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From discrete visco-elasticity to continuum rate-independent plasticity: Rigorous results

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Abstract

We show that continuum models for ideal plasticity can be obtained as a rigorous mathematical limit starting from a discrete microscopic model describing a visco-elastic crystal lattice with quenched disorder. The constitutive structure changes as a result of two concurrent limiting procedures: the vanishing-viscosity limit and the discrete to continuum limit. In the course of these limits a non-convex elastic problem transforms into a convex elastic problem while the quadratic rate-dependent dissipation of visco-elastic solid transforms into a singular rate-independent dissipation of an ideally plastic solid. In order to emphasize ideas we employ in our proofs the simplest prototypical system describing transformational plasticity of shape-memory alloys. The approach, however, is sufficiently general and can be used for similar reductions in the cases of more general plasticity and damage models.

1 Introduction

Phenomenological models involving rate-independent hysteresis appear in various solid mechanics problems ranging from friction to plasticity and damage. Typically, the associated systems of phenomenological equations contain empirical parameters characterizing the failure thresholds and the hardening rates. In sharp contrast to elastic moduli, these measures of out-of-equilibrium behavior can rarely be formally linked to the structure of the underlining microscopic system. The main difficulty originates from the fact that at finite temperature the microscopic dissipation is necessarily rate dependent while the observed macroscopic dissipation is rate independent. This means that the correct coarse graining, implying averaging out of the microscopic time and space scales, must necessarily involve the basic change of the model structure. Essentially one needs to understand the limit transition from quadratic dissipative potentials of Onsager type to singular dissipative potentials used in the description of rate-independent dissipative processes. The main physical ingredients of such a limit were identified in [PuT05], where rate-independent plasticity was obtained as a rheological model. Here we present the first rigorous mathematical analysis of the problem and obtain the corresponding system of partial differential equations in space and time.

The foundations of the general phenomenological theory of rate-independent systems have been laid down in [Hil50, Mor74] (see also [NgR76, FeE89, Hac97, FrM98, OrR99, Pet05]). The universal mathematical features of such models found their most clear manifestation in the general concept of energetic rate-independent systems (ERIS) introduced in [MiT99, MTL02]. The ERIS-based approach has been already used in the description of fracture [DFT05, DeT09], plasticity [DDM06, DD*08, MaM09], delamination [KMR06, RSZ09], damage [FrG06, BMR09, GaL09] and phase transformations [MTL02, Rou02, The02, KMR05].

The microscopic models in all these areas rely on the existence of characteristic defects carrying inelastic deformation (e.g. dislocations, phase boundaries, fracture fronts, etc.) The microscopic dynamics of the individual defects is well understood, however, their interaction is very complex which is the reason why the detailed bridge between the microscopic and the phenomenological models has not been yet built. In this situation simple prototypical meso-scopic models, even extremely schematic ones, still offer an insight and have a considerable heuristic value.

In the framework of plasticity theory the microscopic origin of rate independent dissipation was first studied by using simplified zero-dimensional models describing a single particle on a periodic landscape (e.g. [Pra28, Deh29]). Later such models were applied to a wide range of rate-independent dissipative phenomena from charge density waves and magnetism to superconductivity and phase transitions [PBK79, Fis85, HB*94, CDP*99]. One-dimensional discrete models involving bi-stable snap-springs (soft spins) represent the next level of schematization allowing one to model realistic hysteretic behavior without introducing a periodic landscape [MüV77, FeZ92, PuT00, TrV05, PRTZ09]. Higher-dimensional snap-spring models allow one to study pinning-depinning transition, criticality and power law structure of fluctuations e.g. [Kar98, Zai06, PRTZ08].

Despite the considerable literature on the subject, no attempt has been made so far to bridge the gap between viscous and rate independent plastic systems by rigorous mathematical analysis outside the simplest *zero-dimensional* case leading only to rheological models [ACJ96, Men02, PuT05, Sul09]. In the present paper we prove for the first time some exact convergence results for the *one-dimensional* problem. Although we deal with the simplest nontrivial case, we have to confront all the major problems associated with non-convexity and coarse graining in both space and time. We therefore expect that our technique can be extended to more general systems.

More specifically, we consider a quasi-statically driven discrete chain of bi-stable, visco-elastic snap-springs and derive a coarse-grained model that is equivalent to continuum rate-independent plasticity. The main ingredient of the microscopic model making such reduction possible is the rugged energy landscape. Under slow external loading our system remains in a local equilibrium (metastable state) till it is forced to undergo a fast transition from an unstable state to a new local minimum of the energy. The energy dissipated during the fast transitions can be described in the continuum limit by a dissipation potential that is homogeneous function of degree one. Some formal computations justifying such limit have been presented in [PuT05]. In particular, it was realized that the transition must involve simultaneous averaging over the fast time scale and homogenization over spatial inhomogeneity. In this paper we present the first rigorous analysis of the full dynamics and show that in order to obtain in the limit a spatially nontrivial rate independent plasticity problem it is necessary to regularize the discrete system by introducing quenched disorder. Previously, the disorder in such systems was used to obtain hardening and produce realistic inner hysteresis loops, but only in spatially independent rheological setting [PuT02].

In mathematical terms, our starting point is a system of N ordinary differential equations of the gradient flow type. The system is non-autonomous because the chain is driven through applied displacement on the boundaries (hard device). We identify two main small parameters. The parameter δ is the rate of viscous relaxation on the time scale of the loading. This parameter goes to zero when either driving is quasi-static or the internal relaxation is fast. The second parameter $\varepsilon = 1/N$ is the macroscopic length of the N snap springs and thus gives the scale of the inhomogeneity: it disappears when the internal length is much smaller than the external one. To avoid degeneracy leading to Neishtadt type phenomena [Neĭ88] we introduce small random inhomogeneity, which adds a third small parameter accounting for the dispersion r. We then assume that the random properties of the system are fixed and focus on the study of a particular double limit: first $\delta \to 0$, then $\varepsilon \to 0$. We prove that in this limit the original finite dimensional visco-elastic system reduces to an infinite-dimensional continuum model exhibiting rate-independent hysteretic behavior.

The constitutive structure is changing as a result of two concurrent limiting procedures: the vanishing-viscosity limit and the discrete to continuum limit. In the course of these limits a non-convex elastic energy (in terms of microscopic strains) transforms into a convex elastic energy (in terms of two macroscopic variables, namely the elastic strain and the averaged phase indicators called plastic strain), while the quadratic rate-dependent dissipation of visco-elastic solid transforms (given in terms of the rate of microscopic strains) into a singular rate-independent dissipation of an ideally plastic solid (given in terms of the rate of the plastic strain). As intermediate constructions we encounter jump discontinuities in time and parametric measure-valued solutions in space. The proof involves two main steps. The first is the reduction of a finite-dimensional gradient system of ODEs to a discrete automaton, which gives a quasi-static evolution on the time-dependent set of local energy minima. This automaton is then reformulated as an energetic rate-independent system (ERIS) represented by an energy functional and a dissipation distance. The second step is the limit passage from discrete to continuum in the framework of Γ-convergence of ERIS. Here we exploit the Young measures generated through the disorder and thus are able to pass to the limit in both the energy and the dissipative potential.

In order to emphasize ideas we employ in our proofs the simplest prototypical system describing transformational plasticity of shape-memory alloys. The approach, however, is sufficiently general and can be used for similar reductions in the cases of more general plasticity and damage models.

The paper is organized as follows. In Sections 2 and 3 we set the general dynamic problem for the overdamped ODE system and introduce the regularization through quenched disorder. We then define the macroscopic variables by embedding the discrete system into $\mathrm{L}^2(\Omega)$ where $\Omega =]0,1[$ is the reference configuration of a continuum bar. Most of the rigorous analysis is done under the assumption that Φ is a bi-quadratic and that the body forces are time independent. These assumptions are not essential and are used only to make calculations simpler and the proofs more transparent. In Section 4 we deal with the vanishing-viscosity limit $\delta \to 0$ for fixed ε. We present careful estimates for the viscous solutions comparing them to those of a limiting rate-independent discrete automaton. The main difficulty is to control the phase state of each individual spring, which becomes possible because our disorder and dynamics are consistent with the ordering of the springs. We show that the evolution of the system splits into equilibrium and dissipative stages where the dissipative stages can be replaced by jump discontinuities in isolated moments of time. The limiting ERIS leads to formulations involving incremental minimization problems, which allows us to use direct variational techniques later on.

In Section 5 the limit $\varepsilon = 1/N \to 0$ is obtained through embedding the system into $\mathcal{Q} = \mathrm{L}^2(\Omega)^2$ and controlling the joint Young measures for elastic and plastic strains. The convergence to the limiting plasticity model is interpreted in terms of Γ-convergence of energetic rate-independent systems as first suggested in [MRS08]. In Section 6 we show that in the case of a bi-quadratic potential the more general double limit $(\varepsilon, \delta) \to (0,0)$ with $\delta \leq \kappa_+\varepsilon$ for some $\kappa_* > 0$ produces the same limiting plasticity problem. (However, we do not expect the restriction $\delta \leq \kappa_+\varepsilon$ to be sharp.) In Section 7 we return to the case of general (non necessarily bi-quadratic) potentials Φ and general time dependent body forces. We first study the ordered double limit "$\lim_{\varepsilon\to0}\lim_{\delta\to0}$"

and present a formal calculation showing how the effective dissipation potential and the effective stored-energy density can be obtained from the microscopic elastic potential and the probability distribution of the quenched disorder. We then sketch the proof of the convergence, heavily relying on the corresponding proofs in the case of bi-quadratic potential. Finally, in Section 7.5 we briefly discuss convergence along the generic sequences in the (ε, δ) plane.

2 Preliminaries

Consider a macroscopic interval $[0, 1]$ containing $N-1$ particles at the reference positions $x_j^N = j/N$, $j = 1, \ldots, N-1$. The boundary points $j = 0$ and $j = N$ are assumed to be controlled and undergoing prescribed displacements. The remaining points are linked in series by N identical snap-springs. The discreteness of this mechanical system can be viewed as a schematic representation of an array of obstacles (defects, grain boundaries, etc.).

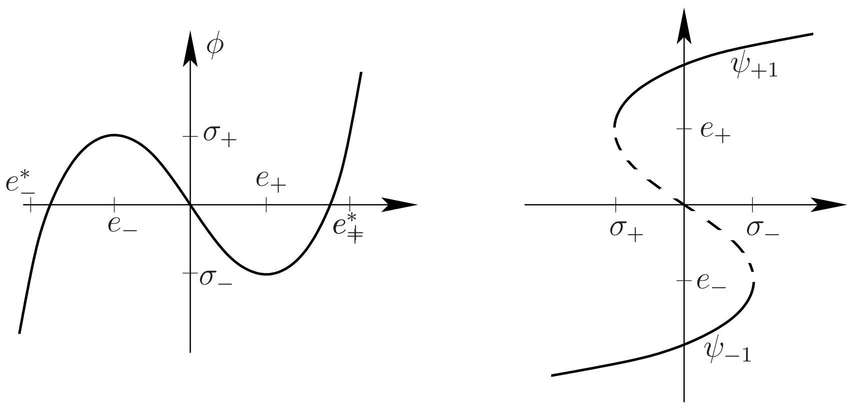

Figure 2.1: Left: Non-monotone stress-strain relation. Right: Two branches ψ_{+1} and ψ_{-1} of the strain-stress relation

The most important ingredient of the model is the bi-stability of the individual elastic elements. To be more precise we write the normalized elastic energy of the chain in the form

$$\widetilde{\mathcal{E}}(\boldsymbol{e}) = \frac{1}{N} \sum_{j=1}^{N} \Phi(e_j) \quad \text{with } \boldsymbol{e} = (e_1, \ldots, e_N) \in \mathbb{R}^N,$$

where e_j is the strain in the jth snap-spring. We assume that the elastic energy of a snap-spring $\Phi : \mathbb{R} \to \mathbb{R}$ is a non-convex two-well potential. This means that the function $\phi = \Phi'$ is decreasing on the interval $]e_-, e_+[$ (spinodal region) and strictly increasing on the two intervals $]-\infty, e_-[$ and $]e_+, \infty[$, representing phase "+" and phase "−", respectively (see Fig. 2.1). We can formally define the corresponding energy wells by setting

$$\sigma_+ := \phi(e_-) > \sigma_- := \phi(e_+).$$

For future convenience we denote by $\psi_{+1} : [\sigma_-, \infty[\to [e_+, \infty[$ and $\psi_{-1} :]-\infty, \sigma_+] \to]-\infty, e_-]$ the inverse functions of $\phi : [e_+, \infty[\to [\sigma_-, \infty[$ and $\phi :]-\infty, e_-] \to]-\infty, \sigma_+]$, respectively. We also define $e_-^* = \psi_{+1}(\sigma_+) > e_+$ and $e_-^* = \psi_{-1}(\sigma_-) < e_-$.

In what follows a prominent role will be played by a particular bi-quadratic potential

$$\Phi_{\rm biq}(e) := \frac{k}{2}\min\{(e+a)^2, (e-a)^2\}, \tag{2.1}$$

giving

$$\phi_{\rm biq}(e) = \begin{cases} k(e+a) & \text{for } e < 0, \\ k(e-a) & \text{for } e > 0. \end{cases}$$

Note that in this case ϕ is not continuous at $e = 0$ where ϕ can take the value either ka or $-ka$. For the bi-quadratic energy $\Phi_{\rm biq}$ we find

$$e_\pm = 0, \quad e_\pm^* = \pm 2a, \quad \sigma_\pm = \pm ka, \quad \psi_{\pm 1}(\sigma) = \frac{1}{k}\sigma \pm a.$$

The chain is loaded by time dependent macroscopic body forces $\widetilde{G}_j(\tau)$ given by

$$\widetilde{G}_j(\tau) = \int_0^{j/N} \widetilde{g}_{\rm ext}(\tau, y)\,{\rm d}y.$$

In addition we impose time-dependent Dirichlet boundary condition (hard device) representing external control of the total average strain $\widetilde{\ell}$, namely

$$\frac{1}{N}\sum_1^N e_j(\tau) = \widetilde{\ell}(\tau). \tag{2.2}$$

It is natural to write the resulting energy function in terms of the relative strains $\widetilde{e}_j = e_j - \widetilde{\ell}(\tau)$. The new unknowns form a vector $\widetilde{\boldsymbol{e}} = (\widetilde{e}_1, \ldots, \widetilde{e}_N) \in \boldsymbol{X}^N$, where $\boldsymbol{X}^N = \{\,(a_1, \ldots, a_N) \in \mathbb{R}^N \mid \sum_1^N a_j = 0\,\}$. In these notations the total energy of the chain can be written as

$$\widetilde{\mathcal{E}}(\tau, \widetilde{\boldsymbol{e}}) = \frac{1}{N}\sum_{j=1}^N \Big(\Phi(\widetilde{e}_j + \widetilde{\ell}(\tau)) - \widetilde{G}_j(\tau)\widetilde{e}_j\Big).$$

In the framework of quasi-static elasticity theory the mechanical problem for the driven chain reduces to parametric minimization of the energy $\widetilde{\mathcal{E}}(\tau, \widetilde{\boldsymbol{e}})$. Due to bi-stability of the individual elastic elements such energy has an exponentially large number of critical points. One can also expect that the corresponding metastable (local minimum) branches $e_j(\tau)$ are not continuous with respect to the parameter τ. In this situation the knowledge of dynamics is necessary to define uniquely the evolution of the system.

Assume that the microscopic dynamics is overdamped (for inertial limit see [YuT10]) and that the dissipation is characterized by a dissipation potential $\mathcal{R}(\dot{\boldsymbol{e}})$ giving

$${\rm D}_{\dot{\widetilde{\boldsymbol{e}}}}\mathcal{R}(\dot{\widetilde{\boldsymbol{e}}}) = -{\rm D}_{\widetilde{\boldsymbol{e}}}\widetilde{\mathcal{E}}(\tau, \widetilde{\boldsymbol{e}}).$$

(We continue to use ${\rm D}_{\boldsymbol{a}}\mathcal{F}$ to denote the (partial) Gateaux derivative of a functional with respect to the variable $\boldsymbol{a}$.) The standard viscous model is characterized by the quadratic dissipation potential

$$\mathcal{R}(\dot{\boldsymbol{e}}) = \frac{\nu}{2N}\sum_{j=1}^N \dot{\widetilde{e}}_j^{\,2},$$

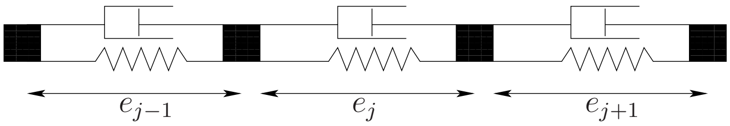

Figure 2.2: Viscoelastic chain with bi-stable springs.

where ν is the viscosity parameter. The resulting dynamics is of gradient-flow type

$$\frac{\nu}{N}\dot{\widetilde{\boldsymbol{e}}} = -\mathrm{D}_{\widetilde{\boldsymbol{e}}}\widetilde{\mathcal{E}}(\tau, \widetilde{\boldsymbol{e}}).$$

We further assume that the loading rate is *small*, i.e.,

$$\widetilde{\ell}(\tau) = \ell(\widetilde{\delta}\tau)$$

where $\ell(\cdot)$ is a given smooth function and $\widetilde{\delta}$ is a measure of loading rate. By introducing the slow time parameter $t = \widetilde{\delta}\tau$ and defining $G(t, y) = \widetilde{G}(\tau, y)$, we obtain

$$\left.\begin{aligned} &\delta \dot{e}_j = -\phi(e_j) - G(t, j/N) + \sigma(t) \text{ for } j = 1, \ldots, N, \\ &\frac{1}{N}\sum_{j=1}^{N} e_j(t) = \ell(t). \end{aligned}\right\} \tag{2.3}$$

(Here we returned to the original stain variables $e_j = \widetilde{e}_j(t) + \ell(t)$ for a better physical interpretation.) The new non-dimensional parameter $\delta = \widetilde{\delta}\nu$ is the ratio of the rate of loading and the rate of viscous relaxation (see also [PuT05]). The function $\sigma : [0, T] \to \mathbb{R}$ representing total stress appears in (2.3) as the Lagrange multiplier associated with the length constraint (2.2).

To gain some insight into the behavior of the system (2.3) subjected to quasi-static loading we perform several numerical experiments. In these experiments we neglect body forces and assume $\phi(e) = e^3 - e$. We also assume that viscosity is small but finite $\delta = 0.015$. The initial data are chosen randomly distributed around the value $e_j(0) \approx -1.3$. In all experiments we prescribe the history of average strain and study the behavior of the average stress $\widehat{\sigma} = \frac{1}{N}\sum_1^N \phi(\varepsilon_j)$.

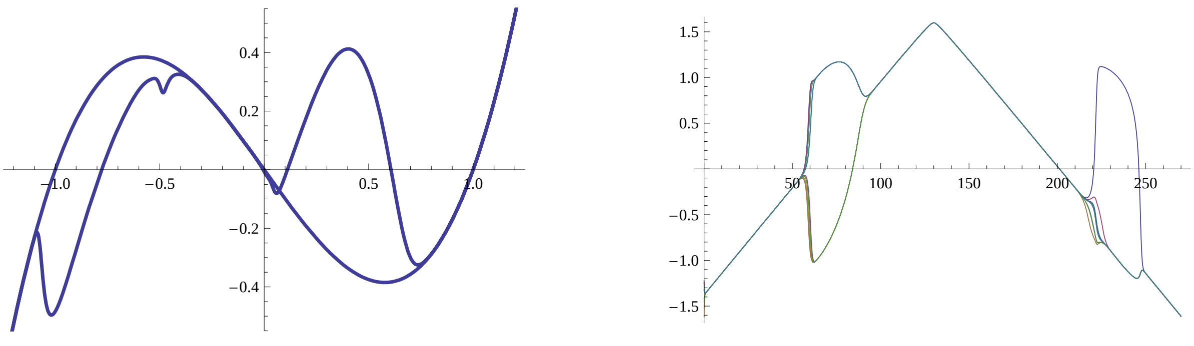

Figure 2.3: Simulation of system (2.3) for $N = 9$. Left: $\widehat{\sigma} = \frac{1}{N}\sum_1^N \phi(e_j)$ versus ℓ. Right: $e_1, \ldots, e_9$ versus t.

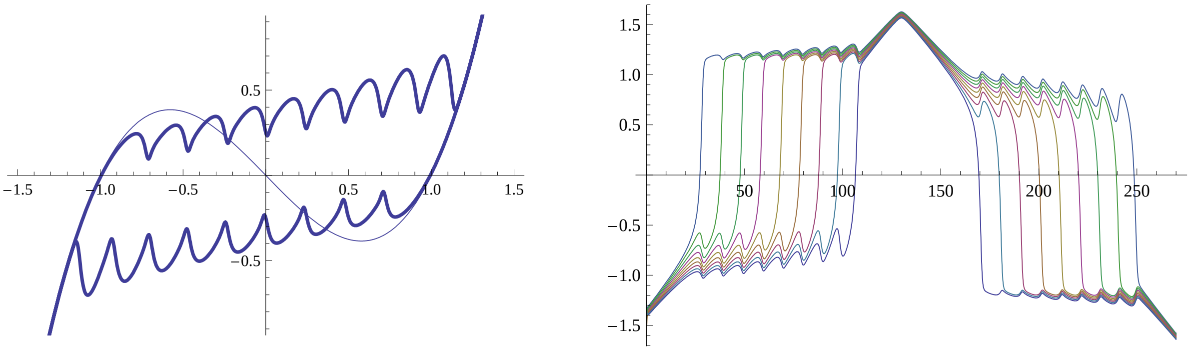

Figure 2.4: Stress and strains for a model with $N = 9$ and linear bias $\mu_j^9 = 0.05(j-5)$. Right: $\widehat{\sigma}$ versus ℓ. Left: $e_1, \ldots, e_9$ versus t.

The first experiment was conducted with a homogeneous chain where all snap-springs were identical. The resulting stress-strain curve and the strains inside individual snap-springs are shown in Fig. 2.3. Observe that we do not obtain a plasticity-like hysteretic behavior. Instead, we detect a "snap" phenomenon, when a large number of springs transform simultaneously forming one big avalanche while the rest of the springs relaxes. As the load subsequently increases, the inhomogeneous state becomes homogeneous again in a smooth way.

We interpret the "snap" behavior as synchronization, which leads to a delayed bifurcation, known as the Neishtadt phenomenon [Neĭ87, Neĭ88]. Indeed, in the stable regime $\ell(t) < e_-$ the strains $e_j^N(t)$ are always close to the quasistatic equilibrium value $\ell(t)$ and the perturbations decay exponentially. More precisely the decay rate is $-\lambda_{\min}/\delta$, where $\lambda_{\min} > 0$ is the smallest eigenvalue of the Hessian of the energy at $\boldsymbol{e} = (\ell, \ldots, \ell)$. Hence, if a solution starts in the stable regime at $t = t_0$ with perturbations of order 1 and reaches the spinodal region at $t = t_1$, the perturbations will be of order $e^{-\lambda_{\min}(t_1-t_0)/\delta}$. Thus, the instability of the steady state $\boldsymbol{e}(t) = (\ell(t), \ldots, \ell(t))$ in the spinodal region needs some time to establish oneself: the unstable eigenvalue will be of the form $\widehat{\lambda}/\delta$, and to obtain perturbations of order 1 we need to wait until t_2 satisfies $\widehat{\lambda}(t_2-t_1)/\delta = \lambda_{\min}(t_1-t_0)/\delta$. The point is that $(t_2-t_1)/(t_1-t_0) = \lambda_{\min}/\widehat{\lambda}$ is independent of δ.

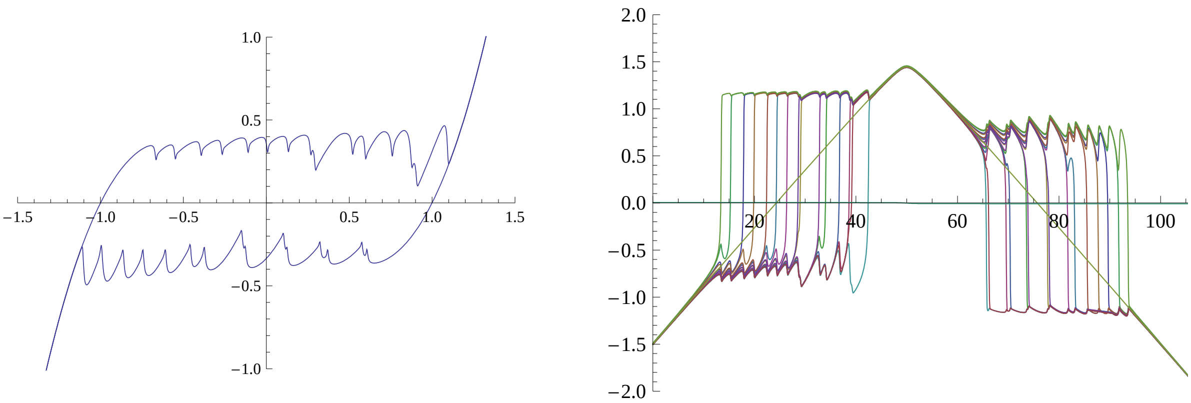

Figure 2.5: Simulation of ODE with $N = 15$ and random bias. Left: $\widehat{\sigma}$ versus ℓ. Right: e_j versus t.

To obtain separation of trajectories of the dynamical system one needs to break the permutational symmetry. The inhomogeneity can be generated through a discrete set of microscopic body forces. This amounts to the following modification of the snap-spring potentials $\Phi_j(e_j) = \Phi(e_j) - \mu_j e_j$, where μ_j with $j = 1, ..., N$ are the biasing forces. The resulting system of the ODEs reads

$$\delta \dot{e}_j = -e_j^3 + e_j + \mu_j + \sigma(t) \text{ for } j = 1, ..., N.$$

In our next numerical experiment we set $\mu_j^9 = 0.05(j-5)$. Such inhomogeneity allows us to generate an unsynchronized response, where each spring transformers at its own critical stress starting from the weakest one, see also [PuT02]. The results are shown in Fig. 2.4. Notice that now, instead of one big "snap", we observe a serious of small "popping" events so that the inhomogeneous system produces realistic plasticity-type behavior (with hardening).

Observe however that plastic deformation (phase transition in our case) propagates through the system in the form of a single front. This is not realistic because we know that (outside very special "easy glide" regimes) plasticity usually develops simultaneously all over the sample. To achieve the stochastic separation of the trajectories we need to assume that parameters μ_j are stochastically independent.

The results of numerical loading-unloading test for the case when μ_j are equi-distributed in the segment $[-0.1, 0.1]$ is presented in Fig. 2.5. We see that the overall behavior of the system is basically the same as in the previous case modulo the dispersion of the "popping" events. The important difference, however, is that now the strain distribution inside the sample is no longer monotone and instead becomes strongly oscillatory making the system macroscopically homogeneous. The ensuing homogeneity at the coarse-grained scale is exactly the property which is necessary to obtain a nontrivial continuum limit.

3 Main results

To formulate the main result we need to introduce random microscopic body forces μ_j representing quenched disorder. We assume that the probability density $f \in \mathrm{L}^1(\mathbb{R})$, which characterizes the distribution of μ_j and satisfies the following natural constraints

$$f \geq 0, \quad \int_{\mathbb{R}} f(\mu)\,\mathrm{d}\mu = 1, \quad \int_{\mathbb{R}} \mu f(\mu)\,\mathrm{d}\mu = 0, \quad \text{and } r^2 = \int_{\mathbb{R}} \mu^2 f(\mu)\,\mathrm{d}\mu > 0. \tag{3.1}$$

The dynamical system

$$\left.\begin{aligned} &\delta \dot{e}_j = -\phi(e_j) + \mu_j - G(t, j/N) + \sigma(t) \text{ for } j = 1, \ldots, N, \\ &\frac{1}{N}\sum_{j=1}^{N} e_j(t) = \ell(t) \end{aligned}\right\} \tag{3.2}$$

depends now on three nondimensional parameters, namely the discreteness level

$$\varepsilon = 1/N > 0,$$

the normalized viscosity

$$\delta > 0,$$

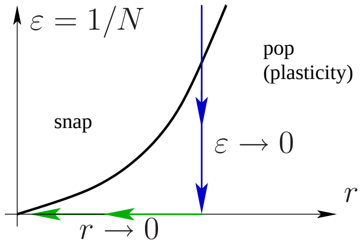

Figure 3.1: Schematic phase diagram in the space of small parameters indicating location of the 'popping' domain which we associate with rate-independent plasticity response.

and the measure of disorder

$$r > 0.$$

As our numerical experiments suggest, one can expect to obtain macroscopic continuum rate-independent plasticity model only in certain triple limit of the form

$$(\varepsilon, \delta, r) \to (0, 0, 0).$$

We have seen that the limit $r \to 0$ at fixed ε, δ may lead to "snap" behavior, and the subsequent driving ε and δ to zero does not save the situation. To obtain the "pop" behavior we need first to assume that $r > 0$ and consider the limit $(\varepsilon, \delta) \to (0, 0)$. We can then continue along the parametric path $r \to 0$ leading to ideal plasticity limit.

At fixed r one can find for each ε and δ a set of solutions of the microscopic problem $\boldsymbol{e}^{\varepsilon,\delta} : [0, T] \to \mathbb{R}^N$. Here the vector $\boldsymbol{e}^{\varepsilon,\delta}(t)$ is defined by

$$\boldsymbol{e}^{\varepsilon,\delta}(t) = (e_j^{\varepsilon,\delta}(t))_{j=1,\dots,N}.$$

It will be convenient to rewrite the original ODE system (3.2) in the form

$$0 = \mathrm{D}_{\dot{\boldsymbol{e}}}\mathcal{R}_{\varepsilon,\delta}(\dot{\boldsymbol{e}}(t)) + \mathrm{D}_{\boldsymbol{e}}\mathcal{E}_\varepsilon(t, \boldsymbol{e}(t)) + \sigma(t)\mathrm{D}_{\boldsymbol{e}}\mathcal{C}_\varepsilon(t, \boldsymbol{e}(t)), \quad \mathcal{C}_\varepsilon(t, \boldsymbol{e}(t)) = 0. \tag{3.3}$$

Here the energy

$$\mathcal{E}_\varepsilon(t, \boldsymbol{e}) = \frac{1}{N}\sum_{j=1}^{N}\big(\Phi(e_j) - h_j^N(t)e_j\big)$$

depends on inhomogeneity through

$$h_j^N(t) = \mu_j^N - G(t, j/N),$$

where we explicitly indicate the dependence of the random terms on the size of the system. The time dependent constraint can be written as

$$\mathcal{C}_\varepsilon(t, \boldsymbol{e}) = \sum_{j=1}^{N}\big(e_j - \ell(t)\big)$$

and the dissipation potential is given by

$$\mathcal{R}_{\varepsilon,\delta}(\dot{\boldsymbol{e}}) = \frac{\delta}{2N}\sum_{j=1}^{N}\dot{e}_j(t)^2.$$

In the vanishing viscosity limit the solutions $\boldsymbol{e}^{\varepsilon,\delta}(t)$ of (3.3) can be expected to stay most of the time close to elastic equilibrium. The corresponding elastic problem reduces to solving the equations

$$0 = -\phi(e_j) + \mu_j^N - G(t, j/N) + \sigma^N(t), \quad \frac{1}{N}\sum_{j=1}^{N} e_j = \ell(t).$$

Since the function $\phi(\cdot)$ is non-monotone, the response $\boldsymbol{e}^{\varepsilon,0} : [0,T] \to \mathbb{R}^N$ is not necessarily single-valued. If we introduce the phase indicators

$$z_j = \operatorname{sign}(e_j) \in \{-1, 0, 1\},$$

specifying three individual sheets of the inverse function $\psi_{z_j}(\cdot)$ (two stable phases and the spinodal region, see Figure 2.1), we can write explicitly

$$e_j = \psi_{z_j}(\sigma^N(t) + \mu_j^N - G(t, j/N)). \tag{3.4}$$

The phase indicators identify individual branches of the equilibrium stress-strain relation and, if the solution remains close to a particular branch, the phase indicators remain unchanged. The discrete variables z_j are the precursors of continuum plastic strain variables, which we introduce in the next section. One can see that if the 'plastic' configuration z_j is given, the elastic strains e_j can be easily recovered from the solution of the convex problem (3.4). This suggests that in the vanishing-viscosity limit the elastic problem can be 'condensed' and the evolution of the system can be reformulated in terms of plastic strains only.

In what follows we show that due to the quenched disorder the phase indicators $\boldsymbol{z}^{\varepsilon,\delta}(t) \in \{-1,0,1\}^N$ and consequentially the strains $\boldsymbol{e}^{\varepsilon,\delta}(t) \in \mathbb{R}^N$ fluctuate in a random fashion. The independence of the random choices at different spatial points leads (due to central limit theorem) to controllable properties of the mean values and thus allows one to construct a coarse-grained theory and explicate the macroscopic properties.

To be more specific, we assume that the quantities varying at the scale ε are microscopic, while those varying at the scale 1 are macroscopic. To define the macroscopic averages we first need to introduce a spatial averaging operator. We begin by embedding the solutions $\boldsymbol{e} \in \mathbb{R}^N$ into $\mathrm{L}^2(\Omega)$ via the characteristic functions

$$\chi_j^N = \chi_{](j-1)/N, j/N[} : x \mapsto \begin{cases} 1 & \text{for } x \in \,](j-1)/N, j/N[\,, \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to define the elastic strain field $\overline{e}^{\varepsilon,\delta} \in \mathrm{L}^2(\Omega)$ as follows

$$\overline{e}^{\varepsilon,\delta}(t,x) = \sum_{j=1}^{N} e_j^{\varepsilon,\delta}(t)\chi_j^N(x).$$

Similarly, we introduce a continuum phase indicator (plastic strain) $\overline{z}^{\varepsilon,\delta} \in \mathrm{L}^2(\Omega)$ via

$$\overline{z}^{\varepsilon,\delta}(t,x) = \sum_{j=1}^{N} \widehat{s}(e_j^{\varepsilon,\delta}(t))\chi_j^N(x).$$

Here

$$\widehat{s}(e) = \begin{cases} -1 & \text{for } e \leq e_-, \\ 0 & \text{for } e_- < e < e_+, \\ +1 & \text{for } e \geq e_+. \end{cases}$$

The discrete-to-continuum limit concerns the asymptotics $\varepsilon = 1/N \to 0$. The strong limits of the above sequences do not exist and our main task is to characterize the weak limits

$$(\overline{e}^{\varepsilon,\delta}(t,\cdot), \overline{z}^{\varepsilon,\delta}(t,\cdot)) \rightharpoonup (\overline{e}(t,\cdot), \overline{z}(t,\cdot)) \text{ in } \mathcal{Q} = \mathrm{L}^2(\Omega)^2.$$

We understand them in the sense that

$$\int_\Omega \overline{e}^{\varepsilon,\delta}(t,x)v_1(x) + \overline{z}^{\varepsilon,\delta}(t,x)v_2(x)\,\mathrm{d}x \to \int_\Omega \overline{e}(t,x)v_1(x) + \overline{z}(t,x)v_2(x)\,\mathrm{d}x$$

for $(\varepsilon,\delta) \to 0$ where the test functions satisfy $v_1, v_2 \in \mathrm{L}^2(\Omega)$. As we show, the limiting mixtures of phases cannot be fully characterized by the value of the average elastic strain $\overline{e}$. The missing information, allowing one to close the coarse-grained description at the macro-scale, is exactly the limit of the indicator function $\overline{z}$.

More precisely, we show that a sequence of limits $\delta \to 0$ and then $\varepsilon \to 0$ allow one to obtain a one-dimensional elasto-plasticity problem in the form

$$0 = \mathrm{D}_{\overline{e}}\overline{\mathcal{E}}(t, \overline{e}, \overline{z}(t)) + \sigma(t) \text{ for } x \in \Omega, \quad \int_\Omega e(t,x)\,\mathrm{d}x = \ell(t); \tag{3.5a}$$

$$0 \in \partial\overline{\mathcal{R}}(\dot{\overline{z}}(t)) + \mathrm{D}_{\overline{z}}\overline{\mathcal{E}}(\overline{e}(t), \overline{z}(t)). \tag{3.5b}$$

Here the macroscopic elastic energy $\overline{\mathcal{E}}$ is given by

$$\overline{\mathcal{E}}(\overline{e}, \overline{z}) = \int_0^1 \big(\overline{\Phi}(\overline{e}(x), \overline{z}(x)) - G(t,x)e(x)\big)\,\mathrm{d}x,$$

where the macroscopic energy density $\overline{\Phi}$ depends on Φ and the probability density f determining the random bias vectors $(\mu_j^N)_{j=1,\dots,N}$. In the bi-quadratic case $\Phi = \Phi_{\mathrm{biq}}$ (see (2.1)) we obtain the explicit formula

$$\overline{\Phi}(\overline{e}, \overline{z}) = \frac{k}{2}\big(\overline{e} - a\,\overline{z}\big)^2 + \overline{H}(\overline{z}), \tag{3.6}$$

where the kinematic hardening function $\overline{H}$ depends on f, see (5.3). In the general case the macroscopic rate independent dissipative potential $\overline{\mathcal{R}}$ takes the form

$$\overline{\mathcal{R}}(\dot{\overline{z}}) = \int_0^1 \overline{R}(\dot{\overline{z}}(x))\,\mathrm{d}x \quad \text{with } \overline{R}(v) = \begin{cases} \rho_+ v & \text{for } v \geq 0, \\ \rho_-|v| & \text{for } v \leq 0, \end{cases}$$

where ρ_+ and ρ_- can be expressed in terms of Φ, see (7.7). In the bi-quadratic case $\Phi = \Phi_{\mathrm{biq}}$ we obtain $\rho_\pm = 2ka^2$.

The most unexpected feature of our result is the fundamental change in the nature of the dynamical system in the limit. Indeed, while (3.3) is an N-dimensional ODE derived from a gradient flow with quadratic dissipation potential, the limit is a rate-independent system, where the dissipation related forces $\partial\overline{\mathcal{R}}(\dot{\overline{z}})$ are homogeneous of degree 0 in $\dot{\overline{z}}$

(as the dissipation potential $R(\cdot)$ is homogeneous function of degree 1). The origin of the change is the 'constructive interference' of micro-elasticity and micro-viscosity in the continuum limit. Notice that both the macroscopic energy and the macroscopic dissipation are affected by these two constitutive components of the microscopic model. Notice also that the memory of the specific nature of the microscopic dissipation has been lost in the macroscopic double limit suggesting that linear viscosity is not the only microscopic dissipative mechanism leading to our rate-independent macro-model.

If introduction of quenched disorder is perceived as an auxiliary technical step, the disorder must be eliminated through yet another limit $r \to 0$. The derivation of the limiting model can follow a well-established path known in classical elasto-plasticity, see e.g. [BMR10]. From the definition (5.3) of the hardening function $\overline{H}_f$ in (3.6) it follows that it depends on f in such a way that $r^2 = \int_{\mathbb{R}} \mu^2 f(\mu)\, \mathrm{d}\mu \to 0$ implies $\overline{H}_f(z) \to 0$ for all $z \in\,]-1,1[$ (while $\overline{H}(z) = \infty$ if $|z| > 1$), see e.g. (5.4). Therefore the limiting model, given again by (3.5) with $\overline{\Phi}$ from (3.6), has the property that $\overline{H}(z) = 0$ for $|z| \le 1$. One can see that the resulting $\overline{\Phi}$ and hence $\overline{\mathcal{E}}$ are only degenerate convex which means that the model is not well-posed: as it is well known in ideal plasticity, several solutions may exist for given initial data.

4 Vanishing-viscosity limit

Suppose that $\varepsilon > 0$ and $r > 0$ are fixed and consider the limit $\delta \to 0$. In fact, the assumption $r > 0$ is not crucial in this section; the only required property of the parameters $\mu_1^N, \dots, \mu_N^N$ is that the effective biases $h_j = \mu_j^N - G(t_0, j/N)$ are pairwise different.

4.1 Energy landscape

We begin with the review of the structure of the elastic energy landscape at the given loads (see also [PuT00]). To this end we fix the time $t = t_0$ and consider the problem of minimizing the energy

$$\mathcal{E}_\varepsilon(t, \boldsymbol{e}) = \frac{1}{N} \sum_1^N \big(\phi(e_j) - h_j e_j\big)$$

under the constraint

$$\frac{1}{N} \sum_1^N e_j = \ell.$$

The critical points of (3.3) can be obtained as solutions of the algebraic equations

$$0 = -\phi(e_j) + h_j + \sigma \text{ for } j = 1, ..., N, \quad \frac{1}{N} \sum_1^N e_j = \ell(t_0). \tag{4.1}$$

Metastable equilibria (local minima of the energy) are selected by the condition of the positive definiteness of the Hessian matrix. For sufficiently large N none of the metastable strains e_j can lie in the spinodal region $]e_-, e_+[$, see [PuT00]. To identify the remaining two phases we define for each j a phase indicator $z_j \in \{-1, +1\}$, such that

$$e_j = \psi_{z_j}(h_j{+}\sigma).$$

A metastable equilibrium corresponding to an indicator vector $\boldsymbol{z} = (z_1, \dots, z_n) \in \{-1, 1\}^N$ exists when the equations

$$\frac{1}{N}\sum_{j=1}^{N} \psi_{z_j}(h_j+\sigma) = \ell \quad \text{and} \quad \begin{cases} h_j+\sigma \geq \sigma_- & \text{if } z_j = 1, \\ h_j+\sigma \leq \sigma_+ & \text{if } z_j = -1 \end{cases}$$

can be satisfied simultaneously. For each metastable branch parameterized by $\boldsymbol{z}$ we can define the equilibrium response functions $\sigma = \sigma(l, \boldsymbol{z})$.

A crucial observation for this work is that, due to imposed inhomogeneity, not all metastable equilibria will be accessible by our dynamics. Indeed, suppose that the bias coefficients h_j are pairwise different and define a subclass of metastable states, which we call *ordered states*, via the condition

$$h_j < h_k \quad \Longrightarrow \quad e_1(t) < e_2(t) < \cdots < e_n(t). \tag{4.2}$$

Then, the knowledge of the set of ordered states is sufficient for the study of the limiting macroscopic problem because the set of ordered states is invariant under the evolution for the viscous and for the limiting inviscid systems (see (4.8) and (DA1)–(DA3) in Definition (4.2)). Moreover, one can see that a system that starts non-ordered will have the tendency to return into an ordered state. For instance, the chain will acquire the ordering if it is ever stretched beyond the transformation thresholds and will then maintain its ordering during all future times. Nevertheless, the system may have an initial nontrivial virgin curve involving some non-ordered states, which our limiting theory would not capture.

Remark 4.1 *The disorder entering through the random microscopic body forces is very special in the sense that it leads to a particular simple structure of the inner hysteresis loops. A somewhat more realistic way of bringing disorder into the model would be through a randomization of the thresholds σ_- and σ_+ as in [PuT02]. This, however, brings additional technical complications, which we would like to avoid here.*

It will be convenient to simplify the ordering condition by using the permutational symmetry of the system. Indeed, without loss of generality we can assume that the biases h_j are ordered as $h_1 < h_2 < \cdots < h_N$, such that (4.2) reduces to the condition

$$e_1(t) < e_2(t) < \cdots < e_N(t). \tag{4.3}$$

In Section 5, however, we need to return to the original ordering condition (4.2) because the strains $(e_j)_{j=1,\dots,N}$ of the springs in a one-dimensional bar $\Omega =]0, 1[$ will be naturally ordered according to the material points $(x = j/N)$.

The class of ordered equilibria in the sense of (4.3) have a simple characterization: for each such state there exists a threshold $\widehat{h}$ such that all j with $h_j \geq \widehat{h}$ are in phase $z_j = +1$ while those with $h_j < \widehat{h}$ are in phase $z_j = -1$. We can then associate with each threshold a particular distribution of snap-springs between the two energy wells

$$z_j = \operatorname{sign}(h_j - \widehat{h}), \tag{4.4}$$

where $\operatorname{sign}(h_j - \widehat{h}) = 1$ for $h_j \geq \widehat{h}$ and $\operatorname{sign}(h_j - \widehat{h}) = -1$ for $h_j < \widehat{h}$. It will also be convenient to introduce the following two functions

$$\overline{h}_+(\widehat{h}) = \min\{\, h_j \mid h_j \geq \widehat{h} \,\}, \quad \overline{h}_-(\widehat{h}) = \max\{\, h_j \mid h_j < \widehat{h} \,\}. \tag{4.5}$$

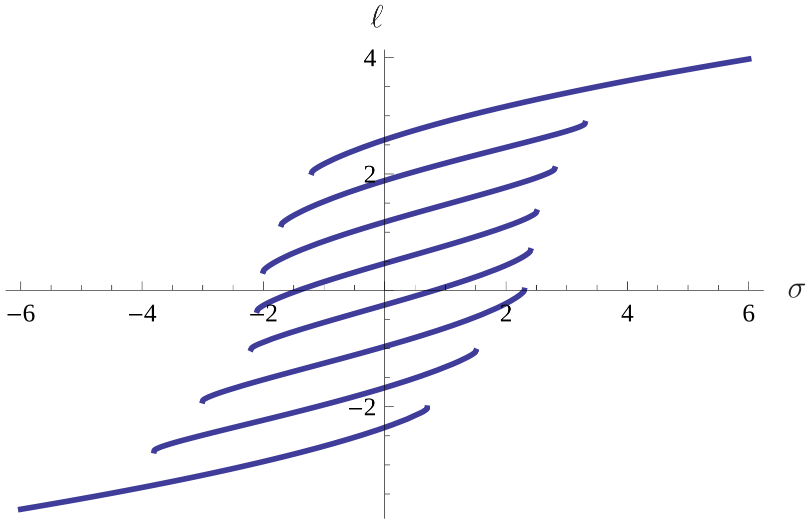

Figure 4.1: Eight monotone stress-strain equilibrium branches $\ell = M(\widehat{h}, \sigma)$ representing ordered choices of the phases.

Notice that $\overline{h}_\pm : \mathbb{R} \to \mathbb{R}$ are nondecreasing piecewise constant functions such that $\overline{h}_-(\widehat{h}) < \widehat{h} \le \overline{h}_+(\widehat{h})$. We shall also define $\overline{h}_+(\widehat{h}) = \infty$ if all $h_j < \widehat{h}$ and $\overline{h}_-(\widehat{h}) = -\infty$ if all $h_j \ge \widehat{h}$.

For each $\widehat{h} \in \mathbb{R}$ we can now define the function $M(\widehat{h}, \cdot) : [\sigma_- - \overline{h}_+(\widehat{h}), \sigma_+ - \overline{h}_-(\widehat{h})] \to \mathbb{R}$ given by the formula

$$M(\widehat{h}, \sigma) = \frac{1}{N} \sum_{j=1}^{N} \psi_{\mathrm{sign}(h_j - \widehat{h})}(h_j + \sigma).$$

It is not hard to see that we can have at most $N+1$ different functions $M(\widehat{h}, \cdot)$. Each of these functions is strictly increasing and has at most one solution for $M(\widehat{h}, \sigma) = \ell$ (see Figure 4.1). Such solutions form equivalence classes defining equilibrium branches

$$\sigma = \sigma(\ell, \xi)$$

where

$$\xi = m/N \tag{4.6}$$

and $m \in \{0, 1, ..., N\}$ is the number of elements j with $\mathrm{sign}(e_j) = 1$. As we see, for ordered states the metastable branch is defined not by the whole vector $\boldsymbol{z}$ but by a single parameter ξ, which is the fraction of the springs in phase $+1$. It will serve as the predecessor of the plastic strain appearing later in the limiting continuum problem.

It is easy to see that one can have at most $N+1$ solutions for each ℓ. For instance, for the case of a bi-quadratic potential Φ_{biq} in (2.1) the functions $M(\widehat{h}, \cdot)$ take the form

$$M(\widehat{h}, \sigma) = \frac{1}{k}\sigma + \frac{1}{kN} \sum_{j=1}^{N} h_j + \frac{a}{N} \sum_{j=1}^{N} \mathrm{sign}(h_j - \widehat{h}),$$

which are $N+1$ parallel lines shifted by the same constant $2a/N$. Under the simplifying assumption that $\sum_1^N h_j = 0$ we find the explicit representation of the equilibrium branches

$$e_j = \ell + a\,\mathrm{sign}(e_j) + h_j/k + a(1-2\xi), \tag{4.7}$$

where ξ is defined by (4.6).

4.2 Jump discontinuities

Suppose now that the body forces h_j remain ordered and constant with $\sum_{j=1}^{N} h_j = 0$, while the total length of the change becomes a function of time $\ell(t)$. The resulting system of ODEs takes the form

$$\delta \dot{e}_j = -\phi(e_j) + h_j + \sigma(t), \quad \frac{1}{N}\sum_{j=1}^{N} e_j(t) = \ell(t), \tag{4.8}$$

where $\ell \in \mathrm{C}^1([0,T])$ is a given datum. We again restrict our attention to ordered states and consider the case of bi-quadratic potential. In this case we can define a unique limiting solution as $\delta \to 0$.

Suppose first that δ is finite. Observe that if all the $e_j(t)$ are ordered and are different from 0, then the solution of the ODE (4.8) can be extended uniquely as differentiable function. Such a differentiable extension will work up to the time t_* when $e_{j_*}(t_*^-) = 0$ for some j_* (here $e_j(s^-) = \lim_{t \nearrow s} e_j(t)$ means the limit from the left), and until that time the solution is unique. If the solution is smoothly extendable, then we choose this as the unique extension, i.e. e_{j_*} does not change sign at t_* (and we ignore the other solution where e_{j_*} would change sign and $\dot{\boldsymbol{e}}$ has a jump at t_*). If there is no extension where $\dot{\boldsymbol{e}}$ is continuous, we can construct a unique differentiable solution on $[t_*, t_*{+}\tau]$ with initial condition $\boldsymbol{e}(t_*)$ that is uniquely determined by choosing $e_{j_*}(\cdot)$ such that its signs differ for $t < t_*$ and $t > t_*$. Concatenating this to the solution on $[0, t_*]$ defines the unique global solution, which is still Lipschitz continuous in time. Observe that the system always remains in the set of ordered states. In the next subsection we prove that the viscous solution $\boldsymbol{e}^\delta(t)$ converges to a solution $\boldsymbol{e}^0(t)$ of a well-defined limit problem. The configurations $\boldsymbol{e}^0(t)$ can be viewed as a time-dependent family of metastable states described in the previous subsection. This family splits into branches and when the branch ends the extension constituting $\boldsymbol{e}^0(t)$ is selected by a suitable jump rule which is the only memory of the viscous dissipative mechanism (see also [PuT05]).

The parameter defining plastic dissipation in the coarse-grained model is the release of energy in a single jump. The energy is defined as follows

$$\boldsymbol{E}(t, \boldsymbol{e}) = \begin{cases} \frac{1}{N}\sum_{j=1}^{N} \big(\Phi(e_j) - h_j e_j\big) & \text{if } \frac{1}{N}\sum_{j=1}^{N} e_j = \ell(t), \\ \infty & \text{else.} \end{cases} \tag{4.9}$$

In the case of the bi-quadratic potential Φ_{biq} the energy release can be calculated explicitly

$$\boldsymbol{E}(t_*, \boldsymbol{e}(t_*^-)) - \boldsymbol{E}(t_*, \boldsymbol{e}(t_*^+)) = \rho_N/N > 0 \quad \text{where } \rho_N = 2ka^2 - 2ka^2/N. \tag{4.10}$$

Here the first term in ρ_N corresponds to the integral $\int_{e_-}^{e_+^*} \sigma_+ {-} \phi(e)\,\mathrm{d}e$, see Lemma 7.2. The second term is due to the relaxation of the stress from $\sigma(t_*^-) = \sigma_\pm$ to $\sigma(t_*^+) = \sigma_\pm \mp 2ak/N$. Because of our special choice of the disorder the critical values e_- and e_+ are not affected by the disorder. For Φ_{biq} both thresholds are equal to 0 and the strains satisfy the following explicit jump relations

$$e_j(t_*^+) = e_j(t_*^-) - a\hat{\Delta}/N \text{ for } j \neq j_*, \quad e_{j_*}(t_*^-) = 0, \quad e_{j_*}(t_*^+) = a\hat{\Delta}(1{-}1/N), \tag{4.11}$$

where $\hat{\Delta} = z(t^+) - z(t^-) \in \{-2, 2\}$.

4.3 The automaton

As we have already mentioned, one can expect the solution $\boldsymbol{e}^\delta$ of the viscous ODE (4.8) to slide along the metastable branches with finitely many well-separated fast jumps from one curve to the next. The limiting dynamics then includes the periods, when the system remains on one of the metastable branch with parameter ξ fixed, and the jumps, when ξ changes and the system switches metastable branches. The resulting dynamical system takes the form of a discrete threshold-type automaton (see [PRTZ08, PRTZ09]).

Definition 4.2 *Given an ordered bias vector $(h_j)_j$ and a loading profile $\ell \in \mathrm{C}^1([0,T])$ a function $\boldsymbol{e} : [0,T] \to \mathbb{R}^N$ is called a solution of the automaton, if the following conditions hold:*

(DA1) *For all $t \in [0,T]$ the state $\boldsymbol{e}(t)$ is an ordered steady state as described in Section 4.1 with $\frac{1}{N}\sum_1^N e_j(t) = \ell(t)$.*

(DA2) *There are at most a finitely many times $0 = t_0 < t_1 < t_2 < \cdots t_L = T$ such that for $l = 1, ..., L$ the function $\boldsymbol{e}|_{]t_{l-1},t_l[}$ has a C^1 extension to $[t_{l-1}, t_l]$.*

(DA3) *At each jump time t_l, $l = 1, ..., L-1$ the following holds:*
(i) the strain is critical, i.e. $e_j(t_l^-) \in \{e_+, e_-\}$,
(ii) the jump conditions (4.11) *hold for $t_* = t_l$, and*
(iii) the energy release $\boldsymbol{E}(t_l, \boldsymbol{e}^0(t_l^-)) - \boldsymbol{E}(t_l, \boldsymbol{e}^0(t_l^+))$ is exactly ρ_N/N,

Notice that the jump conditions in (DA3) are redundant and it would be sufficient to state only (iii), since the special form of ϕ implies that (i) and (ii) must hold. This will be implicitly shown in the proof of Proposition 4.4. Here we stated the redundant conditions to highlight all the special features of the jumps.

Another technical issue is that as in the case of the viscous ODE system (4.8) the solution of the discrete automaton is not unique. A nonuniqueness can occur if a steady state reaches $e_{j_*}(t_*) = 0$ exactly at a moment when ℓ has a local extremum. Then, the phase jump may occur or may not occur. We define a unique extension by asking the solution to stay continuous as long as possible, i.e. we assume that jumps only occur if they are necessary. This additional "rule" for the bi-quadratic problem can be obtained rigorously if one considers an additional limit when a finite spinodal region is asymptotically shrinking to zero.

4.4 An energetic rate-independent system

Before giving the convergence proof for $\delta \to 0$, we show that the automaton (DA1)–(DA3) can be reformulated in terms of an *energetic rate-independent system* (ERIS) in the sense of [Mie05]. This reformulation will serve as a basis of the subsequent continualization of our discrete dynamical system in Section 5.

A general ERIS is given in terms of the state space $\boldsymbol{Q}$, time-dependent energy functional $\boldsymbol{E} : [0,T] \times \boldsymbol{Q} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, and a dissipation distance $\boldsymbol{D} : \boldsymbol{Q} \times \boldsymbol{Q} \to [0,\infty]$. Our state space is $\boldsymbol{Q} = \mathbb{R}^N$ and the energy functional $\boldsymbol{E}$ is defined in (4.9). The new quantity is the dissipation distance $\boldsymbol{D}$, which measures the energy that is dissipated due to fast viscous motion. If the strains vary quasistatically in one of the two wells, there will be no dissipative contribution in the inviscid limit $\delta \to 0$. However, if a strain jumps into the other well (i.e. by changing sign), then the viscous motion is fast, namely of order $1/\delta$ and the energy $\int_{t_1(\delta)}^{t_2(\delta)} \frac{1}{N}\sum_{j=1}^N \delta \dot{e}_j^2(t)\,\mathrm{d}t$ has a finite limit (see also [PuT05]).

We can define the dissipation distance by counting the number of phase jumps as follows:

$$\mathcal{D}(\boldsymbol{e}^0,\boldsymbol{e}^1) = \frac{1}{N}\sum_{j=1}^N D_N(e_j^0,e_j^1), \text{ where } D_N(e^0,e^1) = \begin{cases} \rho_N & \text{if } e^0e^1 < 0 \ \ \text{(phase jump)}, \\ 0 & \text{if } e^0e^1 \geq 0 \ \ \text{(no phase jump)}, \end{cases}$$

where ρ_N is defined in (4.10). Using the triple $(\boldsymbol{Q},\boldsymbol{E},\boldsymbol{D})$ we can further define the notion of *energetic solutions* as follows, see e.g. [Mie05, Mie10]. This notion is especially adapted to solutions that may have jumps like in the present case.

Definition 4.3 *Given a loading $\ell \in \mathrm{C}^1([0,T])$ and a $(h_j)_{j=1,\dots,N} \in \mathbb{R}^N$, a function $\boldsymbol{e} : [0,T] \to \boldsymbol{Q}$ is called an* energetic solution *of the ERIS $(\boldsymbol{Q},\boldsymbol{E},\boldsymbol{D})$, if for all $t \in [0,T]$ we have the stability **(S)** and the energy balance **(E)**:*

$$\begin{aligned} &\textit{(S)} && \infty > \boldsymbol{E}(t,\boldsymbol{e}(t)) \leq \boldsymbol{E}(t,\widetilde{\boldsymbol{e}}) + \boldsymbol{D}(\boldsymbol{e}(t),\widetilde{\boldsymbol{e}}) \text{ for all } \widetilde{\boldsymbol{e}} \in \boldsymbol{Q}, \\ &\textit{(E)} && \boldsymbol{E}(t,\boldsymbol{e}(t)) + \mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[0,t]) = \boldsymbol{E}(0,\boldsymbol{e}(0)) - \int_0^t \Sigma(\boldsymbol{e}(s))\dot{\ell}(s)\,\mathrm{d}s, \end{aligned} \tag{4.12}$$

where $\mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[0,t])$ is the supremum of $\sum_{k=1}^M \boldsymbol{D}(\boldsymbol{e}(\tau_{k-1}),\boldsymbol{e}(\tau_k))$ over all $M \in \mathbb{N}$ and all partitions $0 \leq \tau_0 < \tau_1 < \cdots < \tau_M \leq t$ of $[0,t]$ and $\Sigma(\boldsymbol{e}) = \frac{1}{N}\sum_{j=1}^N \big(\phi(e_j) - h_j\big)$.

Note that the dissipation functional $\mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[r,t])$ gives a counting measure, since it is equal to ρ_N/N times the number of all the phase jumps of $\boldsymbol{e}$ in the time interval $[r,t]$.

The following result states that the evolution given in terms of the discrete automaton is exactly the same as the energetic solutions of $(\boldsymbol{Q},\boldsymbol{E},\boldsymbol{D})$. For this result the ordering property of the solutions is in fact not necessary and it also applies to non-ordered solutions.

Proposition 4.4 *Consider an ordered bias vector $(h_j)_{j=1,\dots,N}$ and that $\ell \in \mathrm{C}^1([0,T])$. Then, an ordered function $\boldsymbol{e} : [0,T] \to \boldsymbol{Q} = \mathbb{R}^N$ is an energetic solution of $(\boldsymbol{Q},\boldsymbol{E},\boldsymbol{D})$ given via* (4.12) *if and only if it satisfies* (DA1)–(DA3) *in Definition 4.2.*

Proof: (S)&(E) ⇒ (DA1)–(DA3).

From (S) we conclude that for each $t \in [0,T]$ the solution satisfies the length constraint and is in equilibrium. For the latter, simply consider variations $\widetilde{\boldsymbol{e}}$ such that $\boldsymbol{D}(\boldsymbol{e}(t),\widetilde{\boldsymbol{e}}) = 0$, i.e. with no additional phase jumps. Then, $\boldsymbol{e}(t)$ is a local minimizer of $\boldsymbol{E}(t,\cdot)$ und thus a stable equilibrium. Thus, (DA1) is established. In particular, we know that $\boldsymbol{e}(t)$ lies in the finite set of stable equilibria. Along these branches the dependence of $\boldsymbol{e}(t)$ on $\ell(t)$ is smooth, see (4.7).

From (E) we conclude that $\mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[0,T])$ is finite. Since $\boldsymbol{D}$ only takes the discrete values $\{\, k\rho_N/N \mid k = 0,1,\dots,N \,\}$ we conclude that the monotone function $\hat{\delta} : [0,T] \to [0,\infty[\,; t \mapsto \mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e},[0,t])$ is piecewise constant with finitely many jump points $t_1 < \cdots < t_{L-1}$, where each jump is an integer multiple of ρ_N/N. Since jumping between the solution branches generates a jump in $\hat{\delta}$, we conclude that on the intervals $]t_{l-1},t_l[$ the solution remains on one branch and hence can be extended smoothly to $[t_{l-1}, t_l]$. Hence (DA2) is established.

(E) implies energy balance on all subintervals, namely $\boldsymbol{E}(t, \boldsymbol{e}(t)) + \mathrm{Diss}_{\boldsymbol{D}}(\boldsymbol{e}, [r, t]) = \boldsymbol{E}(r, \boldsymbol{e}(r)) - \int_r^t \Sigma(\boldsymbol{e}(s))\dot{\ell}(s)\,\mathrm{d}s$. Taking the limits $t \to t_l^+$ and $r \to t_L^-$ we find the jump relation

$$\boldsymbol{E}(t_l, \boldsymbol{e}(t_l^+)) + \boldsymbol{D}(\boldsymbol{e}(t_l^-), \boldsymbol{e}(t_l^+)) = \boldsymbol{E}(t_l, \boldsymbol{e}(t_l^-)). \tag{4.13}$$

However, the choice of ρ_N was exactly such that it corresponds to the energy loss for a jump arising from critical strains $e_{j_*}(t_l^-) \in \{e_-, e_+\}$, which establishes (i). Properties (ii) and (iii) follow from the assumption that all h_j are pairwise disjoint. Then, at most one e_j can have a phase jump.

(DA1)–(DA3) $\Rightarrow$ (S)&(E). From (DA1) we obtain easily (S): Every stable equilibrium is globally stable in the sense of (S), since stability with respect to $\widetilde{\boldsymbol{e}}$ satisfying $\boldsymbol{D}(\boldsymbol{e}(t), \widetilde{\boldsymbol{e}}) = 0$ follows from the equilibrium conditions and convexity of Φ in the two wells. Moreover, ρ_N was chosen as the maximal energy loss when jumping from one branch to a neighboring one. Thus, the energy release $\boldsymbol{E}(t, \boldsymbol{e}(t)) - \boldsymbol{E}(t, \widetilde{e})$ will be always less than $\boldsymbol{D}(\boldsymbol{e}(t), \widetilde{\boldsymbol{e}})$.

Using (DA2) and (DA3) the energy balance (E) is obtained by joining the smooth parts in $]t_{l-1}, \min\{t, t_l\}[$ and the jumps. In the first case set $t_* = \min\{t, t_l\}$, the smoothness gives $\boldsymbol{E}(t_*, \boldsymbol{e}(t_*^-)) = \boldsymbol{E}(t_{l-1}, \boldsymbol{e}(t_{l-1}^+)) - \int_{t_{l-1}}^{t_*} \Sigma(\boldsymbol{e}(s))\dot{\ell}(s)\,\mathrm{d}s$. At the jumps we have (4.13) and (E) follows by addition. ∎

4.5 Convergence proof

We finally prove the convergence for $\delta \to 0$ of the viscous ODE system (4.8) to the automaton (DA1)–(DA3) and consequently to the ERIS system $(\boldsymbol{Q}, \boldsymbol{E}, \boldsymbol{D})$. The proof is constructive and provides explicit error estimates in terms of the small parameter δ and $\varepsilon = 1/N$.

A main point is that there will be different sources of error that need to be estimated in different norms. During the equilibrium phase, when the system slides close to a particular metastable branch, the non-zero viscosity prevents the solution from relaxing to the exact equilibrium state and this gives rise to an error (i) of order δ in all of the components. Two other errors occur during jumps: (ii) one of the strains, namely e_{j_*}, is far away from a stable steady state, while (iii) all the other strains have an error of order ε. The first and the third type of errors is most efficiently measured in the maximum norm $|R|_\infty = \max\{\, |R_j| \mid j = 1, ..., N \,\}$ whereas the second type of errors is better evaluated in the 1-norm $|R|_1 = \sum_1^N |R_j|$.

Under the assumption that body forces are time independent and the potential is bi-quadratic, we have the following result:

Theorem 4.5 *Consider an ordered bias vector $(h_j)_j$ with $\sum_1^N h_j = 0$ and a loading profile $\ell \in \mathrm{C}^{\mathrm{Lip}}([0, T])$ that is piecewise C^1 with $|\dot{\ell}(t)| \geq \lambda > 0$ a.e. in $[0, T]$. Take any ordered steady state $\overline{\boldsymbol{e}}^0 \in \mathbb{R}^N$ associated with $\ell = \ell(0)$. Then, the solution $\boldsymbol{e}^\delta \in \mathrm{C}^{\mathrm{Lip}}([0, T]; \mathbb{R}^N)$ of (4.8) with $\boldsymbol{e}^\delta(0) = \boldsymbol{e}^0$ constructed above converges to the unique solution $\boldsymbol{e}^0 : [0, T] \to \mathbb{R}^N$ with $\boldsymbol{e}^0(0) = \overline{\boldsymbol{e}}^0$ of the discrete automaton (DA1)–(DA3) constructed above, i.e. for almost every $t \in [0, T]$ we have $\boldsymbol{e}^\delta(t) \to \boldsymbol{e}^0(t)$ as $\delta \to 0$.*

Moreover, for each given data k, a, T, and $\ell \in \mathrm{C}^1([0,T])$ there are positive constants C and κ_* such that for all $\delta \in \,]0,1]$ and $N \in \mathbb{N}$ with $\delta N \le \kappa_*$ we have $\boldsymbol{e}^\delta(t) = \boldsymbol{e}^0(t) + R^1(t) + R^2(t)$ with

$$|R^1(t)|_\infty \le C(\delta{+}1/N) \quad \textit{and } |R^2(t)|_1 \le C. \tag{4.14}$$

Proof: To simplify the notations we drop the superscript δ for the viscous solutions but keep the superscript 0 for the limit. Throughout the proof the constant C may vary, but it is always independent of δ, N and the given solutions. We use sometimes constants $C_1, C_2, ...$ to indicate how certain estimates follow from others.

We decompose the time interval into finitely many subintervals on each of which ℓ is monotone. If we allow for a suitable error for the initial condition it is then sufficient to consider only one of these intervals. Indeed, without loss of generality we can assume that ℓ is monotonically increasing on $[0,T]$, however, to be able to concatenate several pieces we allow for a nontrivial shift $\boldsymbol{e}(0){-}\boldsymbol{e}^0(0)$.

From the monotonicity of ℓ and the ordering of the solutions $\boldsymbol{e}$ we obtain jump times $0 < t_1 < \cdots < t_L < T$. For the following it is more convenient to reorder these numbers and to use as the switching times parameters s_j, $j = 1, ..., N$ defined such that $\operatorname{sign} e_j(t) = \operatorname{sign}(t{-}s_j)$. Then, $0 \le s_N \le s_{n-1} \le \cdots s_1 \le T$, where strict inequality holds as soon as the times are different from 0 or T. With $m(t)$ we count the number of $e_j(t)$ and $e^0_j(t)$ bigger than 0, namely $m(t) = N{-}j$ for $t \in \,]s_{j-1}, s_j[$. Similarly, for the solution $\boldsymbol{e}^0$, where $\delta = 0$, we define s^0_j and $m^0(t)$ having exactly the same properties.

For sufficiently small $\delta + 1/N$ we conclude that $m(0) = m^0(0)$. Using m^0 and m the average stresses σ^0 and σ can be calculated as

$$\begin{aligned}
\sigma(t) &= \frac{1}{N}\sum_{j=1}^{N}\big(\phi(e_j(t)) + h_j + \delta\dot{e}_j(t)\big) = k\ell(t) + \delta\dot{\ell}(t) + \frac{ak}{N}(2m(t){-}N),\\
\sigma^0(t) &= k\ell(t) + \frac{ak}{N}(2m^0(t){-}N).
\end{aligned}$$

With these stress histories known, the strains solving (4.8) have the explicit representation

$$e_j(t) = \mathrm{e}^{-kt/\delta}e_j(0) + \int_0^t \mathrm{e}^{-k(t-s)/\delta}\frac{1}{\delta}\big(ak\operatorname{sign}(s{-}s_j) + h_j - \sigma(s)\big)\,\mathrm{d}s, \tag{4.15a}$$

$$e^0_j(t) = a\operatorname{sign}(t{-}s^0_j) + \frac{1}{k}(h_j{+}\sigma^0(t)). \tag{4.15b}$$

We write the difference $\rho_j(t) = e_j(t) - e^0_j(t)$ in the form

$$\begin{aligned}
&\rho(t) = \rho^1_j(t) + \rho^2_j(t) + \rho^3_j(t) + \rho^4_j(t) \text{ with}\\
&\rho^1_j(t) = \mathrm{e}^{-kt/\delta}\rho_j(0), \qquad \rho^2_j(t) = \int_0^t \mathrm{e}^{-k(t-s)/\delta}k\dot{\ell}(s)\,\mathrm{d}s,\\
&\rho^3_j(t) = \int_0^t \mathrm{e}^{-k(t-s)/\delta}\frac{2ak}{\delta N}\big(m^0(t) - m(s)\big)\,\mathrm{d}s,\\
&\rho^4_j(t) = \int_0^t \mathrm{e}^{-k(t-s)/\delta}\frac{ak}{\delta}\big(\operatorname{sign}(t{-}s^0_j) - \operatorname{sign}(s{-}s_j)\big)\,\mathrm{d}s.
\end{aligned}$$

We immediately find $|\rho_j^1(t)| + |\rho_j^2(t)| \leq C(\delta{+}1/N)$ as desired.

To estimate the other terms we need to estimate the difference between s_j and s_j^0. The nontrivial s_j^0 are defined via

$$0 = -a + h_j/k + \ell(s_j^0) + a(2j{-}N)/N, \tag{4.16}$$

which implies $\ell(s_j^0) - \ell(s_{j+1}^0) = (h_{j+1}{-}h_j)/k + 2a/N > 2a/N$. Hence with $C = a\|\dot{\ell}\|_\infty/2$ we find

$$|s_j^0 - s_l^0| \geq \frac{|j{-}l|}{CN} \quad \text{for } j, l = 1, ..., N. \tag{4.17}$$

For the moment we assume a similar estimate

$$|s_j - s_l| \geq \frac{|j{-}l|}{C_m N} \quad \text{for } j, l = 1, ..., N, \tag{4.18}$$

where the constant C_m is still to be determined by choosing $\delta N \leq \kappa_*$ sufficiently small. Using this assumption we can estimate $\dot{e}_j(s_j^-)$ (limit from the left) via the explicit form of e_j in (4.15a). Note that σ is piecewise smooth with jumps of size $O(1/N)$ at each s_l The contributions of the initial condition and the smooth parts are bounded by a constant C_1 independently of δ, N and C_m. Including the terms from the jumps gives the estimate

$$|\dot{e}_j(s_j^-)| \leq C_1 + CC_m\gamma\big(1/(C_m\delta N)\big), \quad \text{where } \gamma(r) = \sum_{l=j+1}^{N} r\mathrm{e}^{-(l-j)r} \leq 1 + r.$$

As the nontrivial s_j are obtained from

$$0 = e_j(s_j) = -ah_j/k + \ell(s_j) + a(2j{-}N)/N + \delta\big(\dot{\ell}(s_j){-}\dot{e}_j(s_j^-)\big),$$

we can compare with (4.16). Using $\lambda \leq \dot{\ell}(t) \leq C$ and $|\dot{e}_j(s_j^-)| \leq C(1{+}C_m)$ we find a constant C such that

$$|s_j - s_j^0| \leq \frac{\delta}{\lambda}\big(C(1{+}C_m) + \|\dot{\ell}\|_\infty\big) =: \delta C_2(1{+}C_m). \tag{4.19}$$

From this we can now derive (4.18) as follows. For nontrival j and l with $j \neq l$ we have

$$\begin{aligned} |s_j{-}s_l| &\geq |s_j^0{-}s_l^0| - |s_j^0{-}s_j| - |s_l^0{-}s_l| \ \geq \ \frac{|j{-}l|}{CN} - 2\delta C_2(1{+}C_m) \\ &\geq \frac{|j{-}l|}{CN}\big(1 - 2\delta NCC_2(1{+}C_m)\big) \ \overset{(*)}{\geq} \ \frac{|j{-}l|}{C_m N}. \end{aligned}$$

To justify $\overset{(*)}{\geq}$ we use $\delta N \leq \kappa_*$ with $\kappa_* := 1/(4C_2 \max\{C, 2C^2\})$ and set $C_m = (2\kappa_* C_2)^{-1/2}$. Thus, (4.18) is finally established.

Using the above estimates between the jump times s_j and s_l^0 we are able to control the difference between $m^0(t)$ and $m(s)$. First assume $m(t) = N{-}j \geq m^0(t) = N{-}l$, then by the definition of m and m^0 we have $s_j \geq s_{l-1}^0$. Thus, we find

$$s_j^0 + \delta C \geq s_j \geq s_{l-1}^0 \geq s_j^0 + \frac{l{-}1{-}j}{CN},$$

which yields $l-j \le 1+\delta NC^2$. Hence, $l-j \le N_* := \lfloor 1+\kappa_* C^2 \rfloor \in \mathbb{N}$. With a similar argument for $m(t) = N-j \le m^0(t) = N-l$ and using (4.17) we obtain

$$|m(s) - m^0(t)| \ \le \ N_* + CN(t-s) \quad \text{for } 0 \le s \le t \le T.$$

Hence, ρ_j^3 can be estimated via

$$|\rho_j^3(t)| \le C(\delta + 1/N) \quad \text{for all } j = 1, ..., N \text{ and } t \in [0, T].$$

Let $s_j^{\min}$ and $s_j^{\max}$ be the minimun and maximum of $\{s_j, s_j^0\}$. Using (4.19) yields

$$|\rho_j^4(t)| \le \begin{cases} 0 & \text{for } s \le s_j^{\min}, \\ 2 & \text{for } s_j^{\min} < s \le s_j^{\max}, \\ 2\mathrm{e}^{-k(t-s_j^{\max})/\delta} & \text{for } s \ge s_j^{\max}. \end{cases}$$

To conclude the theorem we define R^1 via $R_j^1(t) = \rho_j^1(t) + \rho_j^2(t) + \rho_j^3(t)$ and obtain immediately $|R^1(t)|_\infty \le C(\delta + 1/N)$. For $R_j^2(t) = \rho^4(t)$ we use the fact that in a given time t only for a few js there has been a recent jump, namely

$$|\rho^4(t)|_1 = \sum_1^N |\rho_j^4(t)| \le 2\big(N_* + \textstyle\sum_1^N \mathrm{e}^{-k/(C\delta)}\big) \le C_4.$$

Thus, estimate 4.14 is established.

We still have to show the convergence $R^{\delta,1}(t) + R^{\delta,2}(t) \to 0$ for $\delta \to 0$ but N fixed. We now display the dependence on δ again by adding the superscript δ where convenient. We show that this convergence holds for all t in $\mathcal{T} := [0, T] \setminus \{s_1^0, ..., s_N^0\}$, which is a set of full measure.

It is now easy to see that $\rho_j^{\delta,1}(t) + \rho_j^{\delta,2}(t) \to 0$ for all t. To estimate $\rho_j^{\delta,3}$ and $\rho_j^{\delta,4}$ we fix $t \in \mathcal{T}$ and let $\tau = \frac{1}{2}\mathrm{dist}(t, \{s_1^0, ..., s_N^0\})$. Then, for all sufficiently small δ the interval $]t-\tau, t[$ does not contain any s_l^0 or s_l^δ. Whence $m^0(t) = m^\delta(s)$ and $\mathrm{sign}(t-s_j^0) = \mathrm{sign}(s-s_j^\delta)$ for $s \in [t-\tau, t]$, because $s_l^\delta \to s_l^0$, and $\rho_j^{\delta,3}(t) + \rho_j^{\delta,4}(t) \to 0$ follows easily.

Thus, the proof of Theorem 4.5 is complete. ■

5 Continuum limit

We are now interested in the limit $\varepsilon \to 0$, i.e. the number N of elements goes to infinity, which means that we apply the second limiting procedure to the automaton representing the primary inviscid limit of the original ODE system. The main challenge is to replace the automaton type evolution of the plastic variable formulated in terms of *discrete space* and *discrete time* by a dynamical system employing a *continuous time* variable t and *continuous space* variable x. This is feasible because in the limit $\varepsilon \to 0$ the *elastic stages* become progressively shorter while the *plastic jumps* becomes weaker and more frequent (see also [PuT05]). As a result the limiting evolution involves *simultaneous elastic and plastic stages* and the corresponding continuum variables change all the time.

To justify this picture it will be convenient to use the formulation as an energetic system $(\boldsymbol{Q}_N, \boldsymbol{E}_N, \boldsymbol{D}_N)$. The strategy is to embed this system into a system defined on $\mathcal{Q} = \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$, which contains the strains and a plastic variable. For the embedded system we are able to pass to the limit $\varepsilon \to 0$ in the pure rate-independent setting.

5.1 Embedding into physical space

Note that now we are treating a sequence of problems with N as a parameter. Hence, for each N there is a bias vector $\boldsymbol{h}^N$ with components h_j^N, $j = 1, ..., N$. All solutions $\boldsymbol{e}(t)$ we consider satisfy the original ordering condition (4.2), namely

$$h_j < h_k \implies e_j(t) < e_k(t).$$

We define an embedding of $\mathbb{R}^N$ into $\mathrm{L}^2(\Omega)$ via the characteristic functions

$$\chi_j^N \overset{\text{def}}{=} \chi_{](j-1)/N, j/N[} \quad \text{(characteristic function of } \big]\tfrac{j-1}{N}, \tfrac{j}{N}\big[\subset \Omega).$$

The piecewise constant interpolants $\overline{e}^N$ and a plastic variable $\overline{p}^N$ are given by

$$\mathcal{P}_N : \mathbb{R}^N \to \mathcal{Q} := \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega), \quad \mathcal{P}_N(\boldsymbol{e}) := (\overline{e}^N, \overline{p}^N) \text{ with}$$
$$\overline{e}^N(t,x) = \sum_{j=1}^N e_j(t)\chi_j^N(x) \quad \text{and} \quad \overline{p}^N(t,x) = a \sum_{j=1}^N \operatorname{sign}(e_j(t))\chi_j^N(x).$$

For $N \in \mathbb{N}$ we now specify the choice of the random bias coefficients h_j in the form

$$h_j^N = \mu_j^N - G(j/N), \text{ where } G(x) = c + \textstyle\int_0^x g_{\text{ext}}(y)\,\mathrm{d}y \text{ with } \textstyle\int_0^1 G(x)\,\mathrm{d}x = 0,$$

and where the random contributions μ_j^N for $N \in \mathbb{N}$ and $j = 1, ..., N$ are independent, identitically distributed random variables taking values in $\mathbb{R}$. The distribution is given through a density $f \in \mathrm{L}^1(\mathbb{R})$ with compact support and average 0.

5.2 Macroscopic system

To specify the structure of the limiting energy, which incorporates kinematic hardening component, we need to associate to each density f satisfying (3.1) an auxiliary function $\mathcal{F}^*$. We first define

$$F : \mu \mapsto \int_{-\infty}^{\mu} f(y)\,\mathrm{d}y \quad \text{and} \quad \mathcal{F} : \mu \mapsto \int_{-\infty}^{\mu} F(y)\,\mathrm{d}y, \tag{5.1}$$

which gives $\mathcal{F}''(\mu) = f(\mu) \geq 0$. Now, $\mathcal{F}^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined as Legendre transform of $\mathcal{F}$, namely

$$\mathcal{F}^*(\eta) := \sup\{\, \mu\eta - \mathcal{F}(\mu) \mid \mu \in \mathbb{R} \,\}. \tag{5.2}$$

Thus, $\mathcal{F}^*$ is convex as well and satisfies $\mathcal{F}^*(\eta) = \infty$ for $\mu \notin [0,1]$. We can now define the (kinematic) hardening function $H : \mathbb{R} \to \mathbb{R}_\infty$ associated with the density f as

$$H(p) = 2a\mathcal{F}^*\Big((a-p)/(2a)\Big), \tag{5.3}$$

which is convex and satisfies $H(p) = \infty$ for $|p| > a$, by definition.

For the simple example $f(\mu) = \frac{1}{2\mu_*}\chi_{[-\mu_*,\mu_*]}$ we obtain $H(p) = \mu_*(p^2 - a^2)/(2a)$. Consider now a family of densities f_r satisfying $f_r(\mu) = \frac{1}{r} f_1(\frac{\mu}{r})$. Then, we obtain $F_r(\mu) =$

$F_1(\mu/r)$ and $\mathcal{F}_r(\mu) = r\mathcal{F}_1(\mu/r)$. For the Legendre transform this leads to $\mathcal{F}_r^*(\eta) = r\mathcal{F}_1(\eta)$. Thus, we obtain that

$$H_r(p) = rH_1(p) \to 0 \quad \text{for } r \to 0 \text{ and } |p| < a \text{ fixed.} \tag{5.4}$$

By using the definitions above we can now describe the limiting continuum problem. We define an effective macroscopic energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ and the macroscopic dissipation functional $\mathcal{D}$ as follows:

$$\mathcal{E}(t,\overline{e},\overline{p}) = \begin{cases} \mathcal{E}_0(\overline{e},\overline{p}) & \text{for } \int_\Omega e(x)\,\mathrm{d}x = \ell(t), \\ \infty & \text{otherwise,} \end{cases} \quad \text{and} \tag{5.5a}$$

$$\mathcal{D}(\overline{p}_0,\overline{p}_1) = \int_\Omega 2ka|\overline{p}_1(x) - \overline{p}_0(x)|\,\mathrm{d}x, \tag{5.5b}$$

$$\text{where } \mathcal{E}_0(\overline{e},\overline{p}) = \int_\Omega \overline{\Phi}(\overline{e}(x),\overline{p}(x)) + G(x)e(x)\,\mathrm{d}x - \Gamma_f \tag{5.5c}$$

$$\text{with } \overline{\Phi}(\overline{e},\overline{p}) = \frac{k}{2}(\overline{e}-\overline{p})^2 + H(\overline{p}) \text{ and } \Gamma_f = \frac{1}{2k}\int_{\mathbb{R}} \mu^2 f(\mu)\,\mathrm{d}\mu. \tag{5.5d}$$

Here $\overline{\Phi}$ is the continuum energy density depending on the macroscopic elastic and the plastic strain variables.

Using the uniform convexity of H one can show that the macroscopic ERIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$ has a unique energetic solution for each stable initial condition $(\overline{e}^0,\overline{p}^0)$. This solution $(\overline{e},\overline{p})$ is Lipschitz continuous in time and satisfies the following plasticity problem (cf. [Vis94, BrS96, Kre99, Mie05]):

$$k\big(\overline{e}(t,x) - \overline{p}(t,x)\big) + G(x) = \sigma(t), \qquad \int_\Omega \overline{e}(t,y)\,\mathrm{d}y = \ell(t), \tag{5.6a}$$

$$0 \in ka\mathrm{Sign}\big(\dot{\overline{p}}(t,x)\big) + k\big(\overline{p}(t,x) - \overline{e}(t,x)\big) + \partial H(\overline{p}(t,x)), \tag{5.6b}$$

where "Sign" denotes the set-valued function with $\mathrm{Sign}(0) = [-1,1]$ and $\mathrm{Sign}(v) = \{\mathrm{sign}(v)\}$ for $v \neq 0$. Introducing the displacement $u(t,x) = \int_0^x \overline{e}(t,y)\,\mathrm{d}y$ we can rewrite the system in the more classical form

$$\begin{aligned} &-\partial_x\Big(k\big(\partial_x u(t,x) - \overline{p}(t,x)\big)\Big) = g_{\mathrm{ext}}(x), \quad u(t,0) = 0, \quad u(t,1) = \ell(t), \\ &0 \in ka\mathrm{Sign}\big(\dot{\overline{p}}(t,x)\big) + k\big(\overline{p}(t,x) - \partial_x u(t,x)\big) + \partial H(\overline{p}(t,x)). \end{aligned}$$

Note that H_r and Γ_{f_r} are the only terms in $\mathcal{E}$ and $\mathcal{D}$ depending on the probability distribution density f_r. Obviously, Γ_{f_r} is irrelevant for the elasto-plastic evolution, whereas the hardening function H_r is essential. When $r \to 0$ one can show that $\mathcal{F}_r(\mu) \to \max\{0,\mu\}$ and $H_r(p) \to H_0(p) = 0$ for $|p| < a$, see (5.4) for a special case. As we have already mentioned, there is no hardening in the case $H = H_0$, therefore existence of solutions can still be established but uniqueness fails.

5.3 Convergence proof

In this sub-section we prove our second main theorem, which establishes a rigorous relation between the discrete automaton (DA1)–(DA3) and the continuum system (5.6) by using the Γ-convergence for ERIS developed in [MRS08].

More precisely we consider the sequence of discrete ERIS $(\mathbb{R}^N, \boldsymbol{E}_N, \boldsymbol{D}_N)$ described in Section 4.4 with solutions $\boldsymbol{e}_N : [0,T] \to \mathbb{R}^N$ and show that the embedded functions $(\overline{e}^N, \overline{p}^N) = \mathcal{P}_N(\boldsymbol{e}_N) : [0,T] \to \mathcal{Q}$ weakly converge to the unique solution of the macroscopic ERIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, where $\mathcal{P}_N$ is defined in Section 5.1. In fact, we show more, namely that the associated energies and dissipations converge as well. In fact, it is the convergence of the energies and dissipations that allows us to show that the limit is an energetic solution for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$.

Theorem 5.1 *Fix a loading profile $\ell \in \mathrm{C}^1([0,T])$, which is piecewise monotone, and assume that the bias vectors $\boldsymbol{\mu}^N \in \mathbb{R}^N$ are chosen as described above. Define $h_j^N = \mu_j^N - G(j/N) + \lambda^N$ with λ^N such that $\sum_1^N h_j^N = 0$ and take initial conditions $\boldsymbol{e}_0^N \in \mathbb{R}^N$ that are ordered with respect to $\boldsymbol{h}^N$ such that*

$$\mathcal{P}_N(\boldsymbol{e}_0^N) \rightharpoonup (\overline{e}_0, \overline{p}_0) \text{ in } \mathcal{Q} = \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega) \quad \text{and} \quad \boldsymbol{E}_N(0, \boldsymbol{e}_0^N) \to \mathcal{E}(0, \overline{e}_0, \overline{p}_0) < \infty.$$

Then the embeddings of the ordered solutions of $\boldsymbol{e}^N : [0,T] \to \mathbb{R}^N$ of $(\mathbb{R}^N, \mathcal{E}_N, \mathcal{D}_N)$ constructed in Section 4.2 converge to the unique solution $(\overline{e}, \overline{p}) : [0,T] \to \mathcal{Q}$ of $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ with $(\overline{e}(0), \overline{p}(0)) = (\overline{e}_0, \overline{p}_0)$, namely

$$\mathcal{P}_N(\boldsymbol{e}^N(t)) \rightharpoonup (\overline{e}(t), \overline{p}(t)) \text{ in } \mathcal{Q} \quad \text{for all } t \in [0,T].$$

Moreover, we have $\boldsymbol{E}_N(t, \boldsymbol{e}^N(t)) \to \mathcal{E}(t, \overline{e}(t), \overline{p}(t))$ and $\mathrm{Diss}_{\boldsymbol{D}_N}(\boldsymbol{e}^N, [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(\overline{p}, [0,t])$.

Proof: *Step 1:* For the proof we use our precise knowledge of the solutions $\boldsymbol{e}^N$. Note that the ordered states are uniquely determined by the function $m^N(t) : [0,T] \to \{0, ..., N\}$ counting the number of j such that $e_j^N(t)$ is bigger 0. Moreover, we have

$$\sigma^N(t) = k\ell(t) - ak(2m^N(t) - N)/N. \tag{5.7}$$

Thus, $\sigma^N(t)$ also allows us to recover the solution $\boldsymbol{e}^N(t)$ completely as follows. For given t we define $h_+^N(t) > h_-^N(t)$ such that

$$\#\{\, j | h_j \geq h_+^N(t) \,\} = m^N(t),\ h_+^N(t) = \min\{\, h_j^N | h_j^N \geq h_+^N(t) \,\},\ h_-^N(t) = \max\{\, h_l^N | h_l^N < h_+^N(t) \,\}.$$

Along solutions, the values of $h_\pm$ are equal to those of $\overline{h}_\pm$ (cf. (4.5)), but now they depend on $t \in [0,T]$. We have

$$e_j^N(t) = \mathrm{sign}(e_j^N(t))a + \frac{1}{k}(\sigma^N(t) + h_j^N) \text{ and } \mathrm{sign}(e_j^N(t)) = \begin{cases} 1 & \text{for } h_j^N \geq h_+^N(t), \\ -1 & \text{for } h_j^N \leq h_-^N(t). \end{cases} \tag{5.8}$$

Step 2: We only prove that convergence holds along a subsequence. However, since the limit problem has a unique solution, we know a priori that the whole sequence must converge. To find a convergence subsequence we consider the functions σ^N. Since ℓ is piecewise monotone the interval $[0,T]$ can be decomposed into finitely many, let us say P, subintervals where ℓ is monotone. However, each m^N is also monotone in these subintervals. Since the variation of m^N in a montone part is bounded by N, the variation of each m^N is at most PN. Thus, (5.7) shows that the variation of σ^N is bounded by $k\|\dot{\ell}\|_{\mathrm{L}^1} + 2akP$.

Thus, Helly's selection principle allows us to extract a subsequence (not relabeled) such that $\sigma^N(t) \to \sigma^\infty(t)$ for all $t \in [0,T]$. As a consequence we find

$$m^N(t)/N \to \xi^\infty(t) = \frac{k(\ell(t){-}a) - \sigma^\infty(t)}{2ak} \tag{5.9}$$

Step 3: Next we show that this convergence implies the convergence of $(\overline{e}^N, \overline{p}^N) = \mathcal{P}_N(\boldsymbol{e}^N)$ as well as that of the energy and the dissipation. In fact, we show that for each $t \in [0,T]$ the sequence $(\overline{e}^N(t), \overline{p}^N(t))_{N\in\mathbb{N}}$ generates a well-defined Young measure $\nu(t) : \Omega \to \mathrm{Prob}(\mathbb{R}^2)$ (Radon measures on $\mathbb{R}$ with total measure 1). This follows from the independent random choices of μ_j^N using the law of large numbers. It is here, where we exploit the disorder in an essential fashion. Because the biases h_j^N are chosen independently and identically distributed (with density f), the law of large numbers can be applied to any continuous function Ξ to obtain

$$\frac{1}{N}\sum_{j=1}^{N} \Xi(h_j) \quad \to \quad \int_{\mathbb{R}} \Xi(\mu) f(\mu)\,\mathrm{d}\mu. \tag{5.10}$$

In fact, much less than the assumed randomness is sufficient to derive the following conclusions. We only need a type of weak ergodicity that could, e.g., be also generated by quasiperiodic functions.

For a general test function $\Psi \in \mathrm{C}^0(\overline{\Omega} \times \mathbb{R}^2)$ we consider the limit of

$$\psi^N(t) = \int_\Omega \Psi(x, \overline{e}^N(t,x), \overline{p}^N(t,x))\,\mathrm{d}x$$

for $N \to \infty$. Using the definition of $(\overline{e}^N, \overline{p}^N) = \mathcal{P}_N(\boldsymbol{e}^N)$ and defining $\Psi_j^N(e,p) = \frac{1}{N}\int_{(j-1)/N}^{j/N} \Psi(y,e,p)\,\mathrm{d}y$ we find

$$\psi^N(t) = \frac{1}{N}\sum_{j=1}^{N} \Psi_j^N(e_j^N(t), a\,\mathrm{sign}(e_j(t)))$$

Inserting the explicit formula (5.8) for $e_j^N(t)$ we find

$$\psi^N(t) = \frac{1}{N}\sum_{\{\,j\,|\,h_j^N \leq h_-^N(t)\,\}} \Psi_j^N(-a{+}\frac{1}{k}(\sigma^N(t){+}h_j^N), -a) + \frac{1}{N}\sum_{\{\,j\,|\,h_j^N \geq h_-^N(t)\,\}} \Psi_j^N(a{+}\frac{1}{k}(\sigma^N(t){+}h_j^N), a).$$

Recalling $h_j^N = \mu_j^N - G(j/N)$, where all the μ_j^N are independently chosen according to the density distribution f, we can pass to the limit $N \to \infty$. First observe that $h_\pm^N(t)$ converge to $h_\pm^\infty(t)$ defined by

$$h_-^\infty(t) = \sup\{\, h \mid F_G(h) < \xi^\infty(t)\,\} \;\text{ and }\; h_+^\infty(t) = \inf\{\, h \mid F_G(h) > \xi^\infty(t)\,\},$$

where $F_G(h) := \int_\Omega \int_{\eta=-\infty}^{h} f(\eta{+}G(x))\,\mathrm{d}\eta\,\mathrm{d}x \in [0,1]$ and ξ^∞ is defined in (5.9). Note that F_G is a probability distribution with compact support since f has compact support and G

is bounded. Subsequently it suffices to take any $h^\infty(t) \in [h_-^\infty(t), h_+^\infty(t)]$. Using $\sigma^N \to \sigma^\infty$ and the law of large numbers on μ_j^N (cf. (5.10)) we find $\psi^N(t) \to \psi^\infty(t)$ with

$$\begin{aligned}\psi^\infty(t) = &\int_\Omega \int_{-\infty}^{h^\infty(t)} \Psi(x, -a+(\sigma^\infty(t)+h)/k, -a) f(h+G(x))\,\mathrm{d}h\,\mathrm{d}x \\ &+ \int_\Omega \int_{h^\infty(t)}^{\infty} \Psi(x, a+(\sigma^\infty(t)+h)/k, a) f(h+G(x))\,\mathrm{d}h\,\mathrm{d}x.\end{aligned}$$

The Young measure ν is defined via $\int_\Omega \int_{\mathbb{R}^2} \Psi(x,e,p)\nu(t,x,\mathrm{d}e,\mathrm{d}p)\,\mathrm{d}x = \psi^\infty(t)$ giving

$$\begin{aligned}&\int_{\mathbb{R}^2} \widehat{\Psi}(e,p)\nu(t,x,\mathrm{d}e,\mathrm{d}p) \\ &= \int_{\mathbb{R}} \widehat{\Psi}\big(a\,\mathrm{sign}(\mu-\widehat{\mu}(t,x)) + (\sigma^\infty(t)+\mu-G(x))/k, a\,\mathrm{sign}(\mu-\widehat{\mu}(t,x))\big)\, f(\mu)\,\mathrm{d}\mu,\end{aligned}$$

where $\widehat{\mu}(t,x)$ is any solution of $\xi^\infty(t) = F_G(\mu-G(x))$, e.g.

$$\widehat{\mu}(t,x) = h^\infty(t) + G(x). \tag{5.11}$$

Using the identity $\int_{\mathbb{R}} \mathrm{sign}(\widehat{\mu}-\mu) f(\mu)\,\mathrm{d}\mu = 2F(\widehat{\mu})-1$ and the testfunctions $\widehat{\Psi}(e,p) = e$ and $\widehat{\Psi}(e,p) = p$ we obtain the weak limits $\overline{e}(t)$ and $\overline{p}(t)$, respectively, via

$$\begin{aligned}\overline{e}(t,x) &= \int_{\mathbb{R}} \Big(a\,\mathrm{sign}(\mu-\widehat{\mu}(t,x)) + (\sigma^\infty(t)+\mu-G(x))/k\Big) f(\mu)\,\mathrm{d}\mu \\ &= a(2F(\widehat{\mu}(t,x))-1) + (\sigma^\infty(t)-G(x))/k, \\ \overline{p}(t,x) &= a(2F(\widehat{\mu}(t,x))-1).\end{aligned} \tag{5.12}$$

Step 4. For the convergence of the energy we use

$$\begin{aligned}&\boldsymbol{E}_0^N(\boldsymbol{e}^N(t)) = \boldsymbol{E}_1^N(\boldsymbol{e}^N(t)) + \boldsymbol{E}_2^N(\boldsymbol{e}^N(t)), \quad \text{where} \\ &\boldsymbol{E}_1^N(\boldsymbol{e}^N) = \frac{1}{N}\sum_1^N \frac{k}{2}(e_j^N - a\,\mathrm{sign}(e_j^N))^2 \quad \text{and } \boldsymbol{E}_2^N(\boldsymbol{e}^N) = -\frac{1}{N}\sum_1^N h_j^N e_j^N.\end{aligned}$$

Using the explicit form (5.8) of e_j^N we obtain

$$\boldsymbol{E}_1^N(\boldsymbol{e}^N(t)) = \frac{1}{N}\sum_1^N \frac{1}{2k}\big(\sigma^N(t)-G(j/N)+\mu_j^N\big)^2 \to \int_\Omega \frac{1}{2k}\big(\sigma^\infty(t)-G(x)\big)^2\,\mathrm{d}x + \Gamma_f,$$

where Γ_f is defined in (5.5). For $\boldsymbol{E}_2^N$ we proceed as for $\psi^N(t)$ and obtain

$$\begin{aligned}\boldsymbol{E}_2^N(\boldsymbol{e}^N(t)) &= -\frac{1}{N}\sum_{h_j^N \le h_-^N(t)} h_j^N\big(-a + \frac{1}{k}(\sigma^N(t)+h_j^N)\big) - \frac{1}{N}\sum_{h_j^N \ge h_-^N(t)} h_j^N\big(a + \frac{1}{k}(\sigma^N(t)+h_j^N)\big) \\ &\to -\int_\Omega \int_{\mathbb{R}} (\mu-G(x))\Big(a\,\mathrm{sign}(\mu-\widehat{\mu}(t,x)) + (\sigma^\infty(t)-G(x)+\mu)/k\Big) f(\mu)\,\mathrm{d}\mu\,\mathrm{d}x.\end{aligned}$$

Using the representations of the weak limits in (5.12) we obtain

$$\boldsymbol{E}_1^N(\boldsymbol{e}^N(t)) \to \int_\Omega \frac{k}{2}(\overline{e}(t,x) - \overline{p}(t,x))^2\,\mathrm{d}x + \Gamma_f.$$

To compute the limit of the last term $\boldsymbol{E}_2^N(\boldsymbol{e}^N(t))$ we define the auxiliary function

$$\widetilde{F}(\mu) = \frac{1}{2}\int_{\mathbb{R}} y\,\mathrm{sign}(\mu - y) f(y)\,\mathrm{d}y,$$

and denote by $\mu = \widehat{\mu}(\eta) \in [-\infty, \infty]$ any solution of $F(\mu) = \eta \in [0,1]$. Then one can show that the following holds:
(a) For $\eta \in [0,1]$ we have $\mathcal{F}^*(\eta) = \widetilde{F}(\widehat{\mu}(\eta))$.
(b) For all $\mu, \eta \in \mathbb{R}$ we have: $\mu \in \partial\mathcal{F}^*(\eta) \iff \eta = F(\mu)$.
Indeed, the standard Legendre-Fenchel theory gives

$$\eta = \mathcal{F}'(\mu) = F(\mu) \;\Leftrightarrow\; \mu \in \partial\mathcal{F}^*(\eta) \;\Leftrightarrow\; \mu\eta = \mathcal{F}(\mu) + \mathcal{F}^*(\eta).$$

Thus, differentiating $\eta = F(\widehat{\mu}(\eta))$ yields $1 = f(\widehat{\mu}(\eta))\widehat{\mu}'(\eta)$. Moreover, the definition of $\widetilde{F}$ easily gives $\widetilde{F}'(\mu) = \mu f(\mu)$. Thus, the function $J : \eta \mapsto \widetilde{F}(\widehat{\mu}(\eta))$ satisfies $J'(\eta) = \widehat{\mu}(\eta)$ which leads to $J''(\eta) = \widehat{\mu}'(\eta) = 1/f(\widehat{\mu}(\eta))$. By the properties of the Legrendre transform we have $(\mathcal{F}^*)''(\eta) = 1/\mathcal{F}''(\widehat{\mu}(\eta)) = 1/f(\widehat{\mu}(\eta)) = J''(\eta)$.

Finally, using $\widetilde{F}(\pm\infty) = 0$ we obtain $J(0) = J(1) = 0$. The definition of $\mathcal{F}$ gives $\mathcal{F}(\mu) = \max\{0, \mu\} + m(\mu)$ with $0 \le m(\mu) \to 0$ for $|\mu| \to \infty$, which implies $\mathcal{F}^*(0) = \mathcal{F}^*(1) = 0$. Since J and $\mathcal{F}$ coincide at $\eta = 0$ and 1 and have the same second derivative, they are the same on all of $[0,1]$. Thus, (a) and (b) are established.

Based on these properties of the function $\widetilde{F}$ we can now write

$$\boldsymbol{E}_2^N(\boldsymbol{e}^N(t)) \to \int_\Omega 2a\widetilde{F}(\widehat{\mu}(t,x)) + G(x)\overline{e}(t,x)\,\mathrm{d}x - 2\Gamma_f.$$

Then, by using the representation of $\overline{p}$ in (5.12), the definition of H via $\mathcal{F}^*$, and the relation

$$\mu \in \partial H(p) \quad \Leftrightarrow \quad p = a(1 - 2F(\mu))$$

we find

$$H(\overline{p}(t,x)) = 2a\widetilde{F}(\widehat{\mu}(t,x)).$$

The convergence $\boldsymbol{E}^N(t, \boldsymbol{e}^N(t)) \to \mathcal{E}(t, \overline{e}(t), \overline{p}(t))$ is therefore shown.

Step 5. To show the convergence of the dissipation we use that ℓ is piecewise monotone, i.e. there exist times $0 = t_0 < t_1 < \cdots < t_L = t$ such that ℓ is monotone on $[t_{l-1}, t_l]$. As a consequence the solutions $\boldsymbol{e}^N$ and $\overline{p}$ are monotone on these intervals. By the definition of the dissipation functionals $\mathrm{Diss}_{\boldsymbol{D}_N}$ and $\mathrm{Diss}_{\mathcal{D}}$ we then have

$$\mathrm{Diss}_{\boldsymbol{D}_N}(\boldsymbol{e}^N, [0,t]) = \sum_{l=1}^{L} \boldsymbol{D}_N(\boldsymbol{e}^N(t_{l-1}), \boldsymbol{e}^N(t_l)), \quad \mathrm{Diss}_{\mathcal{D}}(\overline{p}, [0,t]) = \sum_{l=1}^{L} \mathcal{D}(\overline{p}(t_{l-1}), \overline{p}(t_l)).$$

Thus, it suffices to show convergence for these time increments only. Without loss of generality we consider the case $\ell(t_{l-1}) < \ell(t_l)$. With $\rho_N \to \rho_\infty = 2ka^2$ we have

$$\boldsymbol{D}_N(\boldsymbol{e}^N(t_{l-1}), \boldsymbol{e}^N(t_l)) = \frac{1}{N}\sum_{j=1}^N \rho_N\big(\operatorname{sign}(e_j^N(t_l)) - \operatorname{sign}(e_j^N(t_{l_1}))\big) = \frac{\rho_N}{N}(m^N(t_l) - m^N(t_{l-1}))$$

$$\to \rho_\infty(\xi^\infty(t_l) - \xi^\infty(t_{l-1})) = \int_\Omega ka(\overline{p}(t_l,x) - \overline{p}(t_{l-1},x))\,\mathrm{d}x = \mathcal{D}(\overline{p}(t_{l-1}), \overline{p}(t_l)).$$

Thus, $\mathrm{Diss}_{\boldsymbol{D}_N}(\boldsymbol{e}^N, [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(\overline{p}, [0,t])$ is established as well.

Step 6: It remains to show that $(\overline{e}, \overline{p})$ is the unique energetic solution for the macroscopic ERIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$. We first consider the energy balance. For all N we have the microscopic energy balance

$$\boldsymbol{E}_N(t, \boldsymbol{e}^N(t)) + \mathrm{Diss}_{\boldsymbol{D}_N}(\boldsymbol{e}^N, [0,t]) = \boldsymbol{E}_N(0, \boldsymbol{e}_0^N) + \int_0^t \sigma^N(s)\dot{\ell}(s)\,\mathrm{d}s.$$

Since all four terms converge to the desired limits for N we immediately obtain the energy balance (E) for the limit $(\overline{e}, \overline{p})$ with respect to the ERIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$.

To establish the stability condition

$$\mathcal{E}(t, \overline{e}(t), \overline{p}(t)) \le \mathcal{E}(t, \widetilde{e}, \widetilde{p}) + \mathcal{D}(\overline{p}(t), \widetilde{p}) \text{ for all } (\widetilde{e}, \widetilde{p}) \in \mathcal{Q},$$

we use the stability of $\boldsymbol{e}^N(t)$ with respect to $(\mathbb{R}^N, \boldsymbol{E}_N, \boldsymbol{D}_N)$. We test the stability using the state $\widetilde{\boldsymbol{e}}^N$, which is defined like $\boldsymbol{e}^N(t)$ but with a different function $\widetilde{G}$ replacing G. We choose an arbitrary $\widetilde{G} \in \mathrm{H}^1(\Omega)$ with $\int_\Omega G(x)\,\mathrm{d}x = 0$ and define the new bias vector $\widetilde{\boldsymbol{h}} = (\widetilde{h}_j^N)_j \in \mathbb{R}^N$ via

$$\widetilde{h}_j^N = \mu_j^N - \widetilde{G}(j/N) + \widetilde{\lambda}^N, \text{ where } \sum_1^N \widetilde{h}_j^N = 0.$$

We define $F_{\widetilde{G}}$ via $F_{\widetilde{G}}(h) = \int_\Omega F(h + \widetilde{G}(x))\,\mathrm{d}x$. Then, for every pair $(\widetilde{\xi}, \widetilde{h})$ satisfying

$$1 - \widetilde{\xi} = F_{\widetilde{G}}(\widetilde{h}) \text{ and } |\widetilde{\sigma} + \widetilde{h}| \le ka, \text{ where } \widetilde{\sigma} = k\ell(t) - ak(2\widetilde{\xi} - 1),$$

where exists a sequence $\widetilde{h}^N$ such that

$$\begin{aligned} &\widetilde{e}_j^N = a\operatorname{sign} e_j^N + \frac{1}{k}(\widetilde{\sigma}^N + \widetilde{h}_j^N) \text{ and } \operatorname{sign}\widetilde{e}_j^N = \begin{cases} 1 & \text{if } \widetilde{h}_j^N \ge \widetilde{h}^N, \\ -1 & \text{if } \widetilde{h}_j^N < \widetilde{h}^N, \end{cases} \\ &\widetilde{\sigma}^N \to \widetilde{\sigma}, \quad \widetilde{h}^N \to \widetilde{h}, \quad (2\widetilde{m}^N - 1)/N \to \widetilde{\xi}, \end{aligned}$$

where $\widetilde{m}^N = \big(N + \sum_1^N \operatorname{sign}\widetilde{e}_j^N\big)/2$, $\widetilde{\sigma}^N = k\ell(t) - ak(2\widetilde{m}^N - 1)/N$.

Repeating the calculations in Step 3 we obtain the convergence $\mathcal{P}_N(\widetilde{\boldsymbol{e}}^N) \rightharpoonup (\widetilde{e}, \widetilde{p})$ in $\mathcal{Q}$, where

$$0 = k(\widetilde{e} - \widetilde{p}) + \widetilde{G} - \widetilde{\sigma} \text{ and } \widetilde{p}(x) = a(1 - 2F(\widetilde{h} + \widetilde{G}(x)). \tag{5.13}$$

Repeating the calculations in Step 4, while carefully distinguishing between the still relevant h_j^N and the artificial $\widetilde{h}_j^N$ which only differ by $\widetilde{G}(j/N) - \widetilde{\lambda}^N - G(j/N) + \lambda^N$, we find the convergence $\boldsymbol{E}_N(t, \widetilde{\boldsymbol{e}}^N) \to \mathcal{E}(t, \widetilde{e}, \widetilde{p})$.

Moreover, we are able to calculate the limit of $\boldsymbol{D}_N(\boldsymbol{e}^N(t), \widetilde{\boldsymbol{e}}^N)$ as follows (using $h^N = h_\pm^N(t)$ and neglecting $\lambda^N, \widetilde{\lambda}^N \to 0$):

$$
\begin{aligned}
\boldsymbol{D}_N(\boldsymbol{e}^N(t), \widetilde{\boldsymbol{e}}^N) &= \frac{\rho_N}{N} \sum_{j=1}^{N} |\operatorname{sign} e_j^N(t) - \operatorname{sign} \widetilde{e}_j^N| \\
&= \frac{\rho_N}{N} \Big(\#\{ j \mid h^N + G(j/N) \leq \mu_j^N < \widetilde{h}^N + \widetilde{G}(j/N) \} \\
&\qquad\quad + \#\{ j \mid \widetilde{h}^N + \widetilde{G}(j/N) \leq \mu_j^N < h^N + G(j/N) \} \Big) \\
&\to \rho_\infty \int_\Omega \Big(\big[F(\widetilde{h} + \widetilde{G}(x)) - F(h^\infty + G(x))\big]^+ + \big[F(h^\infty + G(x)) - F(\widetilde{h} + \widetilde{G}(x))\big]^+ \Big) \, \mathrm{d}x \\
&= 2ka^2 \int_\Omega |F(h^\infty + G(x)) - F(\widetilde{h} + \widetilde{G}(x))| \, \mathrm{d}x \\
&= 2ka^2 \int_\Omega \Big| \frac{1}{2a}(a - \overline{p}(t,x)) - \frac{1}{2a}(a - \widetilde{p}(x)) \Big| \, \mathrm{d}x = ka \int_\Omega \big| \overline{p}(t,x) - \widetilde{p}(x) \big| \, \mathrm{d}x \\
&= \mathcal{D}(\overline{p}(t), \widetilde{p}),
\end{aligned}
$$

where $[a]^+ = \max\{0, a\}$. Hence, we can pass to the limit in the stability condition for $\boldsymbol{e}^N(t)$, namely $\boldsymbol{E}_N(t, \boldsymbol{e}^N(t)) \leq \boldsymbol{E}_N(t, \widetilde{\boldsymbol{e}}^N) + \boldsymbol{D}_N(\boldsymbol{e}^N(t), \widetilde{\boldsymbol{e}}^N)$ and obtain $\mathcal{E}(t, \overline{e}(t), \overline{p}(t)) \leq \mathcal{E}(\overline{e}, \overline{p}) + \mathcal{D}(\overline{p}(t), \widetilde{p})$, where the comparison states $(\widetilde{e}, \widetilde{p})$ are the ones constructed in (5.13). Via the free choice of $\widetilde{G}$ we are able to generate a dense set of $\widetilde{p}$ in $\mathrm{L}^2(\Omega; [-a, a])$. However, the associated strains $\widetilde{e}$ are the equilibrium strains. By the quadratic nature of $\mathcal{E}$, we easily find $\mathcal{E}(t, \widehat{e}, \widetilde{p}) \geq \mathcal{E}(t, \widetilde{e}, \widetilde{p})$ for all $\widehat{e} \in \mathrm{L}^2(\Omega)$. Thus, the stability of $(\overline{e}(t), \overline{p}(t))$ is established, and $(\overline{e}, \overline{p}) : [0, T] \to \mathcal{Q}$ is shown to be an energetic solution for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$. ∎

6 Double asymptotics

Let us now show that in the case of bi-quadratic potential the limit does not change if one performs the double asymptotics $(\varepsilon, \delta) \to (0, 0)$ under the constraint that δ tends to 0 faster than ε. The result is a consequence of the estimates obtained in Theorem 4.5, which allow one to show that the L^2 difference between the viscous solutions and the discrete solutions tends to 0 with $(\varepsilon, \delta) \to (0, 0)$. Since the latter converge weakly, it follows that the former also converge weakly.

Theorem 6.1 *Consider the solutions $\boldsymbol{e}^{\delta,N} : [0, T] \to \mathbb{R}^N$ of the viscous problem (4.8), where $h_j^N = \mu_j^N - G(j/N) + \lambda^N$ is as above. Then, there exists a constant κ_* such that the following holds. If the initial conditions $\boldsymbol{e}^{\delta,N}(0)$ are ordered equilibrium states for given $\ell(0)$ such that*

$$
\mathcal{P}_N(\boldsymbol{e}^{\delta,N}(0)) \rightharpoonup (\overline{e}_0, \overline{p}_0) \text{ in } \mathcal{Q} \quad \text{and } \boldsymbol{E}^N(0, \boldsymbol{e}^{\delta,N}(0)) \to \mathcal{E}(0, \overline{e}_0, \overline{p}_0)
$$

for $(\varepsilon, \delta) \to 0$. Then, for $(\varepsilon, \delta) \to 0$ with $0 < \delta < \kappa_* \varepsilon$ the solutions $\boldsymbol{e}^{\delta,N} : [0, T] \to \mathbb{R}^N$ of the viscous problem (4.8) satisfy

$$
\mathcal{P}_N(\boldsymbol{e}^{\delta,N}(t)) \rightharpoonup (\overline{e}(t), \overline{p}(t)) \text{ in } \mathcal{Q} \text{ for all } t \in [0, T],
$$

where $(\overline{e}, \overline{p})$ *is the unique solution of the plasticity problem* (5.6) *with* $(\overline{e}(0), \overline{p}(0)) = (\overline{e}_0, \overline{p}_0)$.

Proof: The crucial observation is that the definition of the norms $|\cdot|_p$ in $\mathbb{R}^N$ and in $\mathrm{L}^p(\Omega)$ together with the embedding $\mathcal{P}_N$ lead to an additional factor $1/N^{1-1/p}$. For $(\widetilde{e}, \widetilde{p}) = \mathcal{P}_N(\widetilde{\boldsymbol{e}}^N)$ and $(\widehat{e}, \widehat{p}) = \mathcal{P}_N(\widehat{\boldsymbol{e}}^N)$ we have

$$\|\widetilde{e} - \widehat{e}\|_{\mathrm{L}^2(\Omega)} \leq \frac{1}{N^q}|\widetilde{\boldsymbol{e}}^N - \widehat{\boldsymbol{e}}^N|_p \text{ for } p \in [1, \infty] \text{ with } q = \min\{1/2, 1/p\},$$
$$\|\widetilde{p} - \widehat{p}\|_{\mathrm{L}^2(\Omega)} = \frac{2a}{\sqrt{N}}\big(\#\{\, j \mid \operatorname{sign} \widetilde{e}_j^N \neq \operatorname{sign} \widehat{e}_j^N \,\}\big)^{1/2}.$$

If $\delta \leq \kappa_*/N = \kappa_*\varepsilon$, where κ_* is the same as in Theorem 4.5, estimate (4.14) (with $p = \infty$ and $p = 1$ for R_1 and R_2, respectively) yields

$$\|\overline{\boldsymbol{e}}^{\delta,N}(t) - \overline{\boldsymbol{e}}^{0,N}\|_{\mathrm{L}^2(\Omega)} \leq C\big(\delta + 1/N^{1/2}\big).$$

Moreover, the number of different signs between $\boldsymbol{e}^{\delta,N}(t)$ and $\boldsymbol{e}^{0,N}(t)$ is bounded by N_* (independently of δ and N), which leads to the estimate

$$\|\mathcal{P}_N(\boldsymbol{e}^{\delta,N}(t)) - \mathcal{P}_N(\boldsymbol{e}^{0,N}(t))\|_{\mathrm{L}^2(\Omega)} \leq C_2\big(\delta{+}1/N^{1/2}\big) \leq C_3/N^{1/2} = C_3\varepsilon^{1/2},$$

where we have used $\delta \leq \kappa_*/N = \kappa_*\varepsilon$ again. Combining this with the convergence stated in Theorem 5.1 we obtain the desired convergence result. ■

7 General potentials

In the previous sections we have restricted our analysis to the special case of a biquadratic potential Φ_{biq}. Moreover, the loading g_{ext} was assumed to be time independent. Here we drop both assumptions and discuss the necessary changes for generalizing the results to arbitrary loadings and generic stress-strain relations. More precisely, we show that in the case of a general double-well potential and a rather general time dependent body forces the sequence of limits, first $\delta \to 0$ and then $\varepsilon = 1/N \to 0$, leads to basically the same general picture modulo appropriate modification of the hardening function and the dissipative potential in the limiting model.

7.1 Microscopic model

To replace $G(x)$ by a general time-dependent function $G(t, x)$ satisfying $\int_\Omega G(t, x)\,\mathrm{d}x = 0$ for all $t \in [0, T]$ we need to generalize the concept of *ordered states.* Indeed, since the loading may now depend on time, a state that is ordered for t_1 may no longer be ordered for $t_2 > t_1$. Therefore we need to interpret the order condition locally in $(t, x) \in [0, T] \times \Omega$. This is possible, since $G(t, x)$, $\ell(t)$, and $\sigma(t)$ vary only on the macroscopic scale while the bias coefficients fluctuate on the microscopic scale and are independent of time.

Moreover, since the general double-well potential Φ does not allow us to define a plastic strain $p = a\,\mathrm{sign}(e)$ as in the bi-quatratic case, we need to use the microscopic phase indicator variable $z_j \in \{-1, 0, 1\}$ as in Section 2. The threshold $\widehat{\mu}(t, x)$ is now active in a microscopically large but macroscopically small region, which can be defined as follows $|j - xN| \leq \sqrt{N}$. For j in this domain, the condition $\mu_j^N > \widehat{\mu}(t,x)$ then implies $e_j^N(t) \geq e_+$ and $z_j^N(t) = 1$, whereas $\mu_j^N < \widehat{\mu}(t,x)$ implies $e_j^N(t) \leq e_-$ and $z_j^N(t) = -1$.

In the formal proof which follows, the important issue will be to control the evolution of the threshold $\widehat{\mu}(t,x)$. Looking at the dynamics of the discrete automaton in Definition 4.2 we see that phase changes should only occur if the strain is critical. In terms of the macroscopic stress $\overline{\sigma}(t,x) = \sigma(t) - G(t,x)$, we need to have $\sigma_+ = \widehat{\mu} + \overline{\sigma}$, if $\dot{\widehat{\mu}} < 0$, and $\sigma_- = \widehat{\mu} + \overline{\sigma}$, if $\dot{\widehat{\mu}} > 0$. Moreover, the threshold value $\widehat{\mu}(t,x)$ must always satisfy $\overline{\sigma} + \widehat{\mu} \in [\sigma_-, \sigma_+]$.

7.2 Macroscopic energy

As in the special case of bi-quadratic energy, we begin with formally computing the limiting continuum energy and determining the hardening potential.

Notice that the relation (3.4) provides a strong correlation between e_j and μ_j and thus controls the joint Young measures ν generated by $(\overline{e}^N, \overline{z}^N)$, which takes the form

$$\begin{aligned}&\int_{\mathbb{R}^2} \widehat{\Psi}(e,z)\nu(t,x,\mathrm{d}e,\mathrm{d}z)\\ &= \int_{\mathbb{R}} \widehat{\Psi}\big(\operatorname{sign}(\mu - \widehat{\mu}(t,x)), \psi_{\operatorname{sign}(\mu-\widehat{\mu}(t,x))}(\overline{\sigma}(t,x) + \mu)\big) f(\mu)\,\mathrm{d}\mu.\end{aligned}$$

In particular, we can define the macroscopic constitutive relations

$$\widehat{E}(\widetilde{\sigma},\widetilde{\mu}) \overset{\text{def}}{=} \int_{\mathbb{R}} \psi_{\operatorname{sign}(\mu-\widetilde{\mu})}(\widetilde{\sigma} + \mu) f(\mu)\,\mathrm{d}\mu \quad \text{and} \quad \widehat{Z}(\widetilde{\mu}) \overset{\text{def}}{=} \int_{\mathbb{R}} \operatorname{sign}(\mu - \widetilde{\mu}) f(\mu)\,\mathrm{d}\mu, \tag{7.1}$$

such that the limits $\overline{e}$ and $\overline{z}$ satisfy

$$\overline{e}(t,x) = \widehat{E}(\overline{\sigma}(t,x), \widehat{\mu}(t,x)) \text{ and } \overline{z}(t,x) = \widehat{Z}(\widehat{\mu}(t,x)).$$

By $\sigma = \widehat{S}(e,\mu)$ we denote the unique solution σ of $e = \widehat{E}(\sigma,\mu)$. We can now compute the effective potential as a function of $\overline{e}$ and $\widehat{\mu}$ via

$$\widehat{\Phi}(\overline{e},\widehat{\mu}) = \int_M \Big(\Phi\big(\psi_{\operatorname{sign}(\mu-\widehat{\mu})}(\widehat{S}(\overline{e},\widehat{\mu}) + \mu)\big) - \mu\psi_{\operatorname{sign}(\mu-\widehat{\mu})}(\widehat{S}(\overline{e},\widehat{\mu}) + \mu)\Big) f(\mu)\,\mathrm{d}\mu.$$

The joint Young measure $\widehat{\nu}_{(\overline{e},\widehat{\mu})}$ generated by (e_j, μ_j) and associated with the macroscopic pair $(\overline{e},\widehat{\mu})$ has the form

$$\int_{\mathbb{R}^2} \widehat{\Psi}(e,\mu)\nu_{(\overline{e},\widehat{\mu})}(\mathrm{d}e,\mathrm{d}\mu) = \int_M \widehat{\Psi}\big(\psi_{\operatorname{sign}(\mu-\widehat{\mu})}(\widehat{S}(\overline{e},\widehat{\mu}) + \mu), \mu\big) f(\mu)\,\mathrm{d}\mu,$$

where $\widehat{\Psi} \in \mathrm{C}_0(\mathbb{R}^2)$ is an arbitrary test function. In particular, it can by checked that the definitions of $\widehat{S}$ and $\widehat{\Phi}$ are compatible in the sense that $\widehat{S}(\overline{e},\widehat{\mu}) = \partial_{\overline{e}}\widehat{\Phi}(\overline{e},\widehat{\mu})$.

To calculate the partial derivative of $\widehat{\Phi}$ with respect to $\widehat{\mu}$ we introduce the functions

$$\varphi_\pm(\sigma) = \psi_\pm(\sigma)\sigma - \Phi(\psi_\pm(\sigma)), \tag{7.2}$$

which satisfy the relations

$$\varphi_\pm'(\sigma) = \psi_\pm(\sigma), \quad \varphi_+(\sigma) = \sup_{e\geq e_+} \sigma e - \Phi(e), \ \text{ and } \varphi_-(\sigma) = \sup_{e\leq e_-} \sigma e - \Phi(e).$$

For the derivative we obtain (after some elementary calculations involving the chain rule)

$$\begin{aligned}\partial_{\widehat\mu}\widehat\Phi(\overline{e},\widehat\mu) &= \frac{\partial}{\partial\widehat\mu}\Big[\int_{-\infty}^{\widehat\mu} \big(\Phi(\psi_{-1}(\widehat S(\overline{e},\widehat\mu)+\mu)) - \mu\psi_{-1}(\widehat S(\overline{e},\widehat\mu)+\mu)\big) f(\mu)\,\mathrm{d}\mu \\ &\qquad + \int_{\widehat\mu}^{\infty} \big(\Phi(\psi_1(\widehat S(\overline{e},\widehat\mu)+\mu)) - \mu\psi_1(\widehat S(\overline{e},\widehat\mu)+\mu)\big) f(\mu)\,\mathrm{d}\mu\Big] \\ &= \big(\varphi_+(\widehat S(\overline{e},\widehat\mu)+\widehat\mu) - \varphi_-(\widehat S(\overline{e},\widehat\mu)+\widehat\mu)\big) f(\widehat\mu)\end{aligned}$$

Notice that the disorder threshold $\widehat\mu$ enters our formulas as a parametrization and that the energy representation in terms of elastic and plastic variable is still implicit. To abolish the auxiliary variable $\widehat\mu(t,x)$ and to replace it by the continuous internal variable $\overline{z}(t,x) = \widehat Z(\widehat\mu)$ we assume that the latter relation is invertible. We write $\widehat\mu = \widetilde\mu(\overline{z})$ and apply the chain rule in (7.1) to obtain

$$\widetilde\mu'(\overline{z}) = \frac{-1}{2f(\widetilde\mu(\overline{z}))} < 0.$$

We can now define the stored energy density $\overline\Phi$ and the stress $\overline S$ via

$$\overline\Phi(\overline{e},\overline{z}) = \widehat\Phi(\overline{e},\widetilde\mu(\overline{z})) \ \text{ and } \ \overline S(\overline{e},\overline{z}) = \widehat S(\overline{e},\widetilde\mu(\overline{z})),$$

which still satisfy the relation $\partial_{\overline{e}}\overline\Phi = \overline S$. Moreover, we find the identities

$$\partial_{\overline{z}}\overline\Phi(\overline{e},\overline{z}) = \partial_{\widehat\mu}\widehat\Phi\widetilde\mu' = \varphi_-(\overline S(\overline{e},\overline{z})+\widetilde\mu(\overline{z})) - \varphi_+(\overline S(\overline{e},\overline{z})+\widetilde\mu(\overline{z})), \tag{7.3a}$$

$$\partial_{\overline{z}}^2\overline\Phi(\overline{e},\overline{z}) = \Big(\psi_-(\overline S(\overline{e},\overline{z})+\widetilde\mu(\overline{z})) - \psi_+(\overline S(\overline{e},\overline{z})+\widetilde\mu(\overline{z}))\Big)\Big(\partial_{\overline{z}}\overline S(e,\overline{z}) + \frac{1}{f(\widetilde\mu(\overline{z}))}\Big) > 0. \tag{7.3b}$$

Next we show that the function $(\overline{e},\overline{z}) \mapsto \overline\Phi(\overline{e},\overline{z})$ is convex, which is an important property for proving existence and uniqueness of solutions for the associated plasticity problem. For this we introduce the auxiliary functions

$$\widetilde E(\sigma,\overline{z},\mu) = \psi_{\mathrm{sign}(\mu-\widetilde\mu(\overline{z}))}(\sigma+\mu) \ \text{ and } \ E(\sigma,\overline{z}) = \int_{\mathbb{R}} \widetilde E(\sigma,\overline{z},\mu) f(\mu)\,\mathrm{d}\mu,$$

which satisfy $\partial_{\overline{z}}E(\sigma,\overline{z}) = \psi_-(\sigma+\widetilde\mu(\overline{z})) - \psi_+(\sigma+\widetilde\mu(\overline{z}))$. We then have $\sigma = \overline S(\overline{e},\overline{z})$ if and only if $\overline{e} = E(\sigma,\overline{z}) = \widehat E(\sigma,\widetilde\mu(\overline{z}))$. Moreover, we define

$$\overline E(\overline{e},\overline{z},\mu) \overset{\text{def}}{=} \widetilde E(\overline S(\overline{e},\overline{z}),\overline{z},\mu)$$

and find the relations

$$\overline{e} = \int_{\mathbb{R}} \overline E(\overline{e},\overline{z},\mu) f(\mu)\,\mathrm{d}\mu \quad \text{and} \quad \phi(\overline E(\overline{e},\overline{z},\mu)) - \mu = \overline S(\overline{e},\overline{z}). \tag{7.4}$$

Then, the stored-energy density takes the form

$$\overline\Phi(\overline{e},\overline{z}) = \int_{\mathbb{R}} \Big(\Phi(\overline E(\overline{e},\overline{z},\mu)) - \mu\overline E(\overline{e},\overline{z},\mu)\Big) f(\mu)\,\mathrm{d}\mu. \tag{7.5}$$

Lemma 7.1 *The derivatives of $\overline{\Phi}$ take the following form*

$$\partial_{\overline{e}}\overline{\Phi} = \overline{S}, \quad \partial_{\overline{z}}\overline{\Phi} = \varphi_+(\overline{S}(\overline{e},\overline{z}){+}\widetilde{\mu}(\overline{z})) - \varphi_-(\overline{S}(\overline{e},\overline{z}){+}\widetilde{\mu}(\overline{z})),$$

$$\mathrm{D}^2\overline{\Phi} = \begin{pmatrix} \partial_{\overline{e}}\overline{S} & \Delta\partial_{\overline{e}}\overline{S} \\ \Delta\partial_{\overline{e}}\overline{S} & \Delta^2\partial_{\overline{e}}\overline{S} + \frac{\Delta}{f(\widetilde{\mu}(\overline{z}))} \end{pmatrix}$$

where $\partial_{\overline{e}}\overline{S} = \frac{1}{\partial_\sigma E(\overline{S}(\overline{e},\overline{z}),\overline{z})} > 0$ *and* $\Delta = \psi_+(\overline{S}(\overline{e},\overline{z}){+}\widetilde{\mu}(\overline{z})) - \psi_-(\overline{S}(\overline{e},\overline{z}){+}\widetilde{\mu}(\overline{z})) > 0$. *Hence,* $\overline{\Phi}$ *is uniformly convex.*

Proof: The formula for $\partial_{\overline{e}}\overline{\Phi}$ follows by differentiation under the integral and using (7.4). The formula for $\partial_{\overline{z}}\overline{\Phi}$ follows by using $\widetilde{\mu}'(z) = 1/f(\widetilde{\mu}(z))$ and $\overline{E}(e,z,\mu) = \psi_\pm(\overline{S}(e,z)){+}\mu$ for $\mu > \widetilde{\mu}(z)$ and $\mu < \widetilde{\mu}(z)$, respectively.

Differentiating $e = E(\overline{S}(e,z),z)$ with respect to e and using the definition of E we obtain the formula for $\partial_{\overline{e}}\overline{S} = \partial^2_{\overline{e}}\overline{\Phi}$. For the mixed derivative we can use $\varphi'_\pm(\sigma) = \psi_\pm(\sigma)$ to find $\partial_{\overline{e}}\big(\partial_{\overline{z}}\overline{\Phi}\big)$. For $\partial^2_{\overline{z}}\overline{\Phi}$ we differentiate $e = E(\overline{S}(e,z),z)$ with respect to z and find $\partial_z\overline{S}(e,z) = \frac{-\partial_z E}{\partial_\sigma E} = \Delta\partial_e\overline{S}$. Together this gives

$$\partial^2_{\overline{z}}\overline{\Phi} = (\varphi'_+{-}\varphi'_-)\big(\Delta\partial_e\overline{S}{+}\widetilde{\mu}(\overline{z})\big) = \Delta\big(\Delta\partial_e\overline{S}{+}1/f(\widetilde{\mu}(\overline{z}))\big),$$

which is the desired result. ∎

The above calculations can be done explicity for the biquadratic potential Φ_{biq}, see (2.1). We have $\psi_\pm(\sigma) = \sigma/k \pm a$ and find

$$E(\sigma,\overline{z}) = \int_{\mathbb{R}} \big(\sigma/k + a\,\mathrm{sign}(\mu{-}\widetilde{\mu}(\overline{z}))\big) f(\mu)\,\mathrm{d}\mu = \frac{\sigma}{k} + a\overline{z}.$$

Hence, $\overline{S}(\overline{e},\overline{z}) = k(\overline{e}{-}a\overline{z})$, which results in

$$\overline{E}(\overline{e},\overline{z},\mu) = \overline{e} - a\overline{z} + \frac{\mu}{k} + a\,\mathrm{sign}(\mu{-}\widetilde{\mu}(\overline{z})).$$

Inserting this into the definition (7.5) of $\overline{\Phi}$ (with $\Phi = \Phi_{\mathrm{biq}}$) we can use the crucial identity $\Phi_{\mathrm{biq}}(\overline{E}(\overline{e},\overline{z},\mu)) = \frac{k}{2}(\overline{e}{-}a\overline{z}{+}\frac{\mu}{k})^2$. This follows from the stress relation $\overline{S}(\overline{e},\overline{z}) + \widetilde{\mu}(\overline{z}) \in [\sigma_-,\sigma_+] = [-ka,ka]$, which implies $\mathrm{sign}(\mu{-}\widetilde{\mu}(\overline{z})) = \mathrm{sign}\,\overline{E}$. Hence, on the one hand we have $\int_{\mathbb{R}} \Phi_{\mathrm{biq}}(\overline{E}(\overline{e},\overline{z},\mu)) f(\mu)\,\mathrm{d}\mu = \frac{k}{2}(\overline{e}{-}a\overline{z})^2 + 2\Gamma_f$, while on the other hand we have

$$\int_{\mathbb{R}} (-\mu)\overline{E}(\overline{e},\overline{z},\mu) f(\mu)\,\mathrm{d}\mu = -\Gamma_f + a\widetilde{F}(\widetilde{\mu}(z)) = -\Gamma_f + H(z/a).$$

This gives the desired formula in (5.5).

7.3 Macroscopic dissipative potential

We now turn to the analysis of the dynamics of $\overline{z}$, which is strongly linked to that of $\widehat{\mu}$ via $\overline{z} = \widehat{Z}(\widehat{\mu})$. From the above we know that $\overline{\sigma} + \widehat{\mu} \in [\sigma_-,\sigma_+]$ and that $\sigma_+ = \widehat{\mu} + \overline{\sigma}$, if $\dot{\widehat{\mu}} < 0$, and $\sigma_- = \widehat{\mu} + \overline{\sigma}$, if $\dot{\widehat{\mu}} > 0$. These conditions can be formulated as a play operator in the form

$$0 \in \partial\widehat{R}(\dot{\widehat{\mu}}(t,x)) + \widehat{\mu}(t,x) + \overline{\sigma}(t,x), \tag{7.6}$$

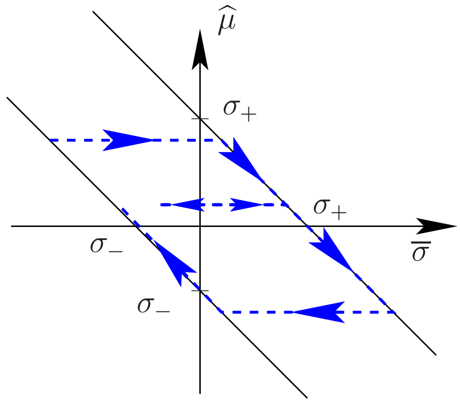

Figure 7.1: Evolution of the play operator generated by Eqn. (7.6)

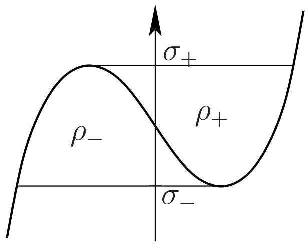

Figure 7.2: Energies dissipates when the system jumps

where the 1-homogeneous friction potential $\widehat{R}:\mathbb{R}\to\mathbb{R}$ is given via

$$\widehat{R}(\dot{\mu}) = -\sigma_{-\,\mathrm{sign}(\dot{\mu})}\dot{\mu} = \begin{cases} -\sigma_-\dot{\mu} & \text{for } \dot{\mu}\geq 0,\\ -\sigma_+\dot{\mu} & \text{for } \dot{\mu}\leq 0.\end{cases}$$

This is a classical hysteresis operator that provides for each $\overline{\sigma}$ a unique solution $\widehat{\mu}$, see [BrS96, Kre99, Vis94] and also Figure 7.1. Note that $\widehat{\mu}+\overline{\sigma}$ always lie in the interval $[\sigma_-,\sigma_+]$. Moreover, $\widehat{\mu}$ can only change if $\widehat{\mu}+\overline{\sigma}$ is either σ_- or σ_+.

To define the macroscopic dissipative potential we introduce the two quantities

$$\rho_+ \stackrel{\text{def}}{=} \int_{e_-}^{\psi_+(\sigma_+)} \sigma_+ - \phi(e)\,\mathrm{d}e > 0 \quad \text{and} \quad \rho_- \stackrel{\text{def}}{=} \int_{\psi_-(\sigma_-)}^{e_+} \phi(e) - \sigma_-\,\mathrm{d}e > 0. \tag{7.7}$$

Recalling $\varphi_\pm$ defined in (7.2) we have the following identities, see also Figure 7.2:

Lemma 7.2 *For the areas enclosed by the the graph of ϕ and the hysteresis loop we have*

$$\rho_+ = \varphi_+(\sigma_+) - \varphi_-(\sigma_+) > 0 \quad \textit{and} \quad \rho_- = \varphi_-(\sigma_-) - \varphi_+(\sigma_-) > 0,$$

Moreover, we have the force relation $\overline{S}(\overline{e},\overline{z}) + \widetilde{\mu}(\overline{z}) = \sigma_\pm \implies \partial_{\overline{z}}\overline{\Phi}(\overline{e},\overline{z}) = \mp\rho_\pm$.

Proof: The integral formulae follow easily using $e_\mp = E_\pm(\sigma_\pm)$ and the definition of $\varphi_\pm$ in (7.2). The second statement follows directly from (7.3a). ∎

The above computations show that the critical thresholds $-\sigma_\pm$ for $\overline{\sigma}+\widehat{\mu}$ are reached if and only if $\partial_z\overline{\Phi}(\overline{e},\overline{z})$ reaches the critical values $\rho_\pm$. Hence, the play operator in (7.6) is equivalent to

$$0\in\partial\overline{R}(\dot{\overline{z}}) + \partial_{\overline{z}}\overline{\Phi}(\overline{e},\overline{z}) \quad \text{with } \overline{R}(\overline{v}) \stackrel{\text{def}}{=} \rho_{\mathrm{sign}(\overline{v})}|\overline{v}|. \tag{7.8}$$

7.4 Plasticity problem

We can now formulate the general macroscopic equations in terms of the variables $\overline{e}$ and $\overline{z}$. Consider the solutions $\boldsymbol{e}^{N,\delta} : [0,T] \to \mathbb{R}^N$ of (3.2). Under the above hypotheses we expect that the embedding $(\overline{e}^{N,\delta}, \overline{z}^{N,\delta}) : [0,T] \to \mathrm{L}^2(\Omega)^2$ converge in the limit "$\lim_{N\to\infty} \lim_{\delta\to 0}$" (weakly in $\mathrm{L}^2(\Omega)^2$) to the solutions $(\overline{e}, \overline{z})$ of the macroscopic elastoplasticity system:

$$0 = \partial_{\overline{e}} \overline{\Phi}(\overline{e}(t,x), \overline{z}(t,x)) - G(t,x) + \sigma(t) \text{ for } x \in \Omega, \quad \int_\Omega e(t,x)\,\mathrm{d}x = \ell(t); \tag{7.9a}$$

$$0 \in \partial \overline{R}(\dot{\overline{z}}(t,x)) + \partial_{\overline{z}} \overline{\Phi}(\overline{e}(t,x), \overline{z}(t,x)). \tag{7.9b}$$

The convergence proof must follow the proof of Theorem 4.5 for the limit $\delta \to 0$ and the proof of Theorem 5.1 for $N \to \infty$. While the former convergence is tedious and lengthy it does not need any substantial new ideas. For the second limit we see easily that by construction and the definition $\overline{\sigma}(t,x) = \sigma(t) - G(t,x)$ the macroscopic equilibrium equation (7.9a) is a direct consequence of (4.1).

For the flow rule (7.9b) one can start from (7.6) which is stated in terms of $\widehat{\mu}$. Since $\widehat{\mu} = \widetilde{\mu}(\overline{z})$, we have the identity $\dot{\widehat{\mu}} = \widetilde{\mu}'(\overline{z})\dot{\overline{z}}$. Since $\widetilde{\mu}'$ is assumed to be strictly negative and the limit problem is rate independent, we can replace $\dot{\widehat{\mu}}$ by $-\dot{\overline{z}}$ in any 0-homogeneous subdifferential. At first sight, $\widehat{R}$ and $\overline{R}$ are not directly related. However, since we are dealing with a simply play operator, we only have to match the thresholds. While (7.6) corresponds to the bounds $\sigma_- \le \widehat{\mu} + \overline{\sigma} \le \sigma_+$, the flow rule (7.9b) corresponds to $-\rho_- \le -\partial_{\overline{z}} \overline{\Phi} \le \rho_+$. Now we can apply the relations derived in Lemma 7.2 to obtain system (7.9).

7.5 Other scalings

In this subsection we briefly discuss how one can study the case when the order of the limits is reversed and we first perform a limit $\varepsilon \to 0$, and then the limit $\delta \to 0$ (see also [PuT05]).

Choose a finite $\delta > 0$. In the case $\mu_j = 0$ for all j (i.e. $r = 0$) the formal pointwise limit $N \to \infty$ leads to the following continuous system

$$\delta \dot{e}(t,x) = -\phi(e(t,x)) - \int_0^x g_{\mathrm{ext}}(t,y)\,\mathrm{d}y + \sigma(t), \quad \int_0^1 e(t,x)\,\mathrm{d}x = \ell(t).$$

Introducing the displacement $u(t,x) = \int_0^x e(t,\xi)\,\mathrm{d}\xi$ and taking the derivative with respect to x we obtain the classical quasistatic visco-elastic problem in space dimension 1:

$$0 = \big(\Phi'(u_x) + \delta \dot{u}_x\big)_x + g_{\mathrm{ext}}(t,x), \quad u(t,x) = 0 \text{ and } u(t,1) = \ell(t). \tag{7.10}$$

In general we cannot expect the convergence of solutions of (3.2) to solutions of (7.10), because of the nonconvexity of Φ.

The limiting behavior may be analyzed by introducing distribution function $F(t,x,\cdot) \in \mathrm{L}^1(\mathbb{R}\times\mathbb{R})$ that account for the fluctuations of the strains e_j^N and the biases μ_j^N via

$$\int_{\mathbb{R}\times\mathbb{R}} F(t,x,\mu,E)\psi(\mu,E)\,\mathrm{d}(\mu,E) = \lim_{N\to\infty} \frac{1}{\#J(x,N)} \sum_{j\in J(x,N)} \psi(\mu_j^N, e_j^N(t)),$$

where $J(x,N) = \{\, j \in \{1, ..., N\} \mid |j - Nx| < N^{1/2} \,\}$. The fluctuations of the initial strain $(e_j^N(0))_j$ may be chosen independently of the bias (μ_j^N) and they do not disappear in finite time because of the viscosity $\delta > 0$. Assuming that the above limits exist we obtain the following transport equation:

$$\delta \partial_t F(t,x,\mu,e) + \Big(-\phi(e) + \mu - G(t,x) + \sigma(t) \Big) \partial_e F(t,x,\mu,e) = 0, \tag{7.11a}$$

$$\int_\Omega \int_{\mathbb{R}} \int_{\mathbb{R}} e\, F(t,x,\mu,e)\, \mathrm{d}(x,\mu,e) = \ell(t), \qquad \int_{\mathbb{R}\times\mathbb{R}} F(t,x,\mu,e)\, \mathrm{d}e = f(\mu). \tag{7.11b}$$

The first constraint in (7.11b) gives the total length of the deformed body, while the second says that the quenched disorder has the bias distribution f, which is independent of t and x. System (7.11) may also be seen as transport equation for a Young measure $\nu_{t,x} \in \mathrm{Prob}(\mathbb{R} \times \mathbb{R})$ and can be treated as in [Tar87, The98, Mie99, BFS01].

The problem can be simplified substantially if we chose initial data such that $F(0,\cdot)$ degenerates to a δ-distribution. This property is preserved by the dynamics and leads to solutions $e = \widetilde{e}(t,x,\mu)$ and $F(t,x,\mu,e) = \delta_{\widetilde{e}(t,x,\mu)}(e) f(\mu)$. Then, (7.11) reduces to a transport equation for $\widetilde{e}$:

$$\begin{aligned} &\delta \partial_t \widetilde{e}(t,x,\mu) = -\phi(\widetilde{e}(t,x,\mu)) + \mu - G(t,x) + \sigma(t), \\ &\int_\Omega \int_{\mathbb{R}} \widetilde{e}(t,x,\mu) f(\mu)\, \mathrm{d}\mu\, \mathrm{d}x = \ell(t). \end{aligned} \tag{7.12}$$

The convergence of the ODE-system in $\mathbb{R}^N$ is now trivial, as the discrete setting can be embedded via functions that are piecewise constant in $x \in \Omega$. Moreover, the right-hand side is locally Lipschitz continuous on $\mathrm{L}^\infty(\Omega \times \mathbb{R})$, and classical continuous dependence on the initial data yields convergence.

The limit $\delta \to 0^+$ forces the solutions to stay in equilibria for all $t \in [0,T]$. This means that for small δ the solution should satisfy $0 \approx -\phi(e(t,x,\mu)) + \mu - G(t,x) + \sigma(t)$. Thus, it should be possible to establish the second convergence for $\delta \to 0^+$ and to obtain the same plasticity limit as in the case $\lim_{\varepsilon\to 0} \lim_{\delta \to 0}$. Again we face the problem that the limiting system is governed by steady states which are non-unique because of the non-monotonicity of ϕ. In the ODE case we were able to derive the corresponding jump rules by hand (see (DA3)), but in the general case the problem remains open.

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