Lower semi–continuity and existence of minimizers in incremental finite-strain elastoplasticity*

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Abstract

We study incremental problems in geometrically nonlinear elastoplasticity. Using the multiplicative decomposition $\mathrm{D}\varphi = F_{\mathrm{el}} F_{\mathrm{pl}}$ we consider general energy functionals of the form

$$\mathcal{I}(\varphi, F_{\mathrm{pl}}) = \int_\Omega U(x, \mathrm{D}\varphi F_{\mathrm{pl}}^{-1}, F_{\mathrm{pl}}, \mathcal{G}(F_{\mathrm{pl}}))\,\mathrm{d}x - \langle \ell, \varphi \rangle,$$

which occur as the sum of the stored energy and the dissipation in one time step. Here $\mathcal{G}(F_{\mathrm{pl}})$ is the dislocation tensor which takes the form $\frac{1}{\det F_{\mathrm{pl}}} \mathrm{curl}_3(F_{\mathrm{pl}})\, F_{\mathrm{pl}}^{\mathsf{T}}$ in dimension $d = 3$.

Imposing the usual constraint $\det F_{\mathrm{pl}} \equiv 1$ and suitable growth and polyconvexity conditions on U we show that the minimum of $\mathcal{I}$ is attained in the natural Sobolev spaces. Moreover, we are able to treat multiple time steps by controlling the stored and dissipated energies. We also address the relation of the incremental problem to the time-continuous energetic formulation of elastoplasticity.

1 Introduction

1.1 From infinitesimal to finite-strain elastoplasticity

Elastoplastic processes play an important role in many engineering applications. Despite the fact that many models are successfully used for numerical simulations, a satisfactory mathematical theory was only developed for the linearized case in the 1970s by J.J.

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Moreau [Mor76]. For further developments including efficient numerical implementations see, e.g., [HR99]. This theory relies on the additive decomposition

$$\varepsilon = \tfrac{1}{2}(\mathrm{D}u + \mathrm{D}u^{\mathsf{T}}) = \varepsilon_{\mathrm{el}} + \varepsilon_{\mathrm{pl}} \tag{1.1}$$

of the linearized strain tensor ε, where $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ denotes the displacement. Moreover, the energy is assumed to be a quadratic functional such that the problem takes the form of a quasi-variational inequality. However, within the last decades it became desirable to predict plastic behavior also under large deformations and corresponding models were developed in the engineering literature [Lee69, SO85, MS92]. These theories are usually based on the multiplicative decomposition

$$F = \mathrm{D}\varphi = F_{\mathrm{el}} F_{\mathrm{pl}}. \tag{1.2}$$

A fundamental difficulty concerning models involving large deformations is that frame-indifference, i.e., invariance under rigid motions, is inconsistent with the convexity assumptions which are in the heart of the infinitesimal theory.

In nonlinear elastostatics, which is governed by the energy functional

$$\int_\Omega W(\mathrm{D}\varphi)\,\mathrm{d}x, \tag{1.3}$$

Ball [Bal77] achieved a breakthrough by identifying polyconvexity as a condition on W which is both physically realistic and mathematically tractable. He showed that if W is polyconvex and satisfies certain coercivity conditions, i.e., lower growth bounds, then the minimum of the energy functionals is always attained when subjected to suitable boundary conditions. Here a function $F \mapsto W(F)$ is called polyconvex if it can be written as a convex function of the minors (subdeterminants) of F; typical examples include Moonley-Rivlin and Ogden materials. The crucial functional analytic property of minors is that they commute with weak convergence, i.e., if $M_s(F)$ is a minor of order s, if $q > s$, and if

$$\mathrm{D}\varphi^{(k)} \rightharpoonup \mathrm{D}\varphi \quad \text{in } \mathrm{L}^q(\Omega; \mathbb{R}^{d\times d}) \quad \text{(weakly)} \tag{1.4}$$

then

$$M_s(\mathrm{D}\varphi^{(k)}) \rightharpoonup M_s(\mathrm{D}\varphi) \quad \text{in } \mathrm{L}^{q/s}(\Omega) \quad \text{(weakly)}, \tag{1.5}$$

see also [Mor52, Res67]. Here and in the following we use the half-arrow $\rightharpoonup$ to denote weak convergence.

1.2 Time-discrete evolution models

The goal of this paper is to establish similar existence results in the context of elastoplastic evolution problems. More precisely, we consider a time-discretized version which leads to a sequence of minimization problems (one for each time step). Such formulations have recently attracted a lot of attention in the engineering literature [OR99, OS99, CHM02, LMD03, Mie03a, NW03]. In the simplest version one considers a multiplicative decomposition of the deformation gradient $\mathrm{D}\varphi = F_{\mathrm{el}}F_{\mathrm{pl}}$ and assumes that the elastic energy depends only on F_{el} and on suitable hardening parameters $p \in \mathbb{R}^m$. This leads to the energy functional

$$\widetilde{\mathcal{E}}(t, \varphi, F_{\mathrm{pl}}, p) = \int_\Omega W(x, \mathrm{D}\varphi(x)\, F_{\mathrm{pl}}(x)^{-1}, p(x))\, \mathrm{d}x - \langle \ell(t), \varphi \rangle$$

where the time-dependent, external loading ℓ is given via

$$\langle \ell(t), \varphi \rangle = \int_\Omega f_{\mathrm{vol}}(t, x) \cdot \varphi(x)\, \mathrm{d}x + \int_{\partial\Omega} f_{\mathrm{surf}}(t, x) \cdot \varphi(x)\, \mathrm{d}a.$$

At the j-th time step one has to solve the problem

$$\widetilde{\mathcal{E}}(t_j, \varphi, F_{\mathrm{pl}}, p) + \mathcal{D}((F_{\mathrm{pl}}, p), (F_{\mathrm{pl}}^{(j-1)}, p_{j-1})) \quad \rightsquigarrow \quad \min, \tag{1.6}$$

where $F_{\mathrm{pl}}^{(j-1)}$ and p_{j-1} are the values of the plastic strain and the hardening parameter at time step $j-1$ and where $\mathcal{D}$ denotes the dissipation distance, which measures the energy dissipated by passing from the state $(F_{\mathrm{pl}}^{(j-1)}, p_{j-1})$ to (F_{pl}, p). For the precise definition of $\mathcal{D}$ see (3.2) and (3.5) below. An important observation is that in general the minimum in (1.6) is not achieved. Minimizing sequences develop fine scale oscillations (microstructures) or concentrations (localization), see [OR99, CHM02, LMD03, Mie03a, BCHH04, Mie04a]. This is due to the fact that the theory involves no intrinsic length scale. Note that Ball's theory no longer applies since the elastic part $F_{\mathrm{el}} = \mathrm{D}\varphi F_{\mathrm{pl}}^{-1}$ is in general not compatible, i.e., cannot be expressed as the gradient of a deformation field ψ.

To introduce a length scale we consider the geometric dislocation tensor $G = \mathcal{G}(F_{\mathrm{pl}})$, which represents the incompatibility of the so-called intermediate configuration F_{pl} relative to the associated surface elements. In Sections 2 and 5 we discuss the general form of the operator $\mathcal{G}$, which is a vector-valued two-form (viz., a tensor of order 3). In dimension $d = 3$ the tensor $G = \mathcal{G}(F_{\mathrm{pl}})$ can be identified with

$$\widehat{G} = \widehat{\mathcal{G}}_3(F_{\mathrm{pl}}) := \frac{1}{\det F_{\mathrm{pl}}} (\mathrm{curl}_3\, F_{\mathrm{pl}})\, F_{\mathrm{pl}}^{\mathsf{T}} \in \mathbb{R}^{3\times 3}. \tag{1.7}$$

We refer to [Sve02, Eqn. (154)] and [CG01, Gur02] and to Sections 2 and 5 for further discussions and note that G in the two latter references means our G^{T}, since our curl_3 acts row by row on 3×3 matrices like in [Sve02], We now include the so-called stored defect energy $\int_\Omega V(x, G(x))\, \mathrm{d}x$ and are thus lead to the functional of the total stored energy:

$$\mathcal{E}(t, \varphi, F_{\mathrm{pl}}, p) = \int_\Omega W(x, \mathrm{D}\varphi(x)\, F_{\mathrm{pl}}(x)^{-1}, p(x)) + V(x, \mathcal{G}(F_{\mathrm{pl}})(x))\, \mathrm{d}x - \langle \ell(t), \varphi \rangle. \tag{1.8}$$

The expression for the stored defect energy is the most general local expression which is invariant under compatible changes in the reference configuration, cf. [Par95, PŠ99].

Going back to (1.6) we see that at each time step we have to minimize a functional of the form

$$\mathcal{I}_j(\varphi, F_{\mathrm{pl}}) = \int_\Omega U_j(x, \mathrm{D}\varphi(x) F_{\mathrm{pl}}^{-1}(x), F_{\mathrm{pl}}(x), \mathcal{G}(F_{\mathrm{pl}})(x))\, \mathrm{d}x - \langle \ell(t_j), \varphi \rangle,$$

where $U_j(x, F_{\rm el}, F_{\rm pl}, G) = \min_{p\in\mathbb{R}^m} \widehat{U}_j(x, F_{\rm el}, p, F_{\rm pl}, G)$ and where U_j depends on j through $(F_{\rm pl}^{(j-1)}, p_{j-1})$. In the following we also make strong use of the standard assumption that the plastic part $F_{\rm pl}$ is volume-preserving, i.e.,

$$\det F_{\rm pl} = 1.$$

1.3 Main results

Our first main result states that a minimizer of the above one-step problem exists if $U_j(x, F_{\rm el}, F_{\rm pl}, G)$ is polyconvex in $(F_{\rm el}, F_{\rm pl})$, convex in G and satisfies suitable growth conditions from below, see Theorem 2.7 for a precise statement.

Note that the control of G does not give us control of the full derivative of $F_{\rm pl}$ or $F_{\rm el}$; in that case lower semicontinuity and existence would be easy. Instead we control only certain special combinations of derivatives and our goal is to show that together with polyconvexity, and the constraint $\det F_{\rm pl} = 1$, this is just enough to obtain lower semicontinuity. This is exactly in the spirit of the Murat-Tartar theory of compensated compactness ([Tar79, Mur78, Mur81], see also [Tar90]) that develops conditions for weak semicontinuity of bilinear expressions if one has control on certain differential expressions of the sequences. In the context of variational problems this is closely related to the notions of quasiconvexity [Mor52] and more generally of A-quasiconvexity [Dac82, FM99, DF02] which define essentially necessary and sufficient conditions for weak lower semicontinuity. The sufficiency statement, however, requires growth conditions which are undesirable in nonlinear elasticity (since they force the energy to remain finite even at infinite elastic compression). We therefore work directly with polyconvexity, and Lemma 2.4 contains the crucial linear algebra calculation.

The functions U_j are defined implicitly and it is therefore not obvious how to translate the hypotheses on U into those on W and those on the dissipation distance $\mathcal{D}$. We thus show by means of an example that the conditions imposed on U in Theorem 2.7 can be satisfied for realistic choices of W and $\mathcal{D}$, see Theorem 3.1 and Section 4.

So far we have only discussed one time step. Let us now address the full incremental problem (IP):

(IP) For a given partition $t_0 = 0 < t_1 < \ldots < t_N = T$ and given initial values $(F_{\rm pl}^{(0)}, p_0)$ find incrementally, for $j = 1, \ldots, N$,

$$(\varphi_j, F_{\rm pl}^{(j)}, p_j) \in \operatorname*{Arg\,min}_{(\varphi, F_{\rm pl}, p)} \Big(\mathcal{E}(t_j, \varphi, F_{\rm pl}, p) + \mathcal{D}((F_{\rm pl}^{(j-1)}, p_{j-1}), (F_{\rm pl}, p))\Big).$$

Here "Arg min" denotes the set of all minimizers. Hence, (IP) consists of N minimization problems which are coupled via the dissipation distance $\mathcal{D}$.

Our second main result, Theorem 3.1, states that under the hypotheses discussed above (and some mild conditions on the loading) the problem (IP) has a solution for arbitrary partitions of $[0, T]$. In addition, we obtain a priori estimates which are independent of the partition. This represents an important first step to address the time-continuous evolution problem.

1.4 The time-continuous evolution problem

Following [MT99, MTL02, Mie02, Mie03b, MR03] the time-continuous, rate-independent evolution for elastic materials with internal variables ("standard generalized materials") can be formulated by energy principles as follows: A triple $(\varphi, F_{\rm pl}, p) : [0,T] \times \Omega \to \mathbb{R}^d \times \mathrm{SL}(\mathbb{R}^d) \times \mathbb{R}^m$ is called an *energetic solution* of the elastoplastic problem associated with $\mathcal{E}(t,\cdot,\cdot)$ and $\mathcal{D}$, if for all $t \in [0,T]$ **stability (S)** and **energy balance (E)** holds:

(S) $\quad \mathcal{E}(t,\varphi(t),F_{\rm pl}(t),p(t)) \le \mathcal{E}(t,\widetilde{\varphi},\widetilde{F}_{\rm pl},\widetilde{p}) + \mathcal{D}((F_{\rm pl}(t),p(t)),(\widetilde{F}_{\rm pl},\widetilde{p}))$ for all $(\widetilde{\varphi},\widetilde{F}_{\rm pl},\widetilde{p})$.

(E) $\quad \mathcal{E}(t,\varphi(t),F_{\rm pl}(t),p(t)) + \mathrm{Diss}((F_{\rm pl},p);[0,t])$
$$= \mathcal{E}(0,\varphi(0),F_{\rm pl}(0),p(0)) - \int_0^t \langle \dot{\ell}(\tau),\varphi(\tau)\rangle \,\mathrm{d}\tau.$$

The dissipation $\mathrm{Diss}((F_{\rm pl},p);[0,t])$ of an internal process $(F_{\rm pl},p) : [0,T]\times\Omega \to \mathrm{SL}(R^d) \times \mathbb{R}^m$ is defined for smooth processes via $\int_0^t \int_\Omega \delta(x,F_{\rm pl},p,\dot{F}_{\rm pl},\dot{p})\,\mathrm{d}x\,\mathrm{d}\tau$ and for general processes via $\sup\Big(\sum_{j=1}^k \mathcal{D}((F_{\rm pl}(\tau_{j-1}),p(\tau_{j-1})),(F_{\rm pl}(\tau_j),p(\tau_j)))\Big)$, where the supremum runs over all partitions of $[0,t]$.

So far, we are not able to provide existence results for (S) & (E) in the present elastoplastic setting for finite strains. The case of infinitesimal-strain elastoplasticity can be formulated and solved via (S) & (E) in a natural way, see [Mie03b, MT04, MM04, Mie04c]. Moreover, for models in phase transformations [MTL02, MR03], in delamination [KMR04], in micromagnetism [Kru02, RK04], and in fracture [FM98, DT02, DFT04] this approach leads to quite general existence results. First positive results for the finite-strain case are given in [FM04]; however, the assumptions there do not cover elastoplasticity.

1.5 Related mathematical work

Existence results in elastoplasticity are mainly restricted to the case of infinitesimal strains which leads to the additive decomposition (1.1). Starting with [Mor74, Mor76, Joh76, Suq81] a full theory was developed for linearized elastoplasticity with and without hardening. Subsequently, a systematic mathematical analysis of the numerical approximations of the solutions was developed, see the monograph [HR99] and for some recent improvements also [AC00, CA03].

More general material models including viscoplasticity and quite general hardening laws are treated in [Alb98, ACZ99, Che01a, Che01b] via the theory of monotone operators. Here again the elastic part is assumed to be linear, i.e., uses infinitesimal strains.

The mathematical theory for the finite-strain case is much less developed. Even for static problems there are only partial results. In fact, a major open problem is still the question which global properties the constitutive laws have to satisfy for local or global existence. First investigations of the relevant convexity and coercivity conditions are done in [Šil01b, Šil01a, Mie03b, Mie04b]. These works show some similarities to the present one, but here we use the regularizing term through $\mathcal{G}(F_{\rm pl})$ and thus we are able to allow for more general constitutive functions.

As was observed in [OR99], one has to expect the formation of microstructures if no regularizing terms are present. For such models the incremental problem (IP) has in general no solution (cf. [CHM02]) and one is forced to study suitably relaxed problems. A general strategy for relaxations of rate-independent evolutionary problems is not yet available, but first approaches are presented in [MR03, BCHH04, Mie04a]. In [CT03] a rigorous relaxation for a model in rigid plasticity with a single-slip system is performed.

Local existence and uniqueness results for smooth solutions of a viscoplastic model with finite-strains is developed in [Nef03a, Nef03b]. This model uses a rotation matrix as an additional internal variable which is interpreted as a grain boundary relaxation. A numerical comparison of different models in finite-strain elastoplasticity is given in [NW03].

2 A lower semi-continuity result

To simplify the notation we write from now on $P = F_{\mathrm{pl}}$. We will usually assume that P takes values in the special linear group

$$\mathrm{SL}(d) = \{\, F \in \mathbb{R}^{d\times d} \mid \det F = 1 \,\}.$$

Throughout we are interested in the cases $d = 2$ and $d = 3$, but we use a general tensor notation to avoid the separate discussion of the two cases. For a general treatment using differential forms we refer to Section 5.

We consider the functional

$$\mathcal{I}(\varphi, P) = \int_\Omega U(x, \mathrm{D}\varphi(x)P(x)^{-1}, P(x), \mathcal{G}(P)(x))\,\mathrm{d}x, \tag{2.1}$$

where $U : \Omega \times \mathbb{R}^{d\times d} \times \mathrm{SL}(d) \times \mathbb{R}^{d\times d\times d} \to \mathbb{R} \cup \{\infty\}$ is assumed to be a *normal integrand* in the sense of [ET76], i.e., there exists a Borel measurable function $\widetilde{U}$ such that $U(x, \cdot)$ coincides with $\widetilde{U}(x, \cdot)$ and that $(F_{\mathrm{el}}, P, G) \mapsto \widetilde{U}(x, F_{\mathrm{el}}, P, G)$ is lower semi-continuous for all $x \in \Omega$ except on a null set.

The geometric dislocation tensor $\mathcal{G}(P)$ is best considered as a vector-valued two-form, and thus is a tensor of order 3. Using the directional derivatives $\mathrm{D}P(x)[v]$ it is defined by its antisymmetric action on a pair of vectors $[a, b] \in \mathbb{R}^d \times \mathbb{R}^d$ via

$$\mathcal{G}(P)(x)[a, b] := \mathrm{D}P(x)[P^{-1}a]P^{-1}b - \mathrm{D}P(x)[P^{-1}b]P^{-1}a \in \mathbb{R}^d.$$

In this section we concentrate on the cases $d = 2$ and 3, where the tensor $\mathcal{G}(P)$ can be identified with $\widehat{\mathcal{G}}_2(P) \in \mathbb{R}^2$ and $\widehat{\mathcal{G}}_3(P) \in \mathbb{R}^{3\times 3}$ given in the form

$$\widehat{\mathcal{G}}_2(P) = \frac{1}{\det P}\begin{pmatrix} \partial_1 P_{12} - \partial_2 P_{11} \\ \partial_1 P_{22} - \partial_2 P_{21} \end{pmatrix} \in \mathbb{R}^2. \tag{2.2}$$

$$\widehat{\mathcal{G}}_3(P) = \frac{1}{\det P}(\mathrm{curl}_3\, P)\, P^{\mathsf{T}} \in \mathbb{R}^{3\times 3}. \tag{2.3}$$

In the latter case "curl_3" acts on a matrix in $\mathbb{R}^{3\times 3}$ by applying the vectorial curl_3 to each row separately and thus generates a matrix in $\mathbb{R}^{3\times 3}$ again. For the derivation of these identifications we refer to Section 5 where the case of general dimension d is treated more elegantly in terms of differential forms.

In general dimensions we define $\mathrm{curl}\, P$ row by row via

$$(\mathrm{curl}\, P)_{ijk} = \partial_j P_{ik} - \partial_k P_{ij}.$$

Thus $A = \mathrm{curl}\, P$ is a tensor of order 3 which is antisymmetric in the following sense:

$$A \in \mathbb{R}^{d\times d\times d}_{\mathrm{anti}} = \{\, (B_{ijk})_{i,j,k=1,\dots,d} \mid B_{ijk} = -B_{ikj} \,\}.$$

In particular, $\mathrm{curl}\, \mathrm{D}\psi \equiv 0$ for all $\psi \in \mathrm{C}^2(\Omega;\mathbb{R}^d)$. As seen above we can identify $\mathrm{curl}\, P$ with a matrix in $\mathbb{R}^{3\times 3}$ and with a vector in $\mathbb{R}^2$ for $d = 3$ and $d = 2$, respectively. We also consider $\mathrm{curl}\, P$ as a vector-valued two-form such that $\mathcal{G}(P)[a, b] = (\mathrm{curl}\, P)[P^{-1}a, P^{-1}b]$.

By $|F|$ we denote the Euclidean norm of a matrix, i.e., $|F|^2 = \sum_{i,j} F_{ij}^2$, and we recall that this norm bounds the Euclidean operator norm, i.e., $|Fa| \le |F||a|$ for all $a \in \mathbb{R}^d$, and that it is submultiplicative, i.e., $|FH| \le |F||H|$ and $|FP^{-1}| \le |F|/|P|$.

We will frequently use the following two estimates. First, for all $A, B > 0$ and all $\varepsilon > 0$ and $r > 1$ we have

$$A/B \ge r\varepsilon^{r/(r-1)} A^{1/r} - (r-1)\varepsilon B^{1/(r-1)}. \tag{2.4}$$

Second, let $q, q_1, q_2 \ge 1$ be such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. Then, for $F, H : \Omega \to \mathbb{R}^{d\times d}$ with $H \in \mathrm{L}^{q_2}(\Omega)$ and $FH^{-1} \in \mathrm{L}^{q_1}(\Omega)$ we have $F \in \mathrm{L}^q(\Omega)$ and

$$\|F\, H^{-1}\|_{q_1} \ge \|F\|_q / \|H\|_{q_2}. \tag{2.5}$$

The first estimate follows from Young's inequality $ab \le \frac{1}{r}a^r + \frac{r-1}{r} b^{r/(r-1)}$ by taking $a = (\varepsilon^{r-1}A)^{1/r}$ and $b = \varepsilon^{r-1}B$ and dividing by $\varepsilon^{r-1}B/r$. The second estimate follows by applying Hölder's inequality to $|F| = |(F\,H^{-1})\,H| \le |F\,H^{-1}|\,|H|$.

Similarly, for tensors A in $\mathbb{R}^{d\times d\times d}_{\mathrm{anti}}$ we let $|A| := (\sum_{i,j,k} A_{ijk}^2)^{1/2}$ and obtain the following estimate between $\mathcal{G}(P)$ and $\mathrm{curl}\, P$.

Lemma 2.1 *For $d \ge 2$ there exists $c_d > 0$ such that*

$$|\mathcal{G}(P)| \ge c_d\, |\mathrm{curl}\, P|\, / |P|^2. \tag{2.6}$$

Moreover, for $d = 2$ and $d = 3$ we have the estimates

$$|\widehat{\mathcal{G}}_2(P)| \ge \frac{|\mathrm{curl}\, P|}{|\det P|} \quad \text{and} \quad |\widehat{\mathcal{G}}_3(P)| \ge \frac{|\mathrm{curl}\, P|}{|\det P|\,|P^{-1}|}\,. \tag{2.7}$$

For a clearer geometric estimate of $\mathcal{G}(P)$ in terms of $\mathrm{curl}\, P$ and P we refer to Remark 2.5.

Proof: By the definition of $\mathcal{G}$ and $\mathrm{curl}\, P$ as bilinear forms we obtain, for all $a, b \in \mathbb{R}^d$,

$$|\mathrm{curl}\, P[a, b]| = |\mathcal{G}(P)[Pa, Pb]| \le C_d |\mathcal{G}(P)||Pa||Pb| \le C_d |\mathcal{G}(P)||P|^2 |a||b|.$$

This implies the first estimate with $c_d = 1/C_d$.

The additional estimates for $d = 2$ and 3 follow immediately from the special form of $\widehat{\mathcal{G}}_2$ and $\widehat{\mathcal{G}}_3$, respectively, see (2.2) and (2.3). ■

With these preparations we are able to derive our coercivity result, which is based on the following growth conditions on the density U. There exist $c > 0$ and $h \in \mathrm{L}^1(\Omega)$ such that for all x, F_{el}, P, G we have

$$U(x, F_{\mathrm{el}}, P, G) \geq c(|F_{\mathrm{el}}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) - h(x), \tag{2.8}$$

where $q_F, q_P, q_G > 1$.

Proposition 2.2 *Assume that* (2.8) *holds with exponents* q_F, q_P *and* q_G *satisfying*

$$\frac{1}{q_\varphi} := \frac{1}{q_F} + \frac{1}{q_P} \leq 1 \quad \textit{and} \quad \frac{1}{q_{\mathrm{C}}} := \frac{\min\{d-2, 2\}}{q_P} + \frac{1}{q_G} \leq 1.$$

Then for any $C_{\mathcal{I}} \in \mathbb{R}$ *there exists* $C > 0$ *such that* $\mathcal{I}(\varphi, P) \leq C_{\mathcal{I}}$ *and* $\det P \equiv 1$ *implies*

$$\|\mathrm{D}\varphi\|_{q_\varphi} + \|P\|_{q_P} + \|P^{-1}\|_{q_P} + \|\mathrm{curl}\, P\|_{q_{\mathrm{C}}} \leq C.$$

Proof: Using (2.8) and Lemma 2.1 with $\det P \equiv 1$ we obtain

$$\begin{array}{rcl} C_{\mathcal{I}} & \geq & \mathcal{I}(\varphi, P) \geq c(\|\mathrm{D}\varphi\, P^{-1}\|_{q_F}^{q_F} + \|P\|_{q_P}^{q_P} + \|P^{-1}\|_{q_P}^{q_P} + +\|\mathcal{G}(P)\|_{q_G}^{q_G}) - \int_\Omega h\,\mathrm{d}x \\ & \geq & c\Big[\|\mathrm{D}\varphi\, P^{-1}\|_{q_F}^{q_F} + \|P\|_{q_P}^{q_P} + \|P^{-1}\|_{q_P}^{q_P} + \|\,|\mathrm{curl}\, P|\,/|P^{\sigma_d}|^{\nu_d}\|_{q_G}^{q_G}\Big] - C_h \end{array}$$

where $\nu_d = \min\{d-2, 2\}$ and $\sigma_d = \mathrm{sign}(d-\frac{5}{2})$. Applying (2.5) to the first and the fourth term on the right-hand side gives

$$C_{\mathcal{I}} \geq c\Big[\|\mathrm{D}\varphi\|_{q_\varphi}^{q_F}/\|P\|_{q_P}^{q_F} + \|P\|_{q_P}^{q_P} + \|P^{-1}\|_{q_P}^{q_P} + \|\mathrm{curl}\, P\|_{q_{\mathrm{C}}}^{q_G}/\|P^{\sigma_d}\|_{q_P}^{\nu_d q_G}\Big] - C_h.$$

Finally, we apply (2.4) to the first and fourth term with r equal to q_F/q_φ and q_G/q_{C}, respectively, and choose ε sufficiently small in both cases to arrive at

$$C_{\mathcal{I}} \geq c\Big[c_1\|\mathrm{D}\varphi\|_{q_\varphi}^{q_\varphi} + \tfrac{1}{2}\|P\|_{q_P}^{q_P} + \tfrac{1}{2}\|P^{-1}\|_{q_P}^{q_P} + c_4\,\|\mathrm{curl}\, P\|_{q_{\mathrm{C}}}^{q_{\mathrm{C}}}\Big] - C_h,$$

with $c_1, c_4 > 0$. This is the desired result. ■

To discuss polyconvexity we denote by $\mathbb{M}_s(P)$ the $\binom{d}{s} \times \binom{d}{s}$ matrix of minors of order s of $P \in \mathbb{R}^{d\times d}$, in particular $\mathbb{M}_0(P) = 1$, $\mathbb{M}_1(P) = P$ and $\mathbb{M}_d(P) = \det P$.

Lemma 2.3 *Assume that* $q_P > d$ *and* $\frac{1}{q_{\mathrm{C}}} < \frac{1}{d} + \frac{1}{q_P}$. *Then, for any sequence* $P^{(k)} : \Omega \to \mathrm{SL}(d)$ *with*

$$\mathrm{curl}\, P^{(k)} \rightharpoonup \widetilde{A} \textit{ in } \mathrm{L}^{q_{\mathrm{C}}}(\Omega) \quad \textit{and} \quad P^{(k)} \rightharpoonup \widetilde{P} \textit{ in } \mathrm{L}^{q_P}(\Omega), \tag{2.9}$$

we have $\widetilde{A} = \mathrm{curl}\, \widetilde{P}$ *and*

(a) $\mathbb{M}_s(P^{(k)}) \rightharpoonup \mathbb{M}_s(\widetilde{P})$ *in* $\mathrm{L}^{q_s}(\Omega; \mathbb{R}^{\binom{d}{s}\times\binom{d}{s}})$ *with* $q_s = q_P/s$.

If in addition $\frac{1}{q_*} := \frac{1}{q_C} + \frac{d-2}{q_P} < 1$, *then*

(b) $\mathcal{G}(P^{(k)}) \rightharpoonup (\mathcal{G}\widetilde{P})$ *in* $\mathrm{L}^{q_*}(\Omega; \mathbb{R}^{d\times d\times d}_{\mathrm{anti}})$.

Proof: The relation $\widetilde{A} = \operatorname{curl} \widetilde{P}$ follows from the theory of distributions.

ad (a). This is a variation of the classical result (1.4), (1.5). The assertion follows from Theorem B.2 in Appendix B (applied with $p_j = q_P$, $f_j^{(k)} = P^{(k)\mathsf{T}} e_j$) and the compact embedding $\mathrm{L}^{q_C}(\Omega) \subset \mathrm{W}^{-1,q_P}(\Omega)$.

ad (b). Here we only proof the case $d \leq 3$ and refer to Proposition 5.1 for the case $d \geq 4$. In the case $d = 2$ there is nothing to be proved, since the constraint $\det P \equiv 1$ and the explicit form of $\mathcal{G}$ via $\widehat{\mathcal{G}}_2$ in (2.2) shows that $\mathcal{G}(P)$ is equal to $\operatorname{curl} P$ whose weak convergence has already been shown.

For the case $d = 3$ we use the special form of $\widehat{\mathcal{G}}_3$ given in (2.3) and $\det P = 1$, i.e., we identify $\mathcal{G}(P)$ with $(\operatorname{curl} P)P^{\mathsf{T}}$.

We apply the L^{q_P}-version of the Helmholtz decomposition to each row of $P^{(k)}$:

$$P^{(k)} = \mathrm{D}\psi^{(k)} + Q^{(k)} \quad \text{with } \operatorname{div} Q^{(k)} = 0,$$

see Proposition A.1. From (2.9) we conclude that $\psi^{(k)} \rightharpoonup \widetilde{\psi}$ in $\mathrm{W}^{1,q_P}(\Omega)$ and that $Q^{(k)} \rightharpoonup \widetilde{Q}$ in $\mathrm{L}^{q_P}(\Omega)$, where $\widetilde{P} = \mathrm{D}\widetilde{\psi} + \widetilde{Q}$ with $\operatorname{div} \widetilde{Q} = 0$. We can apply the third part of Proposition A.1 with $p = \min\{q_C, q_P\} > 1$ and conclude that $Q^{(k)} \to \widetilde{Q}$ (strongly) in $\mathrm{L}^1(\Omega)$. Since $Q^{(k)}$ is bounded in $\mathrm{L}^{q_P}(\Omega)$ we obtain (by Hölder's inequality) $Q^{(k)} \to \widetilde{Q}$ in $\mathrm{L}^q(\Omega)$ for all $q \in [1, q_P)$. Since $\operatorname{curl} Q^{(k)} = \operatorname{curl} P^{(k)}$ we deduce that $\operatorname{curl} Q^{(k)} \rightharpoonup \operatorname{curl} \widetilde{Q}$ in $\mathrm{L}^{q_C}(\Omega)$. Thus $(\operatorname{curl} Q^{(k)})(Q^{(k)})^{\mathsf{T}} \rightharpoonup (\operatorname{curl} \widetilde{Q})\widetilde{Q}^{\mathsf{T}}$ in $\mathrm{L}^{q_*}(\Omega)$.

Moreover, the div-curl lemma (see Theorem B.1) gives

$$(\operatorname{curl} Q^{(k)})(\mathrm{D}\psi^{(k)})^{\mathsf{T}} \rightharpoonup (\operatorname{curl} \widetilde{Q})(\mathrm{D}\widetilde{\psi})^{\mathsf{T}} \text{ in } \mathrm{L}^{q^*}(\Omega),$$

since $\operatorname{div} \operatorname{curl} Q^{(k)} = 0$ and $\operatorname{curl} \mathrm{D}\psi^{(k)} = 0$. Hence, we conclude that

$$\begin{aligned} \mathcal{G}(P^{(k)}) = (\operatorname{curl} P^{(k)})P^{(k)\mathsf{T}} &= (\operatorname{curl} Q^{(k)})(\mathrm{D}\psi^{(k)})^{\mathsf{T}} + (\operatorname{curl} Q^{(k)})Q^{(k)\mathsf{T}} \\ &\overset{\mathrm{L}^{q_*}}{\rightharpoonup} (\operatorname{curl} \widetilde{Q})(\mathrm{D}\widetilde{\varphi})^{\mathsf{T}} + (\operatorname{curl} \widetilde{Q})\widetilde{Q}^{\mathsf{T}} = (\operatorname{curl} \widetilde{P})\widetilde{P}^{\mathsf{T}} = \mathcal{G}(\widetilde{P}). \end{aligned}$$

This proves assertion (b). ■

The final result uses the special structure $F_{\mathrm{el}} = \mathrm{D}\varphi P^{-1}$ of the multiplicative decomposition. Later we will use the important condition $\det P \equiv 1$. For the moment, however, we only assume that $\det P > 0$ for clarity.

Lemma 2.4 *Assume that* $F, P \in \mathbb{R}^{d\times d}$ *with* $\det P \neq 0$. *Then, for* $1 \leq s \leq d$ *all minors of order* s *of the matrix* FP^{-1} *can be written in the form* $\det H / \det P$ *where* H *is a matrix obtained from* P *by replacing* s *rows of* P *by* s *rows of* F.

Proof: For the reader's convenience we give a separate proof for the most common cases $d = 2$ and $d = 3$, since this can be done without using the powerful, but perhaps not so familiar, notation of multilinear algebra.

Proof for $d = 2$. For full 2×2 determinants we use the formula $\det(FP^{-1}) = \det F/\det P$. The assertion for the 1×1 minors follows immediately from the explicit formula for the inverse

$$P^{-1} = \frac{1}{\det P}\begin{pmatrix} P_{22} & -P_{11} \\ -P_{21} & P_{11} \end{pmatrix}.$$

Proof for $d = 3$. We consider the cofactor matrix $\operatorname{cof} F$ and recall that $\operatorname{cof} F_{ij}$ is $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row i and column j. We have the identities

$$\operatorname{cof}(FG) = (\operatorname{cof} F)(\operatorname{cof} G) \quad \text{and} \quad F \operatorname{cof} F^{\mathsf{T}} = \det F\,\mathbf{1}. \tag{2.10}$$

Fix $i, j \in \{1, 2, 3\}$ and let H be the matrix obtained by replacing the j-th row of P by the i-th row of F. Applying (2.10) to H instead of F we get

$$\det H = \sum_k H_{jk}(\operatorname{cof} H)_{jk} = \sum_k F_{ik}(\operatorname{cof} P)_{jk}. \tag{2.11}$$

This yields

$$(FP^{-1})_{ij} = \left(F\frac{1}{\det P}(\operatorname{cof} F)^{\mathsf{T}}\right)_{ij} = \frac{1}{\det P}\sum_k F_{ik}(\operatorname{cof} P)_{jk} = \frac{\det H}{\det P}$$

and thus the assertion for the 1×1 minors. For the 3×3 minor one uses again the identity $\det(FP^{-1}) = \det F/\det P$. Finally, to study the 2×2 minor we note that

$$\operatorname{cof}(FP^{-1}) = (\operatorname{cof} F)(\operatorname{cof} P^{-1}) = (\operatorname{cof} F)(\operatorname{cof} P)^{-1} = \frac{1}{\det P}(\operatorname{cof} F)P^{\mathsf{T}}.$$

Thus

$$\operatorname{cof}(FP^{-1})_{ij} = \frac{1}{\det P}\sum_k (\operatorname{cof} F)_{ik} P_{jk}.$$

Exchanging the roles of F and P (and of i and j) in (2.11) we see that $\operatorname{cof}(FP^{-1})_{ij} = \det H/\det P$ where H is obtained by replacing the rows $\{1, 2, 3\} \setminus j$ of P by the rows $\{1, 2, 3\} \setminus i$ of F. This concludes the proof for case $d = 3$.

Proof for the general case. We first introduce some notations. As before, for a matrix $P \in \mathbb{R}^{d\times d}$ we denote by $\mathbb{M}_s(P)$ the $\binom{d}{s}\times\binom{d}{s}$ matrix of $s\times s$ minors. As indices of the rows and the columns of $\mathbb{M}_s(P)$ we use the (ordered) multi-indices $I = (i_{k_1}, \ldots, i_{k_s})$ with $1 \le i_{k_1} < \cdots < i_{k_s} \le d$, i.e. $\mathbb{M}_s(P)_{IJ} = \det P_{i_1,\ldots,i_s;j_1,\ldots,j_s}$. The formula of Cauchy-Binet gives the product rule $\mathbb{M}_s(PQ) = \mathbb{M}_s(P)\mathbb{M}_s(Q)$. To each multi-index I there is a unique (ordered) complementary index I^* such that $I \cap I^* = \emptyset$ and $I \cup I^* = \{1, \ldots, d\}$. The signature of I is the signature of the permutation $(I, I^*) \in \operatorname{Perm}(\{1, \ldots, d\})$. The cofactor matrix $\mathbb{K}_s(P)$ is defined by $\mathbb{K}_s(P)_{I^*J^*} = \operatorname{sgn} I \operatorname{sgn} J\, \mathbb{M}_s(P)_{IJ}$. Thus $\mathbb{K}_{d-1}(P)_{ij} = (-1)^{i+j}\mathbb{M}_{d-1}(P)_{i^*j^*} = \operatorname{cof} P$ where $\operatorname{cof} P$ is the usual cofactor matrix defined above. Again we have $\mathbb{K}_s(PQ) = \mathbb{K}_s(P)\mathbb{K}_s(Q)$. The general version of (2.10) is (see e.g. [Šil02, App. A])

$$\mathbb{M}_s(P)\,\mathbb{K}_{d-s}(P)^{\mathsf{T}} = \det P\,\mathbf{1} \in \mathbb{R}^{\binom{d}{s}\times\binom{d}{s}}. \tag{2.12}$$

To prove Lemma 2.4 for general d and s we fix two multiindices I and J of length s and consider the matrix

$$H = Q^I_J$$

which is obtained from P by replacing the rows $j_1, \ldots, j_s$ of P by the rows $i_1, \ldots, i_s$ of F. Applying the above formula (2.12) to H instead of P we get (by considering the JJ component)

$$\det H = \sum_K \mathbb{M}_s(H)_{JK} \mathbb{K}_{d-s}(H)_{JK} = \sum_K \mathbb{M}_s(F)_{IK} \mathbb{K}_{d-s}(P)_{JK},$$

where the sum runs over all multi-indices K of length s. On the other hand, we have

$$\mathbb{M}_s(FP^{-1}) = \mathbb{M}_s(F)\mathbb{M}_s(P^{-1}) = \mathbb{M}_s(F)(\mathbb{M}_s(P))^{-1} = \mathbb{M}_s(F)\frac{1}{\det P}\mathbb{K}_{d-s}(P)^{\mathsf{T}}.$$

Evaluating the IJ component of this identity we get

$$\mathbb{M}_s(FP^{-1})_{IJ} = \frac{1}{\det P}\sum_K \mathbb{M}_s(F)_{IK}\mathbb{K}_{d-s}(P)_{JK} = \frac{\det H}{\det P}. \tag{2.13}$$

This concludes the proof of Lemma 2.4. ∎

Remark 2.5 *Using the notations of the previous proof, we may formulate the estimate in Lemma 2.1 in a clearer and stronger way. In fact,* $\mathcal{G}(P)$ *is a composition of* $\operatorname{curl} P$ *and of the mapping* $\mathbb{M}_2(P^{-1})$ *which acts on bilinear forms on* $\mathbb{R}^d$*. Using* $\mathbb{M}_2(P^{-1})^{-1} = \mathbb{M}_2(P) = \det P\, \mathbb{K}_{d-2}(P^{-1})^{\mathsf{T}}$*, we obtain the estimates*

$$|\mathcal{G}(P)| \geq c_d \frac{|\operatorname{curl} P|}{|\mathbb{M}_2(P^{-1})^{-1}|} = c_d\frac{|\operatorname{curl} P|}{|\mathbb{M}_2(P)|} = c_d\frac{|\operatorname{curl} P|}{|\det P|\ |\mathbb{K}_{d-2}(P^{-1})|}.$$

Since $|\mathbb{M}_2(P)| \leq C|P|^2$ *we obtain (2.6). For* $d = 2$ *we have* $\mathbb{K}_{d-2}(P) = \mathbb{K}_0(P) = 1$ *and for* $d = 3$ *we have* $\mathbb{K}_{d-2}(P) = \mathbb{K}_1(P) = SPS$ *with* $S = \operatorname{diag}(1, -1, 1)$*. This gives (2.7).*

We are now able to formulate the main lower semicontinuity result.

Proposition 2.6 *Assume that there exists a normal integrand* $g : \Omega \times \mathbb{R}^\nu \to \mathbb{R} \cup \{\infty\}$ *such that* $g(x, \cdot) : \mathbb{R}^\nu \to \mathbb{R} \cup \{\infty\}$ *is convex and lower semicontinuous and that the density* U *of* $\mathcal{I}$ *takes the form*

$$U(x, F_{\mathrm{el}}, P, G) = g(x, \mathbb{M}_{1,\ldots,d}(F_{\mathrm{el}}), \mathbb{M}_{1,\ldots,d}(P), G).$$

Assume that $q_\varphi, q_P, q_{\mathrm{C}}$ *satisfy*

$$\frac{d-2}{q_P} + \frac{1}{q_{\mathrm{C}}} < 1, \quad \frac{1}{q_{\mathrm{C}}} < \frac{1}{d} + \frac{1}{q_P}, \quad q_P > d \quad \text{and} \quad q_\varphi > d. \tag{2.14}$$

Then, the functional $\mathcal{I}$ *defined in (2.1) is weakly lower semi-continuous on* $\mathrm{W}^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times \mathrm{A}^{q_P,q_{\mathrm{C}}}_{\det}(\Omega)$*, where* $\mathrm{A}^{q_P,q_{\mathrm{C}}}_{\det}(\Omega) := \{\, P \in \mathrm{A}^{q_P,q_{\mathrm{C}}}(\Omega) \mid \det P = 1 \text{ a.e. in } \Omega \,\}$ *with the Banach space*

$$\mathrm{A}^{q_P,q_{\mathrm{C}}}(\Omega) = \{\, P \in \mathrm{L}^{q_P}(\Omega;\mathbb{R}^{d\times d}) \mid \operatorname{curl} P \in \mathrm{L}^{q_{\mathrm{C}}}(\Omega)\,\}.$$

Proof: Consider a sequence $(\varphi^{(k)}, P^{(k)})$ with

$$\begin{array}{rcll} \varphi^{(k)} & \rightharpoonup & \widetilde{\varphi} & \text{in } \mathrm{W}^{1,q_\varphi}(\Omega;\mathbb{R}^d), \\ P^{(k)} & \rightharpoonup & \widetilde{P} & \text{in } \mathrm{L}^{q_P}(\Omega;\mathbb{R}^{d\times d}), \\ \text{and}\quad \operatorname{curl} P^{(k)} & \rightharpoonup & \operatorname{curl}\widetilde{P} & \text{in } \mathrm{L}^{q_\mathrm{C}}(\Omega;\mathbb{R}^{d\times d\times d}_{\mathrm{anti}}). \end{array}$$

Lemma 2.3 guarantees

$$\begin{array}{rcll} \mathbb{M}_{1,\dots,d}(P^{(k)}) & \rightharpoonup & \mathbb{M}_{1,\dots,d}(\widetilde{P}) & \text{in } \mathrm{L}^{q_P/d}(\Omega), \\ G^{(k)} = \mathcal{G}(P^{(k)}) & \rightharpoonup & \mathcal{G}(\widetilde{P}) & \text{in } \mathrm{L}^{q*}(\Omega) \text{ for some } q* > 1. \end{array}$$

From Lemma 2.4 and the constraint $\det P^{(k)} \equiv 1$ we deduce that each component of $\mathbb{M}_s(\mathrm{D}\varphi^{(k)}(P^{(k)})^{-1})$ is the determinant of a $d\times d$ matrix Q^I_J containing s rows of $\mathrm{D}\varphi^{(k)}$ and $d-s$ rows of $P^{(k)}$. In particular, each row of Q^I_J is either curl-free or its curl is bounded in $\mathrm{L}^{q_\mathrm{C}}(\Omega)$. Thus, we can apply Theorem B.2 with $p_j = \min\{q_\varphi, q_P\}$ and obtain that $\mathbb{M}_s(\mathrm{D}\varphi^{(k)}(P^{(k)})^{-1}) \rightharpoonup \mathbb{M}_s(\mathrm{D}\widetilde{\varphi}(\widetilde{P})^{-1})$ in $\mathrm{L}^{\sigma_s}(\Omega)$. Using the weak convergence of $G^{(k)}$ and of the minors, we obtain the desired assertion from the convexity of $g(x,\cdot)$. ■

Combining the coercivity result of Proposition 2.2 and the result on lower semicontinuity of Proposition 2.6, the standard arguments for the direct method in the calculus of variations (cf. [Dac89]) provides the following existence result.

Theorem 2.7 *Let $\mathcal{I}$ be defined as in (2.1) and let U be given as in Proposition 2.6. Moreover, let U satisfy the coercivity estimate (2.8) with*

$$\frac{1}{q_F} + \frac{1}{q_P} < \frac{1}{d}, \quad \frac{1}{q_G} + \frac{\min\{2d{-}4, d\}}{q_P} < 1 \quad \textit{and} \quad \frac{1}{q_G} + \frac{\min\{d{-}3, 1\}}{q_P} < \frac{1}{d}.$$

Let q_C and q_φ be defined via $\frac{1}{q_\mathrm{C}} = \frac{1}{q_G} + \frac{\min\{d-2,2\}}{q_P}$ and $\frac{1}{q_\varphi} = \frac{1}{q_F} + \frac{1}{q_P}$ and let ℓ be any continuous linear functional on $W^{1,q_\varphi}(\Omega;\mathbb{R}^d)$, then the minimization problem

$$\mathcal{I}(\varphi, P) - \langle \ell, \varphi\rangle \to \min_{(\varphi,P)} \tag{2.15}$$

has a solution $(\varphi, P) \in \mathrm{W}^{1,q_\varphi}(\Omega;\mathbb{R}^d) \times \mathrm{A}^{q_P,q_\mathrm{C}}_{\det}(\Omega)$.

Proof: It remains to check that all the corresponding conditions on the exponents q_F, q_P, q_G, q_φ and q_C are satisfied. The two conditions of Proposition 2.2 follow directly from $(2.15)_1$ and from $(2.15)_2$ using $d \geq 2$.

Using the definition of q_C the first two conditions in (2.14) are equivalent to $(2.15)_2$ and $(2.15)_3$. The third condition $q_P > d$ follows from $(2.15)_1$ and the fourth condition q_φ is by definition equivalent to $(2.15)_1$. ■

In the case $d = 2$ the conditions (2.15) take the simple form

$$d = 2: \qquad \frac{1}{q_F} + \frac{1}{q_P} < \frac{1}{2}, \quad \frac{1}{q_G} < 1 \quad \text{and} \quad \frac{1}{q_G} < \frac{1}{d} + \frac{1}{q_P}.$$

By choosing q_P larger but close to 2 and q_F sufficiently large, it is possible to allow for q_G being as close to 1 as we like. For $d = 3$ the conditions take the form

$$d = 3: \qquad \frac{1}{q_F} + \frac{1}{q_P} < \frac{1}{3}, \quad \frac{1}{q_G} + \frac{2}{q_P} < \frac{1}{d} \quad \text{and} \quad \frac{1}{q_G} < \frac{1}{3}.$$

3 The incremental problems in finite-strain elastoplasticity

In this section we want to show that the theory is general enough to allow for the solution of incremental problems with an arbitrary number of time steps. Each time step consists of a minimization problem as discussed above, however, it will depend on the solution of the previous time level and we have to control all quantities such that there are no growth terms with respect to the number of time steps.

We closely follow the approach described in [Mie04b] but the additional regularizing term $\mathcal{G}(P)$ allows us to treat much more general stored-energy densities W. For simplicity, we use the additive form

$$\widetilde{W}(x, F_{\rm el}, p, G) = W_{\rm el}(x, F_{\rm el}) + W_{\rm hard}(x, p) + V(x, G). \tag{3.1}$$

As in [Mie03b, Mie04b] we use the dissipation distance on $\mathrm{SL}(d) \times \mathbb{R}^m$ in the form

$$\begin{aligned} D(x, (P_0, p_0), (P_1, p_0)) :=& \\ &\inf\{\, \textstyle\int_0^1 \delta(x, (P, p), (\dot{P}, \dot{p}))\,\mathrm{d}t \mid (P, p) \in \mathrm{C}^1([0, 1], \mathrm{SL}(d) \times \mathbb{R}^m), \\ &\qquad (P(j), p(j)) = (P_j, p_j) \text{ for } j = 0, 1\,\}, \end{aligned} \tag{3.2}$$

where $\delta : \Omega \times \mathrm{T}(\mathrm{SL}(d) \times \mathbb{R}^m) \to [0, \infty]$ is the dissipation potential. The basic properties of D are the triangle inequality

$$D(x, (P_1, p_1), (P_3, p_3)) \leq D(x, (P_1, p_1), (P_2, p_2)) + D(x, (P_2, p_2), (P_3, p_3))$$

and the plastic indifference

$$D(x, (P_1 P_*, p_1), (P_2 P_*, p_2)) = D(x, (P_1, p_1), (P_2, p_2)),$$

for all $p_1, p_2, p_3 \in \mathbb{R}^m$ and $P_*, P_1, P_2, P_3 \in \mathrm{SL}(d)$.

As an auxiliary object we will need the function H which shows the combined effect of dissipation and energy storage due to hardening:

$$H(x, P; p_{\rm old}) := \min_{p \in \mathbb{R}^m} W_{\rm hard}(x, p) + D(x, (\mathbf{1}, p_{\rm old}), (P, p)). \tag{3.3}$$

Here we assume that $p \mapsto W_{\rm hard}(x, p) + D(x, (\mathbf{1}, p_{\rm old}), (P, p))$ is coercive such that the minimum H is attained for some $p \in \mathbb{R}^m$.

Our assumptions concern measurability, coercivity and convexity of the functions $W_{\rm el}, V$ and H only.

(A1) $W_{\rm el}, V$ and H are normal integrands.

(A2) There exist exponents q_F, q_G and $q_P > 1$ and constants $C, c > 0$ such that

$$W_{\rm el}(x, F_{\rm el}) \geq c|F_{\rm el}|^{q_F} - C, \quad V(x, G) \geq c|G|^{q_G} - C, \quad H(x, P; p) \geq c(|P|^{q_P} + |P^{-1}|^{q_P}) - C$$

for all x, F_{el}, G, P and p.

(A3) Each $W_{\mathrm{el}}(x,\cdot)$ is polyconvex, each $V(x,\cdot)$ is convex, and each $H(x,\cdot;p)$ is polyconvex.

In addition to these constitutive assumptions we impose time-independent Dirichlet data φ_{Dir} on $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ where $\Gamma_{\mathrm{Dir}} \neq \emptyset$ and we impose a time-dependent loading via a volume force f_{vol} and a surface force f_{surf} on $\Gamma_{\mathrm{Neu}} = \partial\Omega\backslash\Gamma_{\mathrm{Dir}}$, i.e.,

$$\langle \ell(t), \varphi\rangle := \int_\Omega f_{\mathrm{vol}}(t,x)\cdot\varphi(x)\,\mathrm{d}x + \int_{\Gamma_{\mathrm{Neu}}} f_{\mathrm{surf}}(t,x)\cdot\varphi(x)\,\mathrm{d}x. \tag{3.4}$$

The space $\mathcal{F}$ of admissible deformations φ is given via

$$\mathcal{F} = \{\, \varphi \in \mathrm{W}^{1,q_\varphi}(\Omega) \mid \varphi = \varphi_{\mathrm{Dir}} \text{ on } \Gamma_{\mathrm{Dir}} \,\},$$

where $\varphi_{\mathrm{Dir}} \in \mathrm{W}^{1,q_\varphi}(\Omega)$ is given and $q_\varphi \in (d,\infty)$ will be chosen later. The space $\mathcal{Z}$ of internal states is defined via

$$\mathcal{Z} := \{\, (P,p) : \Omega \to \mathrm{SL}(d)\times\mathbb{R}^m \mid p \text{ measurable, } P \in \mathrm{L}^{q_P}(\Omega),\ \operatorname{curl} P \in \mathrm{L}^1(\Omega) \,\}.$$

The energy functional $\mathcal{E} : [0,T]\times\mathcal{F}\times\mathcal{Z} \to \mathbb{R}\cup\{\infty\}$ is defined via

$$\mathcal{E}(t,\varphi,P,p) := \int_\Omega W_{\mathrm{el}}(x, \mathrm{D}\varphi\, P^{-1}) + W_{\mathrm{hard}}(x,p) + V(x,\mathcal{G}(P))\,\mathrm{d}x - \langle \ell(t),\varphi\rangle$$

and the dissipation distance $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ reads

$$\mathcal{D}((P_0,p_0),(P_1,p_1)) := \int_\Omega D(x,(P_0,p_0),(P_1,p_1))\,\mathrm{d}x. \tag{3.5}$$

The elastoplastic incremental problem (IP) reads as follows:

(IP) For a given partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ and given initial data $(\varphi_0, P_0, p_0) \in \mathcal{F}\times\mathcal{Z}$ with $\mathcal{E}(t_0,\varphi_0,P_0,p_0) < \infty$ find $(\varphi_j,P_j,p_j) \in \mathcal{F}\times\mathcal{Z}$ incrementally for $j = 1,2,\ldots,N$ as minimizer of the functional

$$\mathcal{K}_j : \begin{cases} \mathcal{F}\times\mathcal{Z} & \to \quad \mathbb{R}\cup\{\infty\}, \\ (\varphi,P,p) & \mapsto \quad \mathcal{E}(t_j,\varphi,P,p) + \mathcal{D}((P_{j-1},p_{j-1}),(P,p)). \end{cases}$$

Theorem 3.1 *Let $d \geq 2$ and let the assumptions (A1), (A2) and (A3) hold with*

$$\frac{1}{q_\varphi} := \frac{1}{q_F} + \frac{1}{q_P} < \frac{1}{d}, \quad \frac{1}{q_G} + \frac{\min\{2d-4, d\}}{q_P} < 1 \quad \textit{and} \quad \frac{1}{q_G} + \frac{\min\{d-3, 1\}}{q_P} < \frac{1}{d}. \tag{3.6}$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $\ell \in \mathrm{C}^1([0,T], (\mathrm{W}^{1,q_\varphi}_{\Gamma_{\mathrm{Dir}}}(\Omega))^*)$, $(\varphi_0,P_0,p_0) \in \mathcal{F}\times\mathcal{Z}$ with $\mathcal{E}(0,\varphi_0,P_0,p_0) < \infty$ and $P_0 \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{d\times d})$. Then, there exists a constant $C_* > 0$, **(IP)** *is solvable for all partitions of $[0,T]$ and all solutions $(\varphi_j,P_j,p_j)_{j=1,\ldots,N} \in (\mathcal{F}\times\mathcal{Z})^N$ satisfy the bound*

$$\begin{gathered} \|\varphi_j\|_{1,q_\varphi} + \|P_j\|_{q_P} + \|P_j^{-1}\|_{q_P} \leq C_* \textit{ for } j = 0,1,\ldots,N, \\ \textstyle\sum_{j=1}^N \mathcal{D}((P_{j-1},p_{j-1}),(P_j,p_j)) \leq C_*. \end{gathered} \tag{3.7}$$

Before going into the proof of this result we use the major advantage of the energetic formulation that the associated incremental problem immediately supplies useful energy estimates, see [Mie04c, Thm. 3.2]. In fact, if a solution (φ_j, P_j, p_j) exists in the j-th step, we may compare its minimization property with the competitor $(\varphi_{j-1}, P_{j-1}, p_{j-1})$, which leads to

$$\begin{aligned}\mathcal{E}(t_j, \varphi_j, P_j, p_j) + \mathcal{D}((P_{j-1}, p_{j-1}), (P_j, p_j)) &\le \mathcal{E}(t_j, \varphi_{j-1}, P_{j-1}, p_{j-1}) \\ &= \mathcal{E}(t_{j-1}, \varphi_{j-1}, P_{j-1}, p_{j-1}) - \int_{t_{j-1}}^{t_j} \langle \dot{\ell}(s), \varphi_{j-1}\rangle \,\mathrm{d}s.\end{aligned} \tag{3.8}$$

Summing these estimates over $j = 1, \ldots, n \le N$ gives

$$\begin{aligned}&\mathcal{D}((P_0, p_0), (P_n, p_n)) \le \textstyle\sum_{j=1}^n \mathcal{D}((P_{j-1}, p_{j-1}), (P_j, p_j)) \\ &\le \mathcal{E}(0, \varphi_0, P_0, p_0) - \mathcal{E}(t_n, \varphi_n, P_n, p_n) - \textstyle\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \langle \dot{\ell}(s), \varphi_{j-1}\rangle \,\mathrm{d}s.\end{aligned} \tag{3.9}$$

We will use these general estimates to derive a priori estimates for the solutions of (IP).

Proof: We first show by induction that $(\varphi_j, P_j, p_j) \in \mathcal{F} \times \mathcal{Z}$ with $\mathcal{D}((P_0, p_0), (P_j, p_j)) < \infty$ exists. This holds for $j = 0$ by the assumptions. In each step $j = 1, \ldots, N$ we have to minimize the functional $\mathcal{K}_j : \mathcal{F} \times \mathcal{Z} \to \mathbb{R} \cup \{\infty\}$ with

$$\mathcal{K}_j(\varphi, P, p) = \int_\Omega f_j(x, \mathrm{D}\varphi, P, p, \operatorname{curl} P) \,\mathrm{d}x - \langle \ell(t_j), \varphi\rangle.$$

Since p appears only locally in the integral (i.e., the integrand at x depends on p only through $p(x)$) we may minimize with respect to p under the integral, i.e. $\inf \mathcal{K}_j(\varphi, P, p) = \inf \mathcal{I}_j(\varphi, P)$ where

$$\mathcal{I}_j(\varphi, P) = \int_\Omega U_j(x, \mathrm{D}\varphi, P, \mathcal{G}(P)) \,\mathrm{d}x - \langle \ell(t_j), \varphi\rangle$$

with

$$U_j(x, F, P, G) = W_{\mathrm{el}}(x, FP^{-1}) + H(x, PP_{j-1}^{-1}(x); p_{j-1}(x)) + V(x, G).$$

Our assumption (A1) shows that U_j is a normal integrand. Moreover, by (A3) it has the form of U imposed in Proposition 2.6 (polyconvexity in F_{el} and P and convexity in G). Note that $P \mapsto H(x, PP_*^{-1}; p)$ is still polyconvex since $\mathbb{M}_s(PP_*^{-1}) = \mathbb{M}_s(P)\mathbb{M}_s(P_*^{-1})$ and since P_* is fixed.

Finally, (A2) provides the necessary coercivity via

$$\begin{aligned}&U_j(x, F_{\mathrm{el}}, P, G) = W_{\mathrm{el}}(x, F_{\mathrm{el}}) + H(x, PP_{j-1}^{-1}(x); p_{j-1}(x)) + V(x, G) \\ &\ge c|F_{\mathrm{el}}|^{q_F} - C + H(x, PP_0^{-1}(x); p_0(x)) - D(x, (P_0, p_0), (P_{j-1}, p_{j-1})) + c|G|^{q_G} - C \\ &\ge c(|F_{\mathrm{el}}|^{q_F} + |P|^{q_P} + |P^{-1}|^{q_P} + |G|^{q_G}) - g_j(x),\end{aligned} \tag{3.10}$$

where $g_j \in \mathrm{L}^1(\Omega)$ because of $P_0 \in \mathrm{L}^\infty(\Omega)$ and $\mathcal{D}((P_0, p_0), (P_{j-1}, p_{j-1})) < \infty$ by the induction hypothesis.

Thus, our existence result in Theorem 2.7 can be applied and the existence of solutions for (IP) is established. To derive the a priori estimates let $e_j = \mathcal{E}(t_j, \varphi_j, P_j, p_j)$, $f_j = \|\varphi_j\|_{1,q_\varphi}$, $\delta_j = \mathcal{D}((P_0, p_0), (P_j, p_j))$ and

$$\Lambda^{(0)} = \max\{\, \|\ell(t)\|_{(\mathrm{W}^{1,q_\varphi})^*} \mid t \in [0, T] \,\}, \quad \Lambda^{(1)} = \max\{\, \|\tfrac{\mathrm{d}}{\mathrm{d}t}\ell(t)\|_{(\mathrm{W}^{1,q_\varphi})^*} \mid t \in [0, T] \,\}.$$

Thus, eqn. (3.9) takes the form

$$e_n + \delta_n \le e_0 + \textstyle\sum_{j=1}^{n} (t_j{-}t_{j-1}) \Lambda^{(1)} f_{j-1}. \tag{3.11}$$

On the other hand, from (2.4), (2.5), and (3.10) we obtain

$$\begin{aligned} e_n + \delta_n &\ge c\left(\|\mathrm{D}\varphi_n P_n^{-1}\|_{q_F}^{q_F} {+} \|P_n\|_{q_P}^{q_P}\right) - C|\Omega| - \langle \ell(t_n), \varphi_n\rangle \\ &\ge \widetilde{c}\|\mathrm{D}\varphi_n\|_{q_\varphi}^{q_\varphi} - \widetilde{C} - \Lambda^{(0)}\|\varphi_n\|_{1,q_\varphi}. \end{aligned} \tag{3.12}$$

Since $q_\varphi > d$ and $\Gamma_{\mathrm{Dir}} \ne \emptyset$ the Poincaré inequality $\|\phi\|_{q_\varphi} \le C\|\mathrm{D}\phi\|_{q_\varphi}$ and (3.12) imply $f_n \le C_0 + C_1(e_n{+}\delta_n)$. Inserting this into (3.11) we arrive at

$$e_n + \delta_n \le e_0 + \Lambda^{(1)} T C_0 + \Lambda^{(1)} C_1 \textstyle\sum_{j=0}^{n-1} (t_{j+1}{-}t_j)(e_j{+}\delta_j).$$

Now, the discrete Gronwall estimate provides the a priori bound $e_n + \delta_n \le C_*$ where C_* is independent of n and of the partition. Using this fact together with (3.10) we easily obtain the first line of (3.7) (proceed as in Proposition 2.2). Using (3.9) again together with $f_j \le C_0 + C_1(e_j{+}\delta_j) \le C_0 + C_1 C_*$ and $e_N \le C_*$ we obtain the second line of (3.7). ■

4 Examples

It remains to provide realistic examples which satisfy all the assumptions of the theorem. For the elastic stored-energy density W_{el} we may take *any* polyconvex function which grows sufficiently fast. For instance, we may choose

$$W_{\mathrm{el}}(F_{\mathrm{el}}) = \begin{cases} c_1|F_{\mathrm{el}}|^\alpha + \frac{c_2}{(\det F_{\mathrm{el}})^\beta} & \text{for } \det F_{\mathrm{el}} > 0, \\ \infty & \text{else.} \end{cases} \tag{4.1}$$

However, every polyconvex function may be used, as long as the coercivity estimate $W_{\mathrm{el}}(x, F_{\mathrm{el}}) \ge c|F_{\mathrm{el}}|^\alpha{-}C$ holds.

The construction of a suitable dissipation mechanism is more difficult, since there are not many cases where D can be calculated from the dissipation potential δ, see [Mie03b]. One good case is that of isotropic hardening with a scalar hardening parameter $p \in [0, \infty)$. We let

$$\delta(x, (P, p), (\dot P, \dot p)) = \begin{cases} |\frac{1}{2}\big[\dot P P^{-1} {+} (\dot P P^{-1})^\mathsf{T}\big]| & \text{for } \dot p \ge |\frac{1}{2}\big[\dot P P^{-1} {+} (\dot P P^{-1})^\mathsf{T}\big]|, \\ \infty & \text{else,} \end{cases}$$

which leads to the dissipation distance

$$\begin{aligned} &D(x, (P_0, p_0), (P_1, p_1)) = \widehat{D}(P_1 P_0^{-1}, p_1{-}p_0) \text{ with} \\ &\widehat{D}(P, p) = \begin{cases} \big|\log(P^\mathsf{T} P)^{1/2}\big| & \text{for } p \ge \big|\log(P^\mathsf{T} P)^{1/2}\big|, \\ \infty & \text{else.} \end{cases} \end{aligned} \tag{4.2}$$

This formula was established in Corollary 6.2 of [Mie02].

Choosing an arbitrary monotone hardening potential $W_{\rm hard} : [0,\infty) \to [0,\infty)$, we find

$$H(P;p_0) = \Phi(p_0 + \big|\log(P^{\sf T}P)^{1/2}\big|) - p_0 \text{ with } \Phi(\rho) = W_{\rm hard}(\rho) + \rho.$$

Note that $H(\cdot;p_0) : \mathrm{SL}(d) \to \mathbb{R}$ is an isotropic tensor function which can be expressed in terms of the singular values $\nu \in (0,\infty)^d$ of P, i.e., $P = Q_1 \mathrm{diag}(\nu) Q_2$ with $Q_1, Q_2 \in \mathrm{SO}(d)$.

In particular, we have

$$\begin{aligned} &|P| = (\mathrm{tr} P^{\sf T}P)^{1/2} = |\nu|, \quad |P^{-1}| = |\tfrac{1}{\nu}| := |(\tfrac{1}{\nu_1},\dots,\tfrac{1}{\nu_d})|, \\ &\big|\log(P^{\sf T}P)^{1/2}\big| = |\log \nu| = |(\log \nu_1, \dots, \log \nu_d)|. \end{aligned} \tag{4.3}$$

Because of the constraint $\det P = \prod_{j=1}^d \nu_j = 1$ we may parametrize the singular values with $d-1$ parameters:

$$\begin{aligned} d = 2: &\quad \nu = (\mu, 1/\mu) \text{ with } \mu \in [1,\infty), \\ d = 3: &\quad \nu = (\mu_1, \mu_2/\mu_1, 1/\mu_2) \text{ with } 1 \le \sqrt{\mu_1} \le \mu_2 \le \mu_1^2. \end{aligned}$$

Proposition 4.1 *Let $d \ge 2$. Then, the function $H(\cdot;p_0)$ satisfies the coercivity estimate $H(P;p_0) \ge c|P|^{q_P} + c|P^{-1}|^{q_P} - C$ if and only if Φ satisfies*

$$\Phi(\rho) \ge \widehat{c}\,\mathrm{e}^{\rho\, q_P/\beta_d} - \widehat{C} \ \textit{where}\ \beta_d = (d/(d{-}1))^{1/2}$$

for some $\widehat{c}, \widehat{C} > 0$.

For $d = 2$, the function $H(\cdot,p_0)$ is polyconvex for all $p_0 \ge 0$ if and only if $\Phi'(\rho) \ge 0$ and $\Phi''(\rho) \ge \frac{\mathrm{e}^\rho - 1}{2(\mathrm{e}^\rho + 1)}\Phi'(\rho)$ for all $\rho \ge 0$.

Remark 4.2 *For $d = 2$ it suffices to choose Φ such that $\Phi'' \ge \frac{1}{2}\Phi' \ge 0$, for instance $\Phi(\rho) = \mathrm{e}^{\gamma\rho}$ with $\gamma \ge 1/2$. We conjecture that polyconvexity also holds for $d = 3$ when using the function $\Phi(\rho) = \mathrm{e}^{\kappa\rho}$ with $\kappa \ge 1$. The function*

$$\begin{cases} \widehat{\mathcal{V}}_3 & \to \quad \mathbb{R}, \\ \mu & \mapsto \quad \mathrm{e}^{\kappa((\log \mu_1)^2 + (\log(\mu_2/\mu_1))^2 + (\log \mu_2)^2)^{1/2}}, \end{cases}$$

where $\widehat{\mathcal{V}}_3 = \{\, \mu \in (0,\infty) \mid 1 \le \sqrt{\mu_1} \le \mu_2 \le \mu_1^2 \,\}$, is monotone in each μ_j but it is not convex. Thus, the necessary conditions of Proposition 6.3 in [Mie03c] are satisfied. There it is also shown that convexity is not necessary.

Proof: For $\sigma \in \mathbb{R}^{d-1}$ we have the estimate $\sum_{j=1}^{d-1} \sigma_j \le \sqrt{d{-}1}|\sigma|$. Thus, each $s \in \mathbb{R}^d$ with $\sum_{j=1}^d s_j = 0$ satisfies $\beta_d s_{\max} = \max\{\, s_j \mid j = 1,\dots,d \,\} \le |s|$ with $\beta_d = (d/(d{-}1))^{1/2}$. This implies

$$\log\Big(\textstyle\sum_{j=1}^d \mathrm{e}^{s_j}\Big) \le \log\big(d\mathrm{e}^{s_{\max}}\big) = s_{\max} + \log d \le \frac{1}{\beta_d}|s| + \log d.$$

Now consider $\nu \in (0,\infty)^d$ with $\prod_{j=1}^d \nu_j = 1$ in the form $\nu_j = \mathrm{e}^{s_j/2}$ such that $\sum_{j=1}^d s_j = 0$. Hence, we can estimate

$$\big|\log \tfrac{1}{\nu}\big| = |\log \nu| = \tfrac{1}{2}|s| \ge \tfrac{\beta_d}{2}\log\Big(\textstyle\sum_{j=1}^d \mathrm{e}^{s_j}\Big) - \tfrac{\beta_d}{2}\log d = \beta_d\big(\log|\nu| - \tfrac{1}{2}\log d\big).$$

Together with (4.3) we conclude that exponential growth of Φ implies the desired polynomial growth of $H(\cdot;p_0)$. The necessity follows by testing with $P = \mathrm{diag}(\mu_1, \mu_2, \mu_2, \ldots, \mu_2)$ where $\mu_2 = \mu_1^{-1/(d-1)}$.

For the case $d = 2$ we can use the necessary and sufficient condition for polyconvexity in the incompressible case which is given in Theorem 6.1 in [Mie03c]. With the above notation for $\mu \in [1,\infty)$ a smooth function $\psi : [1,\infty) \to \mathbb{R}$ generates a polyconvex function

$$P \mapsto \begin{cases} \psi(\mu) & \text{if } P = Q_1 \mathrm{diag}(\mu, 1/\mu) Q_2 \\ \infty & \text{else,} \end{cases}$$

if and only if ψ is nondecreasing and satisfies $\mu(\mu^2{+}1)\psi''(\mu) + 2\psi'(\mu) \geq 0$ for $\mu \geq 1$. Inserting $\psi(\mu) = \Phi(p_0{+}2\log\mu)$ and imposing the conditions for all $p_0 \geq 0$ and $\mu \geq 1$ gives the desired result. ∎

For $d = 2$ we are now able to choose W_{el}, V and W_{hard} such that Theorem 3.1 is applicable. Choose W_{el} as in (4.1) and D as in (4.2). Moreover, let $W_{\mathrm{hard}}(p) = \mathrm{e}^{\gamma p} - p$, then we have

$$q_F = \alpha \quad \text{and} \quad q_P = \gamma\beta_d.$$

The exponent q_G can be chosen independently. Thus, it is easy to satisfy the conditions (3.6) which reduce to

$$\frac{1}{\alpha} + \frac{1}{\gamma\sqrt{2}} < \frac{1}{2} \quad \text{and} \quad \frac{1}{q_G} < \frac{1}{2} + \frac{1}{\gamma\sqrt{2}}.$$

For $d = 2$ this implies the desired existence result. For the case $d = 3$ a similar result will hold, if the conjectured polyconvexity in Remark 4.2 can be established.

5 General dimensions

To emphasize the underlying structure we show that the results in Section 2 can be easily extended to arbitrary dimensions. We first review the definition of the geometric dislocation tensor G. As in [CG01] we distinguish between a material point $x \in \mathbb{R}^d$ and the tangent space at x, in order to capture the incompatibility of $F_{\mathrm{pl}} = P$. In other words, we work with the tangent bundle $T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ rather than with $\mathbb{R}^d$. Still following [CG01] we consider three configurations: the reference configuration, the lattice configuration and the deformed configuration. The mapping from the reference configuration to the lattice configuration is given by

$$\pi : T\mathbb{R}^d \to T\mathbb{R}^d; \quad \pi(x,v) = (x, Pv),$$

while the mapping from the lattice configuration to the deformed configuration is given by

$$\epsilon : T\mathbb{R}^d \to T\mathbb{R}^d; \quad \epsilon(x,w) = (x, F_{\mathrm{el}} w).$$

Their composition gives the usual map from the reference to the deformed configuration (now extended to the tangent bundle), i.e.,

$$\epsilon \circ \psi = dy, \qquad \text{where } dy(x,v) = (y(x), \mathrm{D}y(x)v).$$

We now consider the vector-valued one-form α in the reference configuration defined by

$$\alpha(x)(v) = P(x)v.$$

If Γ is a closed curve which is the boundary of a surface S, the incompatibility of F is measured by the continuous analogue of Burger's vector, namely

$$\int_\Gamma \alpha = \int_S d\alpha,$$

where we used Stokes' theorem. Hence $d\alpha$ measures the incompatibility per unit reference area. To obtain a measure per unit area in the lattice configuration we simply consider the pull-back γ of $d\alpha$ under the map π^{-1}, i.e.,

$$\gamma(x)(w_1, w_2) = (\pi^{-1})_\# d\alpha(x)(w_1, w_2) := d\alpha(x)(F_{\mathrm{pl}}^{-1} w_1, F_{\mathrm{pl}}^{-1} w_2).$$

This definition corresponds to the definition of $\mathcal{G}(P)$ in Section 2, since $d\alpha$ is $\operatorname{curl} P$:

$$\gamma(x)(w_1, w_2) = \mathcal{G}(P)[w_1, w_2] = \operatorname{curl} P[P^{-1}w_1, P^{-1}w_2].$$

However, the abstract formulation using γ shows more easily that γ is invariant under a change of reference configuration and under composition of y with a compatible map from the left, cf. [Par95, PŠ99].

The advantage of the abstract form is even clearer for our main result which establishes weak continuity of $\gamma = \mathcal{G}(P)$ as a function of $P = F_{\mathrm{pl}}$. With the following result we complete the proof of Lemma 2.3(b) for the case $d \geq 4$.

Proposition 5.1 *Suppose that* $\frac{1}{q_*} := \frac{1}{q_{\mathrm{C}}} + \frac{d-2}{q_P} < 1$ *and that*

$$P^{(k)} \rightharpoonup \widetilde{P} \quad \textit{in } \mathrm{L}^{q_P}(\Omega) \quad \textit{and} \quad \operatorname{curl} P^{(k)} \rightharpoonup \widetilde{A} \quad \textit{in } \mathrm{L}^1(\Omega). \tag{5.1}$$

Then,

$$\gamma^{(k)} = \mathcal{G}(P^{(k)}) \rightharpoonup \gamma = \mathcal{G}(\widetilde{P}) \quad \textit{in } \mathrm{L}^{q_*}(\Omega). \tag{5.2}$$

Proof: Let I be a multi-index of length $d-2$ and let $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_{d-2}}$. It suffices to study $\gamma^{(k)} \wedge dx^I$ for all multi-indices I. We fix one such index and denote by $\gamma^{(k)}$, $\alpha^{(k)} = \operatorname{curl} P^{(k)}$ and $\pi^{(k)}$ the mappings associated with $P^{(k)}$ as defined above. Since $\det P^{(k)} = 1$ we have

$$\gamma^{(k)} \wedge dx^I = \pi^{(k)}_\# \gamma^{(k)} \wedge \pi^{(k)}_\# dx^I = d\alpha^{(k)} \wedge \pi^{(k)}_\# dx^I.$$

Using the Helmholtz decomposition $P^{(k)} = \mathrm{D}\psi^{(k)} + Q^{(k)}$, where $Q^{(k)}$ converges strongly in $\mathrm{L}^q(\Omega)$ for all $q < q_P$, we get

$$\gamma^{(k)} \wedge dx^I = d\alpha^k \wedge (\mathrm{D}\psi^{(k)} + Q^{(k)})_\# dx^I.$$

Expanding the pull-back and using $dx^i(\mathrm{D}\psi^{(k)}v) = d\psi_i^{(k)}(v)$ we see that the above expression is a sum of terms of the form

$$d\alpha^{(k)} \wedge d\psi_{j_1}^{(k)} \wedge \ldots \wedge d\psi_{j_{d-2}}^{(k)} \quad \text{or} \quad d\alpha^{(k)} \wedge d\psi_{j_1}^{(k)} \wedge \ldots \wedge d\psi_{j_s}^{(k)} \wedge (\text{conv. seq. in } \mathrm{L}^{q/(d-2-s)}(\Omega)),$$

where $s < d-2$ and where $j_1, \ldots, j_s$ is a subset of the multi-index I. Now, a wedge product of differentials is a sum of determinants and thus weakly continuous in the natural spaces (for a statement in the language of differential forms see e.g. [Iwa98, Thm. 6.1]). This concludes the proof. ■

Finally we show how the special forms (2.2) and (2.3) for $\mathcal{G}(P)$ in dimensions $d = 2$ and 3 are obtained by the usual identifications. Here we return to the notations used in Section 2.

In $\mathbb{R}^3$ we denote by $\{ e_i \mid i = 1, 2, 3 \}$ the standard Euclidean basis and by $\langle \cdot, \cdot \rangle$ the scalar product. Then for $G = \mathcal{G}(P)$ the three two-forms $g_{(i)} = \langle e_i, G[\cdot, \cdot] \rangle$ can be expressed through vector products via suitable vectors $\widehat{g}_{(i)} \in \mathbb{R}^3$. Moreover, curl_3 acts as "$\nabla\times$" on vector fields $h : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$. This gives

$$\langle e_i, G[a, b] \rangle = \langle \widehat{g}_{(i)}, a \times b \rangle \quad \text{and} \quad \langle \mathrm{curl}_3\, h, a \times b \rangle = \mathrm{D}h[a]b - \mathrm{D}h[b]a \quad \text{for all } a, b \in \mathbb{R}^3.$$

Thus, we obtain

$$\begin{aligned} \langle e_i, G[a, b] \rangle &= \langle \mathrm{curl}_3(P^\top e_i), (P^{-1}a) \times (P^{-1}b) \rangle = \langle \mathrm{curl}_3(P^\top e_i), \tfrac{1}{\det P} P^\top (a \times b) \rangle \\ &= \langle \tfrac{1}{\det P} P\, \mathrm{curl}_3(P^\top e_i), a \times b \rangle = \langle \widehat{g}_{(i)}, a \times b \rangle, \end{aligned}$$

where we used the identities $(P^{-1}a) \times (P^{-1}b) = \mathrm{cof}\, P^{-1}(a \times b)$ and $\mathrm{cof}(P^{-1}) = \frac{1}{\det P} P^\top$. Recall that curl_3 also acts on matrices row by row, such that

$$\mathrm{curl}_3\, P = (\mathrm{curl}_3(P^\top e_1) | \,\mathrm{curl}_3(P^\top e_2) | \,\mathrm{curl}_3(P^\top e_3))^\top \in \mathbb{R}^{3\times 3}.$$

Thus, $\widehat{G} = \widehat{\mathcal{G}}_3(P) \in \mathbb{R}^{3\times 3}$ defined in (2.3) consists exactly of the rows $\widehat{g}_{(i)}^\top$, and the identification is done.

The case $d = 2$ is even easier. Each two-form on $\mathbb{R}^2$ is a scalar multiple of the determinant, i.e., $[a, b] \mapsto \det(a|b)$. A simple calculation gives

$$e_i^\top G[a, b] = \mathrm{curl}_2(P^\top e_i) \det(P^{-1}a | P^{-1}b) = \frac{1}{\det P} \mathrm{curl}_2(P^\top e_i) \det(a|b),$$

where $\mathrm{curl}_2\, h = \partial_1 h_2 - \partial_2 h_1$ denotes the two-dimensional curl. Identifying the two-form with the multiple of $\det(a|b)$, we find (2.2).

A Helmholtz decomposition

Proposition A.1 *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. For all $p \in (1,\infty)$ the Helmholtz decomposition*

$$u = \operatorname{grad}\psi + v, \quad \text{with } \operatorname{div} v = 0, \quad \psi \in \mathrm{W}^{1,p}(\Omega) \text{ and } \tfrac{\partial\psi}{\partial\nu} = 0 \text{ on } \partial\Omega,$$

defines bounded linear operators $H_1 : \mathrm{L}^p(\Omega;\mathbb{R}^d) \to \mathrm{W}^{1,p}(\Omega);\ u \mapsto \psi$ and $H_2 : \mathrm{L}^p(\Omega;\mathbb{R}^d) \to \mathrm{L}^p(\Omega;\mathbb{R}^d);\ u \mapsto v$.

Moreover, for each domain $\widetilde{\Omega}$ compactly contained in Ω there exists a positive constant C (depending only on p, Ω and $\widetilde{\Omega}$), such that for all u with $\operatorname{curl} u \in \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d}_{\text{anti}})$ we have

$$\|H_2 u\|_{1,p,\widetilde{\Omega}} \leq C\Big(\|u\|_{p,\Omega} + \|\operatorname{curl} u\|_{p,\Omega}\Big). \tag{A.1}$$

Furthermore, for a sequence u_k with $u_k \rightharpoonup u$ in $\mathrm{L}^p(\Omega;\mathbb{R}^d)$ and $\operatorname{curl} u_k$ bounded in $\mathrm{L}^1(\Omega;\mathbb{R}^{d\times d}_{\text{anti}})$ we have $H_2 u_k \to H_2 u$ (strongly) in $\mathrm{L}^q(\Omega;\mathbb{R}^d)$ for each $q \in [1,p)$.

Proof: The first part of the result is standard, see e.g. [vW90, SS92].

For the second part we use a localization technique. Choose a function $\chi \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$ with $\chi|_{\widetilde{\Omega}} \equiv 1$ and define the multiplication-extension operator $M_\chi : \mathrm{L}^p(\Omega;\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d,\mathbb{R}^d); v \mapsto \chi v$. Using that $v = H_2 u$ satisfies $\operatorname{div} v \equiv 0$ we have

$$\operatorname{curl}(M_\chi v) = w, \quad \operatorname{div}(M_\chi v) = \rho \tag{A.2}$$

with $\|w\|_{p,\mathbb{R}^d} + \|\rho\|_{p,\mathbb{R}^d} \leq (1{+}\|\mathrm{D}\chi\|_\infty)\big[\|v\|_{p,\Omega} + \|\operatorname{curl} v\|_{p,\Omega}\big]$. Employing the L^p multiplier theory on the full space $\mathbb{R}^d$, the solution $M_\chi v$ of (A.2) satisfies the a priori estimate

$$\|\mathrm{D}(M_\chi v)\|_{p,\mathbb{R}^d} \leq c_d\big[\|w\|_{p,\mathbb{R}^d} + \|\rho\|_{p,\mathbb{R}^d}\big],$$

where c_d depends only on the dimension. Combining these estimates and using $M_\chi v|_{\widetilde{\Omega}} = v|_{\widetilde{\Omega}}$, estimate (A.1) is established with $C = 1 + (1{+}\|\mathrm{D}\chi\|_\infty)c_d$.

To prove the last assertion we use again (A.2). From this we deduce that there exist matrix-valued kernels $K(z)$ and $L(z)$ which are homogeneous of degree $-(d{-}1)$ in $z \in \mathbb{R}^d\backslash\{0\}$ and smooth on the unit sphere such that

$$M_\chi v^{(k)}(x) = \int_{\mathbb{R}^d} \big(K(x{-}y)w^{(k)}(y) + L(x{-}y)\rho^{(k)}(y)\big)\,\mathrm{d}y.$$

Since $w^{(k)}$ and $\rho^{(k)}$ are bounded in $\mathrm{L}^1(\Omega)$ it follows that $M_\chi v^{(k)}$ is compact in $\mathrm{L}^1(\Omega)$. Since $v_k \rightharpoonup v$ in $\mathrm{L}^p(\Omega)$ by linearity and continuity of H_2, we deduce that $v_k \to v$ in $\mathrm{L}^1(\widetilde{\Omega})$ for each compactly contained subset $\widetilde{\Omega}$. Now we choose $q \in [1,p)$ and let $r = \frac{qp}{p-q} > 1$ and $\theta = \frac{p-q}{q(p-1)} \in (0,1]$. Using $\|v_k - v\|_p \leq C_*$ and Hölder's inequality we find

$$\begin{aligned}
\|v_k - v\|_{q,\Omega} &\leq \|v_k - v\|_{q,\widetilde{\Omega}} + \|v_k - v\|_{q,\Omega\backslash\widetilde{\Omega}} \\
&\leq \|v_k - v\|^\theta_{1,\widetilde{\Omega}}\|v_k - v\|^{1-\theta}_{p,\widetilde{\Omega}} + \|1\|_{r,\Omega\backslash\widetilde{\Omega}}\|v_k - v\|_{p,\Omega\backslash\widetilde{\Omega}} \\
&\leq \|v_k - v\|^\theta_{1,\widetilde{\Omega}}\, C_*^{1-\theta} + \operatorname{vol}(\Omega\backslash\widetilde{\Omega})^{1/r}\, C_*.
\end{aligned}$$

Making $\operatorname{vol}(\Omega\backslash\widetilde{\Omega})$ small first and making k large second, the strong convergence $v_k \to v$ in $\mathrm{L}^q(\Omega)$ follows. ■

B Weak convergence results

We assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary.

Theorem B.1 ([Mur78, Thm. 2] div–curl lemma) *Let p, q, σ satisfy $\frac{1}{\sigma} = \frac{1}{p} + \frac{1}{q} < 1$. If the sequences $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ satisfy*

$$f_k \rightharpoonup f^* \quad \text{in } \mathrm{L}^p(\Omega;\mathbb{R}^d) \quad \text{and} \quad g_k \rightharpoonup g^* \quad \text{in } \mathrm{L}^q(\Omega;\mathbb{R}^d) \quad \text{for } k\to\infty,$$
$$\{\,\operatorname{curl} f_k \mid k\in\mathbb{N}\,\} \text{ is bounded in } \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d}_{\text{anti}}) \text{ and}$$
$$\{\,\operatorname{div} g_k \mid k\in\mathbb{N}\,\} \text{ is bounded in } \mathrm{L}^q(\Omega),$$

then $f_k \cdot g_k \rightharpoonup f^*\cdot g^*$ in $\mathrm{L}^\sigma(\Omega)$.

Murat shows convergence in the sense of distributions. Together with the L^σ bound this yields weak convergence in L^σ.

Theorem B.2 (Weak convergence of subdeterminants) *Assume $1 \le s \le d$, $p_j > 1$ for $j = 1,\dots,s$ such that $\frac{1}{\sigma} = \frac{1}{p_1} + \cdots + \frac{1}{p_s} < 1$. Assume that the vector-valued functions $f_j^{(k)} : \Omega \to \mathbb{R}^d$ satisfy, for each $j = 1,\dots,s$,*

(1) $\quad f_j^{(k)} \rightharpoonup f_j^* \quad$ *in* $\mathrm{L}^{p_j}(\Omega;\mathbb{R}^d)$ *for* $k\to\infty$,

(2) $\quad \{\,\operatorname{curl} f_j^{(k)} \mid k\in\mathbb{N}\,\}$ *is bounded in* $\mathrm{L}^{q_j}(\Omega;\mathbb{R}^{d\times d}_{\text{anti}})$, *where* $\frac{1}{q_j} \le \min\{\frac{1}{p_j} + \frac{1}{d}, 1\}$.

By $F^{(k)}$ and F^* denote the $(d\times s)$ matrix with columns $(f_j^{(k)})_{j=1,\dots,s}$ and $(f_j^*)_{j=1,\dots,s}$, respectively. Then, for the minors of order s we have

$$\mathbb{M}_s(F^{(k)}) \rightharpoonup \mathbb{M}_s(F^*) \quad \text{in } \mathrm{L}^\sigma(\Omega) \quad \text{for } k\to\infty.$$

Proof: See for example [Iwa98, Thm. 6.1]. Iwaniec states the result in terms of differential forms. To make the connection it suffices to identify each of the R^d-valued functions $a(x) = f_j^{(k)}(x)$ with the one-form $\alpha = \sum_{i=1}^d a_i dx^i$. Then the components of the exterior derivative $d\alpha$ are exactly given by $\operatorname{curl} a$. Iwaniec also considers only the case $\sigma = 1$ and shows distributional convergence.

Alternatively one can reduce the above Theorem first to the standard situation involving minors of gradients by using the Helmholtz decomposition (see above) and expanding the minors into the gradient part and the compact, divergence-free part. Then the theorem follows by induction over the order of minors from the div-curl lemma, since minors are divergence free (see e.g. [Dac89], Chapter 4, Thm. 2.6). ■

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References

[AC00] J. Alberty and C. Carstensen. Numerical analysis of time-depending primal elastoplasticity with hardening. *SIAM J. Numer. Anal.*, 37, 1271–1294 (electronic), 2000.

[ACZ99] J. Alberty, C. Carstensen, and D. Zarrabi. Adaptive numerical analysis in primal elastoplasticity with hardening. *Comput. Methods Appl. Mech. Engrg.*, 171(3-4), 175–204, 1999.

[Alb98] H.-D. Alber. *Materials with Memory*, volume 1682 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.

[Bal77] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4), 337–403, 1976/77.

[BCHH04] S. Bartels, C. Carstensen, K. Hackl, and U. Hoppe. Effective relaxation for microstructure simulations: algorithms and applications. *Comput. Methods Appl. Mech. Engrg.*, 193, 5143–5175, 2004.

[CA03] C. Carstensen and J. Alberty. Averaging techniques for reliable a posteriori FE-error control in elastoplasticity with hardening. *Comput. Methods Appl. Mech. Engrg.*, 192, 1435–1450, 2003.

[CG01] P. Cermelli and M. Gurtin. On the characterization of geometrically necessary dislocations in finite plasticity. *J. Mech. Phys. Solids*, 49, 1539–1568, 2001.

[Che01a] K. Chełmiński. Coercive approximation of viscoplasticity and plasticity. *Asymptot. Anal.*, 26, 105–133, 2001.

[Che01b] K. Chełmiński. Perfect plasticity as a zero relaxation limit of plasticity with isotropic hardening. *Math. Methods Appl. Sci.*, 24, 117–136, 2001.

[CHM02] C. Carstensen, K. Hackl, and A. Mielke. Non–convex potentials and microstructures in finite–strain plasticity. *Proc. Royal Soc. London, Ser. A*, 458, 299–317, 2002.

[CT03] S. Conti and F. Theil. Single-slip elastoplastic microstructures. *MPI MIS Preprint 64*, 2003.

[Dac82] B. Dacorogna. *Weak continuity and weak lower semicontinuity of nonlinear functionals.* Springer-Verlag, Berlin, 1982. Lecture Notes in Math. Vol. 922.

[Dac89] B. Dacorogna. *Direct Methods in the Calculus of Variations.* Springer-Verlag, Berlin, 1989.

[DF02] B. Dacorogna and I. Fonseca. A-B quasiconvexity and implicit partial differential equations. *Calc. Var. PDEs*, 14, 115–149, 2002.

[DFT04] G. Dal Maso, G. Francfort, and R. Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, 2004. To appear.

[DT02] G. Dal Maso and R. Toader. A model for quasi–static growth of brittle fractures: existence and approximation results. *Arch. Rat. Mech. Anal.*, 162, 101–135, 2002.

[ET76] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems.* North Holland, 1976.

[FM98] G. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, 46, 1319–1342, 1998.

[FM99] I. Fonseca and S. Müller. $\mathcal{A}$-quasiconvexity, lower semicontinuity, and Young measures. *SIAM J. Math. Anal.*, 30, 1355–1390 (electronic), 1999.

[FM04] G. Francfort and A. Mielke. Existence results for a class of rate-independent material models with nonconvex elastic energies. *J. reine angew. Math.*, 2004. Submitted 10/04.

[Gur02] M. E. Gurtin. A gradient theory of single–crystal viscoplasticity that accounts for geometrically necessary dislocations. *J. Mech. Phys. Solids*, 50, 5–32, 2002.

[HR99] W. Han and B. D. Reddy. *Plasticity (Mathematical theory and numerical analysis)*, volume 9 of *Interdisciplinary Applied Mathematics.* Springer-Verlag, New York, 1999.

[Iwa98] T. Iwaniec. Nonlinear differential forms. Technical report, Math. Inst. Univ. Jyväskylä, Report 80, 1998. Series of Lectures from the Int. Summer School in Jyväskylä, 207 pp.

[Joh76] C. Johnson. Existence theorems for plasticity problems. *J. Math. Pures Appl. (9)*, 55(4), 431–444, 1976.

[KMR04] M. Kočvara, A. Mielke, and T. Roubíček. A rate–independent approach to the delamination problem. *Math. Mech. Solids*, 2004. In press.

[Kru02] M. Kružík. Variational models for microstructure in shape memory alloys and in micromagnetics and their numerical treatment. In A. Ruffing and M. Robnik, editors, *Proceedings of the Bexbach Kolloquium on Science 2000 (Bexbach, 2000)*. Shaker Verlag, 2002.

[Lee69] E. Lee. Elastic-plastic deformation at finite strains. *J. Applied Mechanics*, 36, 1–6, 1969.

[LMD03] M. Lambrecht, C. Miehe, and J. Dettmar. Energy relaxation of non-convex incremental stress potentials in a strain-softening elastic-plastic bar. *Int. J. Solids Structures*, 40, 1369–1391, 2003.

[Mie02] A. Mielke. Finite elastoplasticity, Lie groups and geodesics on SL(d). In P. Newton, A. Weinstein, and P. Holmes, editors, *Geometry, Dynamics, and Mechanics*, pages 61–90. Springer–Verlag, 2002.

[Mie03a] C. Miehe. Computational micro-to-macro transitions for discretized micro-structures of heterogeneous materials at finite strains based on the minimization of averaged incremental energy. *Comput. Methods Appl. Mech. Engrg.*, 192(5-6), 559–591, 2003.

[Mie03b] A. Mielke. Energetic formulation of multiplicative elasto–plasticity using dissipation distances. *Cont. Mech. Thermodynamics*, 15, 351–382, 2003.

[Mie03c] A. Mielke. Necessary and sufficient conditions for polyconvexity of isotropic functions. *J. Convex Analysis*, 2003. Submitted.

[Mie04a] A. Mielke. Deriving new evolution equations for microstructures via relaxation of variational incremental problems. *Comput. Methods Appl. Mech. Engrg.*, 193, 5095–5127, 2004.

[Mie04b] A. Mielke. Existence of minimizers in incremental elasto–plasticity with finite strains. *SIAM J. Math. Analysis*, 36, 384–404, 2004.

[Mie04c] A. Mielke. Evolution in rate-independent systems. In *Handbook of Differential Equations II*. North-Holland, 2004. Submitted.

[MM04] A. Mainik and A. Mielke. Existence results for energetic models for rate–independent systems. *Calc. Var. PDEs*, 2004. In press.

[Mor52] C. B. Morrey, Jr. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.*, 2, 25–53, 1952.

[Mor74] J.-J. Moreau. On unilateral constraints, friction and plasticity. In *New variational techniques in mathematical physics (Centro Internaz. Mat. Estivo (C.I.M.E.), II Ciclo, Bressanone, 1973)*, pages 171–322. Edizioni Cremonese, Rome, 1974.

[Mor76] J.-J. Moreau. Application of convex analysis to the treatment of elastoplastic systems. In P. Germain and B. Nayroles, editors, *Applications of methods of functional analysis to problems in mechanics*, pages 56–89. Springer-Verlag, 1976. Lecture Notes in Mathematics, 503.

[MR03] A. Mielke and T. Roubíček. A rate–independent model for inelastic behavior of shape–memory alloys. *Multiscale Model. Simul.*, 1, 571–597, 2003.

[MS92] C. Miehe and E. Stein. A canonical model of multiplicative elasto–plasticity. Formulation and aspects of numerical implementation. *Europ. J. Mech. A/Solids*, 11, 25–43, 1992.

[MT99] A. Mielke and F. Theil. A mathematical model for rate–independent phase transformations with hysteresis. In H.-D. Alber, R. Balean, and R. Farwig, editors, *Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engineering"*, pages 117–129. Shaker–Verlag, 1999.

[MT04] A. Mielke and F. Theil. On rate–independent hysteresis models. *Nonl. Diff. Eqns. Appl. (NoDEA)*, 11, 151–189, 2004. Accepted July 2001.

[MTL02] A. Mielke, F. Theil, and V. Levitas. A variational formulation of rate–independent phase transformations using an extremum principle. *Arch. Rational Mech. Anal.*, 162, 137–177, 2002.

[Mur78] F. Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3), 489–507, 1978.

[Mur81] F. Murat. Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 8, 69–102, 1981.

[Nef03a] P. Neff. Finite multiplicative plasticity for small elastic strains with linear balance equations and grain boundary relaxation. *Contin. Mech. Thermodyn.*, 15, 161–195, 2003.

[Nef03b] P. Neff. Local existence and uniqueness for quasistatic finite plasticity with grain boundary relaxation. *J. Nonlinear Science*, 2003. Submitted.

[NW03] P. Neff and C. Wieners. Comparison of models for finite plasticity: a numerical study. *Comput. Vis. Sci.*, 6(1), 23–35, 2003.

[OR99] M. Ortiz and E. Repetto. Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids*, 47(2), 397–462, 1999.

[OS99] M. Ortiz and L. Stainier. The variational formulation of viscoplastic constitutive updates. *Comput. Methods Appl. Mech. Engrg.*, 171(3-4), 419–444, 1999.

[Par95] G. P. Parry. Elastic–plastic decompositions of defect–preserving changes of state in solid crystals. In *Trends in applications of mathematics to mechanics (Lisbon, 1994)*, volume 77 of *Pitman Monogr. Surveys Pure Appl. Math.*, pages 154–163. Longman, Harlow, 1995.

[PŠ99] G. P. Parry and M. Šilhavý. Elastic scalar invariants in the theory of defective crystals. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 455(1992), 4333–4346, 1999.

[Res67] Y. Reshetnyak. On the stability of conformal maps in multidimensional spaces. *Siberian Math. J.*, 8, 69–85, 1967.

[RK04] T. Roubíček and M. Kružík. Mircrostructure evolution model in micromagnetics. *Zeits. angew. math. Physik*, 55, 159–182, 2004.

[Šil01a] M. Šilhavý. Dissipation postulates in finite–deformation plasticity. In *Proc. of 15th AIMETA Congress of Theoretical and Applied Mechanics*, page 10 pp, 2001.

[Šil01b] M. Šilhavý. On the nonconvex potentials for elastic–plastic materials. Preprint Sept., 2001.

[Šil02] M. Šilhavý. Zeroes of the polyconvex hull of powers of the distance and s–polyconvexity. Czech Academy of Science, Preprint, 2002.

[SO85] J. Simo and M. Ortiz. A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive relations. *Comput. Methods Appl. Mech. Engrg.*, 49, 221–245, 1985.

[SS92] C. G. Simader and H. Sohr. A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains. In *Mathematical problems relating to the Navier-Stokes equation*, volume 11 of *Ser. Adv. Math. Appl. Sci.*, pages 1–35. World Sci. Publishing, River Edge, NJ, 1992.

[Suq81] P.-M. Suquet. Sur les équations de la plasticité: existence et régularité des solutions. *J. Mécanique*, 20(1), 3–39, 1981.

[Sve02] B. Svendsen. Continuum thermodynamic models for crystal plasticity including the effects of geometrically necessary dislocations. *J. Mech. Physics Solids*, 50, 1297–1329, 2002.

[Tar79] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, pages 136–212. Pitman, Boston, Mass., 1979.

[Tar90] L. Tartar. H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 115(3-4), 193–230, 1990.

[vW90] W. von Wahl. Abschätzungen für das Neumann-Problem und die Helmholtz-Zerlegung von L^p. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, (2), 29, 1990.