On the energy release rate in finite–strain elasticity

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Abstract

Griffith's fracture criterion describes in a quasistatic setting whether or not a preexisting crack in an elastic body is stationary for given external forces. This fracture criterion can be reformulated in terms of the the energy release rate (ERR), which is the derivative of the deformation energy of the body with respect to a virtual crack extension.

In this note we consider geometrically nonlinear elastic models with polyconvex energy densities and provide a mathematical framework which guarantees that the ERR is well defined. Moreover, without making any assumptions on the asymptotic structure of the elastic fields near the crack tip, we derive rigorously two formulas for the ERR, namely a generalized Griffith formula and the J-integral. For simplicity we consider here a straight crack in a two dimensional domain. The presented techniques are also applicable to smooth interface cracks, for which we give an example in the last section.

Key words: Griffith fracture criterion; energy release rate; finite–strain elasticity.
2000 Mathematics Subject Classification: 74B20, 74R10.

1 Introduction

In this note we study the behavior of elastic bodies with preexisting cracks when subjected to static exterior loadings. Various fracture criteria are discussed in the literature, among which Griffith's classical energy criterion is frequently used in order to predict whether or not a preexisting crack will grow under the applied loads. We use the following version of the Griffith criterion [1, 2, 3]:

$$\begin{array}{c}\textit{A crack is stationary, if the total potential energy}\\ \textit{in the current configuration is minimal compared to}\\ \textit{the total potential energy of all admissible neighboring configurations.}\end{array} \qquad (1.1)$$

Assuming that the possible crack path is known a priori (e.g. an interface crack), the Griffith criterion can be reformulated in terms of the energy release rate. The energy release rate is defined as the derivative of the deformation energy with respect to the crack length.

We model the elastic behavior of the bodies in the framework of finite strain elasticity with polyconvex energy densities which may take the value ∞ as soon as unphysical deformation gradients occur. There is a large number of papers, where formulas for the energy release rate are derived for different nonlinear elastic models. These formulas are the Griffith formula (based on the Eshelby tensor) and the well-known Cherepanov–Rice or J–integral. To obtain these formulas, it is usually assumed that the deformation fields are "smooth enough" or that the stress fields have a certain asymptotic behavior near the crack tip. However, these smoothness assumptions are not justified yet in the nonlinear case and in particular in the finite strain case. A rigorous analysis without making any additional assumptions on the smoothness of the elastic fields was carried out in [4, 5] for a class of power-law models and recently in [6] for models from finite–strain elasticity. It is the purpose of this paper to present and discuss the conditions from [6] on the energy density W, which guarantee that the energy release rate is well defined in the finite–strain case, and to point out the main differences between the linear case and the finite–strain case. We emphasize that we require neither additional smoothness of the deformations nor a certain asymptotic structure of the stresses near the crack tip in order to prove our results.

The paper is organized as follows. In the next section we fix the notation, recall the Griffith fracture criterion and give a definition of the energy release rate. Furthermore, we summarize shortly results from linear elasticity. In section 3 we formulate the assumptions on the elastic energy density W which guarantee the existence of minimizers. In addition we introduce an assumption on the derivative of W, which implies that the modulus of the Eshelby tensor can be bounded by the energy W. We give examples of polyconvex energy densities which satisfy this additional assumption. Section 4 is devoted to the presentation and discussion of our formulae for the energy release rate. Since minimizers may be nonunique, one can give different interpretations of what is meant by "admissible neighboring configuration" in the Griffith criterion: is it sufficient that the different cracked domains are "close" to each other or does one also require that minimizers on domains with extended cracks are "close" to a particular minimizer on the original domain? These different interpretations are reflected in our formulas for the ERR and are an essential difference between the finite-strain case and the case of linear elasticity. This will be discussed in detail in section 4. In section 5 we finally apply our results to an interface crack in three dimensions.

2 The Griffith fracture-criterion

We give now a precise definition of the notions occurring in the Griffith criterion (1.1). Let $\Omega_0 \subset \mathbb{R}^d$ be a body with preexisting crack. The total energy is the sum of the deformation energy and a dissipation energy, which describes the amount of energy which is needed to create the new crack surface. We assume here that the dissipation energy $\mathcal{D}$ is proportional to the area (in 3D) or length (in 2D) of the crack surface: $\mathcal{D}(C_0) = 2\gamma\,|C_0|$. The material dependent constant $\gamma > 0$ is the fracture toughness and C_0 denotes the crack. Let $W : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ be the elastic energy density and $V_{\rm ad}(\Omega_0)$ the set of admissible deformations $\varphi : \Omega_0 \to \mathbb{R}^d$, i.e. $V_{\rm ad}(\Omega_0)$ consists of those deformations which satisfy the Dirichlet boundary conditions. By $f : \Omega_0 \to \mathbb{R}^d$ we denote given volume forces. The deformation energy $I(\Omega_0)$ with respect to the domain Ω_0 is then given by

$$I(\Omega_0) = \min\{\, I(\Omega_0, \varphi)\,;\, \varphi \in V_{\rm ad}(\Omega_0)\,\}, \tag{2.1}$$

where

$$I(\Omega_0, \varphi) = \int_{\Omega_0} W(\nabla\varphi)\,\mathrm{d}x - \langle f, \varphi\rangle.$$

Under suitable assumptions on W and f, which we specify in section 3, problem (2.1) possesses at least one minimizer φ_0. With these notations, the total potential energy $\Pi(\Omega_0)$ of a domain Ω_0 with crack C_0 can be written as

$$\Pi(\Omega_0) = I(\Omega_0) + 2\gamma\,|C_0|\,.$$

We now have to give an interpretation for the notion *admissible neighboring configuration.* In general, the crack path is not known a priori and in the most general case, a domain Ω_*

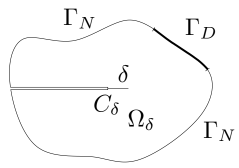

Figure 1: Domain Ω_δ with crack C_δ

with crack C_* defines an admissible neighboring configuration if $\overline{\Omega_*} = \overline{\Omega_0}$ and the crack C_0 is contained in C_*. This general point of view includes the kinking and branching of cracks. DalMaso et al. proposed and investigated a rate independent evolution formulation for crack propagation in nonlinear elastic materials with quasiconvex energy densities, based on this general point of view [7]. In our paper, we have a different point of view. We assume that the crack path is known a priori and we are interested in a (computable) criterion with which one may decide whether or not a given crack is stable. In this section, we consider the simplest situation assuming that the crack is a straight line in a two-dimensional domain and that it can grow straight on, only.

For $\delta \in \mathbb{R}$ let $S_\delta = \{\, x \in \mathbb{R}^2\,;\, x_2 = 0,\, x_1 \leq \delta \,\}$. We make the following assumption on the geometry:

A0 $\tilde{\Omega} \subset \mathbb{R}^2$ is a bounded domain with Lipschitz-boundary and with $0 \in \tilde{\Omega}$. Furthermore, there exists a constant $\delta_0 > 0$ such that $\partial\tilde{\Omega} \cap S_\delta$ is a single point for every $\delta \in [-\delta_0, \delta_0]$. Let $\Omega_\delta = \tilde{\Omega} \backslash S_\delta$ and $C_\delta = \overline{\tilde{\Omega}} \cap S_\delta$ for $|\delta| \leq \delta_0$. The boundary of Ω_δ is split as follows: $\partial\Omega_\delta = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup C_\delta$, where Γ_D and Γ_N are open. Moreover, C_δ, Γ_D and Γ_N are pairwise disjoint, Γ_D and Γ_N are independent of δ and Γ_D is not empty. See fig. 1.

The parts Γ_D and Γ_N are the Dirichlet- and Neumann-boundary, respectively. We call Ω_0 with initial crack C_0 the reference configuration. In our context, the pairs $(\Omega_\delta, C_\delta)$ describe admissible neighboring configurations if $\delta > 0$ is small. In this case, the Griffith criterion reads: if $\Pi(\Omega_0) < \Pi(\Omega_\delta)$ for small $\delta > 0$, then the crack is stationary. This is equivalent to

If $(I(\Omega_0) - I(\Omega_\delta))/\delta < 2\gamma$ *for* $\delta > 0$, *then the crack is stationary.*

Taking the limit $\delta \searrow 0$, we arrive at the definition of the energy release rate:

Definition 2.1 (Energy release rate)**.** The energy release rate $ERR(\Omega_0)$ related to the domain Ω_0 is defined as

$$ERR(\Omega_0) = \lim_{\delta \searrow 0} \delta^{-1}\big(I(\Omega_0) - I(\Omega_\delta)\big).$$

The energy release rate gives the amount of deformation energy which is set free at an infinitesimal extension of the crack. The final formulation of the Griffith criterion reads as follows:

$$\textit{If} \quad ERR(\Omega_0) < 2\gamma, \textit{ then the crack is stationary.} \tag{2.2}$$

For a geometry as described in **A0** and a linear elastic material with energy density $W(\varepsilon(u)) = \frac{1}{2}(A\varepsilon(u)) : \varepsilon(u)$, where $u : \Omega \to \mathbb{R}^2$ is the displacement field, $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ are the linearized strains and A is the fourth order elasticity tensor, the energy release rate can be expressed as follows (if $f = 0$)

$$ERR(\Omega_0) = \int_{\Omega_0} \big(\nabla u_0^\top D_\varepsilon W(\varepsilon(u_0)) - W(\varepsilon(u_0))\mathbb{I}\big) : \nabla \left(\begin{smallmatrix}\theta\\0\end{smallmatrix}\right) \,\mathrm{d}x \tag{2.3}$$

$$= \int_{\Gamma} \big(\nabla u_0^\top D_\varepsilon W(\varepsilon(u_0)) - W(\varepsilon(u_0))\mathbb{I}\big)\vec{n} \cdot \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \,\mathrm{d}s. \tag{2.4}$$

Here, u_0 is the unique minimizer of the corresponding energy functional, $\theta \in C_0^\infty(\tilde{\Omega}, \mathbb{R})$ is an arbitrary function with $\theta = 1$ near the crack tip, Γ an arbitrary path around the crack tip and $\vec{n}$ the interior unit normal vector on Γ. Formula (2.3) is the so–called Griffith formula and (2.4) the Cherepanov–Rice or J–integral. The J–integral and its generalizations was first introduced and investigated by Eshelby [8] and applied in fracture mechanics by Cherepanov [9] and Rice [10]. A mathematical justification of formulas (2.3)–(2.4) was carried out in [11] for traction free crack faces and in [12] for non-interpenetration conditions on the crack faces. In the linear case, the energy release rate can also be expressed in terms of stress intensity factors, see for example [13].

The formulas proposed in literature for nonlinear elastic models have a similar structure and are based on the Griffith integral

$$G(\varphi, \theta) = \int_{\Omega_0} \big(\nabla\varphi^\top DW(\nabla\varphi) - W(\nabla\varphi)\mathbb{I}\big) : \nabla \left(\begin{smallmatrix}\theta\\0\end{smallmatrix}\right) \,\mathrm{d}x - \int_{\Omega_0} \theta f \cdot \partial_{x_1}\varphi \,\mathrm{d}x, \tag{2.5}$$

which involves the Eshelby tensor

$$E(\nabla\varphi) = -\nabla\varphi^\top DW(\nabla\varphi) + W(\nabla\varphi)\mathbb{I}.$$

As already discussed in the introduction, in the nonlinear case the formulas for the ERR are derived under additional smoothness assumptions for minimizers, which are not justified yet, in general. We present now sufficient conditions on the energy density which allow for a rigorous derivation of these formulas without making any additional assumption on the smoothness of the deformation fields.

3 Assumptions on the elastic energy density W

We use the following notation: $\mathbb{M}^{d\times d}$ denotes the set of the real $d \times d$-matrices and $\mathbb{M}_+^{d\times d}$ are those with positive determinant. For elements $A, B \in \mathbb{M}^{d\times d}$ the inner product is given by $A : B = \sum_{k=1}^d \sum_{s=1}^d A_{ks}B_{ks}$ and $|A| = \sqrt{A : A}$. By $\operatorname{cof} A$ we denote the cofactor matrix and $\mathbb{I}$ is the identity in $\mathbb{M}^{d\times d}$.

For a function $W : \mathbb{M}^{d\times d} \to \mathbb{R}$, $DW(A) \in \mathbb{M}^{d\times d}$ is the derivative of W with respect to $A \in \mathbb{M}^{d\times d}$, i.e. $DW(A)_{ks} = \frac{\partial W(A)}{\partial A_{ks}}$ for $1 \leq k, s \leq d$ and $D^2W(A) \in \mathbb{M}^{(d\times d)\times(d\times d)}$ is the Hessian of W with $(D^2W(A))_{ksjr} = \frac{\partial^2 W(A)}{\partial A_{ks}\partial A_{jr}}$, $k, s, r, j \in \{1, \dots, d\}$. Furthermore, $D^2W(A)[B] \in \mathbb{M}^{d\times d}$ with $D^2W(A)[B]_{ks} = \sum_{j=1}^d \sum_{r=1}^d D^2W(A)_{ksjr}B_{jr}$.

Let $d \in \{2, 3\}$, $\Omega \subset \mathbb{R}^d$. For a deformation $\varphi : \Omega \to \mathbb{R}^d$ we denote by $F(x) = \nabla\varphi(x) \in \mathbb{M}^{d\times d}$ the deformation gradient. The elastic energy density W shall be frame indifferent and we require that $W(F) = \infty$ if $\det F \leq 0$ and $W(F_n) \to \infty$ for every sequence $(F_n)_{n\in\mathbb{N}} \subset \mathbb{M}_+^{d\times d}$ with $|F_n| + (\det F_n)^{-1} \to \infty$. These requirements are not satisfiable with convex energy densities but with polyconvex energy densities. Thus our first assumption is that the energy density W is a polyconvex function. We use the notation from Docaorogna's book [14] and define $\mathcal{T}(F) = (F, \det F) \in \mathbb{R}^5$ if $d = 2$ and $\mathcal{T}(F) = (F, \operatorname{cof} F, \det F) \in \mathbb{R}^{19}$ if $d = 3$. Moreover, we set $\tau(d) = 5$ if $d = 2$, $\tau(d) = 19$ if $d = 3$.

A1 Polyconvexity: $W : \mathbb{M}^{d\times d} \to [0, \infty]$ is polyconvex, i.e. there exists a function $g : \mathbb{R}^{\tau(d)} \to [0, \infty]$ which is continuous and convex and $W(F) = g(\mathcal{T}(F))$ for every $F \in \mathbb{M}^{d\times d}$. Moreover, $W(F) = \infty$ if $\det F \leq 0$.

Ogden materials, neo-Hookean and Mooney–Rivlin materials have polyconvex energy densities. In order to be able to apply Ball's existence theorem, we need also a coercivity assumption for W.

A2 Coercivity: There exist constants $\beta \in \mathbb{R}$, $p \geq 2$, $r_1 \geq \frac{p}{p-1}$, $r_2 > 1$, $\alpha_1 > 0$, $\alpha_2, \alpha_3 \geq 0$ with $\alpha_2 > 0$ and $\alpha_3 > 0$ if $p \leq d$ such that for every $F \in \mathbb{M}^{d\times d}$ we have

$$W(F) \geq \alpha_1 |F|^p + \alpha_2 |\operatorname{cof} F|^{r_1} + \alpha_3 |\det F|^{r_2} + \beta. \tag{3.1}$$

Remark 3.1. If $d = 3$ and $p > 3$ in **A3**, then estimate (3.1) implies that there are constants $\tilde{\alpha}_i > 0$ and $\tilde{\beta} \in \mathbb{R}$ such that $W(F) \geq \tilde{\alpha}_1 |F|^p + \tilde{\alpha}_2 |\operatorname{cof} F|^{\frac{p}{2}} + \tilde{\alpha}_3 |\det F|^{\frac{p}{3}} + \tilde{\beta}$ for every $F \in \mathbb{M}^{d\times d}$.

In addition we assume that the forces and boundary data are given according to

A3 Data: $p \geq 2$, $q = \frac{p}{p-1}$, $g_D \in W^{1-\frac{1}{p},p}(\Gamma_D, \mathbb{R}^d)$, $h \in \big(W^{1-\frac{1}{p},p}(\Gamma_N)\big)'$ and $f \in L^q(\tilde{\Omega}, \mathbb{R}^d)$.

For $|\delta| \leq \delta_0$ the set of admissible deformations is defined as

$$V_{\mathrm{ad}}(\Omega_\delta) = \{\, \varphi \in W^{1,p}(\Omega_\delta)\,;\, \varphi\big|_{\Gamma_D} = g_D \,\}.$$

The following existence theorem is due to J. Ball [15], see also [16]:

Theorem 3.2. *Let $p \geq 2$ and let* **A0**–**A3** *be satisfied. For every $\delta \in [-\delta_0, \delta_0]$ there exists an element $\varphi_\delta \in V_{ad}(\Omega_\delta)$ with*

$$I(\Omega_\delta, \varphi_\delta) \leq I(\Omega_\delta, \psi) = \int_{\Omega_\delta} W(\nabla\psi)\,\mathrm{d}x - \int_{\Omega_\delta} f \cdot \psi\,\mathrm{d}x - \langle h, \psi\big|_{\Gamma_N} \rangle_{\Gamma_N} \tag{3.2}$$

for every $\psi \in V_{ad}(\Omega_\delta)$.

As one can see from the formulas for the linear case, the energy release rate is related to a volume integral which involves the Eshelby tensor $E(F) = -F^\top DW(F) + W(F)\mathbb{I}$. Up to now we did not impose any upper bound on W or DW which would guarantee that the Eshelby tensor is integrable for minimizers. We therefore require the following multiplicative stress control condition to be satisfied [17]:

A4 $W : \mathbb{M}^{d\times d} \to [0, \infty]$ is differentiable on $\mathbb{M}^{d\times d}_+$ and there exists a constant $\kappa_1 > 0$ such that for every $F \in \mathbb{M}^{d\times d}_+$

$$\big|F^\top DW(F)\big| \leq \kappa_1(W(F) + 1). \tag{3.3}$$

Condition **A4** was first introduced and discussed in [17, 18]. It follows from this condition that if $I(\Omega_\delta, \varphi) < \infty$ for some $\varphi \in V_{\mathrm{ad}}(\Omega_\delta)$, then the corresponding Eshelby tensor is an element from $L^1(\Omega_\delta)$ and the Griffith integral (2.5) is finite for every $\theta \in C_0^\infty(\tilde{\Omega})$. Note that an upper bound on W of the type $W(F) \leq c(1 + |F|^p)$ or $|DW(F)| \leq c(1 + |F|^{p-1})$ would not be appropriate, since we assume that $W(F) = \infty$ if $\det F \leq 0$.

For technical reasons we need also an assumption on D^2W:

A5 $W : \mathbb{M}^{d\times d} \to [0, \infty]$ is twice differentiable on $\mathbb{M}^{d\times d}_+$ and there exists a constant $\kappa_2 > 0$ such that for every $F \in \mathbb{M}^{d\times d}_+$ and every $B \in \mathbb{M}^{d\times d}$

$$\big|F^\top (D^2W(F)[FB])\big| \leq \kappa_2(W(F) + 1)\,|B|\,. \tag{3.4}$$

This assumption can slightly be weakened, see [6]. Compressible Mooney–Rivlin materials with an energy density of the form

$$W(F) = \begin{cases} a_1 \left|F\right|^2 + a_2 \left|\operatorname{cof} F\right|^2 + a_3(\det F)^2 - a_4 \log(\det F) & \text{for } F \in \mathbb{M}^{3\times3}_+ \\ \infty & \text{else} \end{cases},$$

satisfy **A1**, **A2**, **A4** and **A5** if $a_i > 0$ for every i. Moreover, let

$$W(F) = \begin{cases} W_1(F) + \Gamma(\det F) & \text{for } F \in \mathbb{M}^{d\times d}_+ \\ \infty & \text{else} \end{cases}$$

and assume that $W_1 : \mathbb{M}^{d\times d} \to \mathbb{R}$ is a convex, twice differentiable function of p–growth for some $p > 1$. This means that there exist constants $c, c_i > 0$ such that $c_1 \left|F\right|^p - c_2 \leq W_1(F) \leq c(1 + \left|F\right|^p)$, $\left|DW_1(F)\right| \leq c(1 + \left|F\right|^{p-1})$ and $\left|D^2W_1(F)\right| \leq c(1 + \left|F\right|^{p-2})$. Furthermore, let $\Gamma : (0,\infty) \to [0,\infty)$ be convex, twice differentiable, $\Gamma(s) \to 0$ for $s \to 0$ and $\left|s\Gamma'(s)\right| + \left|s^2\Gamma''(S)\right| \leq c(\Gamma(s) + 1)$ for every $s > 0$ and some constant $c > 0$. Then the assumptions **A1**, **A4** and **A5** are satisfied for W. The proof relies on the identities $D_F(\det F) = \operatorname{cof} F$ and $F^\top \operatorname{cof} F = (\det F)\mathbb{I}$, which imply that $F^\top D_F(\Gamma(\det F)) = \det F\, \Gamma'(\det F)\mathbb{I}$.

The following convergence theorem for Eshelby tensors is proved in [6] and is the key for our further analysis.

Theorem 3.3 (Weak convergence of Eshelby tensors)**.** *[6] Let $\Omega \subset \mathbb{R}^d$ be a bounded, open subset of $\mathbb{R}^d$. Let further $W : \mathbb{M}^{d\times d} \to [0,\infty]$ satisfy assumptions* **A1**, **A4** *and* **A5**, *let $p \geq 1$ and let $(\varphi_n)_{n\in\mathbb{N}_0} \subset W^{1,p}(\Omega)$ be a sequence with*

$$\mathcal{T}(\nabla\varphi_n) \rightharpoonup \mathcal{T}(\nabla\varphi_0) \quad \textit{weakly in } L^1(\Omega),$$
$$J(\nabla\varphi_n) \to J(\nabla\varphi_0) = \int_\Omega W(\nabla\varphi_0)\,\mathrm{d}x < \infty \qquad \textit{for } n \to \infty.$$

Then the Eshelby tensors $E(\nabla\varphi_n)$ converge weakly in $L^1(\Omega)$:

$$E(\nabla\varphi_n) \rightharpoonup E(\nabla\varphi_0) \quad \textit{weakly in } L^1(\Omega).$$

4 Energy release rate in finite strain elasticity

We are now ready to formulate our main theorem on the energy release rate for a two dimensional body with a straight crack.

Theorem 4.1 (Griffith formula)**.** *[6] Let $d = 2$, $p \geq 2$, let* **A0**–**A5** *be satisfied and assume that $\inf_{\varphi \in V_{ad}(\Omega_0)} I(\Omega_0, \varphi) < \infty$. Let finally $\theta \in C_0^\infty(\tilde{\Omega})$ with $\theta = 1$ near the crack tip $(0,0)^\top$. The energy release rate $ERR(\Omega_0)$ is well defined and a generalized Griffith formula is valid:*

$$ERR(\Omega_0) = \max\{\, G(\varphi, \theta)\,;\, \varphi \text{ minimizes } I(\Omega_0, \cdot) \text{ over } V_{ad}(\Omega_0)\,\}, \tag{4.1}$$

where

$$G(\varphi) := G(\varphi, \theta) = -\int_{\Omega_0} E(\nabla\varphi) : \nabla \left(\begin{smallmatrix}\theta \\ 0\end{smallmatrix}\right) \,\mathrm{d}x - \int_{\Omega_0} \theta f \cdot \partial_{x_1}\varphi \,\mathrm{d}x. \tag{4.2}$$

Formulas (4.1) *and* (4.2) *are independent of the choice of θ.*

The energy release rate can also be expressed through the J–integral. Since there is no regularity result available, which would allow us to speak about traces or restrictions of the Eshelby tensor on paths surrounding the crack tip, we have the following theorem for almost every path, only. We define $B_R(x_0) = \{\, x \in \mathbb{R}^d\,;\, |x - x_0| < R\,\}$.

Theorem 4.2 (J–integral)**.** *[6] Let the assumptions from the previous theorem be satisfied. Let $R_0 > 0$ such that $\overline{B_{R_0}(0)} \subset \tilde{\Omega}$. Assume furthermore, that $\partial_{x_1} f = 0$ on $B_{R_0}(0)$. For every minimizer φ_0 of $I(\Omega_0, \cdot)$ and almost every $R \in (0, R_0)$ we have*

$$G(\varphi_0) = -\int_{\partial B_R(0)} \big(E(\nabla\varphi_0)\vec{n}\big) \cdot \vec{e}_1 + (\varphi_0 \cdot f)(\vec{n} \cdot \vec{e}_1) \,\mathrm{d}s, \tag{4.3}$$

where $\vec{n}$ is the interior unit normal vector on $\partial B_R(0)$ and $\vec{e}_1 = (1,0)^\top$ is tangential to the crack.

Let us give some comments on our results. Assume first that the minimizer of $I(\Omega_0, \cdot)$ is unique. In this case, formulas (4.1)–(4.3) coincide with formulae for the ERR proposed in the literature, see e.g. [19, 3, 10], and have the same structure as in the linear case. In the general nonlinear case, however, there may exist several minimizers of $I(\Omega_0, \cdot)$ and it is an open problem, whether $G(\varphi_0) = G(\varphi_1)$ for different minimizers φ_0 and φ_1 of $I(\Omega_0, \cdot)$.

The Griffith criterion as formulated in (2.2) relies on global energy minimization and it is not excluded that the body jumps from one configuration with minimal energy into a different configuration with the same energy and the crack starts to develop from this configuration. If one has $ERR(\Omega_0) < 2\gamma$ for a particular problem, then it is guaranteed that the crack is always stationary, regardless which minimizer is realized. On the other hand, if the Griffith criterion is formulated in the following way

$$G(\varphi_0) < 2\gamma \;\Rightarrow \textit{the crack is stationary} \tag{4.4}$$

for a particularly chosen minimizer φ_0 (e.g. the one calculated with a FEM-code), then there might exist an other minimizer φ_1 with $G(\varphi_1) > 2\gamma$. Criterion (4.4) states that the crack will not propagate even though there might be a further configuration (namely φ_1) with the same energy, in which the crack would grow. Thus, criterion (4.4) can be regarded as a local version of the Griffith criterion, whereas (2.2) is a global one. In fact, it is a modeling assumption whether one trusts in a local fracture criterion of the form (4.4) or whether one prefers a global criterion like (2.2). These two different criteria result from different interpretations of the notion *admissible neighboring configuration* in (1.1). Are two configurations close to each other if Ω_δ is close to Ω_0 ($\rightarrow$ global criterion (2.2)) or is it required in addition that the minimizer φ_δ of $I(\Omega_\delta, \cdot)$ is close to a particularly chosen minimizer φ_0 ($\rightarrow$ local criterion (4.4)).

This discussion might become clearer by considering the following problem. Let φ_0 be an arbitrary minimizer of $I(\Omega_0, \cdot)$. For $\psi \in V_{\mathrm{ad}}(\Omega_\delta)$ we define

$$I_{\varphi_0}(\Omega_\delta, \psi) := I(\Omega_\delta, \psi) + \int_{\Omega_\delta} |\varphi_0 - \psi|^p \, \mathrm{d}x.$$

Obviously, $I_{\varphi_0}(\Omega_0, \varphi_0) = I(\Omega_0, \varphi_0)$ and φ_0 is the unique minimizer of $I_{\varphi_0}(\Omega_0, \cdot)$. Let $I_{\varphi_0}(\Omega_\delta) = \min_{\psi \in V_{ad}(\Omega_\delta)} I_{\varphi_0}(\Omega_\delta, \psi)$. Assuming **A0**–**A5** one can prove the following [6]:

$$\lim_{\delta \to 0} \tfrac{1}{\delta}(I_{\varphi_0}(\Omega_\delta) - I_{\varphi_0}(\Omega_0)) = G(\varphi_0, \theta) = G(\varphi_0) \tag{4.5}$$

for every $\theta \in C_0^\infty(\tilde{\Omega})$ with $\theta = 1$ near the crack tip. Here, G is the expression from (4.2). The original energy $I(\Omega_0, \cdot)$ is modified in such a way that minimizers for the domain with the extended crack are close (in the L^p–sense) to the chosen minimizer φ_0 on Ω_0. Relation (4.5) reveals that $G(\varphi_0)$ can be considered as a local energy release rate.

Proof. Let us give a short sketch of the proofs of theorems 4.1 and 4.2. For a detailed proof we refer to [6]. Analogously to the linear elastic case we introduce the following mapping:

$$T_\delta : \Omega_\delta \to \Omega_0 : \; x \mapsto T_\delta(x) = x - \delta\theta(x) \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \tag{4.6}$$

where $\theta \in C_0^\infty(\tilde{\Omega})$ with $\theta = 1$ near the crack tip. If $|\delta| < \delta_0$ is small enough, then T_δ is a diffeomorphism from Ω_δ to Ω_0 which maps C_δ to C_0, see [20]. Let $\{ \varphi_\delta \, ; \, \delta \in [0, \delta_0] \}$ be minimizers corresponding to $I(\Omega_\delta, \cdot)$. For every $\delta > 0$ we have

$$\begin{aligned} \tfrac{1}{\delta}\big(I(\Omega_0, \varphi_0) - I(\Omega_\delta, \varphi_0 \circ T_\delta)\big) &\le \tfrac{1}{\delta}\big(I(\Omega_0) - I(\Omega_\delta)\big) \\ &\le \tfrac{1}{\delta}\big(I(\Omega_0, \varphi_\delta \circ T_\delta^{-1}) - I(\Omega_\delta, \varphi_\delta)\big). \end{aligned} \tag{4.7}$$

Like in [17, 18] one verifies that

$$\lim_{\delta\searrow 0} \tfrac{1}{\delta}\big(I(\Omega_0,\varphi_0) - I(\Omega_\delta,\varphi_0\circ T_\delta)\big) = G(\varphi_0,\theta). \tag{4.8}$$

Since the minimizer φ_0 was chosen arbitrarily, it follows from (4.7) and (4.8) that

$$\liminf_{\delta\searrow 0} \tfrac{1}{\delta}\big(I(\Omega_0) - I(\Omega_\delta)\big) \geq \sup\{\, G(\varphi_0,\theta)\,;\, \varphi_0 \text{ minimizes } I(\Omega_0,\cdot)\,\}. \tag{4.9}$$

Using the convergence theorem 3.3 for Eshelby tensors one proves that the supremum in (4.9) is attained and can be replaced by "max".

In addition one can show that the sequence $(\varphi_\delta\circ T_\delta^{-1})_{\delta\in(0,\delta_0)}$ is a minimizing sequence for $I(\Omega_0,\cdot)$ and contains a subsequence which converges weakly in $V_{\mathrm{ad}}(\Omega_0)$ to a minimizer φ_0 of $I(\Omega_0,\cdot)$. Using again the convergence theorem for Eshelby tensors 3.3, we obtain for this subsequence that

$$\begin{aligned}\lim_{n\to\infty} \tfrac{1}{\delta_n}\big(I(\Omega_0,\varphi_{\delta_n}\circ T_{\delta_n}^{-1}) - I(\Omega_{\delta_n},\varphi_{\delta_n})\big) &= G(\varphi_0,\theta)\\ &\leq \max\{\, G(\varphi,\theta)\,;\, \varphi \text{ minimizes } I(\Omega_0,\cdot)\,\}. \end{aligned}\tag{4.10}$$

By contradiction it follows that the previous inequality holds true for the whole sequence. Combining (4.7) with (4.9) and (4.10) finishes the proof of theorem 4.1.

Theorem 4.2 is proved with Fubini's theorem and the fundamental lemma of the calculus of variations and uses the fact that $G(\varphi_0,\theta)$ is independent of the function θ if φ_0 is a minimizer of $I(\Omega_0,\cdot)$, see [6]. $\square$

We emphasize that theorems 4.1 and 4.2 are derived without making any smoothness assumptions on the minimizers.

Remark 4.3. The normal stress $h \in (W^{1-\frac{1}{p},p}(\Gamma_N))'$ does not appear explicitly in the formula for $G(\varphi_0,\theta)$. This follows from (4.8) and the properties of the mapping T_δ: for every $x\in\Gamma_N$ and every δ we have $T_\delta(x) = x$, which implies that

$$\langle h,\varphi_0\rangle_{\Gamma_N} - \langle h\circ T_\delta,\varphi_0\circ T_\delta\rangle\big|_{T_\delta(\Gamma_N)} = 0.$$

Remark 4.4. Let φ_0 be a minimizer of $I(\Omega_0,\cdot)$ and assume that there are minimizers φ_δ of $I(\Omega_\delta,\cdot)$ for $\delta\in(0,\delta_0]$ such that the whole sequence $\varphi_\delta\circ T_\delta^{-1}$ converges weakly in $V_{\mathrm{ad}}(\Omega_0)$ to φ_0. Then $ERR(\Omega_0) = G(\varphi_0,\theta)$. This assumption on φ_0 is a weakened version of the assumptions made in [19, 21] and many other references on the dependence of minimizers on the crack parameter δ. In the nonlinear setting with nonconvex energies it is an open problem under which conditions such assumptions are satisfied.

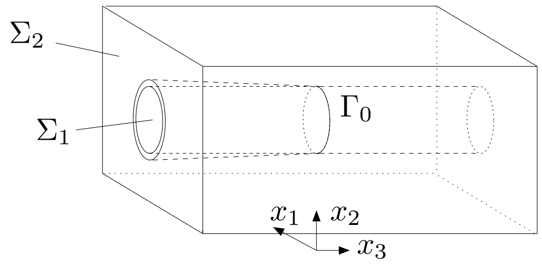

Figure 2: Matrix with embedded fiber and crack front Γ_0

5 A 3D example: Fiber embedded in a matrix

The methods of our proof are not restricted to the two–dimensional case. As a three–dimensional example we consider a compound consisting of a matrix with an embedded fiber of a different material. The geometry is given as follows: Let $\Sigma, \Sigma_1, \Sigma_2 \subset \mathbb{R}^2$ be bounded domains with Lipschitz boundaries, $\Sigma_1 \Subset \Sigma$ and $\Sigma_2 = \Sigma \backslash \overline{\Sigma_1}$. Let furthermore $\Gamma_{12} = \partial \Sigma_1$. The region Σ_1 shall represent the cross section of the fiber, whereas Σ_2 is the cross section of the matrix. For $\delta \in \mathbb{R}$ we define $S_\delta = \{\, x \in \mathbb{R}^3\,;\, (x_1, x_2) \in \Gamma_{12},\, x_3 \leq \delta \,\}$. Let $\tilde{\Omega} = \Sigma \times (-\ell, \ell)$ for some $\ell > 0$ and let the domains Ω_δ be defined as $\Omega_\delta := \tilde{\Omega} \backslash S_\delta$. Furthermore, $C_\delta = \overline{\tilde{\Omega}} \cap S_\delta$ and $\Gamma_\delta = \Gamma_{12} \times \{\delta\}$. We call the domain Ω_0 the reference configuration with crack C_0 and crack front Γ_0. The fiber is defined through $\tilde{\Omega}_1 = \Sigma_1 \times (-\ell, \ell)$, whereas the matrix is given by $\tilde{\Omega}_2 = \Sigma_2 \times (-\ell, \ell)$. We assume that the crack front stays perpendicular to the fiber axis and thus is given by Γ_δ. With these assumptions, the domains Ω_δ with crack C_δ are admissible neighboring configurations of Ω_0 for small parameters $\delta > 0$. Let finally $\partial \tilde{\Omega} = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N are open, independent of δ and $\Gamma_D \neq \emptyset$, see fig. 2.

The fiber and the matrix are assumed to consist of hyperelastic materials with polyconvex elastic energy densities $W_i : \mathbb{M}^{d \times d} \to [0, \infty]$. We assume that the energy densities W_i satisfy the coercivity assumption **A2** with possibly different growth exponents p^i, r_1^i and r_2^i. Let $p = \min\{p^1, p^2\}$, $r_1 = \min\{r_1^1, r_1^2\}$ and $r_2 = \min\{r_2^1, r_2^2\}$. (If $p_i > 3$ we may choose $r_1^i = p^i/2$ and $r_2^i = p^i/3$ due to remark 3.1). The energy density describing the compound is defined as

$$W(x, F) = \begin{cases} W_1(F) & \text{if } x \in \tilde{\Omega}_1, \\ W_2(F) & \text{if } x \in \tilde{\Omega}_2 \end{cases} \tag{5.1}$$

and satisfies for almost every $x \in \tilde{\Omega}$ the coercivity assumption **A2** with p, r_1 and r_2. Assume that the data is chosen according to **A3**. Like in the two-dimensional case we define $V_{\mathrm{ad}}(\Omega_\delta) = \{ \varphi \in W^{1,p}(\Omega_\delta)\,;\, \varphi\big|_{\Gamma_D} = g_D \}$ and consider the following minimization problems for small $\delta \geq 0$:

$$\text{Find } \varphi_\delta \in V_{\mathrm{ad}}(\Omega_\delta) \text{ such that for every } \psi \in V_{\mathrm{ad}}(\Omega_\delta) \text{ we have}$$
$$I(\Omega_\delta, \varphi_\delta) \leq I(\Omega_\delta, \psi) = \int_{\Omega_\delta} W(x, \nabla\psi(x))\,\mathrm{d}x - \int_{\Omega_\delta} f \cdot \psi\,\mathrm{d}x - \langle h, \psi\big|_{\Gamma_N} \rangle_{\Gamma_N}.$$

Again, Ball's theorem guarantees the existence of minimizers $\varphi_\delta \in V_{\mathrm{ad}}(\Omega_\delta)$. Let $I(\Omega_\delta) = \min_{\psi \in V_{\mathrm{ad}}(\Omega_\delta)} I(\Omega_\delta, \psi)$ and let the energy release rate be defined as

$$ERR_{3D}(\Omega_0) = \lim_{\delta \searrow 0} \tfrac{1}{\delta}\big(I(\Omega_\delta) - I(\Omega_0)\big).$$

The criterion for a stationary crack reads now

$$ERR_{3D}(\Omega_0) < 2\gamma\,|\Gamma_0| \;\Rightarrow\; \textit{the crack is stationary.}$$

In order to calculate the energy release rate we introduce the following mapping between Ω_δ and Ω_0:

$$T_\delta : \Omega_\delta \to \Omega_0,\; x \mapsto x - \delta\theta(x)\vec{e}_3,$$

where $\vec{e}_3 = (0,0,1)^\top$ and $\theta \in C_0^\infty(\tilde{\Omega})$ is a function with $\theta = 1$ in an open neighborhood of the crack front Γ_0. Exactly in the same way as in the two-dimensional case one arrives at the following formula for $ERR_{3D}(\Omega_0)$:

Theorem 5.1. *Let $W : \tilde{\Omega} \times \mathbb{M}^{d\times d} \to [0,\infty]$ be polyconvex and coercive as described above and assume that* **A4** *and* **A5** *are satisfied by both, W_1 and W_2. If $\inf_{\psi \in V_{ad}(\Omega_0)} I(\Omega_0, \psi) < \infty$, then*

$$ERR_{3D}(\Omega_0) = \max\{\, G(\varphi, \theta)\,;\, \varphi \textit{ minimizes } I(\Omega_0, \cdot) \,\},$$

where

$$G(\varphi, \theta) = -\int_{\Omega_0} E(\nabla\varphi) : \nabla\left(\theta\vec{e}_3\right)\,\mathrm{d}x - \int_{\Omega_0} \theta f \cdot \partial_{x_3}\varphi\,\mathrm{d}x.$$

Again, $G(\varphi_0, \theta)$ is independent of θ for minimizers φ_0 of $I(\Omega_0, \cdot)$.

6 Conclusions

We showed that the energy release rate is well defined in the framework of finite–strain elasticity provided that the elastic energy density satisfies, in addition to polyconvexity, the stress control assumptions **A4** and **A5**. The energy release rate can be expressed through a generalized Griffith formula and the well–known J–integral. Furthermore, we discussed the main difference between the formulas for the linear elastic case, where the minimizers are unique, and the finite–strain case with possibly several minimizers. We also presented an example for an interface crack in a compound of two different materials.

Up to now, the two sides of the crack are allowed to penetrate into each other, which may lead to unphysical solutions. Without any essential changes, our analysis can be applied to the case where non–interpenetration conditions are included in the spaces $V_{\text{ad}}(\Omega_\delta)$. It only has to be ensured that the mapping T_δ introduced in (4.6) preserves the non–interpenetration conditions. Moreover, the analysis can be extended to smooth, curved cracks, which will be investigated in a forthcoming paper.

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