon_0]$ the property $\mathcal{E}^\kappa(t,q_j)\to\infty$ is trivially satisfied. Coercivity inequality (3.6) with $\varepsilon=0$ follows from (2.6) for all $q\in\mathcal{Q}_{\rm C}$. Thus $\mathcal{E}^\kappa(t,\cdot)$ is coercive both on $\mathcal{Q}_{\rm C}$ and on $\mathcal{Q}$.

In order to show lower semicontinuity we assume that $q_j\xrightarrow{\mathcal{T}}q$. If $q_j\in\mathcal{Q}\backslash\mathcal{Q}_{\rm C}$ for almost all $j\in\mathbb{N}$ then there is an index $j_0\in\mathbb{N}$ such that $q_j\in\mathcal{Q}\backslash\mathcal{Q}_{\rm C}$ for all $j\ge j_0$ and hence $\liminf_{j\to\infty}\mathcal{E}^\kappa(t,q_j)=\infty\ge\mathcal{E}^\kappa(t,q)$. Assume that there is a subsequence $(q_j)_{j\in\mathbb{N}}\subset\mathcal{Q}_{\rm C}$ with $u_j\rightharpoonup u$ in $W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ and $z_j\rightharpoonup z$ in $W^{1,r}(\Omega_{\rm D})$. Let $u_j^\pm$, $u^\pm$ denote the traces of $u_j|_{\Omega_\pm}$, $u|_{\Omega_\pm}$ on $\Gamma_{\rm C}$. Then the compactness of the trace operators $T_{\rm C}:W^{1,r}(\Omega_{\rm D})\to L^r(\Gamma_{\rm C})$ and $T_\pm:W^{1,p}(\Omega_\pm,\mathbb{R}^d)\to L^p(\Gamma_{\rm C},\mathbb{R}^d)$ implies that $T_{\rm C}z_j\,u_j^\pm\to T_{\rm C}z\,u^\pm$ in $L^1(\Gamma_{\rm C},\mathbb{R}^d)$ and $u_j^\pm\to u^\pm$ in $L^p(\Gamma_{\rm C},\mathbb{R}^d)$, each containing a subsequence that converges pointwise a.e. on $\Gamma_{\rm C}$. Hence $[\![u\cdot{\rm n}_1]\!]\ge0$ and $T_{\rm C}z[\![u]\!]=0$ a.e. on $\Gamma_{\rm C}$, i.e. $(u,z)\in\mathcal{Q}_{\rm C}$. Furthermore $\{z\in W^{1,r}(\Omega_{\rm D})\,|\,0\le z\le1 \text{ a.e. on } \Omega_{\rm D}\}$ is a closed subset of $W^{1,r}(\Omega_{\rm D})$. Together with (2.6) one obtains lower semicontinuity of $\mathcal{E}^\kappa(t,\cdot)$ on $\mathcal{Q}_{\rm C}$.

Let now $(q_j)_{j\in\mathbb{N}}\subset L^\kappa_E(t)$. By coercivity (3.6) there are constants $c_1(E),c_2(E)$ such that $\|u_j\|_{W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)}\le c_1(E)$ and $\|z_j\|_{W^{1,r}(\Omega_{\rm D})}\le c_2(E)$. Since $W^{1,p}(\Omega_\pm,\mathbb{R}^d)$ and $W^{1,r}(\Omega_{\rm D})$ are reflexive Banach spaces there are subsequences $u_j\rightharpoonup u$ in $W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ and $z_j\rightharpoonup z$ in $W^{1,r}(\Omega_{\rm D})$. From the lower semicontinuity of $\mathcal{E}^\kappa(t,\cdot)$ on $\mathcal{Q}_{\rm C}$ we now infer $E\ge\liminf_{j\to\infty}\mathcal{E}^\kappa(t,q_j)\ge\mathcal{E}^\kappa(t,q)$, which proves that the sublevels of $\mathcal{E}^\kappa:\mathcal{Q}\to\mathbb{R}_\infty$ are compact in with respect to $\mathcal{T}$. ∎

As a consequence of Proposition 3.4 and Lemma 3.5 we obtain condition (A.1-E1).

Corollary 3.6 *Keep* $\kappa\in(0,\kappa_0]$ *fixed and let the assumptions* (2.5) *and* (2.6) *hold true. Then, for all* $\varepsilon\in(0,\varepsilon_0]$ *the sublevels* $L^{\varepsilon,\kappa}_E(t):=\{q\in\mathcal{Q}\,|\,\mathcal{E}^\kappa_\varepsilon(t,q)\le E\}$ *as well as the sublevels* $L^\kappa_E(t):=\{q\in\mathcal{Q}\,|\,\mathcal{E}^\kappa(t,q)\le E\}$ *are compact and the unions* $\cup_{\varepsilon\in(0,\varepsilon_0]}L^{\varepsilon,\kappa}_E(t)$ *are precompact with respect to the topology* $\mathcal{T}$, *which is defined by* (2.18).

Proof: For all $\varepsilon\in(0,\varepsilon_0]$ and $\kappa\in(0,\kappa_0]$ fixed the weak sequential compactness of the sublevels $L^{\varepsilon,\kappa}_E(t)$ in $W^{1,p}(\Omega,\mathbb{R}^d)\times W^{1,r}(\Omega_{\rm D})$ is due to [TM10, Proposition 3.4], since the composed density $\overline{W}$ from (2.7) satisfies hypotheses (2.6). Since $\mathcal{T}$ is coarser than the weak topology of $W^{1,p}(\Omega,\mathbb{R}^d)\times W^{1,r}(\Omega_{\rm D})$ we conclude the compactness of $L^{\varepsilon,\kappa}_E(t)$ with respect to $\mathcal{T}$. The precompactness of unions of sublevels in $\mathcal{T}$ directly follows from Proposition 3.4 for $t_\varepsilon=t$ and the compactness of $L^\kappa_E(t)$ is due to Lemma 3.5. ∎

In the following we prove the Γ-lim inf-inequality (A.3-C3) for $\mathcal{E}^\kappa_\varepsilon$. The main idea in the proof is to exploit the lower semicontinuity of $\mathcal{E}^\kappa_\varepsilon(t,\cdot)$ on $L^p(\Omega_-^\nu\cup\Omega_-^\nu,\mathbb{R}^{d\times d})\times L^r(\Omega_{\rm D},\mathbb{R}^d)$

for all fixed $\nu \in (0, \varepsilon_0]$. The use of this space is admissible since the lower Γ-limit only has to be verified for stable sequences, so that their energies and hence the damage gradients are uniformly bounded.

Lemma 3.7 (Lower Γ-limit of the energy functionals) *Keep $\kappa \in (0, \kappa_0]$ fixed. Let $(t_\varepsilon, u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, u, z)$ as $\varepsilon \to 0$ and $(u_\varepsilon, z_\varepsilon) \in \mathcal{S}^\kappa_\varepsilon(t_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0]$. Then*

$$\mathcal{E}^\kappa(t, u, z) \leq \liminf_{\varepsilon \to 0} \mathcal{E}^\kappa_\varepsilon(t_\varepsilon, u_\varepsilon, z_\varepsilon)\,. \tag{3.27}$$

Proof: In view of (2.5) it holds $g(t_\varepsilon) \to g(t)$ in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$. Since $(u_\varepsilon, z_\varepsilon) \in \mathcal{S}^\kappa_\varepsilon(t_\varepsilon)$ we find a constant $E > 0$ so that $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, u_\varepsilon, z_\varepsilon) \leq E$ for all $\varepsilon \in (0, \varepsilon_0]$. From Proposition 3.4 then follows that the limit $(u, z) \in \mathcal{Q}_{\mathrm{C}}$. Moreover there is a subsequence $z_\varepsilon \rightharpoonup z$ in $W^{1,r}(\Omega_{\mathrm{D}})$ such that we obtain

$$\liminf_{\varepsilon \to 0} \int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r} |{}_\varepsilon\nabla z_\varepsilon|^r \,\mathrm{d}y \geq \liminf_{\varepsilon \to 0} \int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r} |\nabla_s z_\varepsilon|^r \,\mathrm{d}y \geq \int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r} |\nabla_s z|^r \,\mathrm{d}y = \int_{\Omega_{\mathrm{D}}} \tfrac{\kappa}{r} |\nabla z|^r \,\mathrm{d}y\,, \tag{3.28}$$

where the last equality is due to $\partial_{y_1} z = 0$. Furthermore, we observe that $\int_{\Omega^\nu_- \cup \Omega^\nu_+} W(\cdot)\,\mathrm{d}x$ is weakly sequentially lower semicontinuous on $L^p(\Omega^\nu_- \cup \Omega^\nu_+; \mathbb{R}^{d\times d})$ by (2.6a) and (2.6d). In view of (2.6b) and Proposition 3.4, Item *(1.)* it holds for all $\nu > 0$

$$\liminf_{\varepsilon \to 0} \int_{\Omega^\nu_- \cup \Omega^\nu_+} W(e(u_\varepsilon + g(t_\varepsilon)))\,\mathrm{d}x \geq \int_{\Omega^\nu_- \cup \Omega^\nu_+} W(e(u + g(t)))\,\mathrm{d}x\,. \tag{3.29}$$

Putting together (3.28) and (3.29) we obtain the desired lim inf-estimate as $\nu \to 0$, since $u \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ by Proposition 3.4, Item *(2.)*. ■

Next, we verify the conditions (A.1-E2), (A.1-E3) and (A.3-C1) concerning the time-derivatives of both the approximating and the limit energy functional.

Lemma 3.8 (Properties of $\partial_t \mathcal{E}^\kappa_\varepsilon$, $\partial_t \mathcal{E}^\kappa$) *The functionals $\mathcal{E}^\kappa_\varepsilon$, $\mathcal{E}^\kappa : \mathcal{Q} \to \mathbb{R}_\infty$ satisfy* (A.1-E2). *In particular, $\partial_t \mathcal{E}^\kappa(t, q)$ takes the same form as $\partial_t \mathcal{E}^\kappa_\varepsilon(t, q)$ in* (2.17). *Moreover, $\mathcal{E}^\kappa$ satisfies* (A.1-E2) *and, as $\varepsilon \to 0$,* (A.3-C1) *holds true.*

Proof: Recall $\partial_t \mathcal{E}^\kappa_\varepsilon(t, q)$ from (2.17). Condition (A.1-E2) can be proven by repeating the arguments of [TM10, Theorem 3.7]. The proof mainly uses the stress control (2.6c) to derive a Gronwall estimate for the energy. Furthermore it relies on the assumptions (2.5) for g and on the coercivity inequalities (2.6b). Since $\partial_t \mathcal{E}^\kappa_\varepsilon$ is independent of κ also the constants c_0, c_1 do not depend on κ. Due to the uniform Korn's inequality (2.20) these constants are also independent of $\varepsilon \in (0, \varepsilon_0]$ and hence also apply to the limit energy, so that $\partial_t \mathcal{E}^\kappa(t, q)$ is also given by (2.17).

Conditions (A.1-E3) and (A.3-C1) result from (2.6c). An analogous proof can be found in [TM10, Theorems 3.11, 3.9]. ■

3.2 Conditioned Upper Semicontinuity of Stable Sets

We now verify condition (A.3-C2), saying that the limit of a stable sequence is stable. This will be done by verifying that for all sequences $(t_\varepsilon, q_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]} \subset [0,T] \times \mathcal{Q}$ with $(q_\varepsilon) \in \mathcal{S}^\kappa_\varepsilon(t_\varepsilon)$ and $(t_\varepsilon, q_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, q)$ and for all $(\hat{q}) \in \mathcal{Q}$ there is a sequence $(\hat{q}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]} \subset \mathcal{Q}_{\mathrm{D}}$ satisfying $(\hat{q}_\varepsilon) \xrightarrow{\mathcal{T}} (\hat{q})$ such that

$$\limsup_{\varepsilon\to0} \left(\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, \hat{q}_\varepsilon) + \mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon)\right) \leq \mathcal{E}^\kappa(t, \hat{q}) + \mathcal{R}(\hat{z} - z)\,. \tag{3.30}$$

To gain that $\mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) \to \mathcal{R}(\hat{z} - z)$ we must ensure $\mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) < \infty$ for all $\varepsilon \in (0, \varepsilon_0]$. Moreover, $\hat{u}_\varepsilon \in W^{1,p}(\Omega, \mathbb{R}^d)$ must hold for all $\varepsilon \in (0, \varepsilon_0]$ to assure that $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) < \infty$, whereas the limit $\hat{u} \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$, only. We will construct $(\hat{u}_\varepsilon, \hat{z}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ in such a way that $\mathcal{E}^\kappa_\varepsilon(t_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) \to \mathcal{E}^\kappa(t, \hat{u}, \hat{z})$. This requires an interplay of $\hat{u}_\varepsilon$ and $\hat{z}_\varepsilon$.

The difficulty is to construct $(\hat{u}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ in a way which allows it to prove that

$$\int_{\Omega^\varepsilon_{\mathrm{D}}} \Pi_\varepsilon^{-1} \hat{z}_\varepsilon W(e(\hat{u}_\varepsilon))\,\mathrm{d}x \to 0\,.$$

This construction will be based on reflecting both $\hat{u}_- = \hat{u}|_{\Omega_-}$ and $\hat{u}_+ = \hat{u}|_{\Omega_+}$ at the interface Γ_{C}, i.e. $x_1 = 0$, and on subsequent interpolation on the interval $(-\varepsilon, \varepsilon)$. This method guarantees that $\hat{u}_\varepsilon \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$, in such a way that $\nabla\hat{u}_\varepsilon$ are uniformly bounded for $(x_1, s) \in (-\varepsilon, \varepsilon] \times \left(\Gamma_{\mathrm{C}} \backslash N^{\mathrm{C}}_{\hat{z}}\right)$ and bounded by ε^{-1} on $(-\varepsilon, \varepsilon] \times N^{\mathrm{C}}_{\hat{z}}$, where $N^{\mathrm{C}}_{\hat{z}} := \{s \in \Gamma_{\mathrm{C}} \,|\, T_{\mathrm{C}}\hat{z}(s) = 0\}$.

Lemma 3.9 (Mutual recovery sequences) *Keep $\kappa \in (0, \kappa_0]$ fixed. Let $(\mathcal{Q}, \mathcal{E}^\kappa_\varepsilon, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$ be defined by* (2.12), (2.14), (2.16) *and* (3.1). *Assume that* (2.5) *and* (2.6) *hold true. Moreover, let $\gamma > (p-1)$, $p \in (1, \infty)$ and $r \in (1, \infty)$. Then, for all $(t_\varepsilon, q_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]} \subset [0,T] \times \mathcal{Q}$ with $(t_\varepsilon, q_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, q)$ as $\varepsilon \to 0$ and $q_\varepsilon \in \mathcal{S}^\kappa_\varepsilon(t_\varepsilon)$ and for every $\hat{q} \in \mathcal{Q}$ there is a sequence $(\hat{q}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ such that* (3.30) *holds true.*

Proof: Let $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ and let $(t_\varepsilon, q_\varepsilon) \xrightarrow{\mathcal{T}_T} (t, q)$ as $\varepsilon \to 0$ with $q_\varepsilon \in \mathcal{S}^\kappa_\varepsilon(t_\varepsilon)$. Hence their energies are equibounded and Proposition 3.4 can be applied. Thus, $q \in \mathcal{Q}_{\mathrm{C}}$ with $0 \leq z \leq 1$ a.e. in Ω_{D}, so that $\mathcal{E}^\kappa(t, q)$ is at least finite. For an arbitrary $\hat{q} \in \mathcal{Q}$ we will now construct the mutual recovery sequence $(\hat{q}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ with $\hat{q}_\varepsilon = (\hat{u}_\varepsilon, \hat{z}_\varepsilon)$.

If $\hat{q} \in \mathcal{Q}\backslash\mathcal{Q}_{\mathrm{C}}$, then $\mathcal{E}^\kappa(t_\varepsilon, \hat{q}) = \infty$ for all $\varepsilon \in (0, \varepsilon_0]$ so that (3.30) holds for $\hat{q}_\varepsilon = \hat{q}$. Let now $\hat{q} \in \mathcal{Q}_{\mathrm{C}}$. If $\hat{z} > z$ a.e. in Ω_{D}, then $\mathcal{R}(\hat{z} - z) = \infty$ and (3.30) trivially holds.

Hence, assume $\hat{z} \leq z$ a.e. in Ω_{D}. In order to keep $\mathcal{E}^\kappa_\varepsilon(t, \hat{u}_\varepsilon, \hat{z}_\varepsilon) + \mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon)$ finite, the sequence $(\hat{z}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ has to satisfy $\varepsilon^\gamma \leq \hat{z}_\varepsilon \leq z_\varepsilon$. Furthermore it is required that $\hat{u}_\varepsilon \in \mathcal{U}_{\mathrm{D}}$, i.e. $\hat{u}_\varepsilon \in W^{1,p}(\Omega, \mathbb{R}^d)$ with $\hat{u}_\varepsilon = 0$ on Γ_{Dir}, whereas $\hat{u} \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ with $\hat{u} = 0$ on Γ_{Dir}, $T_{\mathrm{C}}\hat{z}[\![\hat{u}]\!] = 0$ and $[\![\hat{u}\cdot\mathrm{n}_1]\!] \geq 0$ a.e. on Γ_{C}, only. We will first construct $(\hat{z}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ and prove the convergence of the energy terms which solely depend on the damage variable. Then we will construct $(\hat{u}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ in such a way that the interplay of $\hat{u}_\varepsilon$ with $\hat{z}_\varepsilon$ makes the remaining energy terms converge.

Step 1 (Construction of $\hat{z}_\varepsilon$): For every $\varepsilon \in (0, \varepsilon_0]$ we now construct $\hat{z}_\varepsilon$ in such a manner that $\hat{z}_\varepsilon \in \mathcal{Z}_{\mathrm{D}}$ and $\mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) < \infty$, i.e. the property $\varepsilon^\gamma \leq \hat{z}_\varepsilon \leq z_\varepsilon$ a.e. in Ω_{D}

has to be ensured. For this, we adapt the ansatz used in [TM10, Th. 3.14] and we introduce

$$\hat{z}_\varepsilon := \max\left\{\varepsilon^\gamma, \min\{z_\varepsilon, \hat{z}-\delta_\varepsilon\}\right\}, \tag{3.31}$$

where $\delta_\varepsilon = o(\|z_\varepsilon - z\|^r_{L^r(\Omega_\mathrm{D})})$ is determined by Markov's inequality (M) to ensure

$$\mathcal{L}^d\big([|z_\varepsilon - z| > \delta_\varepsilon]\big) \overset{\text{(M)}}{\le} \delta_\varepsilon^{-r}\|z-z_\varepsilon\|^r_{L^r(\Omega_\mathrm{D})}\,\mathrm{d}x \overset{!}{\to} 0\,. \tag{3.32}$$

Here and in the following we use the notation $[f > a] = \{y \in \Omega_\mathrm{D}\,|\,f(y) > a\}$ with a similar meaning for $\ge, <, \le$. Note that $\hat{z}_\varepsilon = \varepsilon^\gamma$ if $\hat{z} - \delta_\varepsilon < \varepsilon^\gamma$ and in particular, if $\hat{z} = 0$. Using a composition lemma for $W^{1,r}$-functions and Lipschitz-functions, see [MM72], one obtains as in [TM10, Th. 3.14]

$$\hat{z}_\varepsilon \in W^{1,r}(\Omega_\mathrm{D}) \quad \text{with} \quad \nabla\hat{z}_\varepsilon(y) = \begin{cases} \nabla\hat{z}(y) & \text{if } y \in A_\varepsilon\,, \\ \nabla z_\varepsilon(y) & \text{if } y \in B_\varepsilon\,, \\ 0 & \text{if } y \in \Omega_\mathrm{D}\backslash(A_\varepsilon \cup B_\varepsilon)\,, \end{cases} \tag{3.33}$$

where $A_\varepsilon = [\varepsilon^\gamma \le \hat{z} - \delta_\varepsilon \le z_\varepsilon]$ and $B_\varepsilon = [z_\varepsilon < \hat{z} - \delta_\varepsilon]$. Because of (3.32) we have $\delta_\varepsilon \to 0$, $\mathcal{L}^d(B_\varepsilon) \to 0$ and one can prove that $\hat{z}_\varepsilon \rightharpoonup \hat{z}$ in $W^{1,r}(\Omega_\mathrm{D})$ as in [TM10, Th. 3.14, step 1]. Because of the compact embedding $W^{1,r}(\Omega_\mathrm{D}) \Subset L^r(\Omega_\mathrm{D})$ we immediately see that $\mathcal{R}(\hat{z}_\varepsilon - z_\varepsilon) \to \mathcal{R}(\hat{z} - z)$.

With the same arguments as in [TM10, Th. 3.14, step 2] we see that

$$\limsup_{\varepsilon\to0}\big(\|\nabla\hat{z}_\varepsilon\|^r_{L^r(\Omega_\mathrm{D})} - \|\nabla z_\varepsilon\|^r_{L^r(\Omega_\mathrm{D})}\big) \le \limsup_{\varepsilon\to0}\|\nabla\hat{z}_\varepsilon\|^r_{L^r(A_\varepsilon)} - \liminf_{\varepsilon\to0}\|\nabla z_\varepsilon\|^r_{L^r(A_\varepsilon\cup C_\varepsilon)}\,,$$

where $\|\nabla\hat{z}\|^r_{L^r(A_\varepsilon)} \le \|\nabla\hat{z}\|^r_{L^r(\Omega_\mathrm{D})}$ for all $\varepsilon \in (0, \varepsilon_0]$. Moreover, to increase the estimate, we may drop the sets C_ε in the $-\liminf$-term. We define $\hat{W}(I, Z) = I|Z|^r$ and introduce $\mathcal{C}(I, z) = \int_{\Omega_\mathrm{D}} \hat{W}(I, \nabla z)\,\mathrm{d}y$, where I stands for the indicator function of a subset in Ω_D. Hence, $\mathcal{C}(I_{A_\varepsilon}, z_\varepsilon) = \|\nabla z_\varepsilon\|_{L^r(A_\varepsilon)} = \|I_{A_\varepsilon}\nabla z_\varepsilon\|_{L^r(\Omega_\mathrm{D})}$. Since $\mathcal{L}^d(A_\varepsilon) \to \mathcal{L}^d(\Omega)$ by (3.32), we have that $I_{A_\varepsilon} \to I_{\Omega_\mathrm{D}}$ strongly in $L^q(\Omega_\mathrm{D})$ for any $q \in [1, \infty)$ and $\nabla z_\varepsilon \rightharpoonup \nabla z$ weakly in $L^r(\Omega_\mathrm{D}, \mathbb{R}^d)$. Hence, by the lower semicontinuity result [Dac00, p. 96, Theorem 3.23] it is $\liminf_{\varepsilon\to0}\mathcal{C}(I_{A_\varepsilon}, z_\varepsilon) \ge \mathcal{C}(\Omega_\mathrm{D}, z) = \|\nabla z\|^r_{L^r(\Omega_\mathrm{D})}$.

Step 2 (Construction of $\hat{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}$): For every $\varepsilon \in (0, \varepsilon_0]$ we now determine $(\hat{u}_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ in such a way that $\hat{u}_\varepsilon \in \mathcal{U}_\mathrm{D}$, see (2.8). Since $(\hat{u}, \hat{z}) \in \mathcal{Q}_\mathrm{C}$ we have $\hat{u} \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$, $\hat{u} = 0$ on Γ_Dir, $T_\mathrm{C}\hat{z}[\![\hat{u}]\!] = 0$ and $[\![\hat{u}\cdot\mathrm{n}_1]\!] \ge 0$ a.e. on Γ_C.

Let $\hat{u}^\pm := \hat{u}|_{\Omega_\pm}$, set $I_\varepsilon^+ := [0, \varepsilon)$ and $I_\varepsilon^- := [-\varepsilon, 0)$. For our construction we reflect $\hat{u}^+|_{I_\varepsilon^+\times\Gamma_\mathrm{C}}$ and $\hat{u}^-|_{I_\varepsilon^-\times\Gamma_\mathrm{C}}$ along the interface $\{0\}\times\Gamma_\mathrm{C}$ and take the additive mean of these functions. Therewith we obtain an interpolated function $\hat{u}^\varepsilon \in W^{1,p}(\Omega_\mathrm{D}^\varepsilon, \mathbb{R}^d)$, which has the form

$$\hat{u}^\varepsilon(x_1, s) := \tfrac{\varepsilon-x_1}{2\varepsilon}\hat{u}^-(\pm x_1, s) + \tfrac{\varepsilon+x_1}{2\varepsilon}\hat{u}^+(\mp x_1, s) \text{ for } x_1 \in I_\varepsilon^\mp, \tag{3.34}$$

i.e. $\hat{u}^\varepsilon(-\varepsilon, s) = \hat{u}^-(-\varepsilon, s)$, $\hat{u}^\varepsilon(\varepsilon, s) = \hat{u}^+(\varepsilon, s)$, $\hat{u}^\varepsilon(0, s) = \frac{1}{2}\big(\hat{u}^+(0, s) + \hat{u}^-(0, s)\big)$. We compose the functions $\hat{u}_\varepsilon \in W^{1,p}(\Omega, \mathbb{R}^d)$ as follows

$$\hat{u}_\varepsilon(x_1, s) := \begin{cases} \hat{u}^\pm(x_1, s) & \text{if } (x_1, s) \in \Omega_\pm^\varepsilon, \\ \hat{u}^\varepsilon(x_1, s) & \text{if } (x_1, s) \in \Omega_\mathrm{D}^\varepsilon. \end{cases} \tag{3.35}$$

By construction it is $\hat{u}_\varepsilon \in W^{1,p}(\Omega, \mathbb{R}^d)$ and, since $\hat{u}_\varepsilon|_{\Omega^\varepsilon_\pm} = \hat{u}|_{\Omega^\varepsilon_\pm}$, we have

$$\int_{\Omega^\varepsilon_\pm} W(e(\hat{u}_\varepsilon + g(t_\varepsilon)))\,\mathrm{d}x = \int_{\Omega^\varepsilon_\pm} W(e(\hat{u} + g(t_\varepsilon)))\,\mathrm{d}x \to \int_{\Omega_\pm} W(e(\hat{u} + g(t)))\,\mathrm{d}x\,, \tag{3.36}$$

where we used (2.5) and the dominated convergence theorem.

Step 3 (Proof of $\int_{\Omega^\varepsilon_{\mathrm{D}}} W_{\mathrm{D}}(e(\hat{u}_\varepsilon), \Pi_\varepsilon^{-1}\hat{z}_\varepsilon)\,\mathrm{d}x \to 0$): From the construction (3.31) recall that $\Pi_\varepsilon^{-1}\hat{z}_\varepsilon(x) = \varepsilon^\gamma$ if $\hat{z}(x) = 0$ for all $\varepsilon \in (0, \varepsilon_0]$. In view of the decomposition $\Omega^\varepsilon_{\mathrm{D}} = A_\varepsilon \cup B_\varepsilon \cup C_\varepsilon$ and (2.6b) we have

$$\begin{aligned}&\int_{\Omega^\varepsilon_{\mathrm{D}}} \Pi_\varepsilon^{-1}\hat{z}_\varepsilon |e(\hat{u}^\varepsilon(x_1, s))|^p\,\mathrm{d}x \\ &\leq \int_{A_\varepsilon} \Pi_\varepsilon^{-1}(\hat{z} - \delta_\varepsilon)|e(\hat{u}^\varepsilon(x_1, s))|^p\,\mathrm{d}x + \int_{B_\varepsilon} |e(\hat{u}^\varepsilon(x_1, s))|^p\,\mathrm{d}x + \int_{C_\varepsilon} \varepsilon^\gamma |e(\hat{u}^\varepsilon(x_1, s))|^p\,\mathrm{d}x\,.\end{aligned}$$

Let $N^{\mathrm{C}}_{\hat{z}} := \{s \in \Gamma_{\mathrm{C}} \,|\, T_{\mathrm{C}}\hat{z}(s) = 0\}$. For $y \in B_\varepsilon = [\hat{z} - \delta_\varepsilon > z_\varepsilon]$ we have $\hat{z}(y) > \varepsilon^\gamma$, which implies that $B_\varepsilon \cap \Gamma_{\mathrm{C}} \subset \Gamma_{\mathrm{C}} \backslash N^{\mathrm{C}}_{\hat{z}}$. Similarly, we find

$$A_\varepsilon = [\varepsilon^\gamma < \hat{z} - \delta_\varepsilon \leq z_\varepsilon] = [\varepsilon^\gamma + \delta_\varepsilon < \hat{z} \leq z_\varepsilon + \delta_\varepsilon] \subset [\varepsilon^\gamma < \hat{z}]\,,$$

i.e. also $A_\varepsilon \cap \Gamma_{\mathrm{C}} \subset \Gamma_{\mathrm{C}} \backslash N^{\mathrm{C}}_{\hat{z}}$. Moreover, $C_\varepsilon = [\hat{z} \leq \varepsilon^\gamma + \delta_\varepsilon]$ and hence $N^{\mathrm{C}}_{\hat{z}} \subset C_\varepsilon \cap \Gamma_{\mathrm{C}}$. Because of this, we can estimate

$$\int_{\Omega^\varepsilon_{\mathrm{D}}} \Pi_\varepsilon^{-1}\hat{z}_\varepsilon |e(\hat{u}^\varepsilon(x_1, s))|^p\,\mathrm{d}x \leq \int_{N^{\mathrm{C}}_{\hat{z}}} \int_{-\varepsilon}^{\varepsilon} \varepsilon^\gamma |e(\hat{u}^\varepsilon)|^p\,\mathrm{d}x_1\,\mathrm{d}s + \int_{\Gamma_{\mathrm{C}} \backslash N^{\mathrm{C}}_{\hat{z}}} \int_{-\varepsilon}^{\varepsilon} |e(\hat{u}^\varepsilon)|^p\,\mathrm{d}x_1\,\mathrm{d}s\,,$$

where $|e(\hat{u}^\varepsilon)|^p \leq 2^{p-1}\big(|\partial_{x_1}\hat{u}^\varepsilon|^p + |\nabla_s \hat{u}^\varepsilon|^p\big)$. For notational simplicity denote by $\hat{u}^\pm$ also their even extensions to Ω by reflection at $x_1 = 0$. In particular, $\hat{u}^\pm \in W^{1,p}(\Omega, \mathbb{R}^d)$. Using that $0 < (\varepsilon \pm x_1)/(2\varepsilon) < 1$ on $I_\varepsilon^- \cup I_\varepsilon^+$ we find

$$\|\nabla_s \hat{u}^\varepsilon\|_{L^p(\Omega^\varepsilon_{\mathrm{D}}, \mathbb{R}^{d-1\times d-1})} \leq 2\|\nabla \hat{u}^-\|_{L^p(I_\varepsilon^- \times \Gamma_{\mathrm{C}}, \mathbb{R}^{d\times d})} + 2\|\nabla \hat{u}^+\|_{L^p(I_\varepsilon^+ \times \Gamma_{\mathrm{C}}, \mathbb{R}^{d\times d})} \to 0\,. \tag{3.37}$$

Moreover, $\partial_{x_1}\hat{u}^\varepsilon = G_1^\varepsilon + G_2^\varepsilon$ with

$$G_1^\varepsilon = \frac{\varepsilon - x_1}{2\varepsilon}\partial_{x_1}\hat{u}^- + \frac{\varepsilon + x_1}{2\varepsilon}\partial_{x_1}\hat{u}^+ \quad \text{and} \quad G_2^\varepsilon = (2\varepsilon)^{-1}(\hat{u}^+ - \hat{u}^-)\,. \tag{3.38}$$

Again, $\|G_1^\varepsilon\|_{L^p(\Omega^\varepsilon_{\mathrm{D}}, \mathbb{R}^d)} \to 0$ as in (3.37), while G_2^ε needs special consideration.

Since $\hat{u} \in W^{1,p}(\Omega \backslash N^{\mathrm{C}}_{\hat{z}})$ it holds for a.e. $s \in \Gamma_{\mathrm{C}} \backslash N^{\mathrm{C}}_{\hat{z}}$ that $\hat{u}^+(0, s) = \hat{u}^-(0, s)$ and hence we find using Hölder's inequality

$$\begin{aligned}|\hat{u}^+(x_1, s) - \hat{u}^-(x_1, s)| &\leq \left|\int_0^{x_1} \partial_\xi \hat{u}^+(\xi, s)\,\mathrm{d}\xi\right| + \left|\int_0^{x_1} \partial_\xi \hat{u}^-(\xi, s)\,\mathrm{d}\xi\right| \\ &\leq C|x_1|^{\frac{p-1}{p}}\Big(\|\partial_{x_1}\hat{u}^+(\cdot, s)\|_{L^p(I_\varepsilon^+, \mathbb{R}^d)} + \|\partial_{x_1}\hat{u}^-(\cdot, s)\|_{L^p(I_\varepsilon^-, \mathbb{R}^d)}\Big)\,.\end{aligned} \tag{3.39}$$

Dividing by 2ε and integrating over $(x_1, s) \in (I_\varepsilon^- \cup I_\varepsilon^+) \times \Gamma_{\mathrm{C}} \backslash N^{\mathrm{C}}_{\hat{z}}$ yields

$$\|G_2^\varepsilon\|^p_{L^p((I_\varepsilon^- \cup I_\varepsilon^+) \times \Gamma_{\mathrm{C}} \backslash N^{\mathrm{C}}_{\hat{z}}, \mathbb{R}^d)} \leq C_\star\Big(\|\partial_{x_1}\hat{u}^+\|^p_{L^p(\Omega^\varepsilon_{\mathrm{D}}, \mathbb{R}^d)} + \|\partial_{x_1}\hat{u}^-\|^p_{L^p(\Omega^\varepsilon_{\mathrm{D}}, \mathbb{R}^d)}\Big) \to 0 \tag{3.40}$$

as $\varepsilon \to 0$, since the constant $C_\star$ is independent of ε.

For $s \in N_{\hat z}^{\mathrm{C}}$ we have in general $\hat u^+(0,s) \neq \hat u^-(0,s)$. Then we find

$$
\begin{aligned}
&|\hat u^+(x_1,s) - \hat u^-(x_1,s)| \\
&\leq \big|[\![\hat u]\!](s)\big| + \left|\int_0^{x_1} \partial_\xi \hat u^+(\xi,s)\,\mathrm{d}\xi\right| + \left|\int_0^{x_1} \partial_\xi \hat u^-(\xi,s)\,\mathrm{d}\xi\right|
\end{aligned} \tag{3.41}
$$

Handling the last terms as in (3.39) leads to

$$
\begin{aligned}
&\|G_2^\varepsilon\|^p_{L^p((I_\varepsilon^- \cup I_\varepsilon^+)\times N_{\hat z}^{\mathrm{C}},\mathbb{R}^d)} \\
&\leq C\varepsilon^{1-p}\big\|[\![\hat u]\!]\big\|_{L^p(\Gamma_{\mathrm{C}},\mathbb{R}^d)} + C_\star\Big(\|\partial_{x_1}\hat u^+\|^p_{L^p(\Omega_{\mathrm{D}}^\varepsilon,\mathbb{R}^d)} + \|\partial_{x_1}\hat u^-\|^p_{L^p(\Omega_{\mathrm{D}}^\varepsilon,\mathbb{R}^d)}\Big),
\end{aligned} \tag{3.42}
$$

where the second term tends to 0 as in (3.40). Using that $\Pi_\varepsilon^{-1}\hat z_\varepsilon = \varepsilon^\gamma$ on $I_\varepsilon^- \cup I_\varepsilon^+ \times N_{\hat z}^{\mathrm{C}}$ with $\gamma > p-1$, we obtain that the term in W_{D} related to the first term in (3.42) will tend to 0 as $\varepsilon \to 0$.

In order to show that also $\int_{\Omega_{\mathrm{D}}^\varepsilon} \varphi(\mathrm{tr}\, e(\hat u^\varepsilon))\,\mathrm{d}x \to 0$ we apply the upper growth estimate in (2.3) and we use that $\big|\big(\mathrm{tr}\, e(\hat u^\varepsilon)\big)^-\big|^{\tilde p} \leq 2^{\tilde p-1}|(\partial_{x_1}\hat u_1^\varepsilon)^-|^{\tilde p} + 2^{\tilde p-1}|\nabla_s \hat u^\varepsilon|^{\tilde p}$ with $\tilde p \in \{\hat p, 1\}$. The integral on $\Omega_{\mathrm{D}}^\varepsilon$ over the second term tends to 0 as in (3.37). For the integral over the first term we use that $(\partial_{x_1}\hat u_1^\varepsilon)^- \leq (G_1^{\varepsilon 1})^- + (G_2^{\varepsilon 1})^-$, where $G_i^{\varepsilon 1}$ denotes the first component of $G_i^\varepsilon \in \mathbb{R}^d$, $i \in \{1,2\}$. We obtain that the integral on $\Omega_{\mathrm{D}}^\varepsilon$ over $|(G_1^{\varepsilon 1})^-|^{\tilde p}$ tends to 0 again as in (3.37). For the term involving $(G_2^{\varepsilon 1})^-$ we use that

$$
\begin{aligned}
&\big(\hat u_1^+(x_1,s) - \hat u_1^-(x_1,s)\big)^- \\
&\leq \big(\hat u_1^+(x_1,s) - \hat u_1^+(0,s)\big)^- + \big([\![\hat u\cdot \mathrm{n}_1]\!](s)\big)^- + \big(\hat u^-(0,s) - \hat u^-(x_1,s)\big)^-,
\end{aligned}
$$

where $\big([\![\hat u\cdot\mathrm{n}_1]\!]\big)^- = 0$ since $\mathcal{E}^\kappa(t,\hat u,\hat z) < \infty$. On the remaining terms we apply integration by parts, Jensen's and Hölder's inequality and find

$$
\|G_2^{\varepsilon 1}\|^{\tilde p}_{L^{\tilde p}((I_\varepsilon^- \cup I_\varepsilon^+)\times N_{\hat z}^{\mathrm{C}},\mathbb{R}^d)} \leq C_\star\Big(\|\partial_{x_1}\hat u^+\|^{\tilde p}_{L^{\tilde p}(\Omega_{\mathrm{D}}^\varepsilon,\mathbb{R}^d)} + \|\partial_{x_1}\hat u^-\|^{\tilde p}_{L^{\tilde p}(\Omega_{\mathrm{D}}^\varepsilon,\mathbb{R}^d)}\Big) \to 0,
$$

due to $\hat p \in (1,p]$ and $\|(\partial_{x_1}\hat u_1^\pm)^-\|^p_{L^p(\Omega,\mathbb{R}^d)} \leq C$ by the equiboundedness of the energies. ■

4 The Second Γ-limit: Griffith-type Delamination

In this section we prove that the gradient delamination models $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})_{\kappa\in(0,\kappa_0]}$ approximate a model $(\mathcal{Q},\mathcal{E},\mathcal{R})$ for Griffith-type delamination as $\kappa \to 0$. Here, $\mathcal{R}:\mathcal{Z}\to[0,\infty]$ is given by (2.16) and

$$
\mathcal{E}(t,q) := \begin{cases} \int_{\Omega_-\cup\Omega_+} W(e(u+g(t)))\,\mathrm{d}x & \text{if } q=(u,z)\in\mathcal{Q}_{\mathrm{G}}, \\ \infty & \text{if } q\in\mathcal{Q}\backslash\mathcal{Q}_{\mathrm{G}}, \end{cases} \tag{4.1}
$$

$$
\mathcal{Z}_{\mathrm{G}} := \{z\in L^\infty(\Omega_{\mathrm{D}})\,|\,0\leq z\leq 1 \text{ and } \partial_{y_1} z = 0 \text{ a.e. in } \Omega_{\mathrm{D}}\}, \tag{4.2}
$$

$$
\mathcal{Q}_{\mathrm{G}} := \Big\{(u,z)\in\mathcal{U}\times\mathcal{Z}_{\mathrm{G}}\,\Big|\,[\![u\cdot\mathrm{n}_1]\!]\geq 0 \text{ and } T_{\mathrm{C}} z[\![u]\!] = 0 \text{ a.e. on } \Gamma_{\mathrm{C}}\Big\}, \tag{4.3}
$$

with $\mathcal{U}$ as in (2.12) and with T_{C} explained by (4.4). For sequences $(u_\kappa, z_\kappa)_{\kappa\in(0,\kappa_0]}$ with equibounded energies there is a subsequence $z_\kappa \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_{\mathrm{D}})$ and due to $\partial_{y_1} z_\kappa = 0$ a.e. in Ω_{D} for all $\kappa\in(0,\kappa_0]$ we find that $z \in L^\infty(\Omega_{\mathrm{D}})$ is constant a.e. in y_1-direction. By the definition of the weak derivative we can verify that $\partial_{y_1} z = 0$ a.e. in Ω_{D} is the weak y_1-derivative of $z \in L^\infty(\Omega_{\mathrm{D}})$. This allows us to define the trace of z on Γ_{C} by

$$T_{\mathrm{C}} z(s) = \tfrac{1}{2}\int_{-1}^{1} z(y_1, s)\,\mathrm{d}y_1\,. \tag{4.4}$$

Then, for all $z \in \mathcal{Z}_{\mathrm{C}}$ from (3.2) definition (4.4) coincides with the trace in the usual sense and for all $v \in \mathcal{Z}_{\mathrm{G}}$ it is

$$\mathcal{R}(v) = \begin{cases} 2\int_{\Gamma_{\mathrm{C}}} -\varrho\, T_{\mathrm{C}} v\,\mathrm{d}s & \text{if } T_{\mathrm{C}} v \le 0 \text{ a.e. on } \Gamma_{\mathrm{C}},\\ \infty & \text{otherwise,} \end{cases} \tag{4.5}$$

so that $(\mathcal{Q}_{\mathrm{G}}, \mathcal{E}, \mathcal{R})$ indeed models delamination along the interface Γ_{C}.

For all $t\in[0,T]$ the stable sets of $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ are given by

$$\begin{aligned}\mathcal{S}^\kappa(t) &:= \{q=(u,z)\in\mathcal{Q} \mid \mathcal{E}^\kappa(t,q)<\infty,\ \mathcal{E}^\kappa(t,q)\le\mathcal{E}^\kappa(t,\tilde q)+\mathcal{R}(\tilde z - z) \text{ for all } \tilde q=(\tilde u,\tilde z)\in\mathcal{Q}\}\,,\\ \mathcal{S}(t) &:= \{q=(u,z)\in\mathcal{Q} \mid \mathcal{E}(t,q)<\infty,\ \mathcal{E}(t,q)\le\mathcal{E}(t,\tilde q)+\mathcal{R}(\tilde z - z) \text{ for all } \tilde q=(\tilde u,\tilde z)\in\mathcal{Q}\}.\end{aligned}$$

Because a function $f \in L^\infty(\Gamma_{\mathrm{C}})$ is only defined $\mathcal{L}^{d-1}$-a.e. on Γ_{C}, its support $\operatorname{supp}^{\mathrm{C}} f$ and its zero set N_f^{C} have to be defined with care. Using the ideas of [Fed69, p. 60] we introduce

$$\begin{aligned}\operatorname{supp}^{\mathrm{C}} f &:= \cap\{A\subset\Gamma_{\mathrm{C}} \mid A \text{ closed},\ \mathcal{L}^{d-1}\big(\{s\in\Gamma_{\mathrm{C}} \mid f(s)\neq 0\}\backslash A\big)=0\}\,,\\ N_f^{\mathrm{C}} &:= \Gamma_{\mathrm{C}}\backslash \operatorname{supp}^{\mathrm{C}} f = \cup\{\mathcal{O}\subset\Gamma_{\mathrm{C}} \mid \mathcal{O} \text{ open},\ \mathcal{L}^{d-1}\big(\mathcal{O}\cap\{s\in\Gamma_{\mathrm{C}} \mid f(s)\neq 0\}\big)=0\}\,.\end{aligned} \tag{4.6}$$

Clearly, $\operatorname{supp}^{\mathrm{C}} f$ is closed and N_f^{C} is open and they are well-defined for equivalence classes $f \in L^\infty(\Gamma_{\mathrm{C}})$. The following lemma is a direct consequence of (4.6), see [Tho10, Lemma 4.3.1].

Lemma 4.1 *Let $f \in L^\infty(\Gamma_{\mathrm{C}})$, $g \in \mathrm{C}^0(\overline{\Gamma_{\mathrm{C}}})$ and let $\operatorname{OS} g := \{s\in\Gamma_{\mathrm{C}} \mid g(s)\neq 0\}$ denote the open support of g. Then*

$$f(s)g(s) = 0 \textit{ for a.e. } s\in\Gamma_{\mathrm{C}} \quad \textit{is equivalent to} \quad \operatorname{supp}^{\mathrm{C}} f\cap \operatorname{OS} g = \emptyset\,. \tag{4.7}$$

The following example emphasizes the interaction of u and z for $(u,z)\in\mathcal{Q}_{\mathrm{G}}$ and shows that the proper definition of N_z^{C} is crucial.

Example 4.2 Let $M\subset\Gamma_{\mathrm{C}}$ be closed and nowhere dense, i.e. M has an empty interior. Let $0 < \mathcal{L}^{d-1}(M) < \mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})$. Such a set can be constructed similarly to Cantor's middle third set, see e.g. [Els02, p. 70 & Exercise 8.9]. Consider $z = 1 - I_M \in L^\infty(\Gamma_{\mathrm{C}})$, i.e. $z=0$ on M and $z=1$ on $\Gamma_{\mathrm{C}}\backslash M$. Then $N_z^{\mathrm{C}} = \emptyset \neq M$. Let $(u,z)\in\mathcal{Q}_{\mathrm{G}}$. Thus, it holds $[\![u]\!]=0$ on $\Gamma_c\backslash M$ and $[\![u]\!]\ge 0$ on M. Because of $p>d$ we have that $[\![u]\!]\in\mathrm{C}^0(\Gamma_{\mathrm{C}})$ and $\{s\in\Gamma_{\mathrm{C}} \mid [\![u]\!]>0\}$ is open. By $\operatorname{int} M=\emptyset$ we conclude that $\{s\in\Gamma_{\mathrm{C}} \mid [\![u]\!]>0\}=\emptyset$, i.e. $[\![u]\!]=0$ on Γ_{C}. Thus, if $z=0$ holds only on a nowhere dense subset of Γ_{C}, then u cannot jump on Γ_{C} at all, although possibly $\mathcal{L}^{d-1}(M)>0$. □

As can be seen from (4.1), the values of $\mathcal{E}(t,u,z)$ are independent of the particular values of z. Moreover Example 4.2 shows that, for $p>d$, only the set N_z is of importance. In the following we prove that the system $(\mathcal{Q},\mathcal{E},\mathcal{R})$ for Griffith-type delamination favours energetic solutions (u,z) with either $z(t,y)=0$ or $z(t,y)=z_0(y)$, where z_0 is a given initial condition.

Lemma 4.3 (Stability of majorants) *Let $(u,z)\in\mathcal{S}(t)$. Consider $\tilde z\geq z$ such that $\{y\in\Omega_{\mathrm{D}}\,|\,\tilde z(y)=0\}=\{y\in\Omega_{\mathrm{D}}\,|\,z(y)=0\}$. Then also $(u,\tilde z)\in\mathcal{S}(t)$.*

Proof: We check the stability condition (1.2 S) for an arbitrary state $(\hat u,\hat z)$. If $\hat z>\tilde z$ on a set of positive measure, then $\mathcal{R}(\hat z-\tilde z)=\infty$ and (1.2 S) is trivially satisfied. Hence it remains to investigate the case $\hat z\leq\tilde z$ a.e. on Ω_{D}.

If $z\leq\hat z\leq\tilde z$ a.e., then we have already $\mathcal{E}(t,\hat u,\hat z)\geq\mathcal{E}(t,u,\tilde z)$, so that (1.2 S) holds for this choice of $(\hat u,\hat z)$. Assume now that $\hat z\leq z\leq\tilde z$. The stability of (u,z) and the fact that $\tilde z\geq z$ then yield

$$\mathcal{E}(t,\hat u,\hat z)=\mathcal{E}(t,u,\tilde z)\leq\mathcal{E}(t,\hat u,\hat z)+\mathcal{R}(\hat z-z)\leq\mathcal{E}(t,\hat u,\hat z)+\mathcal{R}(\hat z-\tilde z)\,.$$

Finally consider $\hat z$ such that $\hat z\leq z\leq\tilde z$ on $A\subset\Omega_{\mathrm{D}}$ and $\tilde z>\hat z>z$ on $\Omega_{\mathrm{D}}\backslash A$ for a set $A\subset\Omega_{\mathrm{D}}$ with $\mathcal{L}^d(A)>0$. We introduce a function $\bar z$ such that $\bar z:=\hat z$ in A and $\bar z:=z$ in $\Omega_{\mathrm{D}}\backslash A$. From the stability of (u,z) we obtain

$$\mathcal{E}(t,u,\tilde z)=\mathcal{E}(t,u,z)\leq\mathcal{E}(t,\hat u,\bar z)+\mathcal{R}(\bar z-z)\leq\mathcal{E}(t,\hat u,\hat z)+\mathcal{R}(\hat z-\tilde z)\,,$$

due to $\mathcal{R}(\bar z-z)=\int_A(z-\hat z)\,\mathrm{d}y\leq\int_A(\tilde z-\hat z)\,\mathrm{d}y\leq\mathcal{R}(\hat z-\tilde z)$. ∎

Proposition 4.4 (Griffith-crack property) *Let $(\mathcal{Q},\mathcal{E},\mathcal{R})$ be given by (2.12), (4.1) and (2.16) such that assumptions (2.5) and (2.6) hold true. Let $(u_0,z_0)\in\mathcal{Q}$ be a given initial value such that $(u_0,z_0)\in\mathcal{S}(0)$. Let $(u,z):[0,T]\to\mathcal{Q}$ be an energetic solution of $(\mathcal{Q},\mathcal{E},\mathcal{R})$. Then $(u,\tilde z)$ is also an energetic solution, where*

$$\tilde z(t,y):=\begin{cases} z_0(y) & \textit{if } z(t,y)>0,\\ 0 & \textit{else.}\end{cases}$$

Moreover, for all $t\in[0,T]$ it is $z(t,\cdot)=\tilde z(t,\cdot)\in L^\infty(\Omega_{\mathrm{D}})$.

Proof: Since $(u(t),z(t))\in\mathcal{S}(t)$ Lemma 4.3 implies that also $(u(t),\tilde z(t))\in\mathcal{S}(t)$. Thus, it remains to verify the energy balance (1.2 E). We have $\mathcal{E}(t,u(t),\tilde z(t))=\mathcal{E}(t,u(t),z(t))$ and $\partial_t\mathcal{E}(t,u(t),\tilde z(t))=\partial_t\mathcal{E}(t,u(t),z(t))$. Moreover, due to the monotonicity of $\tilde z$ and z with $\tilde z\geq z$ it holds that

$$\mathrm{Diss}_{\mathcal{R}}(\tilde z,[0,t])=\mathcal{R}(\tilde z(t)-z_0)\leq\mathcal{R}(z(t)-z_0)=\mathrm{Diss}_{\mathcal{R}}(z,[0,t])\,. \tag{4.8}$$

Hence, the upper energy estimate for $(u,\tilde z):[0,T]\to\mathcal{Q}$ follows. The lower energy estimate, which is a direct consequence of stability (see e.g. [FM06, p. 70] for a proof) then yields equality in (1.2 E). This implies equality in (4.8) and for all $t\in[0,T]$ we conclude that $\tilde z(t,\cdot)=z(t,\cdot)\in L^\infty(\Gamma_{\mathrm{C}})$. ∎

We now state the Γ-convergence result from gradient to Griffith-type delamination.

Theorem 4.5 (Γ-convergence of the delamination problems) *Let the assumptions* (2.5) *and* (2.6) *hold with $p > d$ and $r \in (1,\infty)$. For all $\kappa \in (0,\kappa_0]$, let $q_\kappa : [0,T] \to \mathcal{Q}$ be an energetic solution of $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})$. If the initial values satisfy $q_0^\kappa \xrightarrow{\mathcal{T}} q_0$ and $\mathcal{E}^\kappa(0,q_0^\kappa) \to \mathcal{E}(0,q_0)$, then the delamination problems $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})_{\kappa\in(0,\kappa_0]}$ Γ-converge to the limit delamination problem $(\mathcal{Q},\mathcal{E},\mathcal{R})$ in the sense of Theorem A.1.*

Proof: We proceed as for Theorem 3.1. Since $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ is independent of κ, Remark 2.2 also proves condition (A.2-D2) as $\kappa \to 0$. Furthermore, for all q with finite energy it holds $\partial_t\mathcal{E}(t,q) = \partial_t\mathcal{E}^\kappa(t,q)$ given by (2.17), so that conditions (A.1-E2), (A.1-E3) and (A.3-C1) hold due to Lemma 3.8. The existence of a subsequence $(q_\kappa)_{\kappa\in(0,\kappa_0]}$ of energetic solutions to $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R},q_0^\kappa)$ converging in $\mathcal{T}$ for all $t \in [0,T]$ can be established as for Theorem 3.1. Conditions (A.1-E1), (A.1-E2) and (A.3-C2) will be shown in the subsequent sections. ∎

4.1 Compactness of the Energy Sublevels and Lower Γ-limit

In Lemma 3.5 it has been verified that the sublevels of the functionals $\mathcal{E}^\kappa(t,\cdot)$ are compact in the topology $\mathcal{T}$. In order to complete the proof of (A.1-E1) it remains to show that unions of sublevels with respect to κ are precompact in $\mathcal{T}$. Moreover, we will show that the sublevels of $\mathcal{E}$ are even compact in the weak topology of $\mathcal{Q}$, i.e. in $W^{1,p}(\Omega_- \cup \Omega_+,\mathbb{R}^d)$ for the displacements, which is important for the proof of the Γ-lim inf-inequality.

Theorem 4.6 (Sequences with equibounded energies) *For all $\kappa \in (0,\kappa_0]$ let $\mathcal{E}^\kappa : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ be given by* (3.1) *so that* (2.5) *and* (2.6) *hold. Moreover, let $E \in \mathbb{R}$ and $(t_\kappa)_{\kappa\in(0,\kappa_0]} \subset [0,T]$. Assume that $\mathcal{E}^\kappa(t_\kappa,u_\kappa,z_\kappa) \le E$ for all $\kappa \in (0,\kappa_0]$. Then*

(1.) there is a subsequence $(u_\kappa,z_\kappa) \rightharpoonup (u,z)$ in $\mathcal{Q}$ and hence also $(u_\kappa,z_\kappa) \xrightarrow{\mathcal{T}} (u,z)$ as $\kappa \to 0$,

(2.) for the limit holds $(u,z) \in \mathcal{Q}_{\mathrm{G}}$, see (4.3)*, and $0 \le T_{\mathrm{C}}z \le 1$ a.e. on Γ_{C}.*

Proof: Ad *(1.)*: From $\mathcal{E}^\kappa(t_\kappa,u_\kappa,z_\kappa) \le E$ and coercivity estimate (3.6) we obtain that $(u_\kappa)_{\kappa\in(0,\kappa_0]}$ is equibounded in $W^{1,p}(\Omega_- \cup \Omega_+,\mathbb{R}^d)$. Since $\mathcal{U} \subset W^{1,p}(\Omega_- \cup \Omega_+,\mathbb{R}^d)$ is a reflexive Banach space there is a subsequence $u_\kappa \rightharpoonup u$ in $\mathcal{U}$ and in $W^{1,p}(\Omega_-^\nu \cup \Omega_+^\nu,\mathbb{R}^d)$ for all $\nu \in (0,\varepsilon_0]$. Furthermore, the equiboundedness of $\mathcal{E}^\kappa(t_\kappa,u_\kappa,z_\kappa)$ implies that $\|z_\kappa\|_{L^\infty(\Omega_{\mathrm{D}})} \le 1$ for all $\kappa \in (0,\kappa_0]$. By Banach-Alaoglu's theorem there is a subsequence $z_\kappa \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_{\mathrm{D}})$. This proves that the subsequence $(u_\kappa,z_\kappa)_{\kappa\in(0,\kappa_0]}$ converges to (u,z) both in the weak topology of $\mathcal{Q}$ and in $\mathcal{T}$.

Ad *(2.)*: For the limit (u,z) of the subsequence $(u_\kappa,z_\kappa)_{\kappa\in(0,\kappa_0]} \subset \mathcal{U}\times\mathcal{Z}_{\mathrm{C}}$ from above we now show that $(u,z) \in \mathcal{Q}_{\mathrm{G}}$. Since $\mathcal{U}$ is a Banach space it clearly holds $u \in \mathcal{U}$. For $z_\kappa \overset{*}{\rightharpoonup} z$ in $L^\infty(\Omega_{\mathrm{D}})$ with $z_\kappa \in W^{1,r}(\Omega_{\mathrm{D}})$, $\partial_{y_1}z_\kappa = 0$ and $0 \le z_\kappa \le 1$ a.e. in Ω_{D} it remains to prove that $z \in \mathcal{Z}_{\mathrm{G}}$, see (4.2). We first verify that $0 \le z \le 1$ a.e. in Ω_{D}. Testing the weak*-convergence with $L^1_+(\Omega_{\mathrm{D}}) = \{\varphi \in L^1(\Omega_{\mathrm{D}}) \,|\, \varphi \ge 0 \text{ a.e. in } \Omega_{\mathrm{D}}\}$ yields $0 \le \lim_{\kappa\to0}\int_{\Omega_{\mathrm{D}}}\varphi z_\kappa\,\mathrm{d}y = \int_{\Omega_{\mathrm{D}}}\varphi z\,\mathrm{d}y$ for all $\varphi \in L^1_+(\Omega_{\mathrm{D}})$. To conclude that $z \ge 0$ a.e. on Ω_{D} we assume that $z < 0$ on $A \subset \Omega_{\mathrm{D}}$ with $\mathcal{L}^d(A) > 0$. For the indicator function $I_A : \Omega_{\mathrm{D}} \to \{0,1\}$ of the set A holds $I_A \in L^1_+(\Omega_{\mathrm{D}})$, but $\int_A z\,\mathrm{d}y < 0$, which is a contradiction to $\int_{\Omega_\mathrm{D}} \varphi z \,\mathrm{d}y \geq 0$ for all $\varphi \in L^1_+(\Omega_\mathrm{D})$. Hence it indeed holds that $z \geq 0$ a.e. in Ω_D. With the same arguments we obtain that $0 \leq \lim_{\kappa\to 0} \int_{\Omega_\mathrm{D}} \varphi(1-z_\kappa)\,\mathrm{d}y = \int_{\Omega_\mathrm{D}} \varphi(1-z)\,\mathrm{d}y$ for all $\varphi \in L^1_+(\Omega_\mathrm{D})$, which yields that $z \leq 1$ a.e. in Ω_D.

Now we prove that z is constant a.e. in y_1-direction. For all $\kappa \in (0,\kappa_0]$ we obtain $0 = -\int_{\Omega_\mathrm{D}} \partial_{y_1} z_\kappa \varphi \,\mathrm{d}y = \int_{\Omega_\mathrm{D}} z_\kappa \partial_{y_1}\varphi\,\mathrm{d}y$ for all $\varphi \in \mathrm{C}_0^\infty(\Omega_\mathrm{D})$. Hence by the weak*-convergence it holds $0 = \lim_{\kappa\to 0}\int_{\Omega_\mathrm{D}} z_\kappa \partial_{y_1}\varphi\,\mathrm{d}y = \int_{\Omega_\mathrm{D}} z\partial_{y_1}\varphi\,\mathrm{d}y$ for all $\varphi \in \mathrm{C}_0^\infty(\Omega_\mathrm{D})$. The fundamental lemma of the calculus of variations then yields that z is constant a.e. in y_1-direction. Moreover, since $0 \leq z \leq 1$ a.e. in Ω_D we obtain that $0 = T_\mathrm{C} 0 \leq T_\mathrm{C} z \leq T_\mathrm{C} 1 = 1$.

To show (1.1) we use testfunctions $f \in L^1(\Omega_\mathrm{D})$ with $f(y_1,s) = f(s)$ and we find

$$2\int_{\Gamma_\mathrm{C}} f(s) T_\mathrm{C} z_\kappa(s)\,\mathrm{d}s = \int_{\Omega_\mathrm{D}} f z_\kappa \,\mathrm{d}y_1 \mathrm{d}s \to \int_{\Omega_\mathrm{D}} f z \,\mathrm{d}y_1 \mathrm{d}s = 2\int_{\Gamma_\mathrm{C}} f(s) T_\mathrm{C} z(s)\,\mathrm{d}s.$$

This proves in particular that $0 = \int_{\Gamma_\mathrm{C}} T_\mathrm{C} z_\kappa \big|[\![u_\kappa]\!]\big|\,\mathrm{d}s \to \int_{\Gamma_\mathrm{C}} T_\mathrm{C} z \big|[\![u]\!]\big|\,\mathrm{d}s$, since the compactness of the trace operator $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d) \to L^p(\Gamma_\mathrm{C}, \mathbb{R}^d)$ yields $[\![u_\kappa]\!] \to [\![u]\!]$ strongly in $L^p(\Gamma_\mathrm{C}, \mathbb{R}^d)$. Therefore we find a subsequence which converges pointwise a.e. on Γ_C and hence $0 \leq \lim_{\kappa\to 0}[\![u_\kappa \cdot \mathrm{n}_1]\!] = [\![u \cdot \mathrm{n}_1]\!]$ a.e. on Γ_C. ■

For $t_\kappa = t$ fixed the above theorem states the precompactness of unions of energy sublevels both in the weak topology of $\mathcal{Q}$ and in $\mathcal{T}$. It remains to verify the compactness of the sublevels of the limit functional $\mathcal{E}(t,\cdot)$.

Lemma 4.7 (Properties of the limit energy) *Let $\mathcal{E}$ be given by* (4.1) *such that the assumptions* (2.5) *and* (2.6) *hold true. Then $\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ is coercive and weakly sequentially lower semicontinuous on $\mathcal{Q}$ for all $t \in [0,T]$. In particular,* (3.6) *holds for $\kappa = 0$ and $\Omega_\pm^\nu = \Omega_\pm$. Moreover for all $E \in \mathbb{R}$ the sublevels $L_E(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}(t) \leq E\}$ of the functional $\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ are sequentially compact in the weak topology of $\mathcal{Q}$ and hence in $\mathcal{T}$.*

Proof: Estimate (3.6) is a direct consequence of (2.6b), (2.5) and Korn's inequality (2.20). This estimate together with the fact that $\mathcal{E}(t,u,z) = \infty$ if $\|z\|_{L^\infty(\Gamma_\mathrm{C})} > 1$ proves the coercivity of $\mathcal{E}(t,\cdot)$ on $\mathcal{Q}$. Lower semicontinuity follows from convexity (2.6a) and the closedness of $\mathcal{Q}_\mathrm{G} \cap \{(u,z) \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d) \times L^\infty(\Omega_\mathrm{D}) \,|\, 0 \leq z \leq 1 \text{ a.e. in } \Omega_\mathrm{D}\}$ in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d) \times L^\infty(\Omega_\mathrm{D})$, which can be shown as in the proof of Lemma 3.5 using the ideas of the proof of Theorem 4.6, Item *(2.)* Then the compactness of the sublevels in the weak topology of $\mathcal{Q}$ directly follows from the lower semicontinuity and coercivity as in the proof of Lemma 3.5. Since $\mathcal{T}$ is coarser than the the weak topology of $\mathcal{Q}$ the compactness of the sublevels in $\mathcal{T}$ follows. ■

To establish the Γ-lim inf-estimate for $(\mathcal{Q}, \mathcal{E}^\kappa, \mathcal{R})$ we use that stable sequences have equibounded energies, which yields a subsequence even converging weakly in $\mathcal{Q}$.

Theorem 4.8 (Lower Γ-limit of the energy functionals) *Let $\mathcal{E}^\kappa$ and $\mathcal{E}$ be given by* (3.1) *and* (4.1) *such that the assumptions* (2.5) *and* (2.6) *hold. Let $(t_\kappa, q_\kappa) \xrightarrow{\mathcal{T}_T} (t,q)$ as $\kappa \to 0$ with $q_\kappa \in \mathcal{S}^\kappa(t_\kappa)$ for all $\kappa \in (0,\kappa_0]$. Then*

$$\mathcal{E}(t,q) \leq \liminf_{\kappa\to 0} \mathcal{E}^\kappa(t_\kappa, q_\kappa)\,. \tag{4.9}$$

Proof: Since $q_\kappa = (u_\kappa, z_\kappa) \in \mathcal{S}^\kappa(t_\kappa)$ for all $\kappa \in (0, \kappa_0]$ there is a constant $E > 0$ such that $\mathcal{E}^\kappa(t_\kappa, u_\kappa, z_\kappa) \le E$. Thus, Theorem 4.6 can be applied and yields the existence of a subsequence $(u_\kappa, z_\kappa) \rightharpoonup (u, z)$ in $\mathcal{Q}$ with $(u, z) \in \mathcal{Q}_{\rm G}$.

Due to assumptions (2.6) we obtain that the functional $\int_{\Omega_-\cup\Omega_+} W(\cdot)\,\mathrm{d}x$ is weakly sequentially lower semicontinuous on $W^{1,p}(\Omega_-\cup\Omega_+, \mathbb{R}^d)$. Together with (2.5) we deduce $\liminf_{\kappa\to 0} \int_{\Omega_-\cup\Omega_+} W(e(u_\kappa + g(t_\kappa)))\,\mathrm{d}x \ge \int_{\Omega_-\cup\Omega_+} W(e(u + g(t)))\,\mathrm{d}x$. Furthermore it clearly holds $\liminf_{\kappa\to 0} \frac{\kappa}{r} \int_{\Gamma_{\rm C}} |\nabla_s z_\kappa|^r\,\mathrm{d}s \ge 0$, which establishes (4.9). ■

4.2 Conditioned Upper Semicontinuity of the Stable Sets

We show condition (A.3-C2) by proving the existence of a mutual recovery sequence, i.e. for any sequence $(t_\kappa, q_\kappa) \xrightarrow{\mathcal{T}_T} (t, q)$ with $q = (u, z)$ and with $q_\kappa = (u_\kappa, z_\kappa) \in \mathcal{S}^\kappa(t_\kappa)$ for all $\kappa \in (0, \kappa_0]$ and for all $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ our task is to construct a mutual recovery sequence $(\hat{q}_\kappa)_{\kappa\in(0,\kappa_0]}$ with $\hat{q}_\kappa = (\hat{u}_\kappa, \hat{z}_\kappa)$ such that

$$\limsup_{\kappa\to 0} \big(\mathcal{E}^\kappa(t_\kappa, \hat{q}_\kappa) + \mathcal{R}(\hat{z}_\kappa - z_\kappa) - \mathcal{E}^\kappa(t_\kappa, q_\kappa)\big) \le \mathcal{E}(t, \hat{q}) + \mathcal{R}(\hat{z} - z) - \mathcal{E}(t, q)\,. \quad (4.10)$$

In order to constitute $(\hat{z}_\kappa)_{\kappa\in(0,\kappa_0]} \subset W^{1,r}(\Omega_{\rm D})$ for a given function $\hat{z} \in L^\infty(\Omega_{\rm D})$ we have to mollify $T_{\rm C}\hat{z}$ by a sequence of suitable mollifiers $(\eta_\kappa)_{\kappa(0,\kappa_0]} \subset \mathrm{C}_0^\infty(\mathbb{R}^{d-1})$ in such a way that $\int_{\Gamma_{\rm C}} \frac{\kappa}{r}\big(|\nabla T_{\rm C}\hat{z}_\kappa|^r - |\nabla T_{\rm C} z_\kappa|^r\big)\,\mathrm{d}s$ vanishes. For this, we use mollifiers of the form

$$\tilde{\eta}_1(s) := \begin{cases} c\exp(-1/(1-|y|^2)) & \text{if } |s| \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \tilde{\eta}_\rho(s) := \tfrac{1}{\rho^{d-1}}\tilde{\eta}_1(s/\rho)\,,\ \eta_\kappa = \tilde{\eta}_{\rho(\kappa)}\,, \quad (4.11)$$

where c is defined in such a way that $\|\tilde{\eta}_1\|_{L^1(\mathbb{R}^{d-1})} = 1$ and $\rho(\kappa) \to 0$ as $\kappa \to 0$ suitably.

For $T_{\rm C}\hat{z} \in L^\infty(\Gamma_{\rm C})$ the mollification guarantees that $T_{\rm C}\hat{z}_\kappa \to T_{\rm C}\hat{z}$ in $L^q(\Gamma_{\rm C})$ for all $q \in [1, \infty)$, see [Ada75, p. 29, Lemma 2.18]. Moreover, by [Jan71, p. 33, Theorem 39.1] we have

$$\mathrm{supp}(T_{\rm C}\hat{z} * \tilde{\eta}_\rho) \subset \mathrm{supp}^{\rm C}\,\hat{z} + B_\rho^{\rm C}(0) = \{s + \tilde{s} \,|\, s \in \mathrm{supp}_{\,\rm C}\hat{z},\, \tilde{s} \in B_\rho^{\rm C}(0)\}\,, \quad (4.12)$$

where $B_\rho^{\rm C}(0) \subset \Gamma_{\rm C}$ is the closed ball of radius ρ around 0 and $\mathrm{supp}^{\rm C}\,\hat{z} = \mathrm{supp}\,T_{\rm C}\hat{z}$.

We define $\hat{z}_\kappa(y_1, s) = T_{\rm C}\hat{z}_\kappa(s)$ for a.e. $(y_1, s) \in \Omega_{\rm D}$, so that $\hat{z}_\kappa \in \mathcal{Z}_{\rm G}$.

Since in general $N_{\hat{z}_\kappa}^{\rm C} \not\subset N_{\hat{z}}^{\rm C}$, it is necessary to modify $\hat{u}$ so that the modified functions $\hat{u}_\kappa$ satisfy $[[\![\hat{u}_\kappa]\!] > 0] \subset N_{\hat{z}_\kappa}^{\rm C}$. In order to verify (4.10) we want that $\mathcal{E}^\kappa(t_\kappa, \hat{u}_\kappa, \hat{z}_\kappa) \to \mathcal{E}(t, \hat{u}, \hat{z})$. This can be guaranteed if $\hat{u}_\kappa \to \hat{u}$ strongly in $W^{1,p}(\Omega_-\cup\Omega_+, \mathbb{R}^d)$. In the following we prove the existence of this sequence for the case $p > d$, since the continuity of $[\![\hat{u}]\!]$ on $\overline{\Gamma_{\rm C}}$ then allows us to conclude from Lemma 4.1 that $(\hat{u}, \hat{z}) \in \mathcal{Q}_{\rm G}$ is equivalent to $\mathrm{supp}^{\rm C}\,\hat{z} \cap \mathrm{OS}\,[\![\hat{u}]\!] = \emptyset$. We will apply a Hardy inequality according to [Lew88, p. 190].

Proposition 4.9 *Let $\hat{M} \subset \overline{\Gamma_{\rm C}}$ be closed and let $\Omega_\pm \subset \mathbb{R}^d$ as in Fig. 1. Assume that $p > d$. Let $d_{\hat{M}}(x) := \min_{\hat{x}\in\hat{M}} |x - \hat{x}|$ for all $x \in \overline{\Omega_\pm}$. For all $u \in W_{\hat{M}}^{1,p}(\Omega_\pm, \mathbb{R}^d)$ with $W_{\hat{M}}^{1,p}(\Omega_\pm, \mathbb{R}^d) := \{\tilde{u} \in W^{1,p}(\Omega_\pm, \mathbb{R}^d) \,|\, \tilde{u} = 0 \text{ on } \hat{M} \cup \Gamma_{\rm Dir}\}$ it holds $(u/d_{\hat{M}}) \in L^p(\Omega_\pm, \mathbb{R}^d)$. In particular, there is a constant $C_{\hat{M}} > 0$ such that*

$$\big\|u/d_{\hat{M}}\big\|_{L^p(\Omega_\pm,\mathbb{R}^d)} \le C_{\hat{M}}\, \|\nabla u\|_{L^p(\Omega_\pm,\mathbb{R}^{d\times d})}\,. \quad (4.13)$$

We now construct a sequence $(\hat{u}_\kappa)_{\kappa\in(0,\kappa_0]}$ such that $T_{\mathrm{C}}\hat{z}_\kappa[\![\hat{u}_\kappa]\!] = 0$ a.e. on Γ_{C}. For this, let $\hat{u}_{\mathrm{sym}}(x_1, s) = \frac{1}{2}\big(\hat{u}(x_1, s) + \hat{u}(-x_1, s)\big)$ and $\hat{u}_{\mathrm{anti}}(x_1, s) = \frac{1}{2}\big(\hat{u}(x_1, s) - \hat{u}(-x_1, s)\big)$. Then, $\hat{u}_{\mathrm{sym}} \in W^{1,p}(\Omega, \mathbb{R}^d)$ and $\hat{u}_{\mathrm{anti}} \in W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$, which satisfies $\hat{u}_{\mathrm{anti}}(0, s) = 0$ if and only if $[\![\hat{u}]\!](s) = 0$ for $s \in \Gamma_{\mathrm{C}}$, in particular, $\hat{u}_{\mathrm{anti}} = 0$ on $\hat{M} = \mathrm{supp}^{\mathrm{C}}\,\hat{z}$, i.e. $\hat{u}_{\mathrm{anti}} \in W^{1,p}(\Omega_- \cup \hat{M} \cup \Omega_+, \mathbb{R}^d)$. We use cut-off functions that push $\hat{u}_{\mathrm{anti}}$ to 0 in a suitable neighborhood of $\hat{M}$. Thanks to Proposition 4.9 we can show for $p > d$ that this construction converges strongly in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ as the size of the neighborhood tends to 0.

Corollary 4.10 *Let $p > d$ and $\hat{u}\in W^{1,p}(\Omega_-\cup\hat{M}\cup\Omega_+, \mathbb{R}^d)$ with $\hat{u}=0$ on Γ_{Dir} in the trace sense. With $\xi^{\hat{M}}_\rho(x) := \min\big\{\frac{1}{\rho}\big(d_{\hat{M}}(x) - \rho\big)^+, 1\big\}$ set*

$$\hat{u}^\rho(x_1, s) := \hat{u}_{\mathrm{sym}}(x_1, s) + \xi^{\hat{M}}_\rho(x_1, s)\, \hat{u}_{\mathrm{anti}}(x_1, s)\,. \tag{4.14}$$

Then the following statements hold:

(i) $\hat{u}^\rho \to \hat{u}$ *strongly in* $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$,

(ii) $\hat{u} \in W^{1,p}(\Omega_- \cup \hat{M} \cup \Omega_+, \mathbb{R}^d) \Rightarrow \hat{u}^\rho \in W^{1,p}(\Omega_- \cup (\hat{M} + B_\rho(0)) \cup \Omega_+, \mathbb{R}^d)$ *with* $B_\rho(0) \subset \mathbb{R}^d$,

(iii) $[\![\hat{u} \cdot \mathrm{n}_1]\!] \geq 0 \Rightarrow [\![\hat{u}^\rho{\cdot}\mathrm{n}_1]\!] \geq 0$.

Proof: Recall that $\hat{u}_{\mathrm{sym}} \in W^{1,p}(\Omega, \mathbb{R}^d)$ is fixed in $\hat{u}^\rho$, so that it suffices to verify the statements for $\hat{u}^\rho_{\mathrm{anti}} = \xi^{\hat{M}}_\rho \hat{u}_{\mathrm{anti}}$. From $\xi^{\hat{M}}_\rho$ positive and $[\![\hat{u}{\cdot}\mathrm{n}_1]\!] \geq 0$ it follows $[\![\hat{u}^\rho_{\mathrm{anti}}{\cdot}\mathrm{n}_1]\!] \geq 0$, which proves *(iii)*. Note that

$$\xi^{\hat{M}}_\rho(x) \begin{cases} = 0 & \text{if } d_{\hat{M}}(x) \leq \rho, \\ \in (0, 1) & \text{if } \rho < d_{\hat{M}}(x) \leq 2\rho, \\ = 1 & \text{if } 2\rho < d_{\hat{M}}(x), \end{cases} \quad \text{and} \quad \xi^{\hat{M}}(x) := \begin{cases} 0 & \text{if } x \in \hat{M}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.15}$$

Hence $\hat{u}^\rho_{\mathrm{anti}} = 0$ in $\hat{M} + B_\rho(0)$. This implies $\hat{u}^\rho \in W^{1,p}(\Omega_-\cup(\hat{M} + B_\rho(0))\cup\Omega_+, \mathbb{R}^d)$, so that *(ii)* holds.

It remains to prove *(i)*. From (4.15) we see that $\xi^{\hat{M}}_\rho \to \xi^{\hat{M}}$ pointwise in Ω. With $A_\rho := [d_{\hat{M}}(x) \leq \rho]$, $B_\rho := [\rho < d_{\hat{M}}(x) \leq 2\rho]$ and $C_\rho := [2\rho < d_{\hat{M}}(x)]$ we obtain by the dominated convergence theorem that

$$\|\hat{u}^\rho_{\mathrm{anti}} - \hat{u}_{\mathrm{anti}}\|^p_{L^p(\Omega,\mathbb{R}^d)} = \int_{A_\rho} |\hat{u}_{\mathrm{anti}}|^p \,\mathrm{d}x + \int_{B_\rho} |(\xi^{\hat{M}}_\rho - \xi^{\hat{M}})\hat{u}_{\mathrm{anti}}|^p \,\mathrm{d}x + \int_{C_\rho} |0|^p \,\mathrm{d}x \to 0\,,$$

due to $\mathcal{L}^d\big([d_{\hat{M}}(x) \leq \rho]\big) \to 0$, $\mathcal{L}^d\big([\rho < d_{\hat{M}}(x) \leq 2\rho]\big) \to 0$ and $|\xi^{\hat{M}}_\rho - \xi^{\hat{M}}| \leq 1$ for all $\rho > 0$.

By the chain rule we calculate that $\nabla\hat{u}^\rho_{\mathrm{anti}} = \xi^{\hat{M}}_\rho\nabla\hat{u}_{\mathrm{anti}} + \hat{u}_{\mathrm{anti}}\otimes\nabla\xi^{\hat{M}}_\rho$. Thus,

$$\|\nabla(\hat{u}^\rho_{\mathrm{anti}}{-}\hat{u}_{\mathrm{anti}})\|_{L^p} \leq \|(1{-}\xi^{\hat{M}}_\rho)\nabla\hat{u}_{\mathrm{anti}}\|_{L^p} {+} \|\hat{u}_{\mathrm{anti}}\otimes\nabla\xi^{\hat{M}}_\rho\|_{L^p}$$

where $\|(1 - \xi^{\hat{M}}_\rho)\nabla\hat{u}_{\mathrm{anti}}\|_{L^p(\Omega_-\cup\Omega_+,\mathbb{R}^{d\times d})} \to 0$ again by dominated convergence.

It remains to show that $\|\hat{u}_{\mathrm{anti}}\otimes\nabla\xi^{\hat{M}}_\rho\|_{L^p(\Omega_-\cup\Omega_+,\mathbb{R}^{d\times d})} \to 0$. We obtain that

$$|\nabla\xi^{\hat{M}}_\rho(x)| = \begin{cases} 0 & \text{if } 0 \leq d_{\hat{M}}(x) \leq \rho, \\ 1/\rho & \text{if } \rho < d_{\hat{M}}(x) \leq 2\rho, \\ 0 & \text{if } 2\rho < d_{\hat{M}}(x)\,, \end{cases}$$

i.e. $|\nabla\xi_\rho^{\hat M}|\le 1/\rho$. Since $d_{\hat M}(x)\in[\rho,2\rho]$ it holds $1/\rho\le\frac{2}{d_{\hat M}(x)}$ for all $x\in\Omega$. We conclude

$$\|\hat u_{\mathrm{anti}}\otimes\nabla\xi_\rho^{\hat M}\|^p_{L^p(\Omega_-\cup\Omega_+,\mathbb{R}^{d\times d})}\le 2^p\int_{B_{2\rho}(\hat M)\backslash B_\rho(\hat M)}\left|\frac{\hat u_{\mathrm{anti}}(x)}{d_{\hat M}(x)}\right|^p\,\mathrm{d}x\to 0,$$

since $\|\hat u_{\mathrm{anti}}/d_{\hat M}\|_{L^p(\Omega_-\cup\Omega_+,\mathbb{R}^d)}$ is bounded by Corollary 4.9 and since

$$\mathcal{L}^d\big(B_{2\rho}(\hat M)\backslash B_\rho(\hat M)\big)\to 0 \text{ for } B_{2\rho}(\hat M)\backslash B_\rho(\hat M)=\{x\in\Omega\,|\,\rho<d_{\hat M}(x)\le 2\rho\}\,.$$

■

With these tools at hand we now prove the existence of a mutual recovery sequence under the assumption that $r\in(1,\infty)$. In particular we have to determine the mollifiers η_κ in such a way that their slopes grow sufficiently slow, so that $\int_{\Omega_\mathrm{D}}\frac{\kappa}{r}\big(|\nabla\hat z_\kappa|^r-|\nabla z_\kappa|^r\big)\,\mathrm{d}y$ vanishes. In order to verify this, we will exploit the Lipschitz-continuity of $|\cdot|^r$.

Theorem 4.11 (Mutual recovery sequences) *Let $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})$ and $(\mathcal{Q},\mathcal{E},\mathcal{R})$ be given by* (2.12), (3.1), (2.16) *and* (4.1), *such that the assumptions* (2.5) *and* (2.6) *hold true with $p>d$ and $r\in(1,\infty)$. Then, for all $(t_\kappa,q_\kappa)\xrightarrow{\mathcal{T}_T}(t,q)$ with $q_\kappa\in\mathcal{S}^\kappa(t_\kappa)$ for all $\kappa\in(0,\kappa_0]$ and for every $\hat q\in\mathcal{Q}$ there is a sequence $(q_\kappa)_{\kappa\in(0,\kappa_0]}$ such that* (4.10) *holds.*

Proof: Let $(t_\kappa,u_\kappa,z_\kappa)\xrightarrow{\mathcal{T}_T}(t,u,z)$ with $q_\kappa=(u_\kappa,z_\kappa)\in\mathcal{S}^\kappa(t_\kappa)$ for every $\kappa\in(0,\kappa_0]$. Consider $\hat q=(\hat u,\hat z)\in\mathcal{Q}$. If $\hat q\in\mathcal{Q}\backslash\mathcal{Q}_\mathrm{G}$, then $\mathcal{E}(t_\kappa,\hat q)=\infty$ for all $\kappa\in(0,\kappa_0]$ and (4.10) trivially holds. Hence, assume that $\hat q\in\mathcal{Q}_\mathrm{G}$. Additionally let $0\le\hat z\le z$ a.e. in Ω_D, otherwise $\mathcal{R}(\hat z-z)=\infty$. For every $\kappa\in(0,\kappa_0]$ we now have to construct the mutual recovery sequence $(\hat u_\kappa,\hat z_\kappa)_{\kappa\in(0,\kappa_0]}\subset\mathcal{Q}$ in such a way that $\hat q_\kappa=(\hat u_\kappa,\hat z_\kappa)\in\mathcal{Q}_\mathrm{C}$ and $\mathcal{R}(\hat z_\kappa-z_\kappa)<\infty$ for all $\kappa\in(0,\kappa_0]$. This means in particular that $\hat z_\kappa\in W^{1,r}(\Omega_\mathrm{D})$, whereas $\hat z\in L^\infty(\Omega_\mathrm{D})$, only. Additionally it is required that $\hat z_\kappa\le z_\kappa$ a.e. in Ω_D. The construction of $(\hat z_\kappa)_{\kappa\in(0,\kappa_0]}$ will be done in Step 1. In Step 2 we verify that $\int_{\Omega_\mathrm{D}}\frac{\kappa}{r}\big(|\nabla\hat z_\kappa|^r-|\nabla z_\kappa|^r\big)\,\mathrm{d}s\to 0$. Finally, in Step 3, we specify $\hat u_\kappa$ using Corollary 4.10.

Step 1 (Construction of $\hat z_\kappa$): For all $\kappa\in(0,\kappa_0]$ we now construct $\hat z_\kappa$. We have $\hat z\in L^\infty(\Omega_\mathrm{D})$ with $0\le\hat z\le 1$ being constant a.e. in y_1-direction, whereas $\hat z_\kappa$ has to satisfy $\hat z_\kappa\in W^{1,r}(\Omega_\mathrm{D})$ with $\partial_{y_1}\hat z_\kappa=0$ and $0\le\hat z_\kappa\le 1$. First, we put

$$\zeta(y):=\begin{cases}\hat z(y)/z(y) & \text{if } z(y)>0\,,\\ 0 & \text{if } z(y)=0.\end{cases}\tag{4.16}$$

Due to the assumption $0\le\hat z\le z$ it clearly holds that $0\le\zeta\le 1$ a.e. in Ω_D. We mollify $T_\mathrm{C}\zeta$ by convolution with the sequence $(\eta_\kappa)_{\kappa\in(0,\kappa_0]}\subset\mathrm{C}_0^\infty(\mathbb{R}^{d-1})$ of (4.11), where the dependence of ρ from κ will be specified below. For all $\kappa\in(0,\kappa_0]$ the convolution leads to functions $\tilde\zeta_\kappa=T_\mathrm{C}\zeta*\eta_\kappa$ which satisfy $\tilde\zeta_\kappa\to T_\mathrm{C}\zeta$ strongly in $L^q(\Gamma_\mathrm{C})$ for all $q\in[1,\infty)$ by [Ada75, Lemma 2.18], since $\hat z/z\in L^q(\Omega_\mathrm{D})$. Then we set $\zeta_\kappa(y_1,s)=\tilde\zeta_\kappa(s)$ for all $(y_1,s)\in\Omega_\mathrm{D}$. As the final recovery sequence we introduce

$$\hat z_\kappa:=z_\kappa\zeta_\kappa\quad\text{for all }\kappa\in(0,\kappa_0]\,,\tag{4.17}$$

which satisfies $0\le\hat z_\kappa\le z_\kappa$. Since $z_\kappa\overset{*}{\rightharpoonup}z$ in $L^\infty(\Omega_\mathrm{D})$ by assumption, $\tilde\zeta_\kappa\to T_\mathrm{C}\zeta$ in $L^1(\Gamma_\mathrm{C})$ and thus $\zeta_\kappa\to\zeta$ in $L^1(\Omega_\mathrm{D})$ we have $\hat z_\kappa\rightharpoonup\hat z$ in $L^1(\Omega_\mathrm{D})$, and hence

$$\lim_{\kappa\to0}\mathcal{R}(\hat z_\kappa-z_\kappa)=\lim_{\kappa\to0}\varrho\int_{\Omega_\mathrm{D}}(z_\kappa-\hat z_\kappa)\,\mathrm{d}y=\mathcal{R}(\hat z-z)\,.\tag{4.18}$$

Application of the chain rule yields that $\nabla\hat{z}_\kappa = \nabla(\zeta_\kappa z_\kappa)=\zeta_\kappa\nabla z_\kappa+z_\kappa\nabla\zeta_\kappa \in L^r(\Omega_\mathrm{D},\mathbb{R}^d)$ as well as $\partial_{y_1}\hat{z}_\kappa=0$ due to $\partial_{y_1}z_\kappa=0$ and $\partial_{y_1}\zeta_\kappa=0$.

In order to ensure that $\frac{\kappa}{r}\|\nabla\hat{z}_\kappa\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \to 0$ as $\kappa\to 0$ we now determine the radius $\rho(\kappa)$ for the mollifiers $\eta_\kappa = \tilde{\eta}_{\rho(\kappa)}$ suitably. For $\tilde{\eta}_\rho$ from (4.11) we have

$$\|\nabla(T_\mathrm{C}\zeta*\tilde{\eta}_\rho)\|^r_{L^r(\Gamma_\mathrm{C},\mathbb{R}^{d-1})} \leq \|T_\mathrm{C}\zeta\|_{L^\infty(\Gamma_\mathrm{C})}\|\nabla\tilde{\eta}_\rho\|^r_{L^r(\Gamma_\mathrm{C},\mathbb{R}^{d-1})} \leq \|\nabla\tilde{\eta}_1\|^r_{L^r(\Gamma_\mathrm{C},\mathbb{R}^{d-1})}\rho^{-r(d-1)}\,. \tag{4.19}$$

Hence, $\rho(\kappa)$ has to be chosen in such a way that $\kappa\rho^{-r(d-1)}\to 0$. This is satisfied e.g. for $\rho(\kappa)=\kappa^{1/(2r(d-1))}$. We define $\eta_\kappa=\tilde{\eta}_{\rho(\kappa)}$.

Step 2 (Cancellation argument): Up to now our construction makes sure that $\frac{\kappa}{r}\|\nabla\zeta_\kappa\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \leq C\kappa\rho^{-r(d-1)}\to 0$ as $\kappa\to 0$. Since $\frac{\kappa}{r}\|\nabla z_\kappa\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)}$ is only uniformly bounded by the properties of stable sequences, we conclude that $\frac{\kappa}{r}\|\nabla\hat{z}_\kappa\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)}$ may not vanish completely. However, in the lim sup-estimate (4.10) we can compensate the remaining terms by the term $-\frac{\kappa}{r}\|\nabla z_\kappa\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)}$ that occurs in $\mathcal{E}^\kappa(t_\kappa,u_\kappa,z_\kappa)$. In order to show that these terms indeed cancel out we use the following Lipschitz-estimate for $w(x)=|x|^r$ with $r\in(1,\infty)$ and $x\in\mathbb{R}$, which can be obtained by a Taylor expansion:

$$|w(a)-w(b)| = \left|\int_0^1 w'(b+\alpha(a-b))(a-b)\,\mathrm{d}\alpha\right| \leq 2^{r-1}(|a|^{r-1}+|b|^{r-1})|a-b| \tag{4.20}$$

for all $a,b\in\mathbb{R}$. Using $0\leq\zeta_\kappa\leq 1$ and $0\leq z_\kappa\leq 1$ a.e. in Ω_D, estimate (4.20) and Hölder's inequality imply

$$\begin{aligned}
&\int_{\Omega_\mathrm{D}}\tfrac{\kappa}{r}\big(|\nabla\hat{z}_\kappa|^r-|\nabla z_\kappa|^r\big)\,\mathrm{d}y \leq \int_{\Omega_\mathrm{D}}\tfrac{\kappa}{r}\big((|\nabla\zeta_\kappa|+|\nabla z_\kappa|)^r-|\nabla z_\kappa|^r\big)\,\mathrm{d}y\\
&\leq 2^{r-1}\int_{\Omega_\mathrm{D}}\tfrac{\kappa}{r}\big(2^{r-1}|\nabla\zeta_\kappa|^{r-1}+2^r|\nabla z_\kappa|^{r-1}\big)|\nabla\zeta_\kappa|\,\mathrm{d}y\\
&\leq \tfrac{2^{2r-2}}{r}\kappa\|\nabla\zeta_\kappa\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} + \tfrac{2^{2r-1}}{r}\kappa^{1-1/r}\|\nabla z_\kappa\|^{r-1}_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)}\kappa^{1/r}\|\nabla\zeta_\kappa\|_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)} \;\to 0\,,
\end{aligned}$$

since $\kappa\|\nabla\zeta_\kappa\|^r_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)}\to 0$ by construction and $\kappa^{1-1/r}\|\nabla z_\kappa\|^{r-1}_{L^r(\Omega_\mathrm{D},\mathbb{R}^d)}\leq C$ due to the properties of stable sequences.

Step 3 (Convergence of $\boldsymbol{\mathcal{E}^\kappa(t_\kappa,\hat{q}_\kappa)}$): Because of $\hat{z}_\kappa = z_\kappa\zeta_\kappa$ we find $\operatorname{supp}^\mathrm{C}\hat{z}_\kappa = \operatorname{supp}^\mathrm{C} z_\kappa\cap\operatorname{supp}^\mathrm{C}\zeta_\kappa \subset \operatorname{supp}^\mathrm{C}\zeta + B^\mathrm{C}_{\rho(\kappa)}(0)$. Hence, for $\mathcal{E}^\kappa(t_k,\hat{q}_k)<\infty$ it suffices to show that $[\![\hat{u}_\kappa]\!]=0$ on $\operatorname{supp}T_\mathrm{C}\hat{z}_\kappa$. Since $p>d$ we can apply Corollary 4.10 and set

$$\hat{u}_\kappa := \hat{u}^{\rho(\kappa)} \quad\text{with}\quad \hat{M}=\operatorname{supp}^\mathrm{C}\hat{z}\,. \tag{4.21}$$

where $\rho(\kappa)$ is determined by (4.19). From (4.12), Corollary 4.10 *(ii)* and Lemma 4.1 we infer that $T_\mathrm{C}\hat{z}_\kappa[\![\hat{u}_\kappa]\!]=0$ on Γ_C. By Corollary 4.10 *(i)* we have $\hat{u}_\kappa\to\hat{u}$ and $(\hat{u}_\kappa+g(t_\kappa))\to(\hat{u}+g(t))$ strongly in $W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ by (2.5). Because of (2.6b), a Taylor expansion gives $\int_{\Omega_-\cup\Omega_+}W(e(\hat{u}_\kappa+g(t_\kappa)))\,\mathrm{d}x\to\int_{\Omega_-\cup\Omega_+}W(e(\hat{u}+g(t)))\,\mathrm{d}x$. This finishes the proof of the lim sup-estimate (4.10). ■

5 Simultaneous Convergence

In the Sections 3 and 4 we proved that energetic solutions of the Griffith-type delamination problem $(\mathcal{Q},\mathcal{E},\mathcal{R})$ can be approximated by energetic solutions of the partial damage models $(\mathcal{Q},\mathcal{E}^\varepsilon_\kappa,\mathcal{R})$ via a double limit (first $\varepsilon\to 0$ and then $\kappa\to 0$). That is, we performed the intermediate step of first approximating energetic solutions of the gradient delamination problems $(\mathcal{Q},\mathcal{E}^\kappa,\mathcal{R})$ as $\varepsilon\to 0$. In this section we show that one can merge this double limit passage to a simultaneous convergence. For this, we have to prove the existence of a κ-dependent upper bound $G:(0,\kappa_0]\to(0,\varepsilon_0]$ for the parameter ε. The growth of this function G is on the one hand determined by the assumption $\kappa^{1/(r(d-1))}/\rho(\kappa)\to 0$, which is needed to control the gradient of the mollified delamination variable for the construction of the recovery sequence as $\kappa\to 0$, see formula (4.19). On the other hand it stems from the fact, that the property $\partial_{y_1}z=0$ on Ω_D for the limit $z\in L^\infty(\Omega_\mathrm{D})$ of a sequence $(z^\kappa_\varepsilon)_{\varepsilon\in(0,\varepsilon_0],\kappa\in(0,\kappa_0]}\subset W^{1,r}(\Omega_\mathrm{D})$ with $z^\kappa_\varepsilon\overset{*}{\rightharpoonup}z$ requires that $\varepsilon/\kappa^{1/r}\to 0$ as $(\varepsilon,\kappa)\to(0,0)$, as can be seen from formula (3.6). These two requirements imply that

$$\varepsilon\ll\kappa^{1/r}\ll\kappa^{1/(r(d-1))}\ll\rho(\kappa)\quad\text{for}\quad 0<\kappa<\kappa_0\ll 1\,. \tag{5.1}$$

For the upper bound on ε we choose a function $G:(0,\kappa_0]\to(0,\varepsilon_0]$ with the property

$$G(\kappa)/\kappa^{1/r}\to 0\quad\text{as}\quad\kappa\to 0\,. \tag{5.2}$$

This relation is essential to show the simultaneous limit. Moreover, to obtain this result for sequences $(\varepsilon,\kappa)\to(0,0)$ simultaneously, the crucial step is the construction of a joint mutual recovery sequence. We formalize this construction with the aid of so-called recovery operators, which are defined as follows.

Definition 5.1 (Recovery operators) *A family $(\mathfrak{R}_h)_{h\in(0,h_0]}$ with $\mathfrak{R}_h:\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to\mathcal{Q}$ for all $h>0$ is called a family of recovery operators, if for a given stable sequence $(t_h,q_h)_{h\in(0,h_0]}$ with $(t_h,q_h)\xrightarrow{\mathcal{T}_T}(t,q)$ and any testfunction $\hat q\in\mathcal{Q}$ the sequence $\hat q_h=\mathfrak{R}_h(\hat q,q,q_h)$ provides a mutual recovery sequence, i.e.*

$$\limsup_{h\to 0}\big(\mathcal{E}_h(t_h,\hat q_h)+\mathcal{R}(\hat q_h-q_h)-\mathcal{E}_h(t_h,q_h)\big)\le\mathcal{E}(t,\hat q)+\mathcal{R}(\hat q-q)-\mathcal{E}(t,q)\,.$$

Speaking in this notion the recovery sequence constructed in Lemma 3.9 as $\varepsilon\to 0$ is formed by recovery operators $\mathfrak{R}_\varepsilon=(\mathfrak{R}^\mathcal{U}_\varepsilon,\mathfrak{R}^\mathcal{Z}_\varepsilon):\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to\mathcal{Q}$ with

$$\mathfrak{R}^\mathcal{U}_\varepsilon:\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to\mathcal{U},\ \mathfrak{R}^\mathcal{U}_\varepsilon(\hat q,q,q_\varepsilon)=\hat u_\varepsilon=\hat u_\mathrm{sym}+A_\varepsilon\hat u\,, \tag{5.3}$$
$$\mathfrak{R}^\mathcal{Z}_\varepsilon:\mathcal{Q}\times\mathcal{Q}\times\mathcal{Q}\to\mathcal{Z},\ \mathfrak{R}^\mathcal{Z}_\varepsilon(\hat q,q,q_\varepsilon)=\hat z_\varepsilon=\max\big\{\varepsilon^\gamma,\min\{\hat z-\delta_\varepsilon,z_\varepsilon\}\big\}\,, \tag{5.4}$$

i.e. here, the recovery operators do not depend on all the components of the state $\hat q$, the elements of the stable sequence q_ε and its limit q. In (5.3) it is $\delta_\varepsilon=o(\|z_\varepsilon-z\|_{L^r(\Omega_\mathrm{D})})$. Moreover, for $\hat u\in W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$ we introduced $\hat u_\mathrm{sym}(x_1,s)=\frac12\big(\hat u(x_1,s)+\hat u(-x_1,s)\big)$ and $\hat u_\mathrm{anti}(x_1,s)=\frac12\big(\hat u(x_1,s)-\hat u(-x_1,s)\big)$. Clearly, $\hat u_\mathrm{sym}\in W^{1,p}(\Omega,\mathbb{R}^d)$ and $\hat u_\mathrm{anti}\in W^{1,p}(\Omega_-\cup\Omega_+,\mathbb{R}^d)$. Then, omitting to indicate the dependence of $A_\varepsilon\hat u(x_1,s)$ on $s\in\Gamma_\mathrm{C}$, we set

$$A_\varepsilon\hat u(x_1)=\begin{cases}\frac12\big(\hat u^\pm(x_1)-\hat u^\mp(-x_1)\big) & \text{if } (x_1,s)\in\Omega^\varepsilon_\pm,\\ \frac{\varepsilon-x_1}{4\varepsilon}\big(\hat u^-(\pm x_1)-\hat u^+(\mp x_1)\big)+\frac{\varepsilon+x_1}{4\varepsilon}\big(\hat u^+(\mp x_1)-\hat u^-(\pm x_1)\big) & \text{if } x_1\in I^\mp_\varepsilon,\end{cases}$$

with $\hat u^\pm=\hat u|_{\Omega_\pm}$, $I^-_\varepsilon=(-\varepsilon,0]$ and $I^+_\varepsilon=[0,\varepsilon)$.

The recovery sequence from Lemma 4.11 for $\kappa \to 0$ is similarly formed by recovery operators $\mathfrak{R}_\kappa = (\mathfrak{R}_\kappa^{\mathcal{U}}, \mathfrak{R}_\kappa^{\mathcal{Z}}) : \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ with

$$\mathfrak{R}_\kappa^{\mathcal{U}} : \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q} \to \mathcal{U},\ \mathfrak{R}_\kappa^{\mathcal{U}}(\hat{q}, q, q_\kappa) = \hat{u}_\kappa = \hat{u}_{\mathrm{sym}} + \xi_{\rho(\kappa)}^{\mathrm{supp}^{\mathrm{C}}\hat{z}} \hat{u}_{\mathrm{anti}}\,, \tag{5.5}$$

$$\mathfrak{R}_\kappa^{\mathcal{Z}} : \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q} \to \mathcal{Z},\ \mathfrak{R}_\kappa^{\mathcal{Z}}(\hat{q}, q, q_\kappa) = \hat{z}_\kappa = z_\kappa \eta_{\rho(\kappa)} * T_{\mathrm{C}}(\hat{z}/z)\,, \tag{5.6}$$

where $\kappa\rho(\kappa)^{-r(d-1)} \to 0$ and $\xi_{\rho(\kappa)}^{\mathrm{supp}^{\mathrm{C}}\hat{z}}$ as in Corollary 4.10. Again, we see that these operators do not depend on all the components of $\hat{q}$, q_κ and q.

For the simultaneous limit we now have to compose these two recovery operators $\mathfrak{R}_\varepsilon^\kappa = \mathfrak{R}_\kappa \circ \mathfrak{R}_\varepsilon$ to get a joint mutual recovery sequence by $\hat{q}_\varepsilon^\kappa = \mathfrak{R}_\kappa \circ \mathfrak{R}_\varepsilon(\hat{q}, q, q_\varepsilon^\kappa)$, where $q_\varepsilon^\kappa \in \mathcal{S}_\varepsilon^\kappa(t_\varepsilon^\kappa)$ with $(t_\varepsilon^\kappa, q_\varepsilon^\kappa) \xrightarrow{\mathcal{T}_T} (t, q)$. In particular, we have to specify how the composition $\circ$ has to be understood in our context. From the construction (5.3)-(5.6) we see that the recovery operators $\mathfrak{R}_\varepsilon$ and $\mathfrak{R}_\kappa$ of our problems do not depend on all the components of $\mathcal{Q} \times \mathcal{Q} \times \mathcal{Q}$. Moreover, to get a finite energy it is necessary that the recovery operators map to a subspace of $\mathcal{Q}$, that is $\mathcal{Q}_{\mathrm{C}}$ for $\mathfrak{R}_\varepsilon$ and $\mathcal{Q}_{\mathrm{G}}$ for $\mathfrak{R}_\kappa$, respectively. For the same reason, also $\mathcal{Q} \times \mathcal{Q} \times \mathcal{Q}$ is restricted to subspaces, namely $\mathfrak{R}_\varepsilon : \mathcal{Q}_{\mathrm{C}} \times \mathcal{Q}_{\mathrm{C}} \times \mathcal{Q}_{\mathrm{D}} \to \mathcal{Q}_{\mathrm{D}}$ and $\mathfrak{R}_\kappa : \mathcal{Q}_{\mathrm{G}} \times \mathcal{Q}_{\mathrm{G}} \times \mathcal{Q}_{\mathrm{C}} \to \mathcal{Q}_{\mathrm{C}}$. For the simultaneous limit passage we now want to plug in testfunctions $\hat{q} \in \mathcal{Q}_{\mathrm{G}}$, elements of stable sequences $q_\varepsilon^\kappa \in \mathcal{S}_\varepsilon^\kappa(t_\varepsilon^\kappa) \subset \mathcal{Q}_{\mathrm{D}}$ for all $\varepsilon \in (0, \varepsilon_0]$, $\kappa \in (0, \kappa_0]$ and their limit $q \in \mathcal{Q}_{\mathrm{G}}$ and we need that $\mathfrak{R}_\varepsilon \circ \mathfrak{R}_\kappa$: $\mathcal{Q}_{\mathrm{G}} \times \mathcal{Q}_{\mathrm{G}} \times \mathcal{Q}_{\mathrm{D}} \to \mathcal{Q}_{\mathrm{D}}$. Recall from (5.6) that $\hat{z}_\kappa^\varepsilon(y_1, s) = z_\kappa^\varepsilon(y_1, s)(\eta_{\rho(\kappa)} * T_{\mathrm{C}}(\hat{z}/z))(s)$, i.e. $\eta_{\rho(\kappa)} * T_{\mathrm{C}}(\hat{z}/z) \in \mathrm{C}^\infty(\Gamma_{\mathrm{C}})$ and multiplication with $z_\varepsilon^\kappa \in W^{1,r}(\Omega_{\mathrm{D}})$ leads to $\hat{z}_\kappa^\varepsilon \in W^{1,r}(\Omega_{\mathrm{D}})$. Since $\partial_{y_1} z_\kappa^\varepsilon \neq 0$, in general, we have $\mathfrak{R}_\kappa^{\mathcal{Z}}(\hat{q}, q, q_\varepsilon^\kappa) = \hat{z}_\kappa^\varepsilon \in \mathcal{Z}_{\mathrm{D}}$ with the property $\hat{z}_\varepsilon^\kappa \leq z_\varepsilon^\kappa$. Hence, in view of (5.4) and (5.6), we can define $\mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{Z}} = \mathfrak{R}_\varepsilon^{\mathcal{Z}} \circ \mathfrak{R}_\kappa^{\mathcal{Z}}$ as follows

$$\mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{Z}}(\hat{q}, q, q_\varepsilon^\kappa) = \mathfrak{R}_\varepsilon^{\mathcal{Z}} \circ \mathfrak{R}_\kappa^{\mathcal{Z}}(\hat{q}, q, q_\varepsilon^\kappa) = \mathfrak{R}_\varepsilon^{\mathcal{Z}}\big(\mathfrak{R}_\kappa^{\mathcal{Z}}(\hat{q}, q, q_\varepsilon^\kappa), q, q_\varepsilon^\kappa\big) = \max\{\varepsilon^\gamma, \hat{z}_\varepsilon^\kappa\}\,. \tag{5.7}$$

From (5.3) and (5.5) we see that $\mathfrak{R}_\varepsilon^{\mathcal{U}}(\cdot, q, q_\varepsilon^\kappa) : \mathcal{Q}_{\mathrm{C}} \to \mathcal{U}_{\mathrm{D}}$ and $\mathfrak{R}_\kappa^{\mathcal{U}}(\cdot, \hat{z}, q, q_\varepsilon^\kappa) : \mathcal{U}_{\mathrm{G}} \to \mathcal{U}_{\mathrm{C}}$ are linear operators. Here, we define the composition $\mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{U}} = \mathfrak{R}_\varepsilon^{\mathcal{U}} \circ \mathfrak{R}_\kappa^{\mathcal{U}}$ by

$$\mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{U}}(\hat{q}, q, q_\varepsilon^\kappa) = \mathfrak{R}_\varepsilon^{\mathcal{U}} \circ \mathfrak{R}_\kappa^{\mathcal{U}}(\hat{q}, q, q_\varepsilon^\kappa) = \mathfrak{R}_\varepsilon^{\mathcal{U}}\big(\mathfrak{R}_\kappa^{\mathcal{U}}(\hat{q}, q, q_\varepsilon^\kappa)\big) = \hat{u}_{\mathrm{sym}} + \xi_{\rho(\kappa)}^{\mathrm{supp}^{\mathrm{C}}\hat{z}} A_\varepsilon \hat{u}\,, \tag{5.8}$$

Now we are in a position to show that $\mathfrak{R}_{\varepsilon,\kappa}$ given by (5.7) and (5.8) is a joint mutual recovery operator for the simultaneous limit passage $(\varepsilon, \kappa) \to (0, 0)$.

Corollary 5.2 (Joint mutual recovery operators) *Let* $(\varepsilon, \kappa) \to (0, 0)$ *under the condition that* $0 < \varepsilon \leq G(\kappa)$ *with* $G : (0, \kappa_0] \to (0, \varepsilon_0]$ *satisfying* (5.2). *Assume that* $r, p \in (1, \infty)$ *and* $\gamma \in (p-1, P)$, *such that* (3.12) *and* (3.9) *are satisfied. Let* (2.5) *and* (2.6) *hold. Then, the operators* $\mathfrak{R}_\varepsilon^\kappa = (\mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{U}}, \mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{Z}}) : \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ *defined by* (5.7) *and* (5.8) *form joint mutual recovery operators for the systems* $(\mathcal{Q}, \mathcal{E}_\varepsilon^\kappa, \mathcal{R})$ *and* $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$.

Proof: Let $\hat{q} \in \mathcal{Q}_{\mathrm{G}}$ and $(t_\varepsilon^\kappa, q_\varepsilon^\kappa) \xrightarrow{\mathcal{T}_T} (t, q)$ as $\kappa \to 0$ with $(t_\varepsilon^\kappa, q_\varepsilon^\kappa) \in \mathcal{S}_\varepsilon^\kappa(t_\varepsilon^\kappa)$. Then, $q_\varepsilon^\kappa \in \mathcal{Q}_{\mathrm{D}}$ for all $\kappa \in (0, \kappa_0]$. For the proof we set $\hat{M} = \mathrm{supp}^{\mathrm{C}}\,\hat{z}$ and in the arguments of the recovery operators (5.3)–(5.6) we only indicate the quantities they depend on.

By (5.7) and (5.8) it is $\mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{U}}(\hat{u}, \hat{M}) = \hat{u}_{\mathrm{sym}} + \xi_{\rho(\kappa)}^{\hat{M}} A_\varepsilon \hat{u}$. Hence $\xi_{\rho(\kappa)}^{\hat{M}} A_\varepsilon \hat{u} = 0$ in $B_{\rho(\kappa)}(\hat{M})$, while $\mathrm{supp}\,\mathfrak{R}_\kappa^{\mathcal{Z}}(\hat{z}, z, z_\varepsilon^\kappa) \subset (-1, 1) \times (B_{\rho(\kappa)}^{\mathrm{C}}(\hat{M}))$, so that $\mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{Z}}(\hat{z}, z, z_\varepsilon^\kappa) = \varepsilon^\gamma$ in $(-1, 1) \times \Gamma_{\mathrm{C}} \backslash (B_{\rho(\kappa)}^{\mathrm{C}}(\hat{M}))$. Moreover, we have

$$e\big(\mathfrak{R}_{\varepsilon,\kappa}^{\mathcal{U}}(\hat{u}, \hat{M})\big) = \nabla\hat{u}_{\mathrm{sym}} + \xi_{\rho(\kappa)}^{\hat{M}} e(A_\varepsilon \hat{u}) + \tfrac{1}{2}\big(A_\varepsilon \hat{u} \otimes \nabla\xi_{\rho(\kappa)}^{\hat{M}} + (A_\varepsilon \hat{u} \otimes \nabla\xi_{\rho(\kappa)}^{\hat{M}})^\top\big)\,. \tag{5.9}$$

Recall that the assumption $\kappa^{1/(r(d-1))}/\rho(\kappa) \to 0$ is needed to control the gradient of the mollified delamination variable for the construction of the recovery sequence as $\kappa \to 0$, see formula (4.19). Moreover, to preserve that $\partial_{y_1} z = 0$ on Ω_{D} requires $\varepsilon/\kappa^{1/r} \to 0$ as $\kappa \to 0$, as can be seen from formula (3.6). Thus, relation (5.1) follows. By the assumptions (5.2) and $\varepsilon \le G(\kappa)$ we have ensured that $\varepsilon/\kappa^{1/r} \to 0$. Hence, clearly $\mathfrak{R}^{\mathcal{Z}}_{\kappa,\varepsilon}(\hat{z}, z, z^\kappa_\varepsilon) \overset{*}{\rightharpoonup} \hat{z}$, so that $\mathcal{R}\big(\mathfrak{R}^{\mathcal{Z}}_{\kappa,\varepsilon}(\hat{z}, z, z^\kappa_\varepsilon) - z^\kappa_\varepsilon\big) \to \mathcal{R}\big(\hat{z} - z\big)$. Moreover, both $\Omega^\varepsilon_\pm \cap B_{\rho(\kappa)}(\hat{M}) \neq \emptyset$ and $\Omega^\varepsilon_{\mathrm{D}} \cap B_{\rho(\kappa)}(\hat{M}) \neq \emptyset$.

In the following we omit indicating the dependence of ε and ρ on κ. Using the positivity of W given by (2.6b), the fact that $A_\varepsilon \hat{u}|_{\Omega^\varepsilon_\pm} = \hat{u}_{\mathrm{anti}}|_{\Omega^\varepsilon_\pm}$, Corollary 4.10 *(i)*, (2.5) and the dominated convergence theorem we find

$$\begin{aligned}\int_{\Omega^\varepsilon_\pm} W\big(e\big(\mathfrak{R}^{\mathcal{U}}_{\varepsilon,\kappa}(\hat{u}, \hat{M}) + g(t_\varepsilon)\big)\big)\,\mathrm{d}x &\le \int_{\Omega_\pm} W\big(e\big(\hat{u}_{\mathrm{sym}} + \xi^{\hat{M}}_\rho \hat{u}_{\mathrm{anti}} + g(t_\varepsilon)\big)\big)\,\mathrm{d}x \\ &\to \int_{\Omega_\pm} W\big(e(\hat{u} + g(t))\big)\,\mathrm{d}x\,.\end{aligned}$$

In view of (5.9) we obtain on $\Omega^\varepsilon_{\mathrm{D}}$

$$\begin{aligned}&\int_{\Omega^\varepsilon_{\mathrm{D}}} \Pi^{-1}_\varepsilon \mathfrak{R}^{\mathcal{Z}}_{\kappa,\varepsilon}(\hat{z}, z, z^\kappa_\varepsilon) \widetilde{W}\big(e\big(\mathfrak{R}^{\mathcal{U}}_{\varepsilon,\kappa}(\hat{u}, \hat{M})\big)\big)\,\mathrm{d}x \\ &\le 3^{p-1}\tilde{c} \int_{\Omega^\varepsilon_{\mathrm{D}}} \Big(|\nabla \hat{u}_{\mathrm{sym}}|^p + \Pi^{-1}_\varepsilon \mathfrak{R}^{\mathcal{Z}}_\kappa(\hat{z}, z, z^\kappa_\varepsilon)|e(A_\varepsilon \hat{u}_{\mathrm{anti}})|^p + \big|A_\varepsilon \hat{u} \otimes \nabla \xi^{\hat{M}}_\rho\big|^p\Big)\,\mathrm{d}x\,,\end{aligned} \tag{5.10}$$

where the first term obviously tends to 0 as $\varepsilon \to 0$. For the third term we proceed as in the proof of Corollary 4.10, i.e. with $D_\rho(\hat{M}) = B_{2\rho}(\hat{M}) \backslash B_\rho(\hat{M})$ we have

$$\int_{\Omega^\varepsilon_{\mathrm{D}}} \big|A_\varepsilon \hat{u} \otimes \nabla \xi^{\hat{M}}_{\rho(\kappa)}\big|^p\,\mathrm{d}x \le \int_{\Omega^\varepsilon_{\mathrm{D}} \cap D_\rho(\hat{M})} 2^p \left|\frac{A_\varepsilon \hat{u}_{\mathrm{anti}}}{d_{\hat{M}}(x)}\right|^p \mathrm{d}x \le \int_{\Omega^\varepsilon_{\mathrm{D}} \cap D_\rho(\hat{M})} 2^p \left|\frac{\hat{u}_{\mathrm{anti}}}{d_{\hat{M}}(x)}\right|^p \mathrm{d}x \to 0\,, \tag{5.11}$$

since $\|\hat{u}_{\mathrm{anti}}/d_{\hat{M}}(x)\|_{L^p(\Omega_- \cup \Omega_+, \mathbb{R}^d)}$ is bounded by Proposition 4.9. Moreover, we have used that $(\varepsilon \pm x_1)/(4\varepsilon) \le 1/2$ for $x_1 \in I_\varepsilon$, where $I_\varepsilon = (-\varepsilon, \varepsilon)$.

Furthermore, the second term in (5.10) can be estimated using that

$$\int_{\Omega^\varepsilon_{\mathrm{D}}} \Pi^{-1}_\varepsilon \mathfrak{R}^{\mathcal{Z}}_{\varepsilon,\kappa}(\hat{z}, z, z^\kappa_\varepsilon)|e(A_\varepsilon \hat{u})|^p\,\mathrm{d}x \le \int_{I_\varepsilon \times (\Gamma_{\mathrm{C}} \backslash B^{\mathrm{C}}_\rho(\hat{M}))} \varepsilon^\gamma |e(A_\varepsilon \hat{u})|^p\,\mathrm{d}x + \int_{(I_\varepsilon \times B^{\mathrm{C}}_\rho(\hat{M})) \backslash B_\rho(\hat{M})} |e(A_\varepsilon \hat{u}_{\mathrm{anti}})|^p\,\mathrm{d}x\,.$$

By repeating the estimates (3.37)-(3.42) we conclude that this term tends to 0.

To verify that also $\int_{\Omega^\varepsilon_{\mathrm{D}}} \varphi\big(e\big(\mathfrak{R}^{\mathcal{U}}_{\varepsilon,\kappa}(\hat{u}, \hat{M})\big)\big)\,\mathrm{d}x \to 0$ we use the upper growth estimate in (2.3) and again formula (5.9). Moreover, since $|\operatorname{tr} A| \le |A|$ for all $A \in \mathbb{R}^{d\times d}$, we see that the terms containing $\operatorname{tr} \nabla \hat{u}_{\mathrm{sym}}$ and $\operatorname{tr}(A_\varepsilon \hat{u} \otimes \nabla \xi^{\hat{M}}_\rho)$ tend to 0 with the same arguments as above. To prove that also the term containing $\operatorname{tr} e(A_\varepsilon \hat{u})$ tends to 0 one has to repeat the corresponding arguments in the proof of Lemma 3.9.

Finally, for the gradient of the delamination variable it is

$$\begin{aligned}&\kappa\big(\|{}_\varepsilon\nabla \mathfrak{R}^{\mathcal{Z}}_{\varepsilon,\kappa}(\hat{z}, z, z^\kappa_\varepsilon)\|^r_{L^r(\Omega_{\mathrm{D}}, \mathbb{R}^d)} - \|{}_\varepsilon\nabla z^\kappa_\varepsilon\|^r_{L^r(\Omega_{\mathrm{D}}, \mathbb{R}^d)}\big) \\ &\le \kappa\big(\|{}_\varepsilon\nabla \mathfrak{R}^{\mathcal{Z}}_\kappa(\hat{z}, z, z^\kappa_\varepsilon)\|^r_{L^r(\Omega_{\mathrm{D}}, \mathbb{R}^d)} - \|{}_\varepsilon\nabla z^\kappa_\varepsilon\|^r_{L^r(\Omega_{\mathrm{D}}, \mathbb{R}^d)}\big)\,.\end{aligned}$$

Then formula (4.19) and the cancellation argument lead to the desired result, since $\kappa^{1-1/r}\|{}_{\varepsilon}\nabla z_\varepsilon^\kappa\|^{r-1}_{L^r(\Omega_{\mathrm{D}},\mathbb{R}^d)} \leq C$ due to the properties of stable sequences. ■

The existence of a joint mutual recovery sequence by Corollary 5.2 implies that the limit (t,q) of a stable sequence $(t_\varepsilon^\kappa, q_\varepsilon^\kappa) \xrightarrow{\mathcal{T}_T} (t,q)$ as $(\varepsilon,\kappa)\to(0,0)$, satisfies $q\in\mathcal{S}(t)$, that is $\mathcal{E}(t,q)\leq\mathcal{E}(t,\tilde q)+\mathcal{R}(\tilde z - z)$ for all $\tilde q=(\tilde u,\tilde z)\in\mathcal{Q}$. This yields that $\mathcal{E}(t,q)<\infty$ and hence $q\in\mathcal{Q}_{\mathrm{G}}$. In particular, this means that $q=(u,z)$ satisfies the transmission and the noninterpenetration condition, see (4.3).

Since $q_\varepsilon^\kappa\in\mathcal{S}_\varepsilon^\kappa(t_\varepsilon^\kappa)$ for all $\varepsilon\in(0,\varepsilon_0]$, $\kappa\in(0,\kappa]$ implies the equiboundedness of the corresponding energies one obtains the existence of a subsequence $q_\varepsilon^\kappa \xrightarrow{\mathcal{T}} q$ by Lemma 3.2 and the definition of $\mathcal{T}$. Thus, we may state the following corollary.

Corollary 5.3 *Let the assumptions of Corollary 5.2 hold true. Consider a family* $(t_\varepsilon^\kappa, u_\varepsilon^\kappa, z_\varepsilon^\kappa)_{\varepsilon(0,\varepsilon_0],\kappa\in(0,\kappa_0]}$ *with* $0<\varepsilon\leq G(\kappa)$ *and* G *as in* (5.2)*, such that* $(u_\varepsilon^\kappa, z_\varepsilon^\kappa)\in\mathcal{S}_\varepsilon^\kappa(t_\varepsilon^\kappa)$ *and* $t_\varepsilon^\kappa\to t$*. Then, there is a subsequence* $(u_{\varepsilon_k}^{\kappa_k}, z_{\varepsilon_k}^{\kappa_k}) \xrightarrow{\mathcal{T}} (u,z)$ *as* $(\varepsilon_k,\kappa_k)\to(0,0)$ *and* $(u,z)\in\mathcal{Q}_{\mathrm{G}}$*, so that the transmission and the noninterpenetration condition* (1.1) *are satisfied.*

Moreover, the simultaneous lower Γ-limit can directly be adopted from Lemmata 3.7 and 4.8. Lemma 3.8 concerning the properties of the partial time-derivatives of the energy functionals and Lemma 4.7 on the limit functional and are valid as well. Hence, we are in a position to conclude with the simultaneous convergence result.

Theorem 5.4 (Simultaneous convergence) *Let the assumptions of Corollary 5.2 hold. For all* $\varepsilon\in(0,\varepsilon_0]$, $\kappa\in(0,\kappa_0]$ *let* $q_\varepsilon^\kappa:[0,T]\to\mathcal{Q}$ *denote energetic solutions of the systems* $(\mathcal{Q},\mathcal{E}_\varepsilon^\kappa,\mathcal{R})$ *and the initial values* $q_0^{\varepsilon,\kappa}$*, which satisfy* $\mathcal{E}_\varepsilon^\kappa(0,q_0^{\varepsilon,\kappa})\to\mathcal{E}(0,q_0)$*. Then every subsequence* $\big(q_{\varepsilon_k}^{\kappa_k}(t)\big)_{k\in\mathbb{N}}$ *with* $\varepsilon_k/\kappa_k^{1/r}\to 0$*, which converges for all* $t\in[0,T]$ *with respect to the topology* $\mathcal{T}$*, has an energetic solution of* $(\mathcal{Q},\mathcal{E},\mathcal{R},q_0)$ *as its limit.*

Proof: The stability inequality (1.2 S) for $q:[0,T]\to\mathcal{Q}$ and $(\mathcal{Q},\mathcal{E},\mathcal{R})$ is a direct consequence of Corollary 5.2. To verify the energy balance (1.2 E) one may repeat the arguments of [MRS08, Theorem 3.1]. Alltogether, this implies that $q:[0,T]\to\mathcal{Q}$ is an energetic solution of $(\mathcal{Q},\mathcal{E},\mathcal{R},q_0)$. ■

A Appendix: Abstract Γ-convergence Result

In [MRS08] the theory of Γ-convergence was adapted to the framework of the energetic formulation of rate-independent processes. In the following we introduce sufficient conditions guaranteeing that a subsequence of energetic solutions of the approximating systems $(\mathcal{Q},\mathcal{E}_j,\mathcal{R}_j)$ converges to an energetic solution of the limit system $(\mathcal{Q},\mathcal{E}_\infty,\mathcal{R}_\infty)$. Let the topology for the convergence of the energetic solutions be denoted by $\mathcal{T}$. Then i.e. we want to obtain that $q_j(t)\xrightarrow{\mathcal{T}} q(t)$ for all $t\in[0,T]$.

For all $j\in\mathbb{N}_\infty=\mathbb{N}\cup\{\infty\}$ we introduce the stable sets
$\mathcal{S}_j(t):=\{q\in\mathcal{Q}\,|\,\mathcal{E}_j(t,q)<\infty,\,\forall\tilde q=(\tilde u,\tilde z):\mathcal{E}_j(t,q_j)\leq\mathcal{E}_j(t,\tilde q)+\mathcal{R}_j(\tilde z-z_j)\}.$

In order to ensure the Γ-convergence of the systems $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)_{j\in\mathbb{N}}$ the following conditions have to be satisfied by the energy functionals $\mathcal{E}_j : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ for all $j \in \mathbb{N}_\infty$.

Compactness of energy sublevels: $\forall\, t\in[0,T]\ \forall\, E\in\mathbb{R}$:

$$\forall j \in \mathbb{N}_\infty :\ L_E^j(t) := \{q \in \mathcal{Q} \,|\, \mathcal{E}_j(t,q) \le E\} \text{ is compact wrt. } \mathcal{T}, \qquad \text{(A.1-E1)}$$
$$\bigcup_{j=1}^{\infty} L_E^j(t) \text{ is relatively compact wrt. } \mathcal{T}\,,$$

Uniform control of the power:

$$\exists\, c_0\in\mathbb{R}\ \exists\, c_1{>}0\ \forall j \in \mathbb{N}_\infty \forall\,(t_q,q)\in[0,T]\times\mathcal{Q} \text{ with } \mathcal{E}(t_q,q) < \infty : \qquad \text{(A.1-E2)}$$
$$\mathcal{E}(\cdot,q) \in \mathrm{c}^1([0,T]) \text{ and } |\partial_t\mathcal{E}(t,q)| \le c_1(c_0{+}\mathcal{E}(t,q)) \text{ for all } t\in[0,T]\,,$$

Uniform time-continuity of $\partial_t\mathcal{E}_\infty$:

$$\forall \varepsilon > 0\, \forall E \in \mathbb{R}\ \exists\, \delta > 0\, \forall q \in \mathcal{Q} \text{ with } \mathcal{E}(0,q) < E : \qquad \text{(A.1-E3)}$$
$$|t_1 - t_2| < \delta \quad \Rightarrow \quad |\partial_t\mathcal{E}_\infty(t_1,q) - \partial_t\mathcal{E}_\infty(t_2,q)| < \varepsilon\,.$$

Furthermore the dissipation distances $\mathcal{D}_j : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ with $\mathcal{D}_j(z,\tilde z) = \mathcal{R}_j(\tilde z{-}z)$ for all $z, \tilde z \in \mathcal{Z}$ must fulfill for all $j \in \mathbb{N}_\infty$:

Quasi-distance:

$$\forall j \in \mathbb{N}_\infty\ \forall\, z_1,z_2,z_3 \in \mathcal{Z} :\quad \mathcal{D}_j(z_1,z_2) = 0 \ \Leftrightarrow z_1 = z_2 \ \text{ and} \qquad \text{(A.2-D1)}$$
$$\mathcal{D}_j(z_1,z_3) \le \mathcal{D}_j(z_1,z_2) + \mathcal{D}_j(z_2,z_3)\,,$$

Semi-continuity:

$$\forall j \in \mathbb{N}_\infty :\ \mathcal{D}_j :\ \mathcal{Z}\times\mathcal{Z} \to [0,\infty] \text{ is lower semi-continuous wrt. } \mathcal{T}\,, \qquad \text{(A.2-D2)}$$

Positivity of $\mathcal{D}_\infty$:

$$\forall \text{ compact } A \subset \mathcal{Z}\,,\ \forall (z_j)_{j\in\mathbb{N}} \subset A :$$
$$\min\{\mathcal{D}_j(z_j,z), \mathcal{D}_j(z,z_j)\} \to 0 \quad \Rightarrow \quad z_j \xrightarrow{\mathcal{T}_\mathcal{Z}} z\,, \qquad \text{(A.2-D3)}$$
where $\mathcal{T}_\mathcal{Z}$ is the restriction of $\mathcal{T}$ to the z-component of $q = (u,z)$.

Additionally the following compatibility conditions have to be satisfied:
For all $t_j \to t$ in $[0,T]$, $q_j = (u_j,z_j) \xrightarrow{\mathcal{T}} q = (u,z)$ with $q_j \in \mathcal{S}_j(t_j)$ for all $j \in \mathbb{N}$ it holds

Conditioned continuous convergence of $\partial_t\mathcal{E}_j$:

$$\partial_t\mathcal{E}_j(t_j,q_k) \to \partial_t\mathcal{E}(t,q)\,, \qquad \text{(A.3-C1)}$$

Conditioned upper semi-continuity of stable sets:

$$q \in \mathcal{S}_\infty(t)\,, \qquad \text{(A.3-C2)}$$

Lower Γ*-limit for* $\mathcal{E}_j$:

$$\mathcal{E}(t,q) \le \liminf_{j\to\infty} \mathcal{E}_j(t_j,q_j)\,, \qquad \text{(A.3-C3)}$$

Lower Γ*-limit for* $\mathcal{D}_j$: Let additionally $\hat q_j = (\hat u_j, \hat z_j) \xrightarrow{\mathcal{T}} \hat q = (\hat u, \hat z)$ with $\hat q_j \in \mathcal{S}_j(t_j), j \in \mathbb{N}$, (A.3-C4)

The theorem below states the convergence result. A proof is given in [MRS08, Th. 3.1].

Theorem A.1 (Γ-convergence of $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)_{j\in\mathbb{N}}$) *Let conditions* (A.1), (A.2) *and* (A.3) *hold and for all $j \in \mathbb{N}$ let $q_j : [0, T] \to \mathcal{Q}$ be an energetic solution of $(\mathcal{Q}, \mathcal{E}_j, \mathcal{R}_j)$ in the sense of Def. 1.1. If $q_j(t) \xrightarrow{\mathcal{T}} q(t)$ for all $t \in [0, T]$ and if $\mathcal{E}_j(0, q_j(0)) \to \mathcal{E}_\infty(0, q(0))$ then $q : [0, T] \to \mathcal{Q}$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{R}_\infty)$.*

Moreover, for all $t \in [0, T]$ it is $\mathcal{E}_j(t, q_j(t)) \to \mathcal{E}(t, q(t))$, $\mathrm{Diss}_{\mathcal{R}_j}(q_j, [0, t]) \to \mathrm{Diss}_{\mathcal{R}}(q, [0, t])$ and $\partial_t\mathcal{E}_j(t, q_j(t)) \to \partial_t\mathcal{E}(t, q(t))$ for a.a. $t \in [0, T]$. Furthermore, for $\mathcal{Q}$ being a separable, reflexive Banach space, the energetic solution q is measurable with respect to time.

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