Published in the Proceedings of the IUTAM/ISIMM Symposium on Structure and Dynamics of Nonlinear Waves, A. Mielke & K. Kirchgässner (eds), World Scientific 1995, pp 353-362.

Homoclinic and Heteroclinic Solutions in Two–Phase Flow

Alexander Mielke
Institut für Angewandte Mathematik, Universität Hannover
mielke@ifam.uni–hannover.de

1. Introduction

We consider travelling waves in a system of two fluid layers of infinite extent which are placed between the rigid bottom an top of a channel under the action of gravity. Both fluids are assumed to be irrotatonal, inviscid, and of constant density $\rho_1 \neq \rho_2$. Our aim is to give a rigorous approach to the existence of solitary waves and of bores. Both of these solutions types look at infinity like parallel flows, however the first attains the same limits upstreams an downstreams whereas the second is a front–like solution connecting parallel flows of different heights. These solutions are also called bores and are observed in some rivers when waves coming from the ocean travel upstreams.

There are several approaches to this problem. In the case of no surface tension Turner[14] obtained the two–fluid model by considering stratified fluids with smooth density profiles converging to a piecewise constant one. The stratified fluid model (Long–Yih equation) is a semilinear elliptic problem[9] and can be treated using variational methods, and thus global results can be derived[1].

Surface wave problems and interface problems are described by quasilinear equations giving rise to more delicate phenomena like surface singularities as for the Stokes wave of extreme height. We use the spatial center manifold approach in the form of[12], which is based on the original ideas in [9]. Thus, we are restricted to a local theory, however, all difficulties arising from the quasilinearity are circumvented by this approach.

Depending on the densities ρ_i, upstream velocities $u_{i\infty}$ and the heights h_i of the fluids we define the dimensionless elevation number $E = h_2^2/h_1^2 - \rho_2 u_{2\infty}^2/(\rho_1 u_{1\infty}^2)$, which tells us whether bifurcating solitary waves are waves of elevation ($E > 0$) or waves of depression ($E < 0$). Here we analyze the unfolding of the case $E \approx 0$ which leads to a scenario where the growing branch of solitary waves obtains waves with larger and larger plateaus, see[15] for physical observations of this effect. Suitable translates of these plateau–like solitary waves converge on compact intervals to heteroclinic solutions (bores).

Here we have restricted ourselves to the case of zero surface tension, however the method applies equally well in cases with surface tension[2, 11]. Also the influence of localized perturbations travelling with the same frame speed can be analyzed, see[12].

2. The basic equations

At the inflow ($x \to -\infty$) the fluid layers have height h_1 and h_2 and inflow velocities $u_{1\infty}$ and $u_{2\infty}$, respectively. The interface between the layers is given by the function $y = Y(x)$. Taking $h_1 + h_2, \rho_1, u_{i\infty}$, and $\rho_1 u_{1\infty}^2$ as reference quantities for length, density, velocities (u_i, v_i) and the pressure, we obtain the following equations in dimensionless form:

$$\begin{aligned}
(x,y) \in S_i : \quad & \left\{ \begin{array}{l} u_{ix} + v_{iy} = 0, \\ u_{iy} - v_{iy} = 0, \end{array} \right\} \qquad \text{for } i = 1,2; \\
y = 0 : \quad & v_1 = 0; \quad y = 1 : \ v_2 = 0; \\
y = Y(x) : \quad & \left\{ \begin{array}{ll} \frac{v_1}{u_1} = Y', & \frac{1}{2}(u_1^2 + v_1^2) + \lambda Y + p = C_1 = \text{const}, \\ \frac{v_2}{u_2} = Y', & \frac{\rho}{2}(u_1^2 + v_1^2) + \frac{\rho_2}{\rho_1}\lambda Y + p = C_2 = \text{const}. \end{array} \right.
\end{aligned} \tag{1}$$

The fluid layers occupy the regions S_1 and S_2 given by $0 < y < Y(x)$ and $Y(x) < y < 1$, respectively. The first two equations are mass conservation and irrotationality. On the interface $y = Y$ we have the kinematic constraint and Bernoulli's law for both fluids (the interface is a streamline). From the inflow conditions $(u_i, v_i) \to (1, 0)$ and $Y \to h$ for $x \to -\infty$ we find the constants $C_1 = 1/2 + \lambda h$ and $C_2 = \rho/2 + \rho_2 \lambda h/\rho_1$. The coupling between the layers occurs through the pressure p which can be easily eliminated. Here and further on we use the non–dimensional parameters

$$\lambda = \tfrac{g(h_1+h_2)}{u_{1\infty}^2}, \ \rho = \tfrac{\rho_2 u_{2\infty}^2}{\rho_1 u_{1\infty}^2}, \ h = \tfrac{h_1}{h_1+h_2}, \ \mu = \tfrac{\rho_1 - \rho_2}{\rho_1}\lambda.$$

Often one is interested in waves travelling through fluid layers in rest at infinity. Then, in the moving frame we have $u_{1\infty} = u_{2\infty}$. In any case we have $(u_i, v_i) \to (1, 0)$ for $x \to -\infty$.

We want to transform the system such that it can be written as an abstract differential equation in the form

$$\frac{d}{dx}\varphi = L_\mu \varphi + N(\varphi), \quad \varphi \in X, \tag{2}$$

where $N(\varphi) = \mathcal{O}(\|\varphi\|^2)$. Therefore we introduce the stream function ψ through $(u_i, v_i) = (\psi_y, -\psi_x)$ and $\psi(x, 0) = 0$, $\psi(x, y) \to y$ for $x \to \infty$. The stream function ψ is continuous but not differentiable across the interface, where $\psi(x, Y(x)) = h$. Following[12] we transform the velocities according to $U_i(x, \psi(x, y)) = (u_i^2(x,y)^2 + v_i^2(x,y)^2 - 1)/2$ and $V_i(x, \psi(x,y)) = v_i(x,y)/u_i(x,y)$. Using $u_i = R_i(U_i, V_i) = \sqrt{(1 + 2U_i)/(1 + V_i^2)}$ we find

$$\begin{aligned}
(x,\psi) \in \widetilde{S}_i : \quad & \frac{\partial}{\partial x}\begin{pmatrix} U_i \\ V_i \end{pmatrix} = \begin{pmatrix} V_i R_i & -R_i^3 \\ 1/R_i & V_i R_i \end{pmatrix} \frac{\partial}{\partial \psi}\begin{pmatrix} u_i \\ v_i \end{pmatrix}, \\
\psi = 0 : \quad & V_1 = 0, \quad \psi = 1 : \ V_2 = 0, \\
\psi = h : \quad & V_1 = V_2 = Y', \ U_1 - \rho U_2 + \mu[Y - h] = 0,
\end{aligned}$$

where $\widetilde{S}_1 = I\!R \times (0,h)$ and $\widetilde{S}_2 = I\!R \times (h,1)$. Additionally, we have the relations

$$Y = \int_0^h \tfrac{1}{R_1}\, d\psi = 1 - \int_h^1 \tfrac{1}{R_2}\, d\psi \tag{3}$$

which are a consequence of $Y(x) = \int_0^{Y(x)} dy = \int_0^{\psi(x,Y(x))} \frac{1}{U_1} d\psi = \int_0^h \frac{1}{R_1} d\psi$ and the analoguous consideration for $y \in [Y(x), 1]$.

Again following[12] we introduce the variable $B = U_1(h) - \rho U_2(h)$, and the interfacial conditon reads

$$B + \mu(Y - h) = 0. \tag{4}$$

According to (3) this is a nonlinear condition, since Y has to be expressed through (U, V). We differentiate (4) and use $Y' = V_1(h)$ in order to obtain $B' = dB/dx = -\mu V_1(h)$. With $\varphi = (U_1, V_1, U_2, V_2, B)^T$ the problem takes the form (2) where the basic phase space is $X = L^2(0,h)^2 \times L^2(h,1)^2 \times I\!R$,

$$\begin{aligned} D(L_\mu) = \{\varphi \in H^1(0,h)^2 \times H^1(h,1)^2 \times I\!R : V_1(0) = V_2(1) = 0, V_1(h) = V_2(h),\\ B = U_1(h) - \rho U_2(h)\}, \end{aligned}$$

$$L_\mu \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ B \end{pmatrix} = \begin{pmatrix} -\partial_\psi V_1 \\ \partial_\psi U_1 \\ -\partial_\psi V_2 \\ \partial_\psi U_2 \\ -\mu V_1(h) \end{pmatrix}, \quad \text{and } N(\varphi) = \begin{pmatrix} V_1 R_1 \partial_\psi U_1 - (R_1^3 - 1)\partial_\psi V_1 \\ V_1 R_1 \partial_\psi V_1 - (1 - 1/R_1)\partial_\psi U_1 \\ V_2 R_2 \partial_\psi U_2 - (R_2^3 - 1)\partial_\psi V_2 \\ V_2 R_2 \partial_\psi V_2 - (1 - 1/R_2)\partial_\psi U_2 \\ 0 \end{pmatrix}.$$

Here N is a smooth (analytic) mapping from $D(L_\mu)$ into X, which vanishes quadratically for $\varphi \to 0$. For later use we derive the following spectral properties of L_μ in dependence of $\mu > 0$.

Theorem 2..1 *(a) The spectrum of L_μ consists of discrete eigenvalues. They are exactly the solutions of the dispersion relation*

$$F_\mu(\sigma) = [\mu - \sigma \cot(\sigma h) - \rho\sigma \cot(\sigma(1-h))]\sigma^2.$$

(b) For all μ the operator L_μ has a two–fold eigenvalue 0. For $\mu < \mu_0 := 1/h + \rho/(1-h)$, there are no further eigenvalues on the imaginary axis. For $\mu \ge \mu_0$ there is a pair of purely imaginary eigenvalues $\pm i\omega(\mu)$ with $\omega(\mu_0) = 0$, $\frac{d\omega}{d\mu} > 0$, and $\omega(\mu)/\mu \to 1/(1+\rho)$ for $\mu \to \infty$.
(c) For all μ the estimate $\|(L_\mu + is)^{-1}\|_{X\to X} = \mathcal{O}(1/|s|)$, $s \in I\!R$, holds.

PROOF: The eigenvalue problem reduces to an ordinary differential equation. The homogeneous problem $\sigma\varphi = L_\mu\varphi$ gives $\sigma U_i = -\partial_\psi V_i$ and $\sigma V_i = \partial_\psi U_i$. With $V_1(0) = V_2(1) = 0$ and $V_1(h) = V_2(h)$ this leads to

$$(U_1, V_1, U_2, V_2) = c_0 \left(-\tfrac{\cos\sigma\psi}{\sin\sigma h}, \tfrac{\sin\sigma\psi}{\sin\sigma h}, \tfrac{\cos\sigma(1-\psi)}{\sin\sigma(1-h)}, \tfrac{\sin\sigma(1-\psi)}{\sin\sigma(1-h)} \right).$$

From $-\mu V_1(h) = \sigma B = \sigma[U_1(h) - \rho U_2(h)]$, we find that c_0 has to be 0 unless $F_\mu(\sigma) = 0$, and part (a) is proved. Part (b) is a simple discussion of the zeros of F_μ.

To establish the resolvent estimate, we consider a general $\eta = (f_1, g_1, f_2, g_2, \alpha) \in X$ and $s \in I\!R$. If $F_\mu(is) \neq 0$ the resolvent equation $(L_\mu + is)\varphi = \eta$ is solvable. It reads

$$\begin{aligned}
&-\partial_\psi V_j + isU_j = f_j, \partial_\psi U_j + isV_j = g_j, \quad j = 1, 2; \\
&-\mu V_1(h) + isB = \alpha,\ V_1(0) = V_2(1) = 0,\ V_1(h) = V_2(2),\ B = U_1(h) - \rho U_2(h).
\end{aligned}$$

Using simple integrations by part we find

$$\begin{aligned}
&\textstyle\int_0^h (|f_1|^2 + |g_1|^2) d\psi = \int_0^h (|\partial_\psi U_1|^2 + |\partial_\psi V_1|^2 + s^2 |U_1|^2 + s^2 |V_1|^2) d\psi + 2is \operatorname{Im}(V_1(h)\overline{U_1(h)}\,), \\
&\textstyle\int_h^1 (|f_2|^2 + |g_2|^2) d\psi = \int_h^1 (|\partial_\psi U_2|^2 + |\partial_\psi V_2|^2 + s^2 |U_2|^2 + s^2 |V_2|^2) d\psi - 2is \operatorname{Im}(V_2(h)\overline{U_2(h)}\,).
\end{aligned}$$

Using $V_1(h) = V_2(h)$ and $B = U_1(h) - \rho U_2(h)$ leads to

$$\begin{aligned}
\|(f_1, g_1, \sqrt{\rho}\, f_2, \sqrt{\rho} g_2)\|^2 &= \|\partial_\psi (U_1, V_1, \sqrt{\rho}\, U_2, \sqrt{\rho}\, V_2)\|^2 \\
&\quad + s^2 \|(U_1, V_1, \sqrt{\rho}\, U_2, \sqrt{\rho}\, V_2)\|^2 + 2is \operatorname{Im}(V_1(h)\overline{B}\,).
\end{aligned} \tag{5}$$

Moreover, we have $|sB| = |\alpha + \mu V_1(h)| \leq |\alpha| + \mu |V_1(h)|$ and $|V_1(h)|^2 \leq \delta \|\partial_\psi V_1\|^2 + \|V_1\|^2/\delta$ for any $\delta > 0$. This allows the estimate

$$s^2|B|^2 - 2is\operatorname{Im}(V_1(h)\overline{B}) \leq 2s^2|B|^2 + |V_1(h)|^2 \leq 4\alpha^2 + (5\mu^2+1)\delta\|\partial_\psi V_1\|^2 + (5\mu^2+1)\|V_1\|^2/\delta.$$

Choosing $\delta = 1/(5\mu^2 + 1)$ and inserting the result into (5) gives

$$\begin{aligned}
\min\{1, \rho\}(s^2 - (5\mu^2+1)^2)\|\varphi\|^2 &\leq (s^2 - (5\mu+1)^2) \left[\|(U_1, V_1, \sqrt{\rho}\, U_2 \sqrt{\rho}\, V_2)\|^2 + |B|^2\right] \\
&\leq \|(f_1, g_1, \sqrt{\rho}\, f_2, \sqrt{\rho}\, g_2)\|^2 + 4\alpha^2 \leq \max\{4, \rho\}\|\eta\|^2,
\end{aligned}$$

which is the content of part (c). □

3. Reduction by first integrals

As indicated in Theorem 1, the operator L_μ has a double zero eigenvalue. It corresponds to the two–dimensional family of equilibria given by

$$\varphi = (U_1, V_1, U_2, V_2, B)^T = (\alpha, 0, \gamma, 0, 0)^T, \alpha, \gamma \in I\!R.$$

These are parallel flows with constant speeds $R_1 = \sqrt{1 + 2\alpha}$ and $R_2 = \sqrt{1 + 2\gamma}$ in the lower and upper layer, respectively. From this we find $Y = \int_0^h \frac{d\psi}{R_1} = h/\sqrt{1 + 2\alpha}$ and the height of both layers is $\int_0^h 1/R_1 d\psi + \int_h^1 1/R_2 d\psi = h/\sqrt{1 + 2\alpha} + (1 - h)/\sqrt{1 + 2\gamma}$.

Since all these solutions can be rescaled to the solution $\alpha,\ \gamma = 0$, we see that this family is generated artificially. In fact, one 0 eigenvalue is due to the transformation from (x, y) into (x, ψ) and the other stems from differentiating (4). We have the following two conserved quantities for (2):

$$J_1(\varphi) = B + \mu \int_0^h \tfrac{1}{R_1}\, d\psi, \quad J_2(\varphi) = \int_0^h \tfrac{1}{R_1}\, d\psi + \int_h^1 \tfrac{1}{R_2}\, d\psi. \tag{6}$$

From (4) we know $J_1(\varphi) = \mu h$ and J_2 is the channel height $y(x, 1) = J_2(\varphi(x))$ which equals to 1 by our scaling (cf. (3)).

Additonally there is a third integral J_3 which derives from the variational structure of the problem and invariance with respect to translations in x–direction. In terms of the variables (u_i, v_i) and Y it reads

$$J_3(u_1, v_2, u_2, v_2, Y) = \int_0^Y \tfrac{1}{2}(u_1^2 - v_1^2)dy + \int_Y^1 \tfrac{\rho}{2}(u_2^2 - v_2^2)dy + (C_1 - C_2)Y - \tfrac{\mu}{2}Y^2.$$

(Taking the x–derivative of J_3 along a solutions of (1) easily shows $dJ_3/dx = 0$.) In terms of (U_i, V_i) and $Y = \int_0^h \frac{1}{R_1} d\psi$ the integral J_3 can be expressed as

$$J_3(\varphi) = \int_0^h \tfrac{R_1}{2}(1-V_1^2)d\psi + \int_h^1 \tfrac{\rho R_2}{2}(1-V_2^2)d\psi + (C_1-C_2)Y - \tfrac{\mu}{2}Y^2. \tag{7}$$

In[6] J_3 is called the flow–force per cross–section, and in[13], where the case with of capillary surface waves was treated, it was observed that functions like J_3 can be interpreted as a Hamiltonian function when a properly chosen sympletic structure is employed, see[5, 7] for surface waves and [8] for interfacial waves. In[13] a general theory for elliptic varational problems is developed which allows to reduce the Hamiltonian structure to the center manifold of finite dimension. Although we do not emphasize the Hamiltonian structure in this paper, the function J_3 will still play a major role in our discussion in Section 5.

We now restrict our problem (2) to cut out the artificial double zero eigenvalue. Without loss of generality we restrict our solutions to lie in the manifold $\mathcal{M}_\mu = \{\, \varphi \in D(L_\mu) \;:\; J_1(\varphi) = \mu h,\; J_2(\varphi) = 1 \,\}$, which has codimension 2 and is invariant with respect to (2). To describe the reduced flow in $\mathcal{M}_\mu$ we project $\mathcal{M}_\mu$ locally onto its tangent space at $\varphi = 0$. To find a suitable projection we analyze the kernel of L_μ further. Here we restrict ourselves to one interesting case, namely $\mu \approx \mu_0 = \frac{1}{h} + \frac{\rho}{1-h}$. For $\mu = \mu_0$ we know that $\sigma = 0$ is a four–fold eigenvalue and bifurcations should occur for μ passing through μ_0. The generalized kernel of L_{μ_0} is spanned by

$$\varphi_1 = \begin{pmatrix} -1/h \\ 0 \\ 1/(1-h) \\ 0 \\ -\mu_0 \end{pmatrix}, \; \varphi_2 = \begin{pmatrix} 0 \\ \psi/h \\ 0 \\ (1-\psi)/(1-h) \\ 0 \end{pmatrix}, \; \varphi_3 = \tfrac{1}{6}\begin{pmatrix} 3\psi^2/h - h \\ 0 \\ \kappa(\psi) \\ 0 \\ 0 \end{pmatrix}, \; \varphi_4 = \varphi_3 - \tfrac{\Delta}{3\rho}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -\rho \end{pmatrix},$$

where $\Delta = h + \rho(1-h)$ and $\kappa(\psi) = 2\Delta/\rho + 1 - h - 3(1-\psi)^2/(1-h)$. We have $L_{\mu_0}\varphi_1 = 0$, $L_{\mu_0}\varphi_2 = \varphi_1$, and $L_{\mu_0}\varphi_{3,4} = \varphi_2$. Using the standard scalar product $\langle \cdot, \cdot \rangle$ in X the adjoint L^* of L_{μ_0} is given by

$$\begin{aligned}D(L^*)=\{(U_1,V_1,U_2,V_2,B)\in H^1(0,h)^2\times H^1(h,1)^2\times I\!R: V_1(0)=V_2(1)=0,\\ V_2(h)=\rho V_1(h),\ \mu_0 B+U_1(h)-\rho U_2(h)=0\},\\ L^*(U_1,\dots,B)^T=(-\partial_\psi V_1,\partial_\psi U_1,-\partial_\psi V_2,\partial_\psi U_2,V_1(h))^T.\end{aligned}$$

The generalized kernel of L^* is spanned by

$$\eta_1=\tfrac{3}{\Delta}\begin{pmatrix}-\mu_0\\0\\0\\0\\1\end{pmatrix},\ \eta_2=c_2\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix},\ \eta_3=\tfrac{3}{\Delta}\begin{pmatrix}0\\\psi/h\\0\\\kappa_0(\psi)\\0\end{pmatrix},\ \eta_4=\tfrac{1}{2\Delta}\begin{pmatrix}\kappa_1(\psi)\\0\\\kappa_2(\psi)\\0\\0\end{pmatrix}-c_4(\eta_1+\eta_2),$$

where $\kappa_0(\psi)=\rho(1-\psi)/(1-h)$, $\kappa_1(\psi)=3\psi^2/h-3\Delta-\rho(1-h)$, $\kappa_2(\psi)=\rho(1-h)-3\rho(1-\psi)^2/(1-h)$, $c_2=3\rho/((1-h)\Delta)$, and $c_4=(2h^3+2\rho(1-h)^3)/(3\Delta)$. We have $L^*\eta_{1,2}=0$, $L^*\eta_3=\eta_1+\eta_2$, $L^*\eta_4=\eta_3$, and $\langle\varphi_i,\eta_j\rangle=1$ for $i+j=5$ and 0 else. Moreover, η_1 and η_2 are chosen such that

$$D_\varphi J_1(0)[\widetilde{\varphi}]=\tfrac{\Delta}{3}\langle\eta_1,\widetilde{\varphi}\rangle \quad\text{and } D_\varphi J_2(0)[\widetilde{\varphi}]=\tfrac{(1-h)\Delta}{3\rho}\langle\eta_2,\widetilde{\varphi}\rangle$$

for all $\widetilde{\varphi}$. Thus, the tangent space X_0 of $\mathcal{M}_{\mu_0}$ at $\varphi=0$ is the orthogonal complement of span$\{\eta_1,\eta_2\}$. We define the projection $Q_0: X\to X_0;\ \varphi\to\varphi-\langle\varphi,\eta_1\rangle\varphi_4-\langle\varphi,\eta_2\rangle\varphi_3$ and decompose $\varphi\in X$ into $\varphi=\varphi_0+\nu_3\ \varphi_3+\nu_4\varphi_4$, where $\varphi_0=Q_0\varphi\in X_0$. Then,

$$J_1(\varphi_0+\nu_3\varphi_3+\nu_4\varphi_4)-\mu h=0,\quad J_2(\varphi_0+\nu_3\varphi_3+\nu_4\varphi_4)-1=0 \tag{8}$$

can be solved locally ($\varphi_0,\nu-\nu_0$ small) for $\nu_j=\nu_j(\mu,\varphi)\in I\!R$ by the implicit function theorem. Thus, φ_0 serves as coordinate in the tangent space X_0 and the correction $\nu_3(\mu,\varphi_0)\varphi_3+\nu_4(\mu,\varphi_0)\varphi_4$ takes into account the curvature of $\mathcal{M}_\mu$.

To derive the differential equation for φ_0 we simply apply Q_0 to (2). Since $Q_0L_{\mu_0}(\varphi_0+\nu_3\varphi_3+\nu_4\varphi_4)=Q_0[L_{\mu_0}\varphi_0+(\nu_3+\nu_4)\varphi_2]$ and $Q_0\varphi_2=\varphi_2$ we find

$$\frac{d}{dx}\varphi_0=\mathcal{L}\varphi_0+\mathcal{N}(\mu,\varphi_0) \tag{9}$$

where $\mathcal{N}(\mu,\varphi)=(\nu_3+\nu_4)\varphi_2+Q_0\left[(L_\mu-L_{\mu_0})(\varphi_0+\nu_3\varphi_3+\nu_4\varphi_4)+N(\varphi_0+\nu_3\varphi_3+\nu_4\varphi_4)\right]$ ($\nu_j=\nu_j(\mu,\varphi_0)$) and $\mathcal{L}=Q_0L_{\mu_0}|_{X_0}=L_{\mu_0}|_{X_0}$. Again $\mathcal{N}$ is a smooth (analytic) mapping from a neighborhood of $(\mu_0,0)$ in $I\!R\times D(\mathcal{L})$ into X_0, where $D(\mathcal{L})=D(L_{\mu_0})\cap X_0$.

4. Reduction onto the center manifold

We define the center space projection $Q_1: X_0\to X_0;\varphi_0\mapsto\varphi_0-\langle\varphi_0,\eta_3\rangle\varphi_2-\langle\varphi_0,\eta_4\rangle\varphi_1$ and the splitting $\varphi_0=a\varphi_1+b\varphi_2+\Phi,\ \Phi\in X_1=Q_1X_0$, which transfers (9) into

$$\frac{d}{dx}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}+f_1(\mu,a,b,\Phi),\quad \frac{d}{dx}\Phi=\mathcal{L}_1\Phi+f_2(\mu,a,b,\Phi), \tag{10}$$

where $\mathcal{L}_1 = \mathcal{L}|_{X_1} = Q_1\mathcal{L}|_{X_1}$ and

$$f_1 = \begin{pmatrix} \langle \eta_4, \mathcal{N}(\mu, a\varphi_1 + b\varphi_2 + \Phi)\rangle \\ \langle \eta_3, \mathcal{N}(\mu, a\varphi_1 + b\varphi_2 + \Phi)\rangle \end{pmatrix}, \quad f_2 = Q_1\mathcal{N}(\mu, a\varphi_1 + b\varphi_2 + \Phi)$$

According to Theorem 2..1(c) the operator $\mathcal{L}_1$ has no eigenvalues on the imaginary axis, and satisfies $\|(\mathcal{L}_1 + is)^{-1}\|_{X_1 \to X_1} \le C/(1+|s|)$, for all $s \in I\!R$. Hence, the reduction theorem in[12] is applicable, and there exists a local center manifold M_C which contains all small bounded solutions and can be written as graph over the center space (here $(a,b) \in I\!R^2$).

Theorem 4..1 *For each $k \in I\!N$ there is an $\varepsilon > 0$, a neighborhood $\mathcal{O}_1 \subset D(\mathcal{L}_1) = X_1 \cap D(L_{\mu_0})$ and a reduction function $\mathcal{H} = \mathcal{H}(\mu, a, b) \in \mathcal{C}^k((\mu_0 - \varepsilon, \mu_0 + \varepsilon) \times (-\varepsilon, \varepsilon)^2, \mathcal{O}_1)$, such that the reduced system*

$$\frac{d}{dx}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + g(\mu, a, b), \quad \Phi = \mathcal{H}(\mu, a, b), \tag{11}$$

with $g(\mu, a, b) = f_1(\mu, a, b, \mathcal{H}(\mu, a, b))$, is locally equivalent to (10) in the sense that every small bounded solution of one equation is also a solution of the other equation.

We remark that the problem has a reflection symmetry $x \to -x$. For the differential equation (2) this gives reversibility with respect to the involution

$$T : X \to X; (U_1, V_1, U_2, V_2, B)^T \mapsto (U_1, -V_1, U_2, -V_2, B)^T.$$

This means $TL_\mu = -L_\mu T$ and $N(T\varphi) = -TN(\varphi)$. As a consequence $\varphi = \varphi(x)$ is a solution if and only if $\tilde{\varphi}(x) = T\varphi(-x)$ is one. The reversibility is inherited onto the reduced problem (10) is reversible, i.e., with $T_0(a,b)^T = (a, -b)^T$ we have $g(\mu, T_0(a,b)) = -T_0 g(\mu, a, b)$.

To calculate the coefficients of the leading nonlinear terms of g, we first expand the functions ν_i with respect to $\varphi_0 = a\varphi_1 + b\varphi_2 + \Phi$ and $\delta = \mu - \mu_0$:

$$\nu_3 = \tfrac{9\rho}{2\Delta h(1-h)^2}a^2 + \tfrac{3\rho(5-h)}{2\Delta h^2(1-h)^3}a^3 + \text{h.o.t.}, \;\; \nu_4 = -\tfrac{3}{\Delta}\delta a = \tfrac{9\mu_0}{2\Delta h}a^2 - \tfrac{15\mu_0}{2\Delta h^2}a^3 + \text{h.o.t.},$$

with h.o.t.$= \mathcal{O}(a^4 + b^2 + \|\Phi\|^2_{D(L)} + |a|\,\|\Phi\|_{D(L)} + |\delta|[a^2 + \|\Phi\|_{D(L)}])$. This implies

$$\begin{aligned} f_1 &= \begin{pmatrix} |b|\mathcal{O}(|\delta| + |a| + b^2 + \|\Phi\|_{D(L)}) \\ -\frac{3}{\Delta}\delta a - E_0 a^2 + G_0 a^3 + \text{ h.o.t.} \end{pmatrix}, \\ f_2 &= \mathcal{O}(|a|^3 + b^2 + \|\Phi\|^2 + |a|\,\|\Phi\|_{D(L)} + |\delta|[a^2 + |b| + \|\Phi\|_{D(L)}]), \\ \text{where } E_0 &= \tfrac{9}{2\Delta}\left(\tfrac{1}{h^2} - \tfrac{\rho}{(1-h)^2}\right), \quad G_0 = \tfrac{3}{2\Delta h^2}\left(\tfrac{\rho(5-h)}{1-h)^3} - 5\mu_0\right). \end{aligned}$$

The function f_2 does not contain a term of order a^2 because of

$$\begin{aligned} f_2(\mu, a, 0, 0) &= Q_1\big\{(\nu_3 + \nu_4)\varphi_2 + Q_0[(L_\mu - L_{\mu_0})\varphi + N(a\varphi_1 + \nu_3\varphi_3 + \nu_4\varphi_4)]\big\} \\ &= Q_1 Q_0 N(a\varphi_1 + \mathcal{O}(a^2)) \;=\; \mathcal{O}(|a|^3), \end{aligned}$$

and $N(a\varphi_1) = 0$ for all a. Thus, the reduction function $\Phi = \mathcal{H}(\mu, a, b)$ satisfies the estimate $\|\mathcal{H}(\mu, a, b)\|_{D(L)} = \mathcal{O}[|a|^3 + b^2 + |\delta|(a^2 + |b|)]$, and insertion of $\mathcal{H}$ into (f_1, f_2) yields

$$g(\mu, a, b) = \begin{pmatrix} |b|\mathcal{O}(|\mu-\mu_0| + |a| + b^2) \\ \frac{3}{\Delta}(\mu_0-\mu)a - E_0a^2 + G_0a^3 + \mathcal{O}(a^4 + b^2 + |\mu-\mu_0|a^2) \end{pmatrix}.$$

The reduced system (11) can be rewritten as a second–order equation by solving $a' = b + g_1(\mu, a, b)$ with respect to $b = a'$+h.o.t. and inserting this into $b' = g_2(\mu, a, b)$:

$$a'' - \sigma^2(\mu)a + E(\mu)a^2 - G(\mu)a^3 + M(\mu, a, a') = 0, \tag{12}$$

where $M(\mu, a, a') = M(\mu, a, -a') = \mathcal{O}(a^4 + a'^2)$ and

$$\sigma^2 = \tfrac{3}{\Delta}(\mu_0 - \mu) + \mathcal{O}(|\mu-\mu_0|^2),\ E = E_0 + \mathcal{O}(|\mu-\mu_0|),\ G = G_0 - c(\mu_0)E_0 + \mathcal{O}(|\mu-\mu_0|).$$

5. Homoclinic and heteroclinic solutions

It is well–known that in the case $E_0 \neq 0$ equation (12) has a bifurcation of homoclinic solutions for $\mu_0 - \mu > 0$, wich have the expansion

$$a(\mu, x) = \frac{\sigma^2(\mu)}{E_0\mu}\,\frac{3}{1 + \cosh(\sigma(\mu)x)} + \mathcal{O}((\mu_0 - \mu)^2 e^{-\sigma(\mu)|x|})$$

for $\mu \to \mu_0$, uniformly in $x \in \mathbb{R}$, see[9, 10]. Using (3), the interface Y satisfies

$$Y(x) = h - \textstyle\int_0^h U_1 d\Psi + \mathcal{O}(\|\varphi\|^2_{D(L)}) = h + \frac{2(\mu_0-\mu)}{(\frac{1}{h^2} - \frac{\rho}{(1-h)^2})(1+\cosh(\sigma(\mu)x))} + \mathcal{O}(\ldots).$$

We find that $E_0 > 0$ yields elevation waves ($Y > h$) and $E_0 < 0$ yields depression waves ($Y < h$), which explains the name elevation number for E_0, see[3, 4].

The case of E_0 very small gives rise to new phenomena, especially the existence of heteroclinic solutions, so–called bores. We now consider $\sigma = \sigma(\mu)$ and $E = E(\mu)$ as two independent small parameters. This can be achieved when, in additon to μ, also ρ (or h) is taken as a control parameter. Of course, then also G and M depend on σ and E. Note that $E_0 = 0$ implies $\rho h^2 = (1-h)^2$ and hence $G(\mu_0, E_0) = 6/[\Delta h^3(1-h)] > 0$.

We are only interested in the case $\mu_0 - \mu > 0$ and define the scalings

$$t = \sigma x, \quad z = \sqrt{G(\mu, E)}\, a/\sigma, \quad \alpha = \sqrt{G(\mu, E)}\, E/\sigma. \tag{13}$$

Hence, $\alpha \in \mathbb{R}$ measures the relative size between the elevation number E and the closeness to criticality $\mu_0 - \mu = \Delta\sigma^2/3 \approx 0$. For $z = z(t)$ we obtain the equation

$$\begin{aligned} &\ddot{z} - z + \alpha z^2 - z^3 + \widetilde{M}(\sigma, \alpha, z, \dot{z}) = 0, \\ &\text{with } \widetilde{M}(\sigma, \alpha, z, \dot{z}) = \widetilde{M}(\sigma, \alpha, z, -\dot{z}) = \mathcal{O}(\sigma(z^4 + \dot{z}^2)). \end{aligned} \tag{14}$$

In the limit $\sigma = 0$ this equation can be discussed explicitly. It has the first integral

$$\widetilde{j}(\alpha, z, \dot{z}) = \tfrac{1}{2}\dot{z}^2 - \tfrac{1}{2}z^2 + \tfrac{\alpha}{3}z^3 - \tfrac{1}{4}z^4,$$

and all equilibria lie on the z–axis. For $\alpha \in [0, 3/\sqrt{2})$ there is one equilibrium ($z = 0$), for $\alpha > 3/\sqrt{2}$ there are three (from now on we only treat the case $\alpha \geq 0$, since $\alpha < 0$ can be handled by changing z to $-z$). Moreover, for $\alpha > 3/\sqrt{2}$ there are solutions which are homoclinic to the origin:

$$z_{\text{hom}}(t) = \frac{72}{36\sqrt{2}e^{-t} + 24\alpha + (2\alpha^2 - 9)\sqrt{2}e^t} = \frac{3}{2\alpha + \sqrt{\alpha^2 - 9/2}\cosh(t+c)},$$

where $c = \log 6 - \frac{1}{2}\log(2\alpha^2 - 9)$. We have shifted z_{hom} such that it converges to the heteroclinic solution $z_{\text{het}}(t) = \sqrt{2}/(1 + e^{-t})$ for $\alpha \to 3/\sqrt{2}$.

The persistence of the homoclinic and heteroclinic solution for small $\sigma > 0$ follows by considering the conserved quantity $J_3(\varphi)$ as expressed in (7). We define the restriction of J_3 to the center manifold M_C,

$$j_3(a, b) = J_3(a\varphi_1 + b\varphi_2 + \mathcal{H}(\mu, a, b) + \nu_3(\ldots)\varphi_3 + \nu_4(\ldots)\varphi_4))$$

where $\nu_k(\ldots) = \nu_k(\mu, a\varphi_1 + b\varphi_2 + \mathcal{H}(\mu, a, b))$. Obviously, j_3 is constant on solutions of the reduced problem (11) and even in b. Moreover, scaling (μ, a, b) as above shows that $\widetilde{j}$ is exactly the scaled limit of j_3. Hence, the persistence of the phase portrait for small $\sigma > 0$ is trivial as all solutions are level curves of j_3.

Remark: It is shown in [11] that the case with surface tension leads to a similar equation, where a'' is be replaced by $-\delta_1(\beta)a''$. Here β is the dimensionless Bond number measuring the relative strength of the surface tension. For $\beta > \beta_0$, δ_1 is positive and $\delta_1(\beta) < 0$ for $\beta < 0$. In the latter case homoclinic bifurcation occurs for $\mu_0 - \mu < 0$, and for E_0 small, a scaling similar to (13) yields $z'' - z + \alpha z^2 + z^3 + \mathcal{O}(\sigma) = 0$. The difference to our case is the plus sign in front of the cubic term, which leads to coexistence of elevation and depression waves for open sets in the parameter space, see[11] for details.

We discuss the above results in the original dimensionless parameter space. Recalling $\sigma^2(\mu) = \frac{3}{\Delta}(\mu_0 - \mu) + \mathcal{O}(|\mu - \mu_0|^2)$ and $E = \alpha\sigma\sqrt{G(\mu_0, 0)} + \mathcal{O}(|\mu - \mu_0|)$ the existence domain of solitray waves in a neighborhood of $(\mu, E) = (\mu_0, 0)$ is given by $\mu \in (\mu_0, \mu_0 + \Gamma(E))$, where Γ has the expansion

$$\Gamma(E) = \tfrac{\Delta^2 h^3(1-h)}{81}E^2 + \mathcal{O}(|E|^3) \quad \text{for } E \to 0.$$

For $E > 0$ the solitary waves are waves of elevations and for $E < 0$ waves of depression.

Moreover, we have proved, at least, locally for small elevation number E, a conjecture of C. Amick and R. Turner[3]. Taking E as small but fixed and letting μ vary on $(-\infty, \mu_0]$ we find a branch of bifurcating solitary waves (homoclinic solutions).

In[3] it is shown that this branch is an unbounded connected continuum in the space $H^1(\mathbb{R}, X) \cup \mathcal{C}^1_{bdd}(\mathbb{R}, X)$. The conjecture is that the solutions remain bounded in $\mathcal{C}^1_{bdd}$ while the H^1–norm blows up due to broadening of the plateau[15]. Our local analysis easily shows that the width of the plateau grows like $\log(\Gamma(E) - \mu)$ for $\mu \to \Gamma(E)$, which implies the blowup of the H^1 norm.

References

1. C.J. Amick. Semilinear elliptic eigenvalue problems on an infinite strip with an application to stratified fluids. *Ann. Scuola Norm. Sup. Pisa (4)*, **11**, 441–499, 1984.
2. C.J. Amick, K. Kirchgässner. A theory of solitary water–waves in the presence of surface tension. *Arch. Rational Mech. Analysis*, **105**, 1–49, 1989.
3. C.J. Amick, R.E.L. Turner. A global theory of internal solitary waves in two–fluid systems. *Trans. Amer. Math. Soc.*, **298**, 431–484, 1986.
4. C.J. Amick, R.E.L. Turner. Small internal waves in two–fluid systems. *Arch. Rational Mech. Analysis*, **108**, 111–139, 1989.
5. C. Baesens, R.S. MacKay. Uniformly travelling water waves from a dynamical systems viewpoint: some insight into bifurcations from the Stokes family. *J. Fluid Mechanics* **241**, 333–347, 1992.
6. T.B. Benjamin. Internal waves of finite amplitude and permanent form. *J. Fluid Mech.*, **25**, 241–270, 1966.
7. T.J. Bridges. Spatial Hamiltonian structure, energy flux and the water–wave problem. *Proc. Royal Soc. London* **A439**, 297–315, 1992.
8. T.J. Bridges, P. Christodoulides, F. Dias. Spatial bifurcations of interfacial waves when the phase and group velocities are nearly equal. Preprint: Institut Non–Linéaire de Nice. 1994.
9. K. Kirchgässner. Wave solutions of reversible systems and applications. *J. Diff. Eqns.*, **45**, 113–127, 1982.
10. K. Kirchgässner. Nonlinearly resonant surface waves and homoclinic bifurcation. *Advances Appl. Mechanics*, **26**, 135–181, 1988.
11. P. Kirrmann. Reduktion nichtlinearer elliptischer Systeme in Zylindergebieten unter Verwendung von optimaler Regularität in Hölder–Räumen. Thesis Stuttgart, 1991.
12. A. Mielke. Reduction of quasilinear elliptic equations in cylindrical domains with applications. *Math. Meth. Appl. Sci.*, **10**, 51–66, 1988.
13. A. Mielke. *Hamiltonian and Lagrangian Flows on Center Manifolds with Applications to Elliptic Variational Problems*. Springer–Verlag, 1991. Lect. Notes in Math. Vol. **1489**.
14. R.E.L. Turner. Internal waves in fluids with rapidly varying density. *Ann. Scuola Norm. Sup. Pisa (4)*, **8**, 513–573, 1981.
15. R.E.L. Turner, J.–M. Vanden–Broeck. Braodening of interfacial solitary waves. *Physics of Fluids* **13**, 2486–2490.