Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Complete damage in elastic and viscoelastic media and its energetics

Alexander Mielke[1,2], Tomáš Roubíček[3,4], Jan Zeman[5]

[1] Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstraße 39, 10117 Berlin, Germany, E-Mail: mielke@wias-berlin.de

[2] Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin-Adlershof, Germany

[3] Mathematical Institute, Charles University, Sokolovská 83, CZ–18 675 Praha 8, Czech Republic, E-Mail: roubicek@karlin.mff.cuni.cz

[4] Institute of Information Theory and Automation, Academy of Sciences, Pod vodárenskou věží 4, CZ–18208 Praha 8, Czech Republic

[5] Department of Mechanics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7 CZ-166 29 Praha 6, Czech Republic, E-Mail: zemanj@cml.fsv.cvut.cz

submitted: 21st December 2007

No. 1285

Berlin 2007

Key words and phrases. Inelastic damage,small strain, energetic formulation.

PACS. 46.15.-x, 62.20.-x .

The first author acknowledges the support from C18-subproject of the Research Center "Matheon" (Deutsche Forschungsgemeinschaft). The second author acknowledges partial support of this research from the grants IAA 1075402 (GA AV ČR), LC 06052 and MSM 21620839 and 1M06031 (MŠMT ČR). The work of the third author was supported by the grant No. 106/08/1379 (GA ČR) and by the research project MSM 6840770003 (MŠMT ČR). Support is acknowledged from the European grant MRTN-CT-2004-505226 "Multi-scale modelling and characterisation for phase transformations in advanced materials", too.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax:	+ 49 30 2044975
E-Mail:	preprint@wias-berlin.de
World Wide Web:	http://www.wias-berlin.de/

Abstract

A model for the evolution of damage that allows for complete disintegration is addressed. Small strains and a linear response function are assumed. The "flow rule" for the damage parameter is rate-independent. The stored energy involves the gradient of the damage variable, which determines an internal length-scale. Quasi-static fully rate-independent evolution is considered as well as rate-dependent evolution including viscous/inertial effects. Illustrative 2-dimensional computer simulations are presented, too.

1 Introduction

Damage, as a special sort of *inelastic response* of solid materials, results from microstructural changes under mechanical load. Routine computational simulations based on various models are widely performed in engineering, although mostly without being supported by rigorous mathematical and numerical analysis.

We will consider damage as a *rate-independent* process. This is often, although not always, an appropriate concept and has applications in a variety of industrially important materials, especially to concrete [Fre02, FrN95, Ort87], filled polymers [DPO94], or filled rubbers [GoS91, Mie95, MiK00]. Being rate-independent, it is necessarily an *activated* process, i.e. to trigger a damage the mechanical stress must achieve a certain activation threshold. The mathematical difficulty is reflected in the fact that only local-in-time existence for a simplified scalar model or for a rate-dependent 0- or 1-dimensional model has been obtained, see [BoS04, DMT01, FKNS98, FKS99]. The 3-dimensional situation was investigated in [FrG06, MiR06, MiRo] with the focus to incomplete damage. The main focus of this paper is on *complete damage*, i.e. the material can completely *disintegrate* and its displacement completely loses any sense on such regions. We show how mathematical modeling can be used to derive well-posed models by suppressing the use of the displacement u and formulating everything in terms of stresses and energies. In Sections 2-3 we will neglect all rate dependent processes like viscosity and inertia so that the damage process is quasistatic and fully rate-independent. Eventually, in Sections 4, we will combine a rate-independent damage process with viscosity and inertia which are, of course, rate-dependent.

We consider an anisotropic material but confine ourselves to a materials with *linear elastic response* and an *isotropic damage* using only one scalar damage parameter under *small strains* (as in [BBT01, BoS04, Fre02, FrN96]). Moreover, we use *gradient-of-damage* theory [DBH96, Fre02, FrN95, FrN96, LoA99, Mau92, PMG04, StH03] expressing a certain nonlocality in the sense that damage of a particular spot is to some extent influenced by its surrounding, leading to possible hardening or softening-like effects, and introducing a certain internal length scale eventually preventing damage microstructure development. From the mathematical viewpoint, the damage gradient has a compactifying character and opens possibilities for the successful analysis of the model. Anyhow, some investigations are still possible without gradient of damage, as shown in [FrG06] for incomplete damage, leading to a possible microstructure in the damage profile.

The goal of this article is to survey and further develop basic mathematical tools focused on complete damage.

2 Complete quasistatic damage at small strains

We will consider specific *stored energy* φ quadratic in terms of *small-strain* tensor e, linear in terms of scalar *damage parameter* z, and convex in terms of a *gradient of the damage* g:

$$\varphi(e,z,g) = \frac{1}{2}z\mathbb{C}e : e + \frac{\kappa}{p}|g|^p + \delta_{[0,+\infty)}(z), \tag{1}$$

where $\mathbb{C} \in \mathbb{R}^{d\times d\times d\times d}$ is a positive-definite *elasticity tensor* satisfying $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}$, $d \in \mathbb{N}$ denotes the considered spatial dimension, and $\kappa > 0$ is a so-called *factor of influence*, and $\delta_{[0,+\infty)}$ is the indicator function of the interval $[0,+\infty)$, i.e. $\delta_{[0,+\infty)}(z) = 0$ for $z \geq 0$ while $\delta_{[0,+\infty)}(z) = +\infty$ otherwise. In this section, we consider the first-order gradient of the damage profile ζ, hence we put $\nabla\zeta(t,x)$ in place of the variable $g \in \mathbb{R}^d$. Another ingredient of the damage-evolution model is a specific *dissipated energy*

$$\varrho(\dot{z}) = \begin{cases} -a\,\dot{z} & \text{if } \dot{z} \leq 0, \\ +\infty & \text{elsewhere.} \end{cases} \tag{2}$$

where $a > 0$ determines the phenomenology how much energy (per d-dimensional "volume") is dissipated by accomplishing the damage process, i.e. by decreasing z from 1 to 0. The value $+\infty$ reflects that we consider damage as a *unidirectional process*, i.e. damage can only develop, but the material can never heal. Note that ϱ is degree-1 homogeneous, which is related with the intended *rate-independent* evolution of the damage process. Simultaneously, a is also *activation threshold* determining the level of the "inelastic" driving force $\sigma_{\mathrm{i}} := \varphi'_z(e,z,g) - \mathrm{div}\varphi'_g(e,z,g)$ (with the physical dimension as stress) that triggers the damage process.

As we want to focus on complete damage where the material can completely disintegrate, in the quasi-static case we cannot have loading by dead load as e.g. gravity load. Thus we will consider a "hard-device" loading by time-varying Dirichlet boundary conditions but zero bulk forces. Considering the "elastic" stress tensor

$\sigma_{\rm e} := \varphi'_{\rm e}(e,z,g)$ that should be in quasistatic equilibrium. Altogether formally we consider the problem

$$\mathrm{div}\big(\sigma_{\rm e}\big) = 0, \qquad \sigma_{\rm e} := \varphi'_e(e(u),\zeta,\nabla\zeta) = \zeta\mathbb{C}e(u), \qquad e(u) = \frac{(\nabla u)^\top + \nabla u}{2}, \tag{3a}$$

$$\begin{aligned}\partial\varrho\Big(\frac{\partial\zeta}{\partial t}\Big) + \sigma_{\rm i} + \sigma_{\rm r} \ni 0, \qquad & \sigma_{\rm r} \in N_{[0,+\infty)}(\zeta),\\ & \sigma_{\rm i} := \varphi'_z(e(u),\zeta,\nabla\zeta) - \mathrm{div}\varphi'_g(e(u),\zeta,\nabla\zeta)\\ & \quad = \frac12\mathbb{C}e(u):e(u) - \mathrm{div}(\kappa|\nabla\zeta|^{p-2}\nabla\zeta),\end{aligned} \tag{3b}$$

where $\partial\varrho$ denotes the subdifferential of the convex function ϱ. We also denoted $\sigma_{\rm r}$ a "reaction force" to the constraint $0 \le \zeta$, and $N_{[0,+\infty)} = \partial\delta_{[0,+\infty)}$ denotes the normal cone. In fact, as the evolution of ζ is unidirectional (here non-increasing in time) and ζ will be prescribed at the beginning, see (9) below, it always holds $0 \le \zeta(t,x) \le \zeta_0(x)$. Usually $\zeta_0 = 1$ is considered so even $\zeta \in [0,1]$ a.e. on $Q := (0,T)\times\Omega$.

This is indeed to be understood only formally because in the completely damaged part $\zeta = 0$ and displacements u as well as strain $e(u)$ lose any sense.

Therefore, we will also consider the *regularized stored energy*

$$\varphi_\varepsilon(e,z,g) = \frac12(z{+}\varepsilon)\mathbb{C}e:e + \frac{\kappa}{p}|g|^p + \delta_{[0,+\infty)}(z), \tag{4}$$

and then the *regularized problem*

$$\mathrm{div}\big(\sigma_{\rm e}\big) = 0, \qquad \sigma_{\rm e} = (\zeta{+}\varepsilon)\mathbb{C}e(u), \tag{5a}$$

$$\begin{aligned}\partial\varrho\Big(\frac{\partial\zeta}{\partial t}\Big) + \sigma_{\rm i} + \sigma_{\rm r} \ni 0, \qquad & \sigma_{\rm r} \in N_{[0,+\infty)}(\zeta),\\ & \sigma_{\rm i} = \frac12\mathbb{C}e(u):e(u) - \mathrm{div}(\kappa|\nabla\zeta|^{p-2}\nabla\zeta).\end{aligned} \tag{5b}$$

As we have the displacement well defined if $\varepsilon > 0$, we can easily consider the Dirichlet boundary conditions

$$u|_\Gamma(t,x) = w(t,x) \tag{6}$$

where $\Gamma \subset \partial\Omega$ is a part of the boundary of Ω where the hard-device loading is applied. For simplicity, the remaining boundary conditions are considered as homogeneous Neumann one:

$$\mathbb{C}e(u)\nu = 0 \quad \text{on } \partial\Omega\backslash\Gamma \quad \text{and} \quad \kappa|\nabla\zeta|^{p-2}\frac{\partial\zeta}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \tag{7}$$

where ν denotes the outer unit normal to the boundary $\partial\Omega$ of Ω. Then we define the Gibbs' stored energy

$$\mathcal{G}_\varepsilon(t,u,\zeta) := \begin{cases} \dfrac12(\zeta{+}\varepsilon)\mathbb{C}e(u):e(u) + \dfrac{\kappa}{p}|\nabla\zeta|^p & \text{if } u|_\Gamma = w(t,\cdot) \text{ and} \\ & \text{if } 0 \le \zeta \text{ a.e. on } \Omega, \\ +\infty & \text{elsewhere,}\end{cases} \tag{8}$$

We still prescribe an initial condition ζ_0 for the damage profile:

$$\zeta(0) = \zeta_0. \tag{9}$$

By the definition of the subdifferential $\partial\varrho(\dot z) = \{\sigma{\in}\mathbb{R};\ \forall \tilde z{\in}\mathbb{R} :\ \varrho(\dot z) + (\tilde z{-}\dot z)\sigma \le \varrho(\tilde z)\}$, the inclusion (5b) can equivalently be written as a variational inequality

$$\forall \tilde z:\ \ \varrho\Big(\frac{\partial\zeta}{\partial t}\Big) \le \varrho(\tilde z) + \Big(\tilde z - \frac{\partial\zeta}{\partial t}\Big)\big(\sigma_{\rm i} + \sigma_{\rm r}\big) \tag{10}$$

for a.a. $(t,x) \in Q := (0,T)\times\Omega$, where $\Omega \subset \mathbb{R}^d$ is a considered domain occupied by the body and $T > 0$ a fixed time horizon. This could be used for a definition of a weak solution.

Here, however, we can use homogeneity of ϱ to formulate a more suitable concept of so-called energetic solution. By (2), we have

$$\sigma_{\rm i} + \sigma_{\rm r} \in \partial\varrho\Big(\frac{\partial z}{\partial t}\Big) \subset \partial\varrho(0) = [-a, +\infty). \tag{11}$$

By the definition of the subdifferential $\partial\varrho(0)$ and properties of ϱ, this means $0 = \varrho(0) \le \varrho(\tilde z) - (\sigma_{\rm i} + \sigma_{\rm r})\tilde z$ for any $\tilde z \in \mathbb{R}$. Written for $\tilde z - z$ instead of $\tilde z$, we have $0 \le \varrho(\tilde z{-}z) + (\sigma_{\rm i}{+}\sigma_{\rm r})(\tilde z{-}z)$. Further, by convexity of $\varphi_\varepsilon(e,\cdot,\cdot)$, we have $\varphi_\varepsilon(e,z,g) \le \varphi_\varepsilon(e,\tilde z,\tilde g) - \xi_1(\tilde z{-}z) - \xi_2\cdot(\tilde g{-}g)$ for any $(\xi_1,\xi_2) \in \partial_{(z,g)}\varphi_\varepsilon(e,z,g)$. In particular, we will use it for $\xi_1 = \frac{1}{2}\mathbb{C}e : e + \sigma_{\rm r}$ and $\xi_2 = (\kappa|\nabla\zeta|^{p-2}\nabla\zeta)$. Altogether, substituting $e = e(u)$, $g = \nabla\zeta(x)$ and $z = \zeta(x)$ we have

$$\begin{aligned}\int_\Omega \varphi_\varepsilon\big(e(u(t)),\zeta(t),\nabla\zeta(t)\big)\,\mathrm{d}x &\le \int_\Omega \varphi_\varepsilon\big(e(u(t)),\tilde\zeta,\nabla\tilde\zeta\big) - (\sigma_{\rm i}{+}\sigma_{\rm r})\big(\tilde\zeta{-}\zeta(t)\big)\,\mathrm{d}x\\ &\le \int_\Omega \varphi_\varepsilon\big(e(u(t)),\tilde\zeta,\nabla\tilde\zeta\big) + \varrho\big(\tilde\zeta{-}\zeta(t)\big)\,\mathrm{d}x\\ &\qquad\qquad \forall 0 \le \tilde\zeta{\in}W^{1,p}(\Omega).\end{aligned} \tag{12}$$

If $\zeta(t)$ satisfies (12), we say that $\zeta(t)$ is *partially stable* at t. Summing (5b) tested by $\frac{\partial\zeta}{\partial t}$ with (5a) tested by $\frac{\partial(u-w)}{\partial t}$, using $-(\sigma_{\rm i}{+}\sigma_{\rm r})\frac{\partial\zeta}{\partial t} \ge \varrho(\frac{\partial\zeta}{\partial t})$ for any $-(\sigma_{\rm i}{+}\sigma_{\rm r}) \in \partial\varrho(\frac{\partial\zeta}{\partial t})$, integrating it over the time interval $[0,T]$, and applying by-part integration in time, we obtain formally the Gibbs-type energy balance

$$\mathcal{G}_\varepsilon\big(T,u(T),\zeta(T)\big) + \mathrm{Var}_\varrho(\zeta;0,T) \le \mathcal{G}_\varepsilon\big(0,u(0),\zeta(0)\big) + \int_0^T\!\!\int_\Omega \sigma_{\rm e}{:}e\Big(\frac{\partial w}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{13}$$

where w means an extension of w from (6) onto the whole Ω and the variation Var_ϱ of ζ with respect to ϱ (i.e. total dissipation of energy within the damage process) is, in view of (2), given by a simple formula

$$\mathrm{Var}_\varrho(\zeta;t_1,t_2) = \begin{cases} a\displaystyle\int_\Omega \zeta(t_1,x){-}\zeta(t_2,x)\ \mathrm{d}x & \text{if } \zeta(\cdot,x) \text{ is nondecreasing on } [t_1,t_2] \text{ for a.a. } x\in\Omega,\\ +\infty & \text{otherwise.}\end{cases}$$

Let us denote by "B" and "BV" the Banach space of everywhere defined bounded measurable and bounded-variation functions, respectively. Moreover, let us abbreviate $I := (0,T)$, $\bar{I} := [0,T]$, $Q := I \times \Omega$, and $\Sigma := I \times \Gamma$. It is important that, as $\varphi(e,\cdot,\cdot)$ is convex, (13) together with the partial stability (12) allows us to derive backwards (10). This authorizes us to introduce a definition of a solution:

Definition 2.1 ***(Weak/energetic solution.)*** *We call $(u_\varepsilon,\zeta_\varepsilon)$ with $u_\varepsilon \in \mathrm{B}(\bar{I};W^{1,2}(\Omega;\mathbb{R}^d))$ and $\zeta_\varepsilon \in \mathrm{B}(\bar{I};W^{1,p}(\Omega;\mathbb{R}^d)) \cap \mathrm{BV}(\bar{I};L^1(\Omega))$ a weak/energetic solution to the original problem (5) with the initial condition (9) and the boundary condition (6)-(7) if*

(i) *the partial stability (12) holds for all $t \in [0,T]$, i.e.*

$$\int_\Omega \varphi_\varepsilon\big(e(u_\varepsilon(t)),\zeta_\varepsilon(t),\nabla\zeta_\varepsilon(t)\big)\,\mathrm{d}x \leq \int_\Omega \varphi_\varepsilon\big(e(u_\varepsilon(t)),\tilde{\zeta},\nabla\tilde{\zeta}\big) + \varrho\big(\tilde{\zeta}{-}\zeta_\varepsilon(t)\big)\,\mathrm{d}x \qquad \forall 0 \leq \tilde{\zeta} {\in} W^{1,p}(\Omega). \quad (14)$$

(ii) *the energy inequality (13) holds with $(u_\varepsilon,\zeta_\varepsilon)$ in place of (u,ζ),*

(iii) *(5a) is satisfied in the weak sense, i.e.*

$$\int_Q (\zeta_\varepsilon(t){+}\varepsilon)\mathbb{C}e(u_\varepsilon(t)) : e(v)\,dx\,\mathrm{d}t = 0 \qquad \forall t \in [0,T], \quad v \in W^{1,2}(\Omega;\mathbb{R}^d), \quad v|_\Sigma = 0, \quad (15)$$

(iv) *(6) and (9) hold with $(u_\varepsilon,\zeta_\varepsilon)$ in place of (u,ζ).*

As the force equilibrium (5a) is governed by minimization of the convex functional $\mathcal{G}_\varepsilon(t,\cdot,\zeta_\varepsilon)$ which also governs the evolution of ζ_ε, (5a) and the partial stability (12) is equivalent to the "full" stability

$$\int_\Omega \varphi_\varepsilon(e(u_\varepsilon(t)),\zeta_\varepsilon(t),\nabla\zeta_\varepsilon(t))\,\mathrm{d}x \leq \int_\Omega \varphi_\varepsilon(e(\tilde{u}),\tilde{\zeta},\nabla\tilde{\zeta}) + \varrho(\tilde{\zeta}{-}\zeta_\varepsilon(t))\,\mathrm{d}x \qquad \forall(\tilde{u},\tilde{\zeta}) \in W^{1,2}(\Omega;\mathbb{R}^d) \times W^{1,p}(\Omega), \quad \tilde{u}|_\Gamma = w(t), \quad \tilde{\zeta} \geq 0. \quad (16)$$

The configuration $(u_\varepsilon(t),\zeta_\varepsilon(t))$ is called *stable* at t if it satisfies (16). In fact, under (16), the energy inequality (13) implies even energy equality at any time t, i.e.

$$\mathcal{G}_\varepsilon\big(t,u_\varepsilon(t),\zeta_\varepsilon(t)\big) + \mathrm{Var}_\varrho(\zeta_\varepsilon;0,t) = \mathcal{G}_\varepsilon\big(0,u_\varepsilon(0),\zeta_0\big) + \int_0^t\!\!\int_\Omega \sigma_\mathrm{e} : e\Big(\frac{\partial w}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t \qquad (17)$$

with $\sigma_\mathrm{e} = (\zeta_\varepsilon + \varepsilon)\mathbb{C}e(u_\varepsilon)$. Note that (16) at $t = 0$ in fact qualify through (9) also ζ_0 to be stable. The conditions (16)–(17) leads to a concept introduced in [MiT99, MiT04, MTL02] (see also [Mie05] for a survey)

Definition 2.2 ***(Energetic solution.)*** *We call* $(u_\varepsilon,\zeta_\varepsilon)$ *with* $u_\varepsilon \in \mathrm{B}(\bar{I};W^{1,2}(\Omega;\mathbb{R}^d))$ *and* $\zeta_\varepsilon \in \mathrm{B}(\bar{I};W^{1,p}(\Omega;\mathbb{R}^d)) \cap \mathrm{BV}(\bar{I};L^1(\Omega))$ *an energetic solution to the original problem (5) with the initial condition (9) and the boundary condition (6)-(7) if*

(i) *the stability (16) holds for all* $t \in [0,T]$,

(ii) *the energy balance (17) holds with* $(u_\varepsilon,\zeta_\varepsilon)$ *in place of* (u,ζ) *for all* $t \in [0,T]$, *and*

(iii) *(6) and (9) hold with* $(u_\varepsilon,\zeta_\varepsilon)$ *in place of* (u,ζ).

As already said, Definitions 2.1 and 2.2 are equivalent to each other. Under the hard-device loading $w \in W^{1,1}(I;W^{1/2,2}(\Gamma))$ (and thus considering an extension from $W^{1,1}(I;W^{1,2}(\Omega))$ for (13) or (17)), assuming $p > d$ and stability of ζ_0, the existence of a (weak) energetic solution $(u_\varepsilon,\zeta_\varepsilon)$ is guaranteed for any $\varepsilon > 0$, cf. [MiR06]. The proof consists in limit passage with $\tau \to 0$ for an approximate solution obtained by the implicit time-discretization with a time step τ, which leads to a recursive minimization problem

$$\left.\begin{array}{ll}\text{Minimize} & \displaystyle\int_\Omega \frac{\zeta_{\tau\varepsilon}^k{+}\varepsilon}{2}\mathbb{C}e(\nabla u_{\tau\varepsilon}^k) : e(\nabla u_{\tau\varepsilon}^k) - a\zeta_{\tau\varepsilon}^k + \frac{\kappa}{p}\big|\nabla\zeta_{\tau\varepsilon}^k\big|^p \,\mathrm{d}x \\ \text{subject to} & 0 \le \zeta_{\tau\varepsilon}^k \le \zeta_{\tau\varepsilon}^{k-1}\,, \qquad u_{\tau\varepsilon}^k|_\Gamma = w(k\tau), \\ & u_{\tau\varepsilon}^k \in W^{1,2}(\Omega;\mathbb{R}^d)\,, \quad \zeta_{\tau\varepsilon}^k \in W^{1,p}(\Omega), \end{array}\right\} \tag{18}$$

for $k = 1,...,K := T/\tau$ with $\zeta_{\tau\varepsilon}^0 := \zeta_0$. Having (some) solutions $(u_{\tau\varepsilon}^k,\zeta_{\tau\varepsilon}^k)$ to (18), we assemble the piecewise constant interpolation $(u_{\tau\varepsilon},\zeta_{\tau\varepsilon})$ so that $u_{\tau\varepsilon}|_{((k-1)\tau,k\tau]} = u_{\tau\varepsilon}^k$ for $k = 1,...,T/\tau$. Likewise, we define also $\zeta_{\tau\varepsilon}$. Also, w_τ denotes the piecewise constant interpolation of w. For the right-hand side of (19) below, we assume the prolongation $\zeta_{\tau\varepsilon}(t) = \zeta_{\tau\varepsilon}^0 = \zeta_0$ for $t < 0$, and similarly $w_\tau(t) = w(0)$ and $u_{\tau\varepsilon}(t) = u_{\tau\varepsilon}^0$ for $t < 0$, with $u_{\tau\varepsilon}^0$ minimizing $\mathcal{G}_\varepsilon(0,\cdot,\zeta_0)$. We have the two-sided discrete energy estimate:

$$\begin{aligned}&\int_0^t\!\!\int_\Omega (\zeta_{\tau\varepsilon}{+}\varepsilon)\mathbb{C}e(u_{\tau\varepsilon}{+}w{-}w_\tau) : e\Big(\frac{\partial w}{\partial\theta}\Big)\,\mathrm{d}x\,\mathrm{d}\theta \\ &\le \mathcal{G}_\varepsilon\big(t,u_{\tau\varepsilon}(t),\zeta_{\tau\varepsilon}(t)\big) + \mathrm{Var}_\varrho(\zeta_{\tau\varepsilon};0,t) - \mathcal{G}_\varepsilon(0,u_{\tau\varepsilon}(0),\zeta_0) \\ &\qquad \le \int_0^t\!\!\int_\Omega (\zeta_{\tau\varepsilon}^{\mathrm{R}}{+}\varepsilon)\mathbb{C}e(u_{\tau\varepsilon}^{\mathrm{R}}{+}w{-}w_\tau^{\mathrm{R}}) : e\Big(\frac{\partial w}{\partial\theta}\Big)\,\mathrm{d}x\,\mathrm{d}\theta\end{aligned} \tag{19}$$

holds with $t = k\tau$ for any $k = 1,...,T/\tau$, where $(\cdot)_\tau^{\mathrm{R}}$ denotes functions "retarded" by τ, i.e. $[u_{\tau\varepsilon}^{\mathrm{R}}](t) := u_{\tau\varepsilon}(t-\tau)$, and where w has the meaning of an extension of the boundary conditions into Ω; cf. [MiR06, Lemma 3.3].

We are now going to formulate a suitable solution to the complete damage problem. The essential peculiarity is that displacement u and the strain $e(u)$ are no longer well defined on areas that are completely damaged, i.e. where $\zeta = 0$.

At each time t, we have, however, estimates on the stress $(\zeta_\varepsilon(t){+}\varepsilon)\mathbb{C}e(u_\varepsilon(t))$ in $L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ uniform with respect to $\varepsilon > 0$, where $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ is the set of symmetric

$d\times d$-matrices. Each weak cluster point $\mathfrak{s}$ is called a *realizable stress*. The set of all realizable stresses is weakly compact in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, cf. [BMR07, Proposition 2.8]. A realizable stress $\mathfrak{s}$ that minimizes $\mathfrak{s}\mapsto\frac{1}{2}\int_\Omega\mathfrak{s}:e(w(t))\,\mathrm{d}x$ is called an *effective stress* at a given t. Let us remark that one can also define an *effective strain* $\mathfrak{e}\in L^2_{\rm loc}(\{x{\in}\Omega;\ \zeta(t,x)>0\};\mathbb{R}^{d\times d}_{\rm sym})$ by

$$\mathfrak{e}(t,x)=\mathbb{C}^{-1}\Big(\frac{\mathfrak{s}(t,x)}{\zeta(t,x)}\Big)\qquad\text{for all } t \text{ and a.a. } x{\in}\Omega \text{ such that } \zeta(t,x)>0 \tag{20}$$

where $\mathbb{C}^{-1}$ means the inversion of the mapping $\mathbb{C}:\mathbb{R}^{d\times d}_{\rm sym}\to\mathbb{R}^{d\times d}_{\rm sym}$. It is important that $\mathfrak{e}(t)$ is a corresponding limit of $e(u_\varepsilon(t))$ for $\varepsilon\to0$, cf. [BMR07, Sect. 2.3] for details. Let us define, for a given damage profile ζ, the effective stored energy as the so-called Γ-limit [Dal95] of the collection $\{\mathcal{G}_\varepsilon(t,u,\tilde\zeta)\}_{\varepsilon>0}$:

$$\boldsymbol{g}(t,\zeta):=\liminf_{\substack{\varepsilon\to0+,\ 0\le\tilde\zeta\in W^{1,p}(\Omega)\\ \tilde\zeta\to\zeta\text{ in }W^{1,p}(\Omega)}}\ \min_{u\in W^{1,2}(\Omega;\mathbb{R}^d)}\mathcal{G}_\varepsilon(t,u,\tilde\zeta). \tag{21}$$

The so-called recovery sequence of damage profiles that asymptotically reaches the value $\boldsymbol{g}(t,\zeta)$ involves $\tilde\zeta=(\zeta-\delta)^+$ when $\delta\to0+$ sufficiently slowly with respect to $\varepsilon\to0+$. An important result of [BMR07] is that for each t and ζ there is unique effective equilibrium stress $\mathfrak{s}(t,\zeta)$ (i.e., $\operatorname{div}\mathfrak{s}=0$). Hence, we can write

$$\boldsymbol{g}(t,\zeta)=\int_\Omega\frac{1}{2}\mathfrak{s}(t,\zeta):e(w(t))+\frac{\kappa}{p}|\nabla\zeta|^p\,\mathrm{d}x. \tag{22}$$

Also, we have an important formula for the power of external loading:

$$\frac{\partial\boldsymbol{g}}{\partial t}(t,\zeta)=\int_\Omega\mathfrak{s}(t,\zeta):e\Big(\frac{\partial w}{\partial t}\Big)\,\mathrm{d}x. \tag{23}$$

Our definition for the complete damage is based on the energetic-solution concept as in Definition 2.2.

Definition 2.3 ***(Energetic solution for complete damage.)*** *The process $\zeta:[0,T]\to W^{1,p}(\Omega)$ is called an energetic solution to the problem given by the data φ, ϱ, w, and ζ_0, if, beside (9), also*
(i) $\zeta\in{\rm BV}([0,T];L^1(\Omega))\cap{\rm B}([0,T];W^{1,p}(\Omega))$,
(ii) *it is stable for all $t\in[0,T]$ in the sense that*

$$\boldsymbol{g}\big(t,\zeta(t)\big)\le\boldsymbol{g}(t,\tilde\zeta)+\int_\Omega\varrho\big(\tilde\zeta{-}\zeta(t)\big)\,\mathrm{d}x\qquad\forall0\le\tilde\zeta{\in}W^{1,p}(\Omega), \tag{24}$$

(iii) *and, for any $0\le t_1<t_2\le T$, the energy equality holds:*

$$\boldsymbol{g}\big(t_2,\zeta(t_2)\big)+{\rm Var}_\varrho(\zeta;t_1,t_2)=\boldsymbol{g}(t_1,\zeta(t_1))+\int_{t_1}^{t_2}\!\!\int_\Omega\mathfrak{s}(t,\zeta(t)){:}e\Big(\frac{\partial w}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t, \tag{25}$$

in particular, the function $t\mapsto\int_\Omega\mathfrak{s}(t,\zeta(t)):e(\frac{\partial w}{\partial t}(t))\,\mathrm{d}x$ belongs to $L^1(0,T)$.

Existence of an energetic solution has been proved in [BMR07] by convergence of the above introduced regularization for $\varepsilon \to 0$.

Proposition 2.4 ***(Existence of energetic solutions, convergence of*** $(u_\varepsilon, \zeta_\varepsilon)$***.)*** *Let $p > d$ and $w \in C^1([0,T]; W^{1/2,2}(\Gamma;\mathbb{R}^d))$, Then, there exist a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to 0 and a process $\zeta : [0,T] \to W^{1,p}(\Omega)$ being an energetic solution according to Definition 2.3 such that the following holds for all $t \in [0,T]$:*
(i) $\mathcal{G}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), \zeta_{\varepsilon_n}(t)) \to \boldsymbol{g}(t, \zeta(t))$,
(ii) $\mathrm{Var}_\varrho(\zeta_{\varepsilon_n}; 0, t) \to \mathrm{Var}_\varrho(\zeta; 0, t)$,
(iii) $\zeta_{\varepsilon_n}(t) \to \zeta(t)$ *strongly in* $W^{1,p}(\Omega)$,
(iv) $(\zeta_{\varepsilon_n}(t)+\varepsilon)\mathbb{C}(e(u_{\varepsilon_n}(t))) \rightharpoonup \mathfrak{s}(t,\zeta(t))$ *weakly in* $L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$.

Remark 2.5 (*Quasi-stress.*) In fact, we have bounded in $\mathrm{B}(\bar{I}; L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ not only the stress $(\zeta_\varepsilon{+}\varepsilon)\mathbb{C}e(u_\varepsilon)$ but even $\sqrt{\zeta_\varepsilon{+}\varepsilon}\mathbb{C}e(u_\varepsilon)$, which thus converges (as a subsequence) weakly* in $L^\infty(I; L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ to some χ. Let us call it *quasi-stress.* We have $\mathfrak{s} = \sqrt{\zeta}\chi$ for the corresponding effective stress $\mathfrak{s}$ and, by (20), $\chi = \sqrt{\zeta}\mathbb{C}\mathfrak{e}$ with the effective strain on the part with $\zeta > 0$. Contrary to the stress itself which converges even L^2-strongly to zero on the completely damaged part, cf. [BMR07, Proposition 2.5], $\sqrt{\zeta_\varepsilon{+}\varepsilon}\mathbb{C}e(u_\varepsilon)$ need not converge to zero on this part.

Remark 2.6 (*Large strains.*) Generalization for stored energies that are non-quadratic in terms of strain seems difficult, however. For incomplete damage (or, in other words, $\varepsilon > 0$ fixed) we refer to [MiR06] where such a model was analyzed even at large strains and a unilateral contact.

3 Numerical implementation, 2D computational simulations

In order to arrive at an implementable numerical algorithm, we perform a spatial discretization of the time-incremental minimization problem (18). To that end, we introduce finite-dimensional spaces $U_h \subset W^{1,2}(\Omega;\mathbb{R}^2)$ and $Z_h \subset W^{1,p}(\Omega)$ and consider the following minimization problem:

$$\left.\begin{array}{ll}\text{Minimize} & \displaystyle\int_\Omega \frac{\zeta^k_{\tau h\varepsilon}{+}\varepsilon}{2}\mathbb{C}e(\nabla u^k_{\tau h\varepsilon}) : e(\nabla u^k_{\tau h\varepsilon}) - a\zeta^k_{\tau h\varepsilon} + \frac{\kappa}{p}\big|\nabla\zeta^k_{\tau h\varepsilon}\big|^p \,\mathrm{d}x \\ \text{subject to} & 0 \le \zeta^k_{\tau h\varepsilon} \le \zeta^{k-1}_{\tau h\varepsilon}, \qquad u^k_{\tau h\varepsilon}|_\Gamma = w(k\tau), \\ & u^k_{\tau h\varepsilon} \in U_h\ , \qquad\qquad \zeta^k_{\tau h\varepsilon} \in Z_h \end{array}\right\} \tag{26}$$

for $k = 1,...,K := T/\tau$ with $(u^0_{\tau h\varepsilon}, \zeta^0_{\tau h\varepsilon}) := (u_0, \zeta_0)$, i.e. the discretized incremental problem leads to a non-convex, box-constrained optimization program. Note that the convergence of the fully discrete solution to the solution of the space-time continuous problem is guaranteed thanks to abstract approximation results available in [MiRo].

In the actual numerical implementation, the spatial discretization is performed using the linear conforming finite elements, e.g. [BiS96, Bra07]. Moreover, for computational efficiency, we restrict our attention to $d = 2$ and dare to choose $p = 2$ (which fits with the theory presented in Section 2 only "up to epsilon" as we have required $p > d$).

For a given regularization parameter ε and the time level k, we express the discrete fields in the form

$$u^k_{\tau h\varepsilon}(x) = \boldsymbol{N}^u_h(x)\boldsymbol{u}^k_h, \qquad \zeta^k_{\tau h\varepsilon}(x) = \boldsymbol{N}^\zeta_h(x)\boldsymbol{\zeta}^k_h, \tag{27}$$

where $\boldsymbol{u}^k_h$ and $\boldsymbol{\zeta}^k_h$ denote vectors of the nodal values of displacement and damage parameter fields, respectively (indices $\tau\varepsilon$ are omitted in the sequel for the sake of brevity) and $\boldsymbol{N}^u_h$ and $\boldsymbol{N}^\zeta_h$ denote the operators of piecewise linear basis functions. The discrete problem (26) can now be re-written in a fully algebraic format

$$\left.\begin{array}{ll} \text{Minimize} & \frac{1}{2}\boldsymbol{u}^{k\mathsf{T}}_h\boldsymbol{K}^u_h\left(\boldsymbol{\zeta}^k_h\right)\boldsymbol{u}^k_h + \frac{1}{2}\boldsymbol{\zeta}^{k\mathsf{T}}_h\boldsymbol{K}^\zeta_h\boldsymbol{\zeta}^k_h + \boldsymbol{f}^{\zeta\mathsf{T}}_h\boldsymbol{\zeta}^k_h \\ \text{subject to} & \boldsymbol{0} \le \boldsymbol{\zeta}^k_h \le \boldsymbol{\zeta}^{k-1}_h, \quad \boldsymbol{u}^k_{h,D} = \boldsymbol{w}_D(k\tau) \end{array}\right\} \tag{28}$$

with components of $\boldsymbol{w}_D$ storing the nodal displacements on the Dirichlet part of the boundary. The individual matrices are provided by:

$$\boldsymbol{K}^u_h\left(\boldsymbol{\zeta}_h\right) = \int_{\Omega_h} \boldsymbol{B}^{u\mathsf{T}}_h(x)\left(\left(\varepsilon + \boldsymbol{N}^\zeta_h(x)\boldsymbol{\zeta}_h\right)\boldsymbol{C}(x)\right)\boldsymbol{B}^u_h(x)\,\mathrm{d}x, \tag{29}$$

$$\boldsymbol{K}^\zeta_h = \int_{\Omega_h} \boldsymbol{B}^{\zeta\mathsf{T}}_h\kappa(x)\boldsymbol{B}^\zeta_h(x)\,\mathrm{d}x, \tag{30}$$

$$\boldsymbol{f}^\zeta_h = -\int_{\Omega_h} a(x)\boldsymbol{N}^{\zeta\mathsf{T}}_h(x)\,\mathrm{d}x, \tag{31}$$

where the $\boldsymbol{B}$ operators contain derivatives of the shape functions and $\boldsymbol{C}$ is the Voigt representation of the material stiffness tensor $\mathbb{C}$; see e.g. [BiS96].

The discrete formulation (28) leads to a (usually large-scale) non-convex program. Nevertheless, recognizing that the objective function is quadratic separately in $\boldsymbol{u}^k_h$ and $\boldsymbol{\zeta}^k_h$ and exploiting the formal similarity between the ε-regularized damage model and the Francfort-Marigo variational approach to fracture [BFM00], the problem (28) can be efficiently solved employing a variant of the alternate minimization algorithm proposed recently by Bourdin in [Bou07, Bou]. In the current context, the incremental version of algorithm is briefly summarized in Table 1. In each internal iteration, the minimization problem with respect to $\boldsymbol{u}$ (Step 4) reduces to the solution of a sparse system of linear equations, while the subsequent sparse box-constrained problem is solved using a reflective Newton method introduced in [CoL96].

The convergence of the alternate minimization was studied by Bourdin in [Bou07], where it was shown that the algorithm converges to a critical point of the discretized problem in a finite number of iterations. Of course, there is no guarantee that the critical point is a global minimizer of the non-convex objective function, which is a crucial assumption of the theoretical framework. This obstacle

Table 1: Conceptual implementation of the optimization algorithm for time level k and an initial value of interval variable $\boldsymbol{\zeta}^{(0)}$.

Set $j = 0$
repeat
  Set $j = j + 1$
  Solve $\boldsymbol{u}^{(j)} = \arg \min\limits_{\boldsymbol{u}_D = \boldsymbol{w}_D(k\tau)} \dfrac{1}{2} \boldsymbol{u}^{\mathsf{T}} \boldsymbol{K}_h^u \big(\boldsymbol{\zeta}^{(j-1)}\big) \boldsymbol{u}$
  Solve $\boldsymbol{\zeta}^{(j)} = \arg \min\limits_{\boldsymbol{0} \leq \boldsymbol{\zeta} \leq \boldsymbol{\zeta}_h^{k-1}} \dfrac{1}{2} \boldsymbol{u}^{\mathsf{T}(j)} \boldsymbol{K}_h^u (\boldsymbol{\zeta}) \boldsymbol{u}^{(j)} + \dfrac{1}{2} \boldsymbol{\zeta}^{\mathsf{T}} \boldsymbol{K}_h^{\zeta} \boldsymbol{\zeta} + \boldsymbol{f}_h^{\zeta\,\mathsf{T}} \boldsymbol{\zeta}$
until $\|\boldsymbol{\zeta}^{(j)} - \boldsymbol{\zeta}^{(j-1)}\|_\infty \leq \delta$
Set $\boldsymbol{u}_h^k = \boldsymbol{u}^{(j)}$, $\boldsymbol{\zeta}_h^k = \boldsymbol{\zeta}^{(j)}$

can be, for example, resolved by resorting to the global stochastic optimization approaches [HJK00, IKLK04, MLZS00]. Such techniques, however, require very large number of iterations and as such are applicable only to very inexpensive objective functions. Fortunately, it is possible to construct a feasible numerical approach exploiting the two-sided energetic estimates (19).

To that end, consider the discretized version of (19)

$$
\begin{aligned}
- \quad & \eta + \sum_{j=1}^{k} \int_{(j-1)\tau}^{j\tau} \int_{\Omega_h} (\zeta_{\tau h\varepsilon}^{j} + \varepsilon) \mathbb{C} e(u_{\tau h\varepsilon}^{j} + w - w_\tau) : e\Big(\frac{\partial w}{\partial \theta}\Big) \, \mathrm{d}x \, \mathrm{d}\theta \\
\leq \quad & \mathcal{G}_\varepsilon\big(k\tau, u_{\tau h\varepsilon}^{k}, \zeta_{\tau h\varepsilon}^{k}\big) + \mathrm{Var}_\varrho(\zeta_{\tau h\varepsilon}; 0, k\tau) - \mathcal{G}_\varepsilon(0, u_{\tau h\varepsilon}^{0}, \zeta_{h\varepsilon}^{0}) \\
\leq \quad & \eta + \sum_{j=1}^{k} \int_{(j-1)\tau}^{j\tau} \int_{\Omega_h} (\zeta_{\tau h\varepsilon}^{j-1} + \varepsilon) \mathbb{C} e(u_{\tau h\varepsilon}^{j-1} + w - w_\tau^{\mathrm{R}}) : e\Big(\frac{\partial w}{\partial \theta}\Big) \, \mathrm{d}x \, \mathrm{d}\theta
\end{aligned}
\tag{32}
$$

where η is an energy tolerance parameter introduced for the numerical implementation. The previous condition is used to detect local minimizers: if the result of the alternate minimization algorithm $\boldsymbol{\zeta}_h^k$ fails to verify the inequality (32), the algorithm is restarted from the previous time level with $\boldsymbol{\zeta}_h^k$ used as an initial value for the minimization algorithm instead of $\boldsymbol{\zeta}_h^{k-1}$. This procedure is repeated until an admissible solution is found, see Table 2 for additional details. It is worth noting that the resulting algorithm shares similar features with the backtracking scheme introduced by Bourdin [Bou07] in the framework of variational fracture theories.

Performance of the proposed algorithm will be illustrated on two benchmark problems inspired by [SAS04]: an inhomogeneous and a pre-notched specimen, see Figure 1. The corresponding geometric and material data together with the algorithm parameters are gathered in Figure 1 and Table 3, respectively. Both structures are assumed to be in the plane stress state and are subject to a proportional-in-time axially symmetric hard-device loading. In both cases, the spatial discretization was performed using the unstructured mesh generator T3D [Ryp98] and the problem size was reduced using symmetries of the specimens. The analyzed time interval $[0, 1]$ was decomposed into 100 identical time steps (a physical dimension of time is omitted in

Table 2: Conceptual implementation of the time stepping procedure.

1	:	Set $k = 1$, $\boldsymbol{\zeta}_h^{-1} = \mathbf{0}, \boldsymbol{\zeta}_h^0 = \mathbf{0}$, $\boldsymbol{\zeta}^{(0)} = \mathbf{0}$
2	:	**repeat**
3	:	Determine $\boldsymbol{\zeta}_h^k$ using the alternate minimization algorithm for time t_k and initial value $\boldsymbol{\zeta}^{(0)}$.
4	:	Set $\boldsymbol{\zeta}^{(0)} = \boldsymbol{\zeta}_h^k$
5	:	**if** two-sided inequality (32) is satisfied
6	:	Set $k = k + 1$
7	:	**else**
8	:	Set $k = k - 1$
9	:	**end**
10	:	**until** $k \leq K$

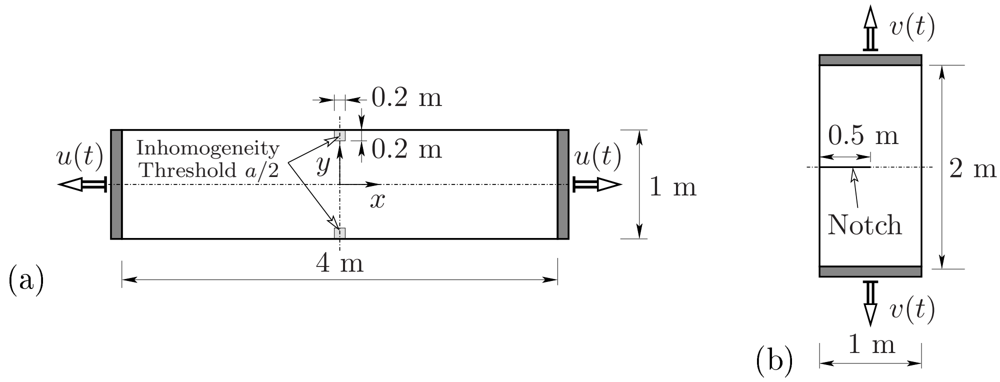


Figure 1: Scheme of simulated experiments; (a) inhomogeneous specimen, (b) pre-notched specimen

the sequel because of rate-independence). Finally, for the inhomogeneous specimen, the damage localization is triggered by pre-existing imperfections introduced by a reduced activation threshold in the shaded area on the axis of symmetry.

The resulting energetics for the inhomogeneous specimen is displayed in Figure 2 for a representative choice of the ε and h parameters. Clearly, in its basic version, the discrete solution obtained by the alternate minimization algorithm fails to provide an appropriate energetic solution to the problem. The two-sided inequality is satisfied only in the initial stage, where the specimen stays mainly elastic. At time $t \approx 0.61$, the damage propagates simultaneously through the specimen, as manifested by the drop of the sum of the globally dissipated and the Gibbs energy, see Figure 2(a). Even after this instant, however, this quantity increases, which is the consequence of the non-zero value of regularization parameter ε. Moreover, the damage profile still evolves in the subsequent time levels, leading to the increase in the dissipated energy balanced by the contribution of the Gibbs energy.

With the backtracking option active, however, the algorithm detects the local minimizer at $t \approx 0.61$ and, following the dotted line in Figure 2(a), returns to the time level where the incremental two-sided inequality is satisfied. After the backtrack-

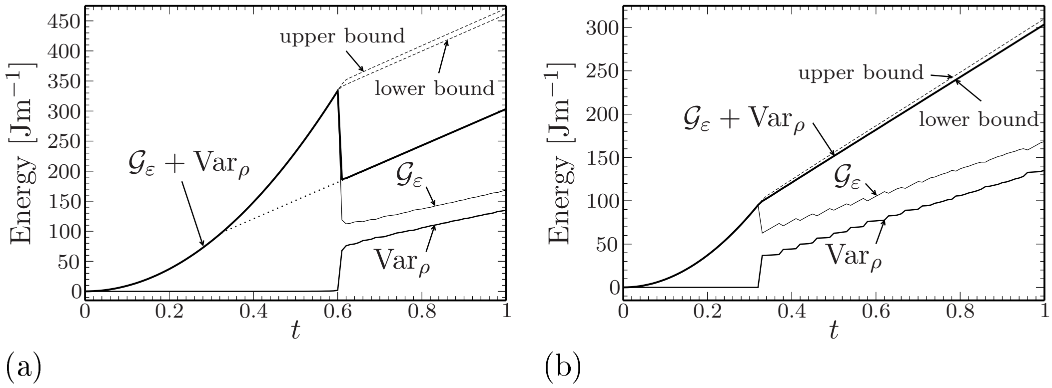


Figure 2: Global energetics of the inhomogeneous specimen ($\varepsilon = 5 \cdot 10^{-2}$, $h = 0.03$ m); (a) Without backtracking (energy balance fails), (b) with backtracking (an approximate energetic solution)

ing stage is completed, the alternate minimization algorithm is capable of finding an approximate energetic solution, cf. Fig. 2(b). As further illustrated by Fig. 3, evolution of the damage profile for the algorithm with backtracking is more gradual when compared with the basic variant.

Additional numerical tests summarized in Figures 4 demonstrate the "mesh-independent" behavior of the method, i.e. the fact that the global energetic response is almost independent of the discretization parameter h. The influence of the energy regularization parameter ε, however, is much stronger, cf. Figure 4(b). As $\varepsilon \to 0$, the algorithms tries to reproduce the one-dimensional optimal damage profile $\zeta(x, y) \approx |x|^\alpha$, derived in [BMR07].

The same set of numerical experiments was executed for the pre-notched specimen leading to the results appearing in Figures 5, 6 and 7. When compared to the inhomogeneous specimen, the global response shows similar trends for algorithms with and without backtracking.

It is further confirmed by Figure 8 that the numerical results are almost independent

Table 3: Parameter of the damage model and incremental algorithm

Young's modulus, E	27 GPa
Possion's ratio, ν	0.2
Factor of influence, κ	10 Jm^{-2}
Activation threshold, a (see [FrN96])	500 Jm^{-3}
Maximal prescribed displacement for the inhomogeneous specimen	$5 \cdot 10^{-4}$ m
Maximal prescribed displacement for the pre-notched specimen	$2.25 \cdot 10^{-4}$ m
Time step, τ	0.01
Damage profile tolerance, δ	10^{-6}
Two-sided energy inequality tolerance, η	10^{-3}

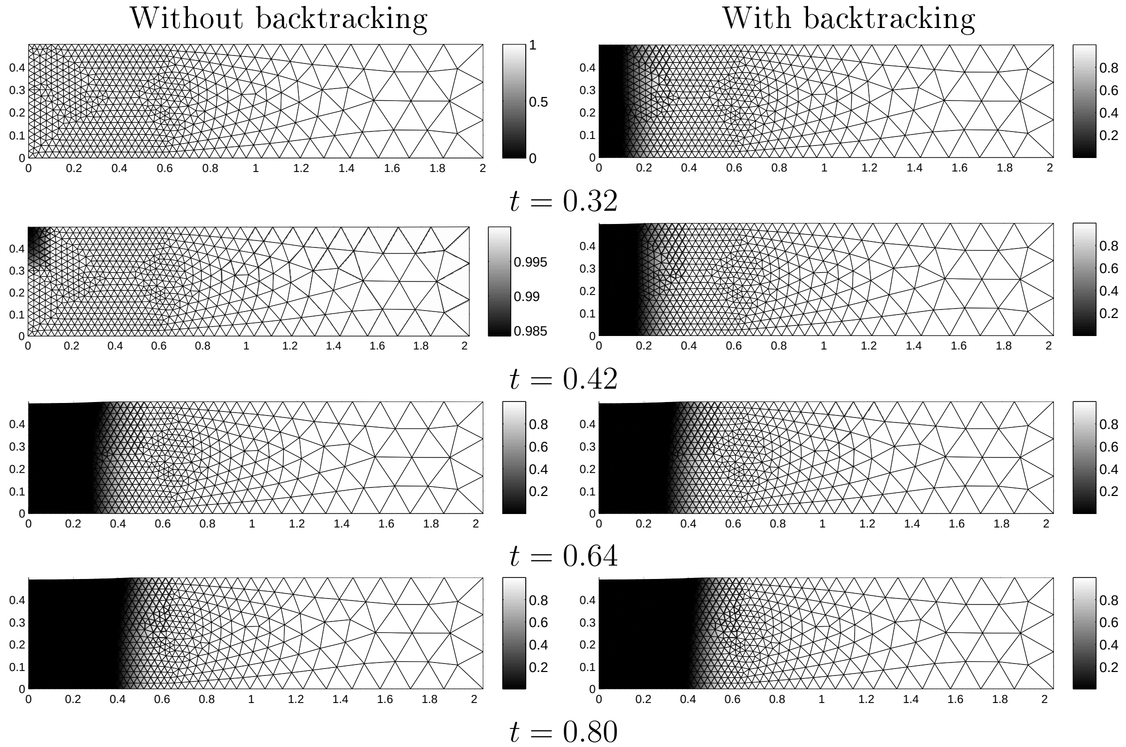
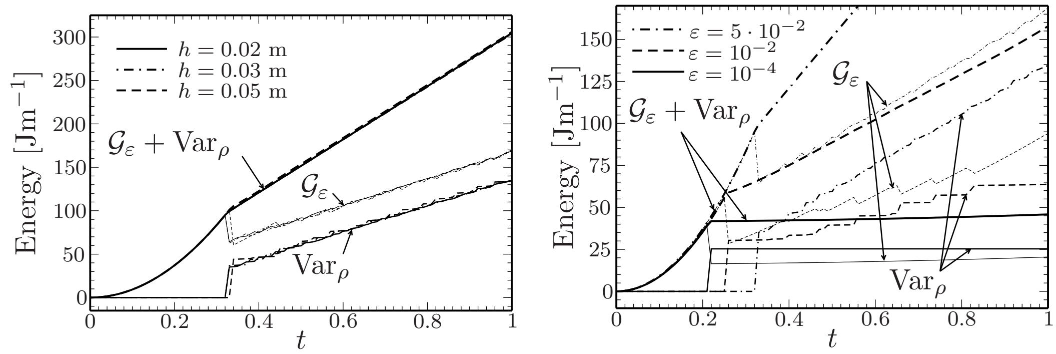


Figure 3: Time evolution of ζ field for the inhomogeneous specimen $h = 0.03$ m, $\varepsilon = 5 \cdot 10^{-2}$, displacements are scaled by a factor 100 and only a quarter of the specimen is shown.

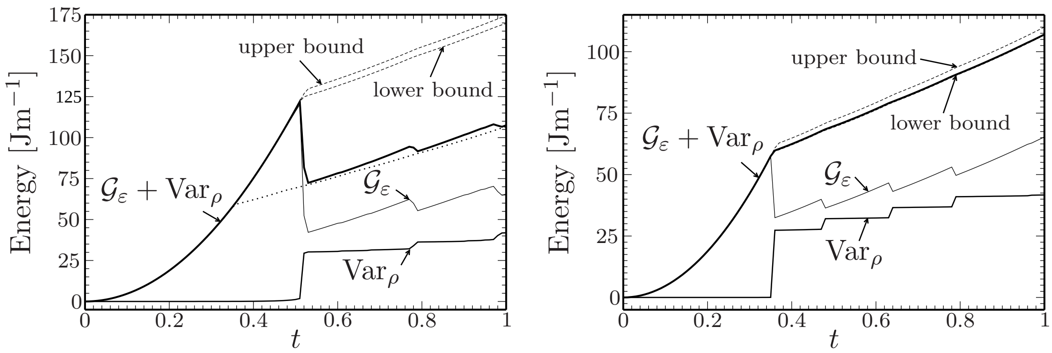
of the spatial discretization parameter h, which is considered to be an essential requirement for any damage model in the engineering literature. The extent of damage zone depends on the value of the regularization parameter ε (related to a “residual” energy after the complete damage). As $\varepsilon \to 0$, however, the width of the localized damage zone, displayed in Figure 9, remains still finite and insensitive to spatial discretization.

Remark 3.1 (*Clapeyron principle.*) Similarly to [KMR06], it can be observed that the work of external load is approximately equally distributed to the dissipated energy Var_ρ and the stored energy $\mathcal{G}_\varepsilon$ after the damage initiation; the effect known as the Clapeyron principle for slowly loaded bodies with viscous damping, cf. [FoT03]. The deviation from the ideal 1 : 1 ratio depends mainly on the energy regularization parameters ε, see Figures 4 and 7, which makes a certain portion of the stored energy “unavailable” to the damage process. In addition, due to the localized character of damage, only a part of the work of the external load can contribute to the dissipative processes (analogously to the beginning of the loading program where no damage occurs).



(a) (b)

Figure 4: Convergence of the approximate energetic solution for the inhomogeneous specimen; (a) $h \to 0$ m, $\varepsilon = 5 \cdot 10^{-2}$, (b) $\varepsilon \to 0, h = 0.02$ m; mesh with $h = 0.05$ m contains 493 triangular elements, $h = 0.03$ m corresponds to 1,193 elements and $h = 0.02$ m to 1,549 elements.



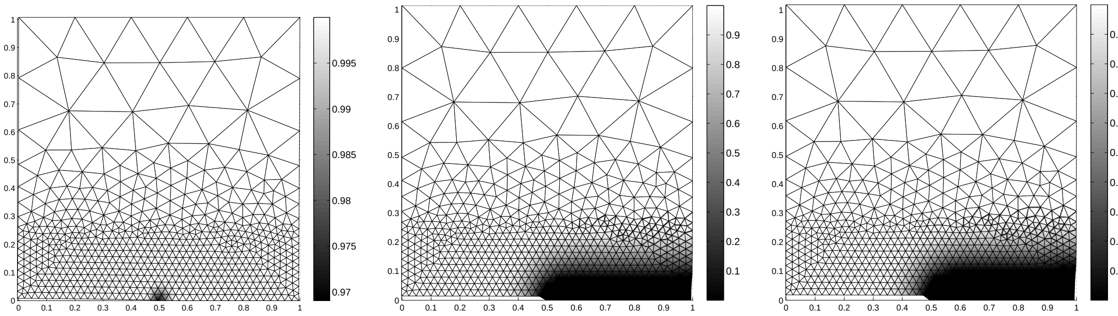
(a) (b)

Figure 5: Global energetics for the pre-notched specimen ($\varepsilon = 10^{-2}$, $h = 0.03$ m); (a) Without backtracking (energy balance fails), (b) with backtracking (an approximate energetic solution)

4 Damage in viscoelastic media with inertia

Finally we include also some rate-dependent phenomena, in particular viscosity and inertia. Combination with viscosity has been addressed in Maxwellian rheology (even with plasticity) in [FeS03] and in the Kelvin-Voigt rheology in [HSS01, PPS07, SHS06, CFKSV06].

We will consider linear viscosity in the Kelvin-Voigt rheology, i.e. the total stress σ is composed from the elastic contribution $\sigma_{\rm e} := \zeta\mathbb{C}e(u)$ as before and now also the viscous contribution $\sigma_{\rm v} := \zeta\mathbb{D}e(\frac{\partial u}{\partial t})$ where $\mathbb{C}$ is a positive-definite elasticity tensor as before and $\mathbb{D}$ is a positive-definite viscosity tensor satisfying $\mathbb{D}_{ijkl} = \mathbb{D}_{jikl} = \mathbb{D}_{klij}$. Note that, like the elastic response, it is natural to assume that also the viscous response depends on the damage ζ and vanishes in the completely damaged. This substantially differs from previous studies [FeS03, HSS01, PPS07, SHS06] which con-



$t = 0.34$ $t = 0.64$ $t = 0.80$

Figure 6: Snapshots of the time evolution of the ζ field for the pre-notched specimens; $h = 0.03$ m, $\varepsilon = 10^{-2}$, displacements are scaled by a factor 100 and only a half of the specimen is displayed.

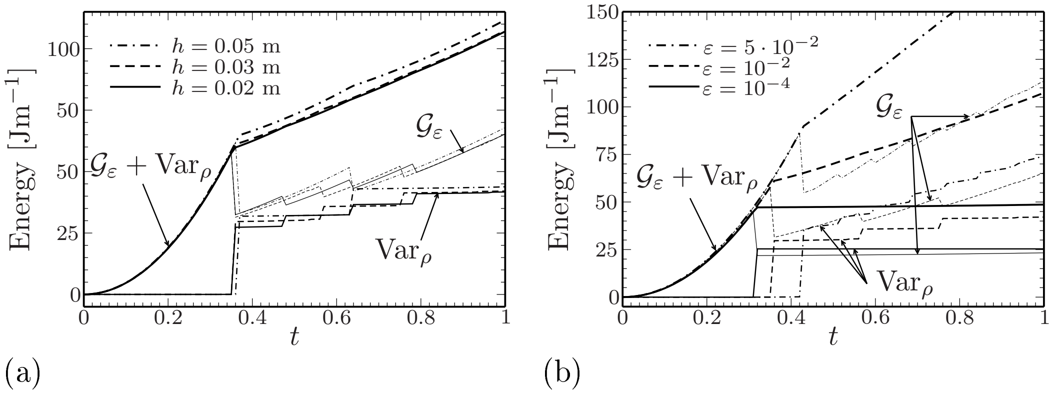


Figure 7: Convergence study for the pre-notched specimen; (a) $h \to 0$ m, $\varepsilon = 10^{-2}$, (b) $\varepsilon \to 0, h = 0.02$ m; mesh with $h = 0.05$ m contains 377 triangular elements, $h = 0.03$ m corresponds to $1,229$ elements and $h = 0.02$ m to $1,773$ elements.

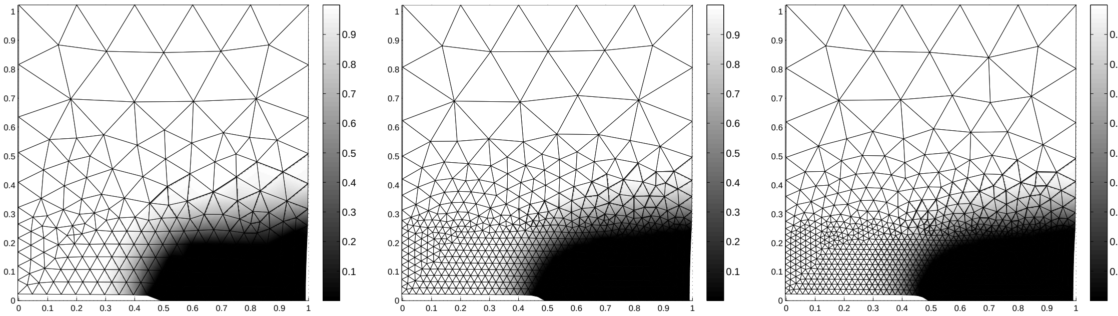
sidered viscosity unchanged even in damaged material. Like in [PPS07, SHS06], we also consider inertia related to the mass density ρ. Naturally, contrary to the visco-elastic response, the inertial effects are independent of damage because the mass is not destroyed by damaging inter-atomic links. Thus the rate-independent evolution of the damage is now coupled with rate-dependent evolution of the displacement. Due to the inertial effects, we can now impose dead loading by a bulk force f. For simplicity, we then do not consider any hard-device loading, i.e. we impose only the boundary conditions (7) with $\Gamma = \emptyset$. Altogether, formally, we consider

$$\rho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\big(\sigma_\mathrm{v} + \sigma_\mathrm{e}\big) = f, \qquad \sigma_\mathrm{v} = \zeta\mathbb{D}e\big(\frac{\partial u}{\partial t}\big), \qquad \sigma_\mathrm{e} = \zeta\mathbb{C}e(u), \tag{33a}$$

$$\partial\varrho\Big(\frac{\partial\zeta}{\partial t}\Big) + \sigma_\mathrm{i} + \sigma_\mathrm{r} \ni 0, \qquad \sigma_\mathrm{r} \in N_{[0,+\infty)}(\zeta)$$

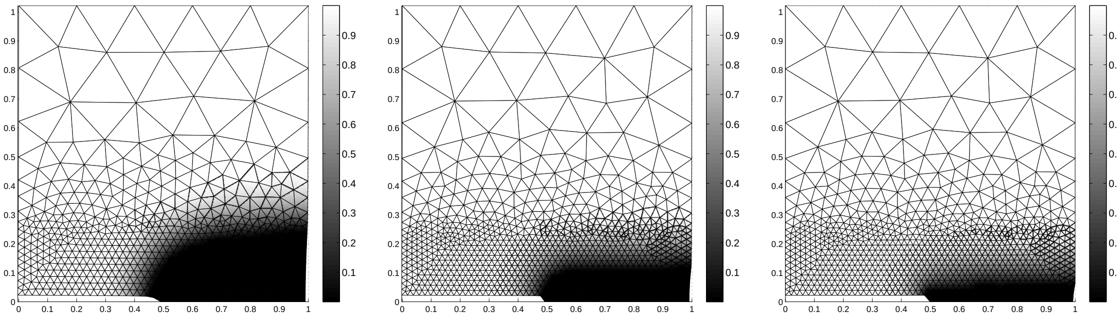
$$\sigma_\mathrm{i} := \frac{1}{2}\mathbb{C}e(u){:}e(u) - \mathrm{div}(\kappa|\nabla^k z|^{p-2}\nabla\zeta). \tag{33b}$$

Of course, now we must prescribe also the initial condition on the displacement and



$h = 0.05$ m $\qquad$ $h = 0.03$ m $\qquad$ $h = 0.02$ m

Figure 8: Examples of the ζ field distribution for $t = 1$ ($\varepsilon = 5 \cdot 10^{-2}$, $h \to 0$ m); displacements are scaled by a factor 100 and only a half of the specimen is displayed.



$\varepsilon = 5 \cdot 10^{-2}$ $\qquad$ $\varepsilon = 10^{-2}$ $\qquad$ $\varepsilon = 10^{-4}$

Figure 9: Examples of the ζ field distribution for $t = 1$ ($h = 0.02$ m, $\varepsilon \to 0$); displacements are scaled by a factor 100 and only a half of the specimen is displayed.

the velocity, so altogether we have

$$\begin{aligned}
&u(0,\cdot) = u_0 \in W^{1,2}(\Omega;\mathbb{R}^d), \qquad \frac{\partial u}{\partial t}(0,\cdot) = \dot{u}_0 \in L^2(\Omega;\mathbb{R}^d),\\
&\zeta(0,\cdot) = \zeta_0 \in W^{1,p}(\Omega).
\end{aligned} \tag{34}$$

We assume naturally $0 \le \zeta_0 \le 1$.

Similarly as before, let us take $\varepsilon > 0$ and consider the regularized problem:

$$\rho\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\Big((\zeta{+}\varepsilon)\mathbb{D}e\big(\frac{\partial u}{\partial t}\big) + (\zeta{+}\varepsilon)\mathbb{C}e(u)\Big) = f, \tag{35a}$$

$$\partial\varrho\Big(\frac{\partial \zeta}{\partial t}\Big) + \frac{1}{2}\mathbb{C}e(u) : e(u) - \mathrm{div}(\kappa|\nabla z|^{p-2}\nabla\zeta) + N_{[0,+\infty)}(\zeta) \ni 0. \tag{35b}$$

Its weak solution, let us denote it by $(u_\varepsilon, \zeta_\varepsilon)$, can be obtained by rather standard methods. The force equilibrium (35a) in the weak form looks as

$$\int_0^T \bigg(\Big\langle \rho\frac{\partial^2 u_\varepsilon}{\partial t^2}, v\Big\rangle + \int_\Omega (\zeta_\varepsilon{+}\varepsilon)\Big(\mathbb{D}e\big(\frac{\partial u_\varepsilon}{\partial t}\big) + \mathbb{C}e(u_\varepsilon)\Big) : e(v) - f\cdot v\,\mathrm{d}x\bigg)\,\mathrm{d}t = 0 \tag{36}$$

for all $v \in L^2(I;W^{1,2}(\Omega;\mathbb{R}^d))$ with $\langle\cdot,\cdot\rangle$ standing for the duality between $W^{1,2}(\Omega;\mathbb{R}^d)^*$ and $W^{1,2}(\Omega;\mathbb{R}^d)$. Like (14) and (13), we have now the "partial stability"

$$\int_\Omega \frac{\zeta_\varepsilon(t){+}\varepsilon}{2}\mathbb{C}e(u_\varepsilon(t)):e(u_\varepsilon(t))+\frac{\kappa}{p}|\nabla\zeta_\varepsilon(t)|^p\,\mathrm{d}x \\ \le \int_\Omega \frac{\tilde\zeta{+}\varepsilon}{2}\mathbb{C}e(u_\varepsilon(t)):e(u_\varepsilon(t))+\frac{\kappa}{p}|\nabla\tilde\zeta|^p+\varrho\big(\tilde\zeta{-}\zeta_\varepsilon(t)\big)\,\mathrm{d}x \quad \forall 0\le\tilde\zeta\in W^{1,p}(\Omega) \tag{37}$$

for any $t\in[0,T]$ with a from (2), and an energy inequality

$$\int_\Omega \frac{\rho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2+\mathcal{G}_\varepsilon(T,u_\varepsilon(T),\zeta_\varepsilon(T))\,\mathrm{d}x+\mathrm{Var}_\varrho(\zeta_\varepsilon;0,T) \\ +\int_Q(\zeta_\varepsilon{+}\varepsilon)\mathbb{D}e\big(\frac{\partial u_\varepsilon}{\partial t}\big):e\big(\frac{\partial u_\varepsilon}{\partial t}\big)\,\mathrm{d}x\,\mathrm{d}t \\ \le \int_\Omega\frac{\varrho}{2}|\dot u_0|^2+\mathcal{G}_\varepsilon(0,u_0,\zeta_0)+\int_Q f{\cdot}\frac{\partial u_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t; \tag{38}$$

here we used $\zeta_0=1$ from (34) and, for coming from (13) to (38), we relied on (36) for all $v:=\frac{\partial u_\varepsilon}{\partial t}\in L^2(I;W^{1,2}(\Omega;\mathbb{R}^d))$. Note that $e(u_\varepsilon(T))$ is well defined because $\frac{\partial u_\varepsilon}{\partial t}\in L^2(I;W^{1,2}(\Omega;\mathbb{R}^d))$ just due to the regularization by $\varepsilon>0$.

Now, as no minimization of stored energy applies, we unfortunately do not have at our disposal the formula like $\frac12\int_\Omega\sigma_\mathrm{e}:e(w)\,\mathrm{d}x$ for the stored energy, cf. (22). To avoid usage of $e(u)$ on the fully damaged parts, the stored energy $\int_\Omega\frac12\zeta\mathbb{C}e(u):e(u)\,\mathrm{d}x$ can alternatively be written as $\int_\Omega\frac12\chi_\mathrm{e}{:}\mathbb{C}^{-1}\chi_\mathrm{e}\,\mathrm{d}x$ where we have denoted $\chi_\mathrm{e}:=\sqrt{\zeta}\mathbb{C}e(u)$ and, as above, $\mathbb{C}^{-1}$ means the inversion of the mapping $\mathbb{C}:\mathbb{R}^{d\times d}_\mathrm{sym}\to\mathbb{R}^{d\times d}_\mathrm{sym}$. As in Remark 2.5, let us call χ_e an *elastic quasi-stress*; its physical dimension is again Pa=J/m^3 as a standard stress. Similarly, to avoid usage of $e(\frac{\partial u}{\partial t})$, we introduce the *viscous quasi-stress* $\chi_\mathrm{v}:=\sqrt{\zeta}\mathbb{D}e(\frac{\partial u}{\partial t})$.

Also, let us denote the corresponding quasi-stresses for (35), i.e.

$$\chi_{\mathrm{e},\varepsilon}=\sqrt{\zeta_\varepsilon{+}\varepsilon}\,\mathbb{C}e(u_\varepsilon)\quad\text{and}\quad\chi_{\mathrm{v},\varepsilon}=\sqrt{\zeta_\varepsilon{+}\varepsilon}\,\mathbb{D}e\big(\frac{\partial u_\varepsilon}{\partial t}\big). \tag{39}$$

Then, in terms of these quasi-stresses, (36) rewrites to

$$\int_0^T\bigg(\Big\langle\rho\frac{\partial^2u_\varepsilon}{\partial t^2},v\Big\rangle+\int_\Omega\sqrt{\zeta_\varepsilon{+}\varepsilon}\big(\chi_{\mathrm{v},\varepsilon}{:}\mathbb{D}^{-1}e(v)+\chi_{\mathrm{e},\varepsilon}{:}\mathbb{C}^{-1}e(v)\big)-f\cdot v\,\mathrm{d}x\bigg)\,\mathrm{d}t=0. \tag{40}$$

Moreover, (37) and (38) can be written as

$$\int_Q\frac12\chi_{\mathrm{e},\varepsilon}{:}\mathbb{C}^{-1}\chi_{\mathrm{e},\varepsilon}+\frac{\kappa}{p}|\nabla\zeta_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}t\le\int_Q\frac12\frac{\tilde\zeta{+}\varepsilon}{\zeta_\varepsilon{+}\varepsilon}\chi_{\mathrm{e},\varepsilon}{:}\mathbb{C}^{-1}\chi_{\mathrm{e},\varepsilon} \\ +\frac{\kappa}{p}|\nabla\tilde\zeta|^p-\varrho\big(\tilde\zeta{-}\zeta_\varepsilon\big)\,\mathrm{d}x\,\mathrm{d}t\qquad\forall 0\le\tilde\zeta\in W^{1,p}(\Omega) \tag{41}$$

to be satisfied for all $t \in I$ and

$$\int_\Omega \frac{\rho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}(T)\Big|^2 + \frac{1}{2}\chi_{\mathrm{e},\varepsilon}(T){:}\mathbb{C}^{-1}\chi_{\mathrm{e},\varepsilon}(T) + \frac{\kappa}{p}|\nabla\zeta_\varepsilon(T)|^p + \delta_{[0,+\infty)}(\zeta_\varepsilon(T))\,\mathrm{d}x$$
$$+ \mathrm{Var}_\varrho(\zeta_\varepsilon;0,T) + \int_Q \chi_{\mathrm{v},\varepsilon}{:}\mathbb{D}^{-1}\chi_{\mathrm{v},\varepsilon}\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_\Omega \frac{\varrho}{2}|\dot u_0|^2 + \frac{1{+}\varepsilon}{2}\mathbb{C}e(u_0){:}e(u_0) + \frac{\kappa}{p}|\nabla\zeta_0|^p\,\mathrm{d}x + \int_Q f\cdot\frac{\partial u_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t. \tag{42}$$

We derive a-priory estimates that are independent of $\varepsilon > 0$ by testing (35a) by $\frac{\partial u_\varepsilon}{\partial t}$. It is essential to use $\frac{\partial \zeta_\varepsilon}{\partial t} \le 0$ and symmetry and positive definiteness of $\mathbb{C}$ to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\Big((\zeta_\varepsilon{+}\varepsilon)\mathbb{C}e(u_\varepsilon):e(u_\varepsilon)\Big) = (\zeta_\varepsilon{+}\varepsilon)\mathbb{C}e(u_\varepsilon):e\big(\frac{\partial u_\varepsilon}{\partial t}\big) + \frac{1}{2}\frac{\partial\zeta_\varepsilon}{\partial t}\mathbb{C}e(u_\varepsilon):e(u_\varepsilon)$$
$$\le (\zeta_\varepsilon{+}\varepsilon)\mathbb{C}e(u_\varepsilon):e\big(\frac{\partial u_\varepsilon}{\partial t}\big). \tag{43}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\varrho}{2}\Big|\frac{\partial u_\varepsilon}{\partial t}\Big|^2 + \frac{\zeta_\varepsilon{+}\varepsilon}{2}\mathbb{C}e(u_\varepsilon):e(u_\varepsilon)\,\mathrm{d}x$$
$$+ \int_\Omega (\zeta_\varepsilon{+}\varepsilon)\mathbb{D}e\big(\frac{\partial u_\varepsilon}{\partial t}\big):e\big(\frac{\partial u_\varepsilon}{\partial t}\big)\,\mathrm{d}x \le \int_\Omega f\cdot\frac{\partial u_\varepsilon}{\partial t}\,\mathrm{d}x. \tag{44}$$

Assuming $f \in L^1(I;L^2(\Omega;\mathbb{R}^d))$, by Gronwall's inequality we obtain the bounds

$$\Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \le C, \tag{45a}$$
$$\big\|\sqrt{\zeta_\varepsilon{+}\varepsilon}\,\mathbb{C}e(u_\varepsilon)\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \le C_{\mathrm{e}}, \tag{45b}$$
$$\Big\|\sqrt{\zeta_\varepsilon{+}\varepsilon}\,\mathbb{D}e\big(\frac{\partial u_\varepsilon}{\partial t}\big)\Big\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \le C_{\mathrm{v}}. \tag{45c}$$
$$\big\|\zeta_\varepsilon\big\|_{\mathrm{BV}(\bar I;L^1(\Omega))\cap L^\infty(I;W^{1,p}(\Omega))} \le C, \tag{45d}$$

with some constants C, C_{e}, and C_{v}. In other words, $\|\chi_{\mathrm{v},\varepsilon}\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} \le C_{\mathrm{v}}$, and $\|\chi_{\mathrm{e},\varepsilon}\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \le C_{\mathrm{e}}$. From this, for $0 < \varepsilon \le 1$, we also obtain

$$\Big\|\frac{\partial^2 u_\varepsilon}{\partial t^2}\Big\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^d)^*)+L^1(I;L^2(\Omega;\mathbb{R}^d))}$$
$$= \sup_{\|v\|_{\mathbf{Y}}\le 1}\int_Q \sqrt{\zeta_\varepsilon{+}\varepsilon}\big(\chi_{\mathrm{v},\varepsilon}{:}\mathbb{D}^{-1}e(v) + \chi_{\mathrm{e},\varepsilon}{:}\mathbb{C}^{-1}e(v)\big) - f{\cdot}v\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \sup_{\|v\|_{\mathbf{Y}}\le 1} 2\int_Q \chi_{\mathrm{v},\varepsilon}{:}\mathbb{D}^{-1}e(v) + \chi_{\mathrm{e},\varepsilon}{:}\mathbb{C}^{-1}e(v) - f{\cdot}v\,\mathrm{d}x\,\mathrm{d}t$$
$$\le 2|\mathbb{D}^{-1}|C_{\mathrm{v}} + 2|\mathbb{C}^{-1}|C_{\mathrm{e}} + 2\|f\|_{L^1(I;L^2(\Omega;\mathbb{R}^d))}. \tag{46}$$

where $\|u\|_{\mathbf{Y}} = \|u\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^d))} + \|u\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))}$.

Unfortunately, it does not seem that any estimate for $\frac{\partial\chi_{\mathrm{e},\varepsilon}}{\partial t}$ is available, which brings troubles by defining values of $\chi_{\mathrm{e},\varepsilon}$ at particular times in the limit. In the spirit of Definitions 2.1 and 2.3 but balancing Helmholtz stored energy (since the by-part integration in time of the outer loading is no longer necessary and advantageous) and in view of the estimates (45), we can exploit the above relations (36), (39), (41), and (42) when putting $\varepsilon = 0$ for a definition of a weak/energetic solution to the complete-damage problem in the following way:

Definition 4.1 ***(Weak/energetic solution.)*** *We call* $(u, \chi_{\mathrm{e}}, \chi_{\mathrm{v}}, \zeta, \mathfrak{E})$ *with*

$$u \in W^{1,\infty}(I; L^2(\Omega; \mathbb{R}^d)), \tag{47a}$$
$$\chi_{\mathrm{e}} \in L^\infty(I; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{47b}$$
$$\chi_{\mathrm{v}} \in L^2(Q; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{47c}$$
$$\zeta \in \mathrm{BV}(\bar{I}; L^1(\Omega)) \cap \mathrm{B}(\bar{I}; W^{1,p}(\Omega)), \tag{47d}$$
$$\mathfrak{E} \in \mathrm{BV}(\bar{I}) \tag{47e}$$

such that

$$e\Big(\frac{\partial u}{\partial t}\Big) \in L^2_{\mathrm{loc}}\big(\{(t,x) \in Q;\ \zeta(t,x) > 0\}; \mathbb{R}^{d\times d}_{\mathrm{sym}}\big), \tag{48a}$$
$$\frac{\partial^2 u}{\partial t^2} \in L^2(I; W^{1,2}(\Omega; \mathbb{R}^d)^*) + L^1(I; L^2(\Omega; \mathbb{R}^d)) \tag{48b}$$

a weak/energetic solution to the problem (33) with the initial conditions (34) and the homogeneous Neumann boundary condition, i.e. (7) with $\Gamma = \emptyset$, *if*

$$\int_0^T \bigg(\Big\langle \rho\frac{\partial^2 u}{\partial t^2}, v\Big\rangle + \int_\Omega \sqrt{\zeta}\big(\chi_{\mathrm{v}}{:}\mathbb{D}^{-1}e(v) + \chi_{\mathrm{e}}{:}\mathbb{C}^{-1}e(v)\big) - f\cdot v\,\mathrm{d}x\bigg)\,\mathrm{d}t = 0 \tag{49}$$

for all $v \in L^2(I; W^{1,2}(\Omega; \mathbb{R}^d))$, *if the "partial stability"*

$$\int_A \frac{1}{2}\chi_{\mathrm{e}}{:}\mathbb{C}^{-1}\chi_{\mathrm{e}} + \frac{\kappa}{p}|\nabla\zeta|^p\,\mathrm{d}x\,\mathrm{d}t \le \int_A \frac{1}{2}\frac{\tilde\zeta}{\zeta}\chi_{\mathrm{e}}{:}\mathbb{C}^{-1}\chi_{\mathrm{e}} + \frac{\kappa}{p}|\nabla\tilde\zeta|^p + \varrho(\tilde\zeta - \zeta)\,\mathrm{d}x\,\mathrm{d}t \qquad \forall 0 \le \tilde\zeta \in L^p(I; W^{1,p}(\Omega)) \tag{50}$$

and

$$\chi_{\mathrm{e}} = \sqrt{\zeta}\,\mathbb{C}e(u) \text{ and } \chi_{\mathrm{v}} := \sqrt{\zeta}\,\mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big) \text{ on any open } A \subset Q \text{ on which } \zeta(t,x) > 0, \tag{51}$$

and if the energy inequality holds, i.e.

$$\mathfrak{E}(T) + \int_\Omega \frac{\rho}{2}\Big|\frac{\partial u}{\partial t}(T)\Big|^2 + \delta_{[0,+\infty)}(\zeta(T))\,\mathrm{d}x + \mathrm{Var}_\varrho(\zeta; 0, T) + \int_Q \chi_{\mathrm{v}}{:}\mathbb{D}^{-1}\chi_{\mathrm{v}}\,\mathrm{d}x\,\mathrm{d}t \le \mathfrak{E}(0) + \int_\Omega \frac{\varrho}{2}|\dot{u}_0|^2\,\mathrm{d}x + \int_Q f\cdot\frac{\partial u}{\partial t}\,\mathrm{d}x\,\mathrm{d}t. \tag{52}$$

with $\mathfrak{E}(0) = \int_\Omega \frac{1}{2}\mathbb{C}e(u_0) : e(u_0) + \frac{\kappa}{p}|\nabla\zeta_0|^p \,\mathrm{d}x$ *and, for all* $t_1 \in I$,

$$\int_0^{t_1} \mathfrak{E}(t)\,\mathrm{d}t \geq \int_0^{t_1}\int_\Omega \frac{1}{2}\chi_\mathrm{e}{:}\mathbb{C}^{-1}\chi_\mathrm{e} + \frac{\kappa}{p}|\nabla\zeta|^p \,\mathrm{d}x\,\mathrm{d}t. \tag{53}$$

Remark 4.2 Let us comment this definition especially at the point that we claim much less information on the completely damaged part than we did in the quasistatic evolution in Section 2, which is related with what we are able to prove. As a consequence, we also cannot prove full energy balance as an equality. Anyhow, the granted a-priory estimates (45) and (46) give certain solid base for engineering calculations and Definition 4.1 then indicates what information we can surely read for the limit when ε approaches zero. In fact, we have bounds also on some other derived quantities, e.g. $(\zeta_\varepsilon{+}\varepsilon)\frac{\partial}{\partial t}(\mathbb{C}e(u_\varepsilon){:}e(u_\varepsilon))$ which equals to $\chi_{\mathrm{e},\varepsilon}{:}\mathbb{D}^{-1}\chi_{\mathrm{v},\varepsilon}$ which is bounded due to (45b,c) in $L^2(I;L^1(\Omega))$.

Proposition 4.3 *Let* $p > d$ *and* $f \in L^1(I;L^2(\Omega;\mathbb{R}^d))$, $u_0 \in W^{1,2}(\Omega;\mathbb{R}^d)$, $\dot u_0 \in L^2(\Omega;\mathbb{R}^d)$, *and* $\zeta_0 \in W^{1,p}(\Omega)$, $0 \leq \zeta_0 \leq 1$. *Then there exists a weak/energetic solution in accord to Definition 4.1.*

Proof. By (45b,c), we can choose a subsequence such that $\chi_{\mathrm{e},\varepsilon} \overset{*}{\rightharpoonup} \chi_\mathrm{e}$ in $L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ and $\chi_{\mathrm{v},\varepsilon} \rightharpoonup \chi_\mathrm{v}$ in $L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Though the obtained χ_e need not be well defined at particular time levels, the stored energy $\mathfrak{E}_\varepsilon : t \mapsto \int_\Omega \frac{1}{2}\chi_\mathrm{e}(t){:}\mathbb{C}^{-1}\chi_\mathrm{e}(t)\,\mathrm{d}x$ itself is well defined and measurable because its sum with the kinetic energy has a bounded variation which is seen from (44) and (45c). By Helly's principle, we choose a subsequence so that also $\mathfrak{E}_\varepsilon(t) \to \mathfrak{E}(t)$ for all $t \in [0,T]$.

The limit passage in (40) uses $\zeta_\varepsilon \to \zeta$ in $L^q(Q)$ with any $1 \leq q < +\infty$, which follows by a generalized Aubin-Lions' theorem [Rou05, Cor.7.9] from the estimate ζ_ε in $L^\infty(I;W^{1,p}(\Omega)) \cap \mathrm{BV}(\bar I;L^1(\Omega))$, and also it uses $\chi_{\mathrm{e},\varepsilon} \overset{*}{\rightharpoonup} \chi_\mathrm{e}$ in $L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ and $\chi_{\mathrm{v},\varepsilon} \rightharpoonup \chi_\mathrm{v}$ in $L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$.

The limit passage in (39) uses also the bounds of $e(u_\varepsilon)$ and $e(\frac{\partial u_\varepsilon}{\partial t})$ in $L^2(K;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ on any compact cylinder K of the form $[0,t]{\times}K_0$ on which $\zeta > 0$. Here we use a very special structure of the problem that $K_0 \subset \Omega$ such that $\zeta(t) > 0$ on K_0 implies that, for any $\delta > 0$, there is ε_0 such that for any $0 < \varepsilon \leq \varepsilon_0$ we have $\zeta_\varepsilon(t) + \varepsilon \geq \delta$ for all $x \in K_0$; here we used that $W^{1,p}(\Omega)$ is embedded into $C(\bar\Omega)$ because $p > d$. Thus also $\zeta_\varepsilon + \varepsilon \geq \delta$ for all $(t,x) \in K = [0,t]{\times}K_0$ because $\zeta_\varepsilon(\cdot,x)$ is nonincreasing. Then we can pass to the limit in (39) and cover A in (51) by cylinders of the form K above.

The limit passage in the "partial" stability condition (41) in the term

$$\int_Q \frac{1}{2}\frac{\tilde\zeta{+}\varepsilon}{\zeta_\varepsilon{+}\varepsilon}\chi_{\mathrm{e},\varepsilon}{:}\mathbb{C}^{-1}\chi_{\mathrm{e},\varepsilon}\,\mathrm{d}x\,\mathrm{d}t = \int_Q \frac{1}{2}(\tilde\zeta{+}\varepsilon)\mathbb{C}e(u_\varepsilon) : e(u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t$$

is more difficult than in the usual "full" stability (16) in the rate-independent case. We must do it simultaneously with the left-hand-side term

$$\int_Q \frac{1}{2}(\zeta_\varepsilon{+}\varepsilon)\mathbb{C}e(u_\varepsilon) : e(u_\varepsilon)\,\mathrm{d}x.$$

Let us take $0 \le \tilde\zeta \le \zeta$ and, following [BMR07, Proposition 2.10], put $\tilde\zeta_\delta := (\tilde\zeta - \delta)^+$. Then, for any fixed $\delta > 0$, we have $\tilde\zeta_\delta(t) \le \zeta_\varepsilon(t)$ if $\varepsilon > 0$ is small enough (depending on t, however); recall that $p > d$ so that $W^{1,p}(\Omega) \subset C(\bar\Omega)$ compactly. Simultaneously $\tilde\zeta_\delta(t) \to \tilde\zeta(t)$ in $W^{1,p}(\bar\Omega)$. Indeed, let us consider an open ϵ-neighbourhood $\mathcal{O}_\epsilon(t)$ of a compact set $N(t) := \{x{\in}\bar\Omega;\ \tilde\zeta(t,x) = 0\}$. Then, for $\delta > 0$ small enough, $\tilde\zeta_\delta > 0$ on $\bar\Omega\backslash\mathcal{O}_\epsilon(t)$. For a.a. $x \in \mathcal{O}_\epsilon(t)\backslash N(t)$, we have either $\tilde\zeta_\delta(x) = 0$ or $\tilde\zeta_\delta(t,x) = \tilde\zeta(t,x) - \delta$ and also $\nabla\tilde\zeta_\delta(t,x) = 0$ or $\nabla\tilde\zeta_\delta(t,x) = \tilde\zeta(t,x)$. Hence, for $\delta > 0$ small enough,

$$\begin{aligned}\int_\Omega \big|\nabla\tilde\zeta_\delta(t) - \nabla\tilde\zeta(t)\big|^p \,\mathrm{d}x &= \int_{\mathcal{O}_\epsilon(t)\backslash N(t)} \big|\nabla\tilde\zeta_\delta(t) - \nabla\tilde\zeta(t)\big|^p \,\mathrm{d}x \\ &\le \int_{\mathcal{O}_\epsilon(t)\backslash N(t)} \big|\nabla\tilde\zeta(t)\big|^p \,\mathrm{d}x. \end{aligned}\tag{54}$$

Yet, the last expression can be pushed to zero with $\epsilon \to 0$ because $|\nabla\tilde\zeta(t)|^p \in L^1(\Omega)$ is absolutely continuous for a.a. $t \in [0,T]$. Then also $\int_0^T\int_\Omega \big|\nabla\tilde\zeta_\delta(t) - \nabla\tilde\zeta(t)\big|^p \,\mathrm{d}x\,\mathrm{d}t \to 0$ by the Lebesgue dominated-convergence theorem; the common integrable majorant is $t \mapsto \|\nabla\tilde\zeta(t)\|^p_{L^p(\Omega;\mathbb{R}^d)}$.

Then, by the "partial" stability for ζ_ε, we have

$$\begin{aligned}&\int_Q \varrho(\zeta_\varepsilon - \tilde\zeta_\delta)\,\mathrm{d}x\,\mathrm{d}t \\ &\ge \int_Q \Big(\frac{\zeta_\varepsilon{+}\varepsilon}{2} - \frac{\tilde\zeta_\delta{+}\varepsilon}{2}\Big)\mathbb{C}e(u_\varepsilon) : e(u_\varepsilon) + \frac{\kappa}{p}|\nabla\zeta_\varepsilon|^p - \frac{\kappa}{p}|\nabla\tilde\zeta_\delta|^p \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q \frac{1}{2}\Big(1 - \frac{\tilde\zeta_\delta{+}\varepsilon}{\zeta_\varepsilon{+}\varepsilon}\Big)\chi_{\mathrm{e},\varepsilon} : \mathbb{C}^{-1}\chi_{\mathrm{e},\varepsilon} + \frac{\kappa}{p}|\nabla\zeta_\varepsilon|^p - \frac{\kappa}{p}|\nabla\tilde\zeta_\delta|^p \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{55}$$

Now we use that $(\tilde\zeta_\delta{+}\varepsilon)/(\zeta_\varepsilon{+}\varepsilon) = \tilde\zeta_\delta/\zeta$ converges strongly in any $L^q(K)$, $q < +\infty$, and weakly* in $L^\infty(K)$ on every compact cylinder K of the form $[0,t] \times K_0$ where $\zeta > 0$, as already used above. Then, by the weak lower semicontinuity, we obtain

$$\int_K \varrho(\zeta - \tilde\zeta_\delta)\,\mathrm{d}x\,\mathrm{d}t \ge \int_K \frac{1}{2}\Big(1 - \frac{\tilde\zeta_\delta}{\zeta}\Big)\chi_{\mathrm{e}} : \mathbb{C}^{-1}\chi_{\mathrm{e}} + \frac{\kappa}{p}|\nabla\zeta|^p - \frac{\kappa}{p}|\nabla\tilde\zeta_\delta|^p \,\mathrm{d}x\,\mathrm{d}t. \tag{56}$$

Then we pass $\delta \to 0$ and use $\tilde\zeta_\delta \to \tilde\zeta$ weakly* in $L^\infty(Q)$ because we proved already strong convergence in $L^p(I;W^{1,p}(\Omega))$ and bounds in $L^\infty(Q)$. When covering A involved in (50) by cylinders of the form K, we obtain just (50).

Limit passage in (42) is then by weak lower-semicontinuity. Here we use also that that $\mathfrak{E}_\varepsilon(t) \to \mathfrak{E}(t)$ and the weak lower semicontinuity, hence we get also (53). $\square$

References

[BBT01] E. Benvenuti, G. Borino, A. Tralli: A thermodynamically consistent non-local formulation for damaging materials, *Euro. J. Mech. A/Solids* **21** (2001), 535-553.

[BiS96] Z. Bittnar and J. Šejnoha, *Numerical methods in structural mechanics*, ASCE Press and Thomas Telford, New York and London, 1996.

[BoS04] E. Bonnetti, G. Schimperna: Local existence for Frémond model of damage in elastic materials. *Cont. Mech. Thermodyn.* **16** (2004), 319–335.

[BMR07] G. Bouchitté, A. Mielke, T. Roubíček: A complete-damage problem at small strains. *Zeit. für angew. Math. Phys.*, in print. http://dx.doi.org/10.1007/s00033-007-7064-0.

[Bou07] B. Bourdin: Numerical implementation of the variational formulation of brittle fracture, *Interface Free Bound.* **9** (2007), 411–430.

[Bou] B. Bourdin: The variational formulation of brittle fracture: numerical implementation and extensions. In: *IUTAM Symp. Discretization Meth. for Evolving Discontinuities.* (T. Belytschko, A. Combescure, R. de Borst, eds.), Springer, 2007. (to appear)

[BFM00] B. Bourdin, G.A. Francfort, J.-J. Marigo: Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids* **48** (2000), 797-826.

[Bra07] D. Braess: *Finite elements: Theory, fast solvers, and applications in solid mechanics*, third ed., Cambridge University Press, Cambridge, 2007.

[CFKSV06] M. Campoa, J.R. Fernándeza, K.L. Kuttler, M. Shillor, J.M. Viaño: Numerical analysis and simulations of a dynamic frictionless contact problem with damage. *Comput. Meth. Appl. Mech. Eng.* **196** (2006), 476–488.

[CoL96] T.F. Coleman and Y. Li: A reflective Newton method for minimizing a quadratic function subject to bounds on some of the variables. *SIAM J. Optim.* **6** (1996), 1040–1058.

[Dal95] G. Dal Maso: *An Introduction to* Γ*-convergence.* Birkhäuser, 1995.

[DBH96] R. De Borst, A. Benallal, O.M. Heeres: A gradient-enhanced damage approach to fracture. *J. de Physique IV* **6** (1996) C6, 491-509.

[DMT01] A. DeSimone, J.-J. Marigo, L. Teresi: A damage mechanism approach to stress softening and its application to rubber. *Eur. J. Mech. A/Solids* **20** (2001), 873-892.

[DPO94] E.A. De Souza Neto, D. Peric, D.R.J. Owen: A phenomenological three-dimensional rate-independent continuum damage model for highly filled polymers: Formulation and computational aspects. *J. Mech. Phys. Solids* **42** (1994), 1533-1550.

[FeS03] J.R. Fernández, M. Sofonea: Variational and numerical analysis of the Signorini's conctact problem in viscoplasticity with damage. *J. Appl. Math.* **2** (2003), 87–114.

[FrG06] G.A. Francfort, A. Garroni: A variational view of partial brittle damage evolution. *Arch. Rat. Mech. Anal.* **182** (2006), 125–152.

[Fre02] M. Frémond: *Non-smooth thermomechanics.* Springer, Berlin, 2002.

[FKNS98] M. Frémond, K.L. Kuttler, B. Nedjar, M. Shillor: One-dimensional models of damage. *Adv. Math. Sci. Appl.* **8** (1998), 541-570.

[FKS99] M. Frémond, K.L. Kuttler, M. Shillor: Existence and uniqueness of solutions for a dynamic one-dimensional damage model. *J. Math. Anal. Appl.* **229** (1999), 271-294.

[FrN95] M. Frémond, B. Nedjar: Damage in concrete: the unilateral phenomenon. *Nucl. Eng. Des.* **156** (1995), 323–335.

[FrN96] M. Frémond, B. Nedjar: Damage, gradient of damage and principle of virtual power. *Int. J. Solid Struct.* **33** (1996), 1083–1103.

[FoT03] R. Fosdick, L. Truskinovsky: About Clapeyron's theorem in linear elasticity. *J. Elast.* **72** (2003), 145–172.

[GoS91] S. Govindjee, J.C. Simo: A micro-mechanically based continuum damage model for carbon black-filled rubbers incorporating Mullins' effect. *J. Mech. Phys. Solids* **39** (1991), 87–112.

[HSS01] W. Han, M. Shillor, M. Sofonea: Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage. *J. Comput. Appl. Math.* **137** (2001), 377–398.

[HJK00] J. Haslinger, D. Jedelský, T. Kozubek, J. Tvrdík: Genetic and random search methods in optimal shape design problems, *J. Glob. Optim.* **16** (2000), 109–131.

[IKLK04] A. Ibrahimbegović, C. Knopf-Lenoir, A. Kučerová, P. Villon: Optimal design and optimal control of structures undergoing finite rotations and elastic deformations, *Int. J. Numer. Methods Eng.* **61** (2004), 2428–2460.

[KMR06] M. Kočvara, A. Mielke, T. Roubíček: A rate-independent approach to the delamination problem. *Math. Mech. Solids* **11** (2006), 423-447.

[LoA99] E. Lorentz, S. Andrieux: A variational formulation for nonlocal damage models. *Int. J. of Plasticity* **15** (1999), 119-138

[MLZS00] K. Matouš, M. Lepš, J. Zeman, M. Šejnoha: Applying genetic algorithms to selected topics commonly encountered in engineering practice, *Comput. Meth. Appl. Mech. Eng.* **190** (2000), 1629–1650.

[Mau92] G.A. Maugin: *The thermodynamics of plasticity and fracture.* Cambridge Univ. Press, Cambridge, 1992.

[Mie95] C. Miehe: Discontinuous and continuous damage evolution in Ogden-type large-strain elastic materials. *Eur. J. Mech., A* **14** (1995), 697-720.

[MiK00] C. Miehe, J. Keck: Superimposed finite elastic-viscoelastic-plastoelastic stress response with damage in filled rubbery polymers. Experiments, modelling and algorithmic implementation. *J. Mech. Phys. Solids* **48** (2000), 323-365.

[Mie05] A. Mielke: Evolution of rate-independent systems. In: *Handbook of Differential Equations: Evolutionary Diff. Eqs.* (Eds. C.Dafermos, E.Feireisl), pp. 461–559, Elsevier, Amsterdam, 2005.

[MiR06] A. Mielke, T. Roubíček: Rate-independent damage processes in nonlinear elasticity. *Math. Models and Meth. in Appl. Sci.* **16** (2006), 177-209.

[MiRo] A. Mielke, T. Roubíček: Numerical approaches to rate-independent processes and applications in inelasticity. (Preprint No.1169, WIAS, Berlin, 2006.) *Math. Modelling Numer. Anal.*, submitted.

[MiT99] A. Mielke, F. Theil: A mathematical model for rate-independent phase transformations with hysteresis. In: *Models of continuum mechanics in analysis and engineering.* (Eds.: H.-D.Alber, R.Balean, R.Farwig), Shaker Ver., Aachen, 1999, pp.117-129.

[MiT04] Mielke, A., Theil, F.: On rate-independent hysteresis models. *Nonlin. Diff. Eq. Appl.* **11** (2004), 151–189.

[MTL02] A. Mielke, F. Theil, V.I. Levitas: A variational formulation of rate-independent phase transformations using an extremum principle. *Archive Rat. Mech. Anal.* **162** (2002), 137–177.

[Ort87] M. Ortiz: An analytical study of the localized failure modes of concrete. *Mech. Mater.* **6** (1987), 159–174.

[PPS07] C. Padovani, G. Pasquinelli, M. Šilhavý, Processes in masonry bodies and the dynamical significance of collapse. *Math. Mech. Solids* (2007), in print.

[PMG04] R. H. J. Peerlings, T. J. Massart, M. G. D. Geers: A thermodynamically motivated implicit gradient damage framework and its application to brick masonry cracking. *Comput. Meth. Appl. Mech. Eng.* **193** (2004), 3403–3417.

[SAS04] A. Simone, H. Askes, L. J. Sluys: Incorrect initiation and propagation of failure in non-local and gradient-enhanced media, *Int. J. Solid Struct.* **41** (2004), 351–363.

[Rou05] T. Roubíček: *Nonlinear partial differential equations with applications.* Birkhäuser, Basel, 2005.

[Ryp98] D. Rypl: *Sequential and parallel generation of unstructured 3D meshes*, Ph.D. thesis, CTU in Prague, 1998, http://mech.fsv.cvut.cz/ dr/t3d.html.

[SHS06] M. Sofonea, W. Han, M. Shillor: *Analysis and Approximation of Contact Problems and Adhesion or Damage.* Chapman & Hall, New York, 2006.

[StH03] H. Stumpf, K. Hackl: Micromechanical concept for the analysis of damage evolution in thermo-viscoelastic and quasi-bittle materials. *Int. J. Solids Structures* **40** (2003), 1567–1584.