Dispersive and long-time behavior of oscillations in lattices

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We study the long-time dynamics of oscillations in lattices of infinitely many particles interacting via harmonic springs and adumbrate how decay rates for the displacements are proved analytically applying methods for oscillatory integrals. After recapitulating the results of an infinite chain with nearest-neighbour interaction, we highlight the differences to the case of next-nearest-neighbour interaction. Finally we give an outlook on the two dimensional lattice with nearest-neighbour interaction where more complicated singularities occur.

1 Introduction

Infinite dimensional Hamiltonian systems on lattices are of great importance in the challenging field of multiscale problems as microscopic models (cf. [Mie06, GHM06]) as well as for the numerical analysis of partial differential equations (cf. [IZ05]). In both cases it is important to understand if (and when) the qualitative and quantitative properties of solutions are consistent with that of the corresponding continuous version.

This short note aims to the particular question, whether stability results hold for dispersive systems on lattices analogously to those known for PDE's (see e.g. [Rac92]). We outline how to obtain l^p-estimates for solutions of linearized systems on lattices. In PDE theory the proof of nonlinear dispersive stability is based on estimates of this form and similar arguments might also be used for the discrete case, cf. [GHM06].

2 Harmonic lattices, plane waves and dispersion relations

In the following we restrict ourselves to the scalar valued linear setting and consider the d-dimensional lattice $\mathbb{Z}^d$ of atoms with finite-range interaction. Thus, for $x_j(t) \in \mathbb{R}$ we have the set of coupled ODE's

$$\ddot{x}_j = -\sum_{0<|k|\leq n} a_k(x_j - x_{j+k}) \ , \quad j \in \mathbb{Z}^d \ , \tag{1}$$

where $a_k = a_{-k} \in \mathbb{R}$. The system is Hamiltonian with energy $\mathcal{H}(\mathbf{x}, \dot{\mathbf{x}}) = \sum_{j\in\mathbb{Z}^d}\big(\frac{1}{2}\dot{x}_j^2 + \frac{1}{4}\sum_{0<|k|\leq n} a_k(x_j - x_{j+k})^2\big)$. Note that the formulation of the $\mathbb{R}^m$-valued case on general Bravais lattices is not essentially different, see [Mie06]. For simplicity we will state everything in terms of the absolute amplitude, but of course similar results hold also for relative deviations as well as in terms of the energy per particle.

A fundamental feature of such harmonic lattices is the presence of *plane wave* solutions of the form $x_j(t) = A\,\mathrm{e}^{\mathrm{i}(\omega t + \theta\cdot j)}$ for $j \in \mathbb{Z}^d$, $A \in \mathbb{C}\backslash\{0\}$ and $\theta \in \mathbb{T}^d$. A travelling wave of this form exists if and only if the *dispersion relation*

$$\omega^2 = 2\sum_{k=1}^{n} a_k\big(1 - \cos(k\cdot\theta)\big) =: \mathbb{A}(\theta) \tag{2}$$

holds. Dispersion occurs if and only if there exists $\theta \in \mathbb{T}^d$ such that $\mathbf{D}^2\omega(\theta) \neq 0$. As it is well known for PDE's the properties of the dispersion relation are essential for the asymtotic behavior of solution of (1).

3 Asymptotic analysis and l^p-estimates

Since the system is linear it is possible to solve (1) explicitly using the Fourier transform on the commutative group $\mathbb{Z}^d$ which is defined by

$$\mathcal{F}:\ \hat{x}(t,\theta) = \sum_{j\in\mathbb{Z}^d} x_j(t)\,\mathrm{e}^{-ij\cdot\theta} \ , \qquad \mathcal{F}^{-1}:\ x_j(t) = \frac{1}{2\pi}\int_{\mathbb{T}^d} \hat{x}(t,\theta)\,\mathrm{e}^{ij\cdot\theta}\,\mathrm{d}\theta \ . \tag{3}$$

Applying the Fourier transform to the system (1) the right hand side, which is basically a linear combination of shift operators, turns into the multiplication operator $\mathbb{A}(\theta)$ in Fourier space. Written as a first order system the obtained ODE reads as $\frac{\partial}{\partial t}\hat{x}(t,\theta) = \omega(\theta)\hat{p}(t,\theta)$, $\frac{\partial}{\partial t}\hat{p}(t,\theta) = -\omega(\theta)\hat{x}(t,\theta)$. Thus, the explicit solution of (1) is determined by the Green's function

$$G_j(t) = \frac{1}{2\pi}\int_{\mathbb{T}^d} \begin{pmatrix} \cos(\omega(\theta)t) & \frac{1}{\omega(\theta)}\,\sin(\omega(\theta)t) \\ -\omega(\theta)\,\sin(\omega(\theta)t) & \cos(\omega(\theta)t) \end{pmatrix} \mathrm{e}^{\mathrm{i}j\cdot\theta}\,\mathrm{d}\theta \ . \tag{4}$$

To obtain a priori estimates for the solutions of (1) we need the asymptotics of $G_j(t)$ for $t \to \infty$. The components of $G_j(t)$ are basically of the form $\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}(j\cdot\theta \pm \omega(\theta)t)} \mathrm{d}\theta$. Thus, they are typical examples of oscillatory integrals and the *stationary phase method* may be applied to calculate the asymptotics, cf. [Hör90, Ste93].

In [Fri03] the harmonic chain with nearest-neighbour interaction, that is $\mathbb{A}(\theta) = 2(1-\cos\theta)$, is studied in detail using these methods. The amplitudes decay $\sim t^{-1/3}$ near the macroscopic wave fronts $j \approx \pm t$ and $\sim t^{-1/2}$ in the inner region, cf. the first three plots of Figure 1. Note that the macroscopic wave front corresponds to $\theta \approx 0$ with $\lim_{\theta\to 0} \omega'(\theta) = 1$, $\lim_{\theta\to 0} \omega''(\theta) = 0$ and $\lim_{\theta\to 0} \omega'''(\theta) \neq 0$; whereas in the inner region we have $\theta \neq 0$ and hence $\omega''(\theta) \neq 0$. As a second example let us consider the harmonic chain with next-nearest-neighbour interaction, that is with symbol $\mathbb{A}(\theta) = 2(a_1+a_2) - 2a_1\cos(\theta) - 2a_2\cos(2\theta)$, where $a_1, a_2 > 0$. The time evolution of this system (with $a_1 = a_2 = 1/5$) is plotted again in Figure 1. In this example a second wave front occours which is due to the fact that there exists $\theta_1 \in (0,\pi)$ such $\omega''(\theta_1) = 0$. The asymptotics for this case are stated precisely in the following theorem. Note that the result depends mainly on the properties of $\omega(\theta)$, thus, the asymptotics depend essentially on the dispersion relation.

Theorem 3.1 *Consider* (1) *with initial condition* $(x_j(0), \dot{x}_j(0)) = (\delta_{0,j}, 0)$ *and assume that there exists just one* $\theta_1 \neq 0$ *such that* $\omega''(\theta_1) = 0$ *and furthermore* $\omega'''(\theta_1) \neq 0$. *Then*

(i) $\exists\, C > 0\ \forall\, t > 0 :\ |x_j(t)| \ \le\ C \cdot t^{-1/3}$ *for all* $j \in \mathbb{Z}$,

(ii) $\forall\, \delta > 0\ \exists\, C_\delta > 0\ \forall\, t > 0 :\ |x_j(t)| \ \le\ C \cdot t^{-1/2}$ *for* $|j| \notin [(\omega'(0)-\delta)t\,, (\omega'(0)+\delta)t] \cup [(\omega'(\theta_1)-\delta)t\,, (\omega'(\theta_1)+\delta)t]$.

If we do not care about the spatial resolution of the wave fronts and formulate the result in terms of l^p-spaces we obtain the *a priory estimate*

$$\exists\, C > 0\ \forall \text{ initial conditions } (x_0, \dot{x}_0) \in l_1\ \forall\, t > 0 : \quad \|(x(t), \dot{x}(t))\|_{l^\infty} \le \tfrac{C}{(1+t)^{1/3}}\ \|(x_0, \dot{x}_0)\|_{l^1} \ .$$

Estimates of this form are the key ingredient to prove dispersive stability of nonlinear systems, cf. [GHM06], and they are of great importance for the analysis of numerical schemes for dispersive nonlinear wave equations, cf. [IZ05].

To finish this note we want to give an outlook on the scalar valued square lattice. We consider the system (1) with $\mathbb{A}(\theta_1, \theta_2) = 4 - 2\cos(\theta_1) - 2\cos(\theta_2)$. Although the details are more complicated, it is possible to qualify the asymptotics in the same way as above. On the basis of the degeneracy of the dispersion relation it is possible to distinguish three different regions: The four peaks that decay $\sim t^{-3/4}$, the inner wave fronts where the amplitude behaves $\sim t^{-5/6}$ and the remaining region inside the outer wave front where we have $\sim t^{-1}$, see Figure 1 to the right.

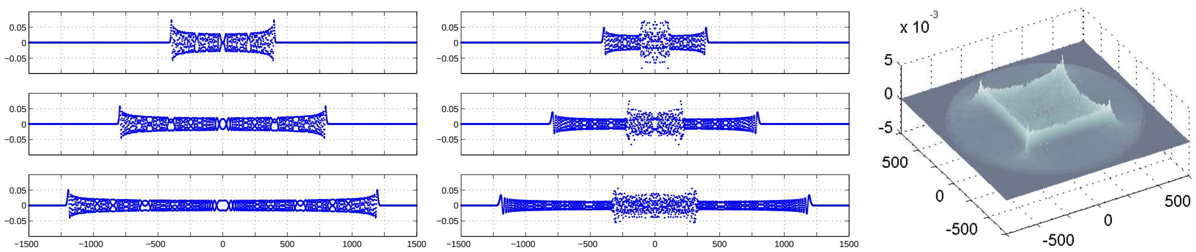

Fig. 1 Time evolution of $x_j(0) = \delta_{0,j}$ for the harmonic chain with NN, next-NN interaction and the square lattice.

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