A rate-independent approach to the delamination problem

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Abstract. We study delamination processes for elastic bodies glued together by an adhesive as an activated, rate-independent process. The adhesive is assumed to absorb a specific amount of energy during the delamination process. A solution is defined by energetic principles of stability and balance of stored and dissipated energies with the work of external loading, realized here through displacement on parts of the boundary. Starting from a time discretization, we construct solutions via a rigorous limiting analysis. Moreover, we provide computer simulations for some model problems using a further finite-element spatial discretization.

Key Words. Inelastic damage, variational inequality, numerical approximation, finite elements, convergence analysis, simulations.

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1. Introduction

Laminate structures are widely used in many industrial branches and the study of their failure modes becomes more and more important, see e.g. [15]. In this paper, we will concentrate on one such mode, namely delamination of layered laminate structures. *Delamination* is a progressive separation of bonded laminate and, simultaneously, degradation of the used adhesive. As such, it is a critical failure mode of layered composite structures because it can degrade the laminate to such a degree that it becomes useless in service. Therefore, delamination is usually considered an unwilling effect. Yet, by another viewpoint, laminated materials have been shown to have energy absorption properties superior to conventional metallic structures. The *energy dissipated* during delamination is a typical example. Therefore, laminated composites are more and more used when designing energy absorbing elements in vehicles [7]. In this situation, proper modeling of delamination and computing of the corresponding energies is of vital importance.

The mechanism of delamination is very complex and involves phenomena like debonding and unilateral contact with nonmonotone friction. Delamination is also often connected with the damage of layers. Here we consider delamination as a fracture-like process that can run along a-priori known surfaces between homogeneous isotropic elastic bodies that are in frictionless unilateral contact. We consider it as an *activated and rate-independent process*, based on the philosophy that a specific energy is needed to cut the macromolecular structure of the adhesive, no matter how fast or slow this process is.

Nowadays there is an extensive literature on modeling and simulating delamination phenomenon. Vast majority of the papers deals with modeling and computational technique and the authors do not attempt at proving the existence of a solution. The problem is generally approached either by using fracture mechanics (see, e.g., [5, 23, 28, 29]) or by introducing special constitutive laws for the interface material in the spirit of damage mechanics, or simply quasistatically.

The first approach is typically based on the notion of the energy-release rate and the delamination is activated when the computed energy-release rate reaches a prescribed threshold. This may lead to extensive numerical calculations when modeling large structures, as the energy release rate must be recomputed at each point where delamination may occur [6]. These models are closely related to fracture models, see e.g. [18]. Other techniques are based on special "interface" finite elements introduced at delamination regions; see [1] for a comprehensive review. In these elements, a rate-independent plasticity model with softening is used [29]. Yet another method allows for discontinuities in displacements to be modeled independently of the finite element mesh [30]. Once again, all these techniques are focused on numerical simulation of delamination.

In the second approach, delamination is described by a *damage variable* reflecting the destruction of the bonds in the a-priori known delamination surface. Proposed by Frémond [8, 9], this approach was developed in [13, 25, 26, 27], cf. also [10, Chap.14]. In this model, a damage variable, taking values between 0 and 1, indicates the state of delamination on the interface boundary and the adhesive contact is considered as a (possibly nonlinear) viscous one. Although the quite typically occurring effects of activation (i.e., no delamination is in progress under small loading even lasting long time) cannot be hit by this, the viscous approach has some application, too. We also refer to [15] for mechanical models for different failure mechanisms in delamination problems for laminated composites.

The third, static approach was proposed by Panagiotopoulos [24] who formulated the problem of equilibrium positions via hemivariational inequalities, a generalization of variational inequalities for nonmonotone operators, cf. also [2]. Thus, this method has limited applications only to processes with simple time–dependent loadings.

In this paper we focus on the second, damage-variable approach. However, as already announced, we prefer to consider the delamination as rate independent, *plastic-like* process (as in the models of the first approach), which automatically involves an activation phenomena. Contrary to the these models, we will support our model by a rigorous analysis based on the apparatus developed for rate-independent process recently in [20, 21, 22, 19]. Also, contrary to [13, 25, 26, 27], we understand the energy spent in damage of the adhesive not as a stored energy but as a dissipated energy, which avoids speaking about "local subdifferential" of nonconvex functions, see [25, 27]. Having in mind applications to impact on a rigid body, we consider loading through (unilateral) Dirichlet boundary conditions, which brings additional difficulties in the formulation of the correct energetics through the work of the (unknown) reaction force.

In Section 2, we formulate the model based on minimization of the stored energy competing with the activation of the delamination process and introduce a suitable definition of its solution. Then, in Section 3, we perform a rigorous analysis by semi-discretization in time, and in Section 4 we provide a numerical approximation via space discretization and present computer simulations of model examples together with a calculated energetic balance.

2. The model

Let us first specify our notation as far as geometry concerns. The elastic body will occupy a reference domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, assumed to be bounded with the Lipschitz boundary $\Gamma = \partial\Omega$. The body is loaded by time-dependent (possibly unilateral) Dirichlet boundary conditions (so called "hard device"), which is used to describe a frictionless (possibly unilateral) contact during an impact of another, completely rigid body on a part of the boundary Γ. The rest of the boundary is assumed to be free. The domain Ω itself is divided into a finite number of subdomains Ω_α, $\alpha = 1, ..., m$. The boundary between the subdomains Ω_α and Ω_β is denoted by $\Gamma_{\alpha\beta} := \Gamma_\alpha \cap \Gamma_\beta$ with $\Gamma_\alpha := \partial\Omega_\alpha$ and $\Gamma_\beta := \partial\Omega_\beta$ being the boundary of the subdomain Ω_α and Ω_β, respectively; of course, the case $\Gamma_{\alpha\beta} = \emptyset$ is not excluded and it indicates that the particular subdomains are not adjacent to each other, cf. Figure 1.

For simplicity and especially for efficiency of calculations, we will confine ourselves to *small displacements* and *isotropic materials* though the model can quite equally be formulated in terms of large deformations. Also, we consider the delamination process temperature-independent or so slow that the produced heat is transferred out to keep temperature constant, which allows us to restrict the model to the *isothermal* situation and speak about stored energy instead of free energy.

2.1. **Stored energy.** The state of the system will be considered as $q = (u_\alpha, \zeta_{\alpha\beta})_{\alpha,\beta=1,...,m,\alpha>\beta}$ where $u_\alpha : \Omega_\alpha \to \mathbb{R}^n$ is the (small) *displacement* in the subdomain Ω_α and $\zeta_{\alpha\beta} : \Gamma_{\alpha\beta} \to [0, 1]$ a *damage parameter* indicating how much of the adhesive

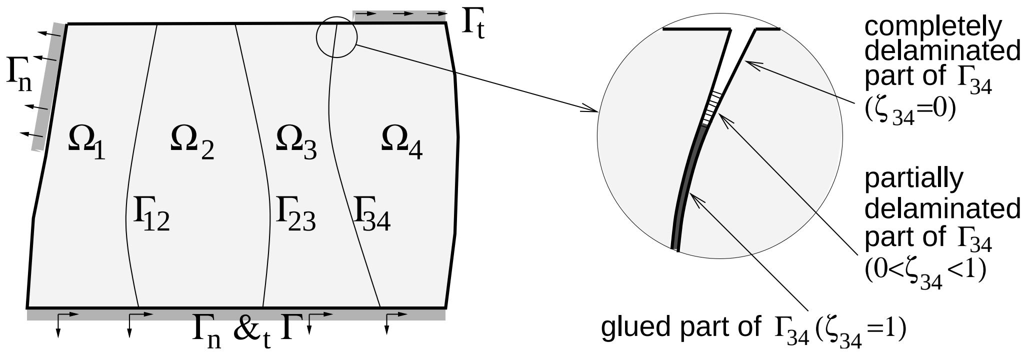

FIGURE 1. *An example of a model geometry ($m = 4$) and an interpretation of the model of delaminating the adhesive gluing the adjacent subdomains (here Ω_3 and Ω_4).*

is effective: 1 means 100% of the adhesive glues at $x \in \Gamma_{\alpha\beta}$, 0 means that the surface is completely delaminated at the current point $x \in \Gamma_{\alpha\beta}$, and $0 < \zeta_{\alpha\beta}(x) < 1$ means that some portion of macromolecules of the adhesive is already cut while the rest is still effective. Often, we will write shortly $u = (u_\alpha)_{\alpha=1,\ldots,m}$ and $\zeta = (\zeta_{\alpha\beta})_{\alpha,\beta=1,\ldots,m,\alpha>\beta}$.

The matrix $b_{\alpha\beta} : \Gamma_{\alpha\beta} \to \mathbb{R}^{n\times n}$ reflects the elastic properties of the adhesive. We adopt, in fact, a dimensional reduction of the n-dimensional adhesive layer to an $(n-1)$-dimensional surface $\Gamma_{\alpha\beta}$.

Considering an isotropic linear material, the elastic energy $V_\alpha(u_\alpha)$ stored in the volume Ω_α is

$$V_\alpha(u_\alpha) := \int_{\Omega_\alpha} \mu_\alpha|e(u_\alpha)|^2 + \frac{\lambda_\alpha}{2}(\operatorname{div} u_\alpha)^2 \mathrm{d}x, \quad e(u) := \frac{1}{2}(\nabla u^\top + \nabla u), \tag{2.1}$$

with $\mu_\alpha > 0$ and $\lambda_\alpha \geq 0$ the *Lamé constants* of the isotropic material in the αth domain, while the energy $V_{\alpha\beta}(\zeta_{\alpha\beta}, u_\alpha - u_\beta)$ stored in the adhesive on the surface $\Gamma_{\alpha\beta}$ is

$$V_{\alpha\beta}(\zeta_{\alpha\beta}, v) := \frac{1}{2}\int_{\Gamma_{\alpha\beta}} \zeta_{\alpha\beta}(x) v^\top b_{\alpha\beta}(x) v \,\mathrm{d}S. \tag{2.2}$$

We further prescribe a unilateral contact on $\Gamma_{\alpha\beta}$ which allows for a general "non-monotone" loading causing that some, already delaminated parts of the surfaces $\Gamma_{\alpha\beta}$ can be pushed together without being glued. It also excludes a penetration of the adjacent domains Ω_α and Ω_β. This leads to the complementarity conditions

$$\begin{gathered}(u_\alpha - u_\beta)|_{\Gamma_{\alpha\beta}} \cdot \nu_{\alpha\beta} \geq 0, \quad \nu_{\alpha\beta}^\top(\sigma_\alpha - \sigma_\beta)\nu_{\alpha\beta} \geq 0,\\ (u_\alpha - u_\beta)|_{\Gamma_{\alpha\beta}} \nu_{\alpha\beta}\, \nu_{\alpha\beta}^\top(\sigma_\alpha - \sigma_\beta)\nu_{\alpha\beta} = 0,\\ \tau^\top\sigma_\alpha\nu_{\alpha\beta} = \tau^\top\sigma_\beta\nu_{\alpha\beta} = 0 \text{ for } \tau \text{ with } \tau^\top\nu_{\alpha\beta} = 0,\end{gathered} \tag{2.3}$$

prescribed on each $\Gamma_{\alpha\beta}$, where $\sigma_\alpha := 2\mu_\alpha e(u_\alpha) + \lambda_\alpha(\operatorname{div} u_\alpha)\delta$ with $\delta = [\delta_{ij}]$ denoting the identity matrix (hence σ_α is the stress corresponding to the energy (2.1)) and where $\nu_{\alpha\beta}$ denotes the unit normal to the surface $\Gamma_{\alpha\beta}$ oriented from Ω_α to Ω_β.

As for the boundary conditions on Γ, we assume a Dirichlet, time-dependent loading on some part of Γ, while a unilateral frictionless contact is assumed on some other part of Γ and the rest is assumed to be free. This enables us to describe possible applications of a partly fixed, laminated elastic body Ω subjected to an impact of another, rigid body touching Ω from outside on the part of Γ where the unilateral contact is prescribed, cf. Figure 7 below. To describe such a general situation in a simple way, we introduce a closed, convex cone $D(x) \subset \mathbb{R}^n$ depending on $x \in \Gamma$, and assume the boundary conditions on Γ in the complementarity form as

$$\left.\begin{array}{rcl} u_\alpha|_{\Gamma_\alpha} & \succeq_D & w(t), \\ \sigma_\alpha \nu & \succeq_D^* & 0, \\ (\sigma_\alpha \nu)\cdot(u|_{\Gamma_\alpha} - w) & = & 0, \end{array}\right\} \text{on } \Gamma_\alpha \cap \Gamma, \tag{2.4}$$

where $\nu = \nu(x)$ is the unit outward normal to Γ_α, $\succeq_D$ is the ordering induced by D in the sense $w \succeq_D 0$ if and only if $w(x) \in D(x)$ for a.a. $x \in \Gamma$, and $\succeq_D^*$ is the dual ordering induced by the negative polar cone to D in the sense $s \succeq_D^* 0$ if and only if $s(x)\cdot w \geq 0$ for all $w \in D(x)$ for a.a. $x \in \Gamma$, and $w = w(t,x)$ is the Dirichlet loading. For formal reasons, it is advantageous to consider w prescribed also on the delamination surfaces $\Gamma_{\alpha\beta}$ and to handle both (2.3) and (2.4) in a unified manner. Then, the particular choices that can be described by a cone $D \subset \mathbb{R}^n$ depending on $x \in \Gamma$ are, e.g.,

- $D(x) = \mathbb{R}\nu$ for fixing tangential displacement at $x \in \Gamma$ (let us denote this part of the boundary by $\Gamma_{\rm t} \subset \Gamma$),
- $D(x) = \{v \in \mathbb{R}^n;\ v\cdot\nu = 0\}$ for fixing the normal displacement at $x \in \Gamma$ (let us denote this part by $\Gamma_{\rm n} \subset \Gamma$),
- $D(x) = \{v \in \mathbb{R}^n;\ v = a\nu;\ a \geq 0\}$ for a unilateral contact in the normal displacement at $x \in \Gamma$, (let us denote this part by $\Gamma_{\rm c} \subset \Gamma$).

Possibly, $\Gamma_{\rm n}$, $\Gamma_{\rm t}$ and $\Gamma_{\rm c}$ can overlap like on Figures 1, 2, or 4. Obviously, $D(x) = \{0\}$ for $x \in \Gamma_{\rm t} \cap \Gamma_{\rm n}$.

For notational simplicity, we will abbreviate

$$\mathcal{Q} := \mathcal{U} \times \mathcal{Z}, \qquad \mathcal{U} := \prod_{\alpha=1}^{m} H^1(\Omega_\alpha), \qquad \mathcal{Z} := \prod_{\alpha>\beta} L^1(\Gamma_{\alpha\beta}), \tag{2.5}$$

$$\mathcal{H} := \prod_{\alpha=1}^{m} H^{1/2}(\Gamma_\alpha; \mathbb{R}^n). \tag{2.6}$$

All spaces $\mathcal{U}$, $\mathcal{Z}$, and $\mathcal{H}$ are assumed to be equipped with their standard norms. Since all our functions $\zeta \in \mathcal{Z}$ lie in the set $Z_{[0,1]} := \{\zeta \in \mathcal{Z};\ \zeta_{\alpha\beta} \in [0,1] \text{ a.e. on } \Gamma_{\alpha\beta}\}$ and we will simply write $0 \leq \zeta \leq 1$ if $\zeta \in Z_{[0,1]}$. Of course $Z_{[0,1]}$ is a bounded subset of $\prod L^\infty(\Gamma_{\alpha\beta})$; however, it is important to work with the L^1 norm which is not equivalent to that of L^∞. Let us further denote by

$$L_\Gamma : \mathcal{U} \to \mathcal{H} : u \mapsto (u_\alpha|_{\Gamma_\alpha})_{\alpha=1}^m \tag{2.7}$$

the operator producing the collection of traces. To treat the variety of (possibly unilateral) boundary conditions, let us consider the (convex, closed) cone $\mathcal{D}\subset\mathcal{H}$ given by

$$\mathcal{D}:=\big\{v=(v_\alpha)_{\alpha=1}^m\in\mathcal{H};\quad \forall_{\text{(a.a.)}}\,x\in\Gamma\cap\Gamma_\alpha:\quad v_\alpha(x)\in D(x),\quad \forall_{\text{(a.a.)}}\,x\in\Gamma_{\alpha\beta}:\quad \nu_{\alpha\beta}\cdot(v_\alpha{-}v_\beta)\geq 0\big\}. \tag{2.8}$$

Now, $\succeq$ denotes the ordering of $\mathcal{H}$ by the cone $\mathcal{D}$.

The overall (Gibbs-type) stored energy is then

$$G(t,q):=\begin{cases}\displaystyle\sum_{\alpha=1}^m\Big(V_\alpha(u_\alpha)+\sum_{\beta=1}^{\alpha-1}V_{\alpha\beta}(\zeta_{\alpha\beta},u_\alpha{-}u_\beta)\Big) & \text{if } L_\Gamma u\succeq w(t,\cdot)\ \text{and}\ \zeta\in Z_{[0,1]},\\ +\infty & \text{otherwise.}\end{cases} \tag{2.9}$$

Note that the time-dependence occurs only through the contraint via the Dirichlet loading $w(t,\cdot)$. Also, we will often write

$$G(t,q)\equiv G(t,u,\zeta)=V(u,\zeta)+\delta_{L_\Gamma^{-1}(w(t)+\mathcal{D})\times Z_{[0,1]}}(u,\zeta), \tag{2.10}$$

where V is the overall "elastic" stored energy

$$V(u,\zeta):=\sum_{\alpha=1}^m\Big(V_\alpha(u_\alpha)+\sum_{\beta=1}^{\alpha-1}V_{\alpha\beta}(\zeta_{\alpha\beta},u_\alpha{-}u_\beta)\Big), \tag{2.11}$$

and $\delta_C(q)=0$ denotes the indicator function of a convex set C. Of course, L_Γ is not injective so that L_Γ^{-1} in (2.10) is a set-valued mapping. To avoid a situation when a completely delaminated subdomain Ω_α is not fixed with respect to rigid-body motions, we will assume:

$$\forall\alpha=1,...,m:\quad \text{meas}_{n-1}\big(\Gamma_{\rm t}\cap\Gamma_{\rm n}\cap\Gamma_\alpha\big)\neq\emptyset. \tag{2.12}$$

By Korn's inequality, (2.12) ensures coercivity of $G(t,\cdot,\zeta)$ uniformly with respect to $\zeta\in Z_{[0,1]}$ and to $w(t)$ ranging over bounded sets in $\mathcal{H}$.

2.2. **Dissipation.** Dissipative mechanisms are routinely described by Rayleigh's (pseudo)*potential of dissipative forces*, here denoted by R, as a function of the rate of $q=q(t)$.

We will consider the material in the particular subdomains Ω_α as purely elastic. The only dissipation of energy we will consider can occur in the adhesive and, on the atomistic level, it is related with cutting the macromolecular chains composing the adhesive. To describe this process, we allow for the simplification that this process can be described with good accuracy by a single phenomenological parameter $d_{\alpha\beta}=d_{\alpha\beta}(x)$ having the meaning of a specific energy (per area, i.e. in physical units Jm^{-2}) needed to delaminate the surface $\Gamma_{\alpha\beta}$ at a point $x\in\Gamma_{\alpha\beta}$, i.e. the energy needed to switch $\zeta_{\alpha\beta}(x)$ from 1 to 0. This energy is irreversibly dissipated to the structural change of the adhesive on the surface $\Gamma_{\alpha\beta}$. In other words, the delamination process in our model is *rate independent*, in particular it is an *activated process*. The specific dissipation then includes only the rate of damage coefficients $(\zeta_{\alpha\beta})$ but not of the displacement (u_α), and has the form

$$\big[\varrho(\dot{q})\big]_{\alpha\beta}(x) := \begin{cases} -d_{\alpha\beta}(x)\dot{\zeta}_{\alpha\beta}(x) & \text{if } \dot{\zeta}_{\alpha\beta}(x) \le 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.13}$$

where the variable $\dot{q} = (\dot{u}; \dot{\zeta})$ stands for the rate of q. The mentioned irreversibility of the delamination process is related with the phenomenon that, if once delaminated, the surface cannot be glued back and it is reflected by the non-symmetry $\varrho(\dot{q}) \neq \varrho(-\dot{q})$, cf. (2.13). The consequence of the assumed rate-independency is that ϱ is homogeneous of degree 1. In particular, ϱ is nonsmooth at 0, which is related to the activation phenomena.

The overall (non-symmetric) dissipation potential is then defined as

$$R(\dot{q}) \;:=\; \sum_{\alpha=1}^{m}\sum_{\beta=1}^{\alpha-1}\int_{\Gamma_{\alpha\beta}} \big[\varrho(\dot{q})\big]_{\alpha\beta}(x)\mathrm{d}S. \tag{2.14}$$

The above formula is to be understood in the sense that $R(\dot{q}) = +\infty$ if $\varrho(\dot{q})$ is not finite a.e. on $\bigcup_{\alpha>\beta}\Gamma_{\alpha\beta}$. The important property of R is that it satisfies the triangle inequality, i.e.

$$\forall q_1, q_2, q_3 \in \mathcal{Q}: \quad R(q_1 - q_3) \le R(q_1 - q_2) + R(q_2 - q_3), \tag{2.15}$$

which follows immediately from convexity and the homogeneity of degree 1.

2.3. **Solution processes and their energetics.** When the boundary data w varies in time, i.e. $w = w(t)$, the state response will expectedly vary as well. We will write $q = q(t)$ for $t \in [0, T]$ with $T > 0$ as a fixed time horizon.

Following [20] (see also [21, 22]), we say that the process $q = q(t)$ is *stable* if

$$\forall \tilde{q} \in \mathcal{Q}: \quad G\big(t, q(t)\big) \le G(t, \tilde{q}) + R\big(\tilde{q} - q(t)\big) \tag{2.16}$$

for all $t \in [0, T]$. The philosophy of (2.16) is that the gain of Gibbs' energy $G(t, q(t)) - G(t, \tilde{q})$ at any other state $\tilde{q}$ is not larger than the dissipation $R(\tilde{q} - q(t))$; cf. [22] for discussion.

To proceed further, we must define the reaction force $F = F(t)$ on the "hard-device" loading $w(t) \in \mathcal{H}$. As R involves only ζ but not u, the stability (2.16) of $q(t) = (u(t), \zeta(t))$ implies that $u(t)$ is the global minimizer of $G(t, \cdot, \zeta(t))$. By the definition (2.9) of G, it implies that the displacement $u = (u_\alpha)$ solves, for $\zeta = (\zeta_{\alpha\beta}) \ge 0$ and w fixed, the following minimization problem

$$\begin{cases} \text{Minimize} & V(u, \zeta) \\ \text{subject to} & L_\Gamma u \succeq w \equiv w(t)\ , \\ & u \in \mathcal{U}\ , \end{cases} \tag{2.17}$$

where the inequality in (2.17) refers to the ordering of $\mathcal{H}$ by the cone $\mathcal{D}$, cf. (2.8). Then we define the *reaction force* as the quantity that balances the equation created by "forgetting" the (possibly unilateral) conditions on $\bigcup_{\alpha=1}^m \Gamma_\alpha$, i.e. for any u solving (2.17) we define $F \in \mathcal{H}^*$ by the formula

$$\begin{aligned}\langle F, v_\Gamma\rangle \;:=\; \textstyle\sum_{\alpha=1}^m \Bigg(&\int_{\Omega_\alpha} 2\mu_\alpha e(u_\alpha){:}e(v_\alpha) + \lambda_\alpha(\operatorname{div} u_\alpha)(\operatorname{div} v_\alpha)\,\mathrm{d}x \\ &+\sum_{\beta=1}^{\alpha-1}\int_{\Gamma_{\alpha\beta}} \zeta_{\alpha\beta}(x)(u_\alpha{-}u_\beta)^\top b_{\alpha\beta}(x)(v_\alpha{-}v_\beta)\,\mathrm{d}S\Bigg), \\ &\text{for all} \quad v\in\mathcal{U} \quad \text{with } L_\Gamma v = v_\Gamma,\end{aligned} \tag{2.18}$$

where $e(u){:}e(v) := \sum_{i,j=1}^n e_{ij}(u)e_{ij}(v)$. As the minimization problem (2.17) depends on (w,ζ), so does F. The important points are that the minimizer u of the uniformly convex functional $V(\cdot,\zeta)$ on the convex set of admissible u's in (2.17) is defined uniquely, and that u in (2.18) is just this minimizer so that the right-hand side of (2.18) depends only on the trace $L_\Gamma v$:

Lemma 2.1. *The formula (2.18) determines $F \in \mathcal{H}^*$ uniquely, and the mapping $(w,\zeta) \mapsto (u,F) : \mathcal{H}\times\mathcal{Z} \to \mathcal{U}\times\mathcal{H}^*$ is (norm×weak,norm×weak)-continuous.*

Proof. Let us abbreviate the trilinear form on the right-hand side of (2.18) by

$$a(\zeta,u,v) := \langle V_u'(u,\zeta), v\rangle, \tag{2.19}$$

cf. (2.9). As $u=(u_\alpha)\in L_\Gamma^{-1}(\mathcal{D}+w)$ in (2.18) is the (unique) solution to (2.17), it satisfies the variational inequality (in variable u)

$$\forall v \in L_\Gamma^{-1}(\mathcal{D}+w): \qquad a(\zeta,u,v-u)\ge 0. \tag{2.20}$$

Take $v_1,v_2\in\mathcal{U}$ such that $L_\Gamma v_1 = L_\Gamma v_2$. Then $v:=v_1-v_2+u\in L_\Gamma^{-1}(\mathcal{D}+w)$ and putting it into (2.20), we get $a(\zeta,u,v_1-v_2)\ge 0$. Doing the same for $v:=v_2-v_1+u$, we can see that altogether $a(\zeta,u,v_1-v_2)=0$. By the definition (2.18) of $L_\Gamma^*F = a(\zeta,u,\cdot)$, we have proved that $a(\zeta,u,\cdot)$ indeed depends only on the collection of traces $L_\Gamma v$ of $v\in\mathcal{U}$. Hence it defines a functional $F:\mathcal{H}\to\mathbb{R}$. Obviously, $a(\zeta,u,\cdot)$ is linear and hence this functional is linear, too. Moreover, the functional F is continuous, as can be seen from the estimate

$$\begin{aligned}||F||_{\mathcal{H}^*} := \sup_{\substack{||v_\Gamma||_{\mathcal{H}}\le 1\\ v_\Gamma\in\mathcal{H}}} \langle F, v_\Gamma\rangle \;&=\; \sup_{\substack{||v_\Gamma||_{\mathcal{H}}\le 1\\ v_\Gamma\in\mathcal{H}}}\;\inf_{\substack{v\in\mathcal{U}\\ L_\Gamma v = v_\Gamma}} a(\zeta,u,v) \\ &\le\; \sup_{||v||_{\mathcal{U}}\le N} a(\zeta,u,v) \;<\; +\infty.\end{aligned} \tag{2.21}$$

Here we used definition (2.18) and N denotes the bound from the a-priori estimate $||v||_{\mathcal{U}}$ of the solution v to the boundary-value problem $A_\zeta v = 0$, $L_\Gamma v = v_\Gamma$, with $v_\Gamma\in\mathcal{H}$, $||L_\Gamma v||_{\mathcal{H}}\le 1$. Moreover, $A_\zeta:\mathcal{U}\to\mathcal{U}^*$ is defined by $\langle A_\zeta u, v\rangle = a(\zeta,u,v)$; in other words, $A_\zeta u = V_u'(u,\zeta)$.

Now, take a sequence $\{(w_k,\zeta_k)\}_{k\in\mathbb{N}}\subset\mathcal{H}\times\mathcal{Z}$ converging to (w,ζ) in (norm×weak)-topology. Let u_k be the minimizer of (2.17) corresponding to (w_k,ζ_k), i.e.

$$\forall v\in L_\Gamma^{-1}(\mathcal{D}+w_k): \qquad a(\zeta_k,u_k,v-u_k)\ge 0. \tag{2.22}$$

By a-priori estimates, we can select a subsequence weakly converging in $\mathcal{U}$, and show that its limit solves (2.20). Having the limit u identified uniquely, we can see that even the whole sequence $\{u_k\}_{k\in\mathbb{N}}$ converges to it weakly. This is, in fact, a standard procedure, the only peculiarity here is the limit passage $\int_{\Gamma_{\alpha\beta}}[\zeta_k]_{\alpha\beta}u_k^\top b_{\alpha\beta}(v-u_k)\,\mathrm{d}S \to \int_{\Gamma_{\alpha\beta}}\zeta_{\alpha\beta}u^\top b_{\alpha\beta}(v-u)\,\mathrm{d}S$ which uses simply the compactness of the trace operator $u \mapsto u|_{\Gamma_{\alpha\beta}} : \mathcal{U} \to L^2(\Gamma_{\alpha\beta})$.

Now we want to prove $u_k \to u$ in the norm of $\mathcal{U}$. Let us take $\bar{u} \in L_\Gamma^{-1}(w)$ and $\bar{u}_k \in L_\Gamma^{-1}(w_k)$. As $w_k \to w$ in $\mathcal{H}$, we can also assume that $\bar{u}_k \to \bar{u}$ in $\mathcal{U}$ by considering a continuous selection from a continuous set-valued mapping L_Γ^{-1}. Then $v := u + \bar{u}_k - \bar{u} \in L_\Gamma^{-1}(\mathcal{D} + w_k)$ and we can put this v into (2.22). Likewise, $v := u_k + \bar{u} - \bar{u}_k$ is a legal test function for (2.20). Adding the created inequalities, we obtain, after some algebra,

$$a(\zeta, u_k - u, u_k - u) \;\le\; a(\zeta_k, u_k - u, \bar{u}_k - \bar{u}) + a(\zeta_k - \zeta, u, \bar{u}_k - \bar{u}) - a(\zeta_k - \zeta, u_k, u_k - u). \tag{2.23}$$

All the terms on the right-hand side in (2.23) can easily be shown to converge to 0. Hence we get $u_k \to u$ from the coercivity $a(\zeta, v, v) \ge \varepsilon||v||_{\mathcal{U}}^2$ for some $\varepsilon > 0$.

Let us denote by F_k the reaction force that corresponds to (w_k, ζ_k). Fixing v in (2.18), we can easily see that $\langle F_k, v_\Gamma\rangle = a(\zeta_k, u_k, v) \to a(\zeta, u, v) = \langle F, v_\Gamma\rangle$. Hence $F_k \to F$ weakly. □

As the unilateral Dirichlet loading w evolves in time (in a prescribed manner), so will do the (unknown) damage parameter ζ and therefore also the reaction force $F = F(w, \zeta)$. Hence, we can now agree that the process $q = q(t) = (u(t), \zeta(t))$ is called to satisfy the *energy inequality* if, for all $t, s \in [0, T]$, $s < t$,

$$V\big(q(t)\big) + \mathrm{Var}_R(q; s, t) \le V\big(q(s)\big) + \int_s^t \Big\langle F, \frac{\mathrm{d}w}{\mathrm{d}\theta}\Big\rangle\,\mathrm{d}\theta, \qquad F = F\big(\zeta(\theta), w(\theta)\big), \tag{2.24}$$

where the R-variation of the function $q(\cdot)$ over the time interval $[s, t]$ is defined standardly, without using explicitly any time derivative, as

$$\mathrm{Var}_R(q; s, t) := \sup \sum_{i=1}^{j} R\big(q(t_i) - q(t_{i-1})\big) \tag{2.25}$$

with the supremum taken over all $j \in \mathbb{N}$ and over all partitions of $[s, t]$ in the form $s = t_0 < t_1 < \ldots < t_{j-1} < t_j = t$. Of course, in our special situation with R defined via (2.13) and (2.14) the R-variation takes the form

$$\mathrm{Var}_R(q; s, t) = \sum_{\alpha<\beta} \int_{\Gamma_{\alpha\beta}} d_{\alpha\beta}(\zeta_{\alpha\beta}(s,\cdot) - \zeta_{\alpha\beta}(t,\cdot))\,\mathrm{d}S,$$

whenever for a.e. $x \in \Gamma_{\alpha\beta}$ the function $\zeta_{\alpha\beta}(\cdot, x)$ are nonincreasing on $[s, t]$. In all other cases the R-variation will be $+\infty$.

The particular terms in (2.24) thus represent

- stored energy at time t,
- the energy dissipated by delamination during the time interval $[s, t]$,

- stored energy at the initial time s, and
- work done by external "hard-device" loading during the time interval $[s,t]$.

Furthermore, for Z a Banach space, let us agree to understand the space $\mathrm{BV}([0,T];Z)$ of functions with bounded variations as containing measurable Z-valued functions defined *everywhere* on $[0,T]$, and normed by $||z||_{\mathrm{BV}([0,T];Z)} := ||z||_{L^1(0,T;Z)} + \mathrm{Var}_{\|\cdot\|_Z}(u;0,T)$. Also the space $L^\infty(0,T;Z)$ will be understand as containing measurable Z-valued functions defined *everywhere*.

Having in mind an initial-value problem with an initial configuration $q_0 \in \mathcal{Q}$, our problem is thus determined by the stored energy V, dissipation potential R, the loading w, and the initial condition q_0.

Definition 2.2. *A process $q = q(t)$, $t \in [0,T]$, is called the solution process to the problem given by the quadruple (V,R,w,q_0) if, $q = (u,\zeta) \in L^\infty(0,T;\mathcal{U}) \times \mathrm{BV}([0,T];\mathcal{Z})$, it is stable in the sense (2.16) for all $t \in [0,T]$, and if it satisfies (2.24) for all $t,s \in [0,T]$, $t > s$, and if $q(0) = q_0$.*

For further analysis, we will need the following stability and sensitivity results about the value function of the linear/quadratic problem (2.17):

Lemma 2.3. *Denote by $\mathfrak{v} = \mathfrak{v}(w,\zeta) = \min V\big(L_\Gamma^{-1}(\mathcal{D}+w),\zeta\big)$ the minimum in problem (2.17). The mapping $\mathfrak{v} : \prod_{\alpha>\beta} L^1(\Gamma_{\alpha\beta}) \times \mathcal{H} \times Z_{[0,1]} \to \mathbb{R}$ is (norm$\times$weak)-continuous and $\mathfrak{v}(\cdot,\zeta)$ is Gâteaux differentiable, the directional derivatives are (norm$\times$weak)-continuous, and we have the formula*

$$\forall\, \tilde{w} \in \mathcal{H} : \qquad \left[\frac{\partial \mathfrak{v}}{\partial w}(w,\zeta)\right](\tilde{w}) = \big\langle F(w,\zeta), \tilde{w}\big\rangle. \tag{2.26}$$

Remark 2.4. The formula (2.26) expresses *d'Alembert's virtual-work principle*: varying the obstacle w by $\tilde{w}$ against the reaction force F performs a work equal to the variation of the stored energy $\mathfrak{v}$. This also justifies the formula (2.18) for the reaction force.

Proof. Since $(w,\zeta) \mapsto u$ has been proved to be (norm$\times$weak,norm)-continuous in Lemma 2.1 and since the trilinear form $(\zeta,u,v) \mapsto a(\zeta,u,v)$ is certainly (weak$\times$norm$\times$norm)-continuous, the mapping $\mathfrak{v} : (w,\zeta) \mapsto \frac{1}{2}a(\zeta,u,u)$ is (norm$\times$weak)-continuous, as claimed.

For $\zeta \in Z_{[0,1]}$ fixed, we have to investigate sensitivity of the value of (2.17) with respect to the unilateral boundary data w. To use standard results, we first transform this problem to another one with homogeneous boundary data. This is again a standard procedure: considering $\bar{u} \in \mathcal{U}$ such that $L_\Gamma \bar{u} = w$, we can see that u solves (2.17) if and only if $u = u_0 + \bar{u}$ with $u_0 \in L_\Gamma^{-1}(\mathcal{D})$ solving the variational inequality

$$\forall v \in L_\Gamma^{-1}(\mathcal{D}) : \qquad a(\zeta,u_0,v-u_0) \geq \langle f, v-u_0\rangle, \tag{2.27}$$

where the trilinear form a means, as before, the right-hand-side of (2.18), and where $f \in \mathcal{U}^*$ is defined by $\langle f,v\rangle := a(\zeta,-\bar{u},v)$. The important fact is that, now, the convex set $L_\Gamma^{-1}(\mathcal{D})$ is fixed. We can also write $f = -A_\zeta \bar{u}$; for A_ζ see the the proof of Lemma 2.1.

This transforms the problem to a standard variational inequality on a Hilbert space, here on $\mathcal{U}$.

Sensitivity of the energy related to (2.27) with respect to perturbations of the right-hand side f can be found, e.g., in [14, Appendix V]. It is shown that the directional derivative $\mathrm{D}J(f,\tilde f)$ of the energy

$$J(f) := \min_{v\in L_\Gamma^{-1}(\mathcal{D})} \frac{1}{2}a(\zeta,v,v) - \langle f,v\rangle, \tag{2.28}$$

at the point f in the direction $\tilde f$ is given by the expression

$$\mathrm{D}J(f,\tilde f) = -\langle \tilde f, v\rangle \tag{2.29}$$

where v is the minimizer of (2.28), i.e. in our case $v = u_0 = u - \bar u$. Taking into account $\langle f,v\rangle = a(\zeta,-\bar u,v)$, we can express this functional in terms of $\bar w$, i.e. $\mathfrak{J}(\bar u) := J(-A_\zeta \bar u) = \min_{v\in L_\Gamma^{-1}(\mathcal{D})} a(\zeta,\frac{1}{2}v+\bar u,v)$ and, using (2.29) we get its directional derivative in the form

$$\mathrm{D}\mathfrak{J}(\bar u,\hat w) = a(\zeta,\hat w,u_0) = a(\zeta,\hat w,u-\bar u). \tag{2.30}$$

However, we are interested in the derivative of the functional $j(\bar w) := \frac{1}{2}a(\zeta,u,u)$ where u solves (2.17) with $w = L_\Gamma \bar w$; then obviously $\mathfrak{v}(w,\zeta) = j(\bar u)$. Obviously, $j(\bar u) = \frac{1}{2}a(\zeta,u,u) = \frac{1}{2}a(\zeta,u_0+\bar u,u_0+\bar u) = \frac{1}{2}a(\zeta,u_0,u_0) + a(\zeta,\bar u,u_0) + \frac{1}{2}a(\zeta,\bar u,\bar u) = \mathfrak{J}(\bar u) + \frac{1}{2}a(\zeta,\bar u,\bar u)$. As $a(\zeta,\cdot,\cdot)$ is symmetric, the directional derivative of $\bar u \mapsto a(\zeta,\bar u,\bar u)$, denoted by $\mathrm{D}_{\bar u}\big[a(\zeta,\bar u,\bar u)\big](\hat w)$, equals to $2a(\zeta,\hat w,\bar u)$. Then, using (2.30) and (2.18), we get

$$\begin{aligned} \mathrm{D}j(\bar u,\hat w) &= a(\zeta,\hat w,u_0) + \mathrm{D}_{\bar u}\Big[\frac{1}{2}a(\zeta,\bar u,\bar u)\Big](\hat w) \\ &= a(\zeta,\hat w,u-\bar u) + a(\zeta,\bar u,\hat w) \\ &= a(\zeta,\hat w,u) = \langle F(\zeta,w),\hat w\rangle. \end{aligned} \tag{2.31}$$

In particular, the directional derivative in (2.31) depends only on traces of $\bar u$ and of $\hat w$ on Γ, as in agreement also with $\mathfrak{v}(w,\zeta) = j(\bar u)$. Hence, (2.31) proves the formula (2.26) for the directional derivative of $\mathfrak{v}$. Obviously, the right-hand side of (2.26) depends linearly and, thanks to (2.21), also continuously on the variation $\tilde w$, hence $F(w,\zeta)$ represents the Gâteaux differential of $\mathfrak{v}(\cdot,\zeta)$. For any $\tilde w$ fixed, the claimed continuity of the directional derivative, i.e. the left-hand side of (2.26), follows from the (norm×weak,weak)-continuity of F claimed in Lemma 2.1. □

3. Analysis by a semi-discretization in time

We will prove the existence of a solution process quite constructively by a semi-discretization in time, using the implicit Euler scheme. To construct approximate solutions, we consider a time step $\tau > 0$, assuming T/τ integer and also that $\tau \to 0$ in such a way that the equidistant partitions will be nested; for example, the reader can think about a sequence of time steps $\tau = 2^{-k}T$ for $k \in \mathbb{N}$. For $\tau > 0$ fixed, this equi-distant partition of the interval $[0,T]$ leads to the following recursive increment formula: we put

$q_\tau^0 = q_0$ a given initial condition, and, for $k = 1, ..., T/\tau$ we define q_τ^k to be any solution of the minimization problem

$$\left\{ \begin{array}{ll} \text{Minimize} & V(q) + R(q - q_\tau^{k-1}) \\ \text{subject to} & L_\Gamma u \succeq w_\tau^k, \quad \zeta \in Z_{[0,1]} \\ & q \equiv (u, \zeta) \in \mathcal{Q} := \mathcal{U} \times \mathcal{Z}, \end{array} \right. \tag{3.1}$$

where $w_\tau^k = w(k\tau)$. In view of (2.24), it is natural to assume

$$w \in W^{1,1}\big(0, T; \mathcal{H}\big). \tag{3.2}$$

In particular, this ensures the continuity of $t \mapsto w(t) \in \mathcal{H}$, so that the values $w(k\tau)$ are well-defined. As we want to address the initial-value problem, we have to prescribe an initial state $q_0 \in \mathcal{Q}$ and, having the loading regime w specified, it is natural to assume q_0 stable, i.e.

$$\forall \tilde{q} \in \mathcal{Q}: \qquad G(0, q_0) \leq G(0, \tilde{q}) + R(\tilde{q} - q_0). \tag{3.3}$$

The solution to (3.1) will be denoted by q_τ^k, and then we assemble the piecewise constant interpolation $q_\tau \in L^\infty(0, T; \mathcal{Q})$ so that $q_\tau|_{((k-1)\tau, k\tau]} = q_\tau^k$ for $k = 1, ..., T/\tau$. Likewise, w_τ denotes the piecewise constant interpolation of w. For the left-hand side of (3.6) below, we assume the prolongation $[(w_\tau, \zeta_\tau)](t) = (w_\tau^0, \zeta_\tau^0)$ for $t < 0$.

Lemma 3.1. *Let $\mu_\alpha > 0$, $\nu_\alpha \geq 0$, $b_{\alpha\beta} \geq 0$, $d_{\alpha\beta} \geq 0$ for $\alpha, \beta = 1, ..., m$, and the assumptions (2.12), (3.2), and (3.3) be valid. Then the approximate solution q_τ does exist and satisfies stability, i.e.,*

$$\forall \tilde{q} \in \mathcal{Q}: \qquad G_\tau\big(t, q_\tau(t)\big) \leq G_\tau(t, \tilde{q}) + R\big(q_\tau(t) - \tilde{q}\big) \tag{3.4}$$

for all $t \in (0, T]$, where G_τ is piecewise constant approximation of G defined by

$$G_\tau(t, q) = V(q) + \delta_{L_\Gamma^{-1}(w_\tau(t) + \mathcal{D}) \times Z_{[0,1]}}(q). \tag{3.5}$$

Also, it satisfies the two-sided discrete energy estimate

$$\begin{aligned} \int_0^t \Big\langle F(w(\theta), \zeta_\tau(\theta)), \frac{\mathrm{d}w}{\mathrm{d}\theta}(\theta) \Big\rangle \,\mathrm{d}\theta \ &\leq \ V\big(q_\tau(t)\big) + \mathrm{Var}_R(q_\tau; 0, t) - V(q_0) \\ &\leq \ \int_0^t \Big\langle F(w(\theta), \zeta_\tau(\theta - \tau)), \frac{\mathrm{d}w}{\mathrm{d}\theta}(\theta) \Big\rangle \,\mathrm{d}\theta, \end{aligned} \tag{3.6}$$

with $F(\cdot, \cdot)$ from Lemma 2.1 and with $t = k\tau$ for any $k = 1, ..., T/\tau$.

Further, there exist constants C_1, C_2 and C_3 which are independent of the time step τ such the following a-priori estimates hold:

$$\|u_\tau\|_{L^\infty(0,T;\mathcal{U})} \leq C_1, \quad \textit{and} \tag{3.7}$$

$$\|\zeta_\tau\|_{\mathrm{BV}([0,T];\mathcal{Z})} \leq C_2, \tag{3.8}$$

$$\|\mathfrak{G}_\tau\|_{\mathrm{BV}([0,T])} \leq C_3 \quad \textit{with} \ \ \mathfrak{G}_\tau(t) := G_\tau\big(t, q_\tau(t)\big). \tag{3.9}$$

Proof. Existence of a solution $q_\tau^k \in \mathcal{Q}$ to (3.1) follows recursively for $k = 1, ..., T/\tau$ by coercivity and weak compactness arguments, realizing also that $w_\tau^k \in \mathcal{H}$ due to (3.2) so that the set of pairs (u, ζ) admissible for (3.1) is always nonempty. Hence, q_τ does exist.

Since the elasticity problem is convex we know that $q_\tau^k = (U(w(k\tau), \zeta_\tau^k), \zeta_\tau^k)$, where $U : (w, \zeta) \mapsto u$ is defined in Lemma 2.1. Moreover, with the definition in Lemma 2.3 we have $V(q_\tau^k) = \mathfrak{v}(w(k\tau), \zeta_\tau^k)$, and by (2.26) we have that (3.6) is equivalent to

$$\int_0^t \frac{\partial \mathcal{E}}{\partial \theta}\big(\theta, \zeta_\tau(\theta)\big)\mathrm{d}\theta \leq \mathcal{E}\big(t, \zeta_\tau(t)\big) + \mathrm{Var}_R\big(q_\tau; 0, t\big) - \mathcal{E}(0, \zeta_0) \leq \int_0^t \frac{\partial \mathcal{E}}{\partial \theta}\big(\theta, \zeta_\tau(\theta - \tau)\big)\mathrm{d}\theta, \tag{3.10}$$

where $\mathcal{E}(t, \zeta) := \mathfrak{v}(w(t), \zeta)$.

As to the discrete stability condition, as in [22, Thm.3.4], by using successively that q_τ^k is a minimizer (cf. (3.1)) and the triangle inequality (2.15) for R, we obtain

$$\begin{aligned} G_\tau(k\tau, q_\tau^k) \;&\leq\; G_\tau(k\tau, \tilde{q}) + R(\tilde{q} - q_\tau^{k-1}) - R(q_\tau^k - q_\tau^{k-1}) \\ &\leq\; G_\tau(k\tau, \tilde{q}) + R(\tilde{q} - q_\tau^k) \end{aligned} \tag{3.11}$$

for any $k = 1, ..., K = T/\tau$. In view of the definition of q_τ and G_τ, it just means (3.4).

The proof of the energy inequality (3.10) follows as in [22, eqn. (2.12)]. Since ζ_τ^k minimizes the "condensed" energy

$$\zeta \mapsto \mathfrak{v}(w(k\tau), \zeta) + \Xi(\zeta - \zeta_\tau^{k-1}), \qquad \text{where } \; \Xi(\zeta) := R(u, \zeta), \tag{3.12}$$

over $Z_{[0,1]}$, we deduce, by inserting $\zeta = \zeta_\tau^{k-1}$, the estimate

$$\begin{aligned} &\mathfrak{v}(w(k\tau), \zeta_\tau^k) - \mathfrak{v}\big(w((k-1)\tau), \zeta_\tau^{k-1}\big) + \Xi\big(\zeta_\tau^k - \zeta_\tau^{k-1}\big) \\ &\quad \leq \mathfrak{v}\big(w(k\tau), \zeta_\tau^{k-1}\big) - \mathfrak{v}\big(w((k-1)\tau), \zeta_\tau^{k-1}\big) \\ &\quad = \mathcal{E}\big(k\tau, \zeta_\tau^{k-1}\big) - \mathcal{E}\big((k-1)\tau, \zeta_\tau^{k-1}\big) = \int_{(k-1)\tau}^{k\tau} \frac{\partial \mathcal{E}}{\partial \theta}\big(\theta, \zeta_\tau^{k-1}\big)\mathrm{d}\theta. \end{aligned} \tag{3.13}$$

As to the left-hand part of (3.10), like in [21, Theorem 4.1] by the stability (3.11) written for ζ_τ^{k-1}, we can see that ζ_τ^{k-1} minimizes the functional $\zeta \mapsto \mathfrak{v}((k-1)\tau, \zeta) + \Xi(\zeta - \zeta_\tau^{k-1})$, and therefore by inserting $\zeta = \zeta_\tau^k$, we find

$$\begin{aligned} &\mathfrak{v}\big(w(k\tau), \zeta_\tau^k\big) - \mathfrak{v}\big(w((k-1)\tau), \zeta_\tau^{k-1}\big) + \Xi\big(\zeta_\tau^k - \zeta_\tau^{k-1}\big) \\ &\quad \geq \mathfrak{v}\big(w(k\tau), \zeta_\tau^k\big) - \mathfrak{v}\big(w((k-1)\tau), \zeta_\tau^k\big) \\ &\quad = \mathcal{E}\big(k\tau, \zeta_\tau^k\big) - \mathcal{E}\big((k-1)\tau, \zeta_\tau^k\big) = \int_{(k-1)\tau}^{k\tau} \frac{\partial \mathcal{E}}{\partial \theta}\big(\theta, \zeta_\tau^k\big)\mathrm{d}\theta. \end{aligned} \tag{3.14}$$

By summing (3.13) and (3.14) for $k = 1, ..., t/\tau$, we obtain (3.10). Hence, (3.6) is established because, by the chain rule and by (2.26), we find

$$\frac{\partial \mathcal{E}}{\partial \theta}(\theta, \zeta) = \Big\langle \frac{\partial \mathfrak{v}}{\partial w}(w(\theta), \zeta), \frac{\mathrm{d}w}{\partial \theta}(\theta) \Big\rangle = \Big\langle F(w(\theta), \zeta), \frac{\mathrm{d}w}{\partial \theta}(\theta) \Big\rangle. \tag{3.15}$$

Estimate (3.7) follows from boundedness of $w(t)$ uniformly in time, and from the uniform coercivity of $V(\cdot, \zeta)$ for all $\zeta \in Z_{[0,1]}$.

Estimate (3.8) with $C_2 = T{+}1$ follows simply from that fact that $\zeta_\tau(\cdot,x):[0,T]\to[0,1]$ is ultimately nondecreasing for a.a. $x\in\bigcup_{\alpha>\beta}\Gamma_{\alpha\beta}$.

Finally, using (3.13) and (3.14) together with (3.15), we get

$$\big|\mathfrak{v}(w(k\tau),\zeta_\tau^k)-\mathfrak{v}\big(w((k{-}1)\tau),\zeta_\tau^{k-1}\big)\big|\le\Xi\big(\zeta_\tau^k{-}\zeta_\tau^{k-1}\big) + \max_{t,\theta\in[0,T]}\big\|F(w(t),\zeta(\theta))\big\|_{\mathcal{H}^*}\int_{(k-1)\tau}^{k\tau}\Big\|\frac{\mathrm{d}w}{\mathrm{d}t}\Big\|_{\mathcal{H}}\mathrm{d}t \tag{3.16}$$

for $k=1,...,T/\tau$. As $\mathfrak{G}_\tau(t)=V(q_\tau(t))=\mathfrak{v}\big(w(k\tau,\zeta_\tau^k\big)$, (3.16) yields

$$||\mathfrak{G}_\tau||_{\mathrm{BV}([0,T])}\le\big\|V(q_\tau(\cdot))\big\|_{L^1(0,T)} + \mathrm{Var}_R(q_\tau;0,T)+\max_{t,\theta\in[0,T]}\big\|F(w(t),\zeta(\theta))\big\|_{\mathcal{H}^*}\Big\|\frac{\partial w}{\partial t}\Big\|_{L^1(0,T;\mathcal{H})}. \tag{3.17}$$

The important point now is that $F(w(t),\zeta(\theta))$ is bounded in $\mathcal{H}^*$ uniformly for $t,\theta\in[0,T]$, which follows, through (2.21), from the L^∞-bound of $||w(\cdot)||_{\mathcal{H}^*}$ due to (3.2) and from the already proved L^∞-bounds of both $||u_\tau(\cdot)||_{\mathcal{U}}$ and $||\zeta_\tau(\cdot)||_{\mathcal{Z}}$. Then the right-hand estimate (3.6) together with the assumption (3.2) yield a bound C_3 of the right-hand side of (3.17), as claimed in (3.9). □

Moreover, let us define the *stable set* at time t via

$$S(t):=\big\{q\in\mathcal{Q};\quad\forall\tilde q\in\mathcal{Q}:\ G(t,q)\le G(t,\tilde q)+R(\tilde q-q)\big\}. \tag{3.18}$$

The following property of R and the closed-graph property of the set-valued mapping $t\mapsto S(t)$ has been proved in [17]:

Lemma 3.2. (See [17, Lemma 4.2 and Theorem 3.4].) *The dissipation potential R has the property*

$$\forall\tilde q\in\mathcal{Q}\quad\forall\{q_k\}_{k\in\mathbb{N}}\subset\mathcal{Q},\ q=\mathop{\text{w-lim}}_{k\to\infty}q_k\quad\exists\{\tilde q_k\}_{k\in\mathbb{N}}\subset\mathcal{Q}: \quad \tilde q=\mathop{\text{w*-lim}}_{k\to\infty}\tilde q_k\quad\&\quad\lim_{k\to\infty}R(q_k-\tilde q_k)=R(q-\tilde q). \tag{3.19}$$

Moreover, if $q_k\in S(t_k)$, $t_k\to t$ and $q_k\to q$ weakly, then $q\in S(t)$.

Sketch of the proof. As to (3.19), we refer to [17, Lemma 4.2] for an explicit construction of $\tilde\zeta_k$.

Due to (3.2), $w(t_k)\to w(t)$ in $\mathcal{H}$ and, by $q_k\to q$ weakly, in particular $\zeta_k\to\zeta$ weakly in $\mathcal{Z}$. Then, by Lemma 2.3, we have

$$G(t_k,q_k)=\mathfrak{v}\big(w(t_k),\zeta_k\big)\to\mathfrak{v}\big(w(t),\zeta\big)=G(t,q). \tag{3.20}$$

Analogously, we also have $G(t_k,\tilde q_k)\to G(t,\tilde q)$ for $\tilde q_k$ from (3.19). Then, starting from $q_k\in S(t_k)$ and using the mentioned $\tilde q_k$ from (3.19), we have

$$\begin{aligned}G(t,q)&=\lim_{k\to\infty}G(t_k,q_k)\le\liminf_{k\to\infty}G(t_k,\tilde q_k)+R(q_k-\tilde q_k)\\&=\lim_{k\to\infty}G(t_k,\tilde q_k)+\lim_{k\to\infty}R(q_k-\tilde q_k)=G(t,\tilde q)+R(q-\tilde q).\end{aligned} \tag{3.21}$$

□

Using the above results we can prove the convergence:

Proposition 3.3. *Let the assumptions of Lemma 3.1 be valid. Then there is a subsequence* $\{q_\tau\}_{\tau>0}$*, denoted for simplicity by the same index* τ*, and a limit process* $q : [0,T] \to \mathcal{Q}$ *such that:*

(i) $\lim_{\tau\to0} u_\tau(t) = u(t)$*, i.e. norm convergence in* $\mathcal{U}$ *for all* $t \in [0,T]$*, and* $u \in L^\infty(0,T;\mathcal{U})$*,*

(ii) $\text{w-}\lim_{\tau\to0} \zeta_\tau(t) = \zeta(t)$*, i.e. weak convergence in* $\mathcal{Z}$ *for all* $t \in [0,T]$*, and* $\zeta \in \mathrm{BV}([0,T];\mathcal{Z})$*,*

(iii) $\lim_{\tau\to0} G_\tau(t, q_\tau(t)) = G(t, q(t))$ *for all* $t \in [0,T]$*.*

Moreover, every such limit process q *is a solution process according to Definition 2.2; in particular,* $q(t)$ *is stable in the sense of (2.16) and the energy inequality (2.24) holds even as an equality for every* s *and* t *with* $0 \le s < t \le T$*.*

Proof. For clarity, let us divide it into four steps.

Step 1: The points (i)–(iii). By the a-priori estimate (3.9) and Helly's selection principle, we can still select a further subsequence and a function $\mathfrak{G} \in \mathrm{BV}([0,T])$ such that $\lim_{\tau\to0} G_\tau(t, q_\tau(t)) = \mathfrak{G}(t)$ for all $t \in [0,T]$. Furthermore, taking into account the a-priori estimate (3.8) and using a generalized Helly's selection principle for Banach-space valued functions (see [21, Theorem 6.1] or [17]), we can make the selection in such a way that, for some $\zeta \in \mathrm{BV}(0,T;\mathcal{Z})$, $\zeta_\tau(t) \to \zeta(t)$ weakly in $\mathcal{Z}$ for all $t \in [0,T]$.

Fixing $t \in [0,T]$, we have also $w_\tau(t) \to w(t)$ in $\mathcal{H}$. As $u_\tau(t)$ is the unique minimizer of problem (2.17) with $(\zeta, w) = (\zeta_\tau(t), w_\tau(t))$, we can use Lemma 2.1 to see that $u_\tau(t) \to u(t)$ in $\mathcal{U}$. This $u(t)$ is determined uniquely and continuously by $(w(t), \zeta(t))$, and, as $t \mapsto (w(t), \zeta(t))$ is measurable, so is $t \mapsto u(t)$. By the a-priori estimates (3.7), u belongs to $L^\infty(0,T;\mathcal{U})$.

By (2.16) and in view of the definition of G_τ, we can write $G_\tau(t, q_\tau(t)) = G(\vartheta(t,\tau), q_\tau(t))$ for some $\vartheta(t,\tau) \in [t,T]$ such that $\lim_{\tau\to0} \vartheta(t,\tau) = t$; in fact, $\vartheta(t,\tau)$ is $\min_{k\in\mathbb{N}\cup\{0\}}\{k\tau \ge t\}$. Like in (3.20), we now have $G_\tau(t, q_\tau(t)) = \mathfrak{v}(w(\vartheta(t,\tau)), \zeta_\tau(t)) \to \mathfrak{v}(w(t), \zeta(t)) = G(t, q(t))$. Comparing it what we got by Helly's selection principle, we can see that $\mathfrak{G}(t) = G(t, q(t))$ for all $t \in [0,T]$, which proves (iii).

Step 2: $q(t) \in S(t)$ *for all* t*.* Let us fix t. As $q_\tau(t) \in S(\vartheta(t,\tau))$ with $\vartheta(\cdot,\cdot)$ from Step 1, by using Lemma 3.2, we can see that $q(t) \in S(t)$.

Step 3: The energy (in)equality (2.24) for $s = 0$ *and a.a.* $t \in [0,T]$*.* One can pass to the limit in (3.6) considered with t as some grid-point belonging to some partition of $[0,T]$ so that (3.6) is at our disposal for each finer partition (for the limit passage, we will therefore consider only those partitions, i.e. with τ small enough with respect to this t). Note that the set of such t's is dense in $[0,T]$. Again, we use $\lim_{\tau\to0} G_\tau(t, q_\tau(t)) = G(t, q(t))$. From the pointwise converge of $\zeta_\tau(\cdot)$ and from the definition (2.25) of $\mathrm{Var}_{[0,t]}R(\cdot)$, we get $\liminf_{\tau\to0} \mathrm{Var}_{[0,t]}R(q_\tau) \ge \mathrm{Var}_{[0,t]}R(q)$. We already showed in Step 1 that $\zeta_\tau(\theta) \to \zeta(\theta)$ in $\mathcal{Z}$ for any $\theta \in [0,t]$. Then, using the notation $F := F(w,\zeta)$ in accord with Lemma 2.1, this lemma says, in particular, that $F(w(\theta), \zeta_\tau(\theta)) \to F(w(\theta), \zeta(\theta))$ weakly in $\mathcal{Z}^*$, and hence also $\langle F(w(\theta),\zeta_\tau(\theta)),\frac{\mathrm{d}}{\mathrm{d}\theta}w(\theta)\rangle \to \langle F(w(\theta),\zeta_\tau(\theta)),\frac{\mathrm{d}}{\mathrm{d}\theta}w(\theta)\rangle$ for any $\theta \in [0,t]$. Moreover, we have a common integrable majorant (even an L^∞-bound) for $\{\theta \mapsto \langle F(w(\theta),\zeta_\tau(\theta)),\frac{\mathrm{d}}{\mathrm{d}\theta}w(\theta)\rangle\}_{\tau>0}$, cf. the arguments at the end of the proof of Lemma 3.1. Therefore, by Lebesgue's dominated-convergence theorem,

$$\lim_{\tau\to 0}\int_0^t\Big\langle F\big(w(\theta),\zeta_\tau(\theta)\big),\frac{\mathrm{d}w}{\mathrm{d}\theta}\Big\rangle\,\mathrm{d}\theta = \int_0^t\Big\langle F\big(w(\theta),\zeta(\theta)\big),\frac{\mathrm{d}w}{\mathrm{d}\theta}\Big\rangle\,\mathrm{d}\theta. \tag{3.22}$$

Likewise, the reduced work of external loading $\int_0^t\big\langle F(w(\theta),\zeta_\tau(\theta-\tau)),\frac{\mathrm{d}}{\mathrm{d}\theta}w(\theta)\big\rangle\,\mathrm{d}\theta$ occurring on the right-hand side of (3.6) converges to the same limit as $\int_0^t\big\langle F(w(\theta),\zeta_\tau(\theta)),\frac{\mathrm{d}}{\mathrm{d}\theta}w(\theta)\big\rangle\,\mathrm{d}\theta$, i.e. to $\int_0^t\big\langle F(w(\theta),\zeta(\theta)),\frac{\mathrm{d}}{\mathrm{d}\theta}w(\theta)\big\rangle\,\mathrm{d}\theta$; here we need that the shifted $\zeta_\tau(t-\tau)$ has the same weak limit as $\zeta_\tau(t)$ for a.a. $t\in[0,T]$. This is indeed true because $\zeta_\tau(t-\tau)\to\zeta(t)$ weakly in $\mathcal{Z}$ provided t is a point of continuity of $\zeta(\cdot)$, i.e. for a.a. $t\in[0,T]$ because BV-functions are a.e. continuous. Then, we can pass to the limit in both inequalities in (3.6), proving thus

$$\mathfrak{m}(t) := G\big(t,q(t)\big) - G\big(0,q_0\big) + \mathrm{Var}_R(q;0,t) + \int_0^t\Big\langle F\big(w(\theta),\zeta(\theta)\big),\frac{\mathrm{d}w}{\mathrm{d}\theta}\Big\rangle\,\mathrm{d}\theta = 0 \tag{3.23}$$

at each t of the form $k\tau\in[0,T]$, $k=1,...,T/\tau$, τ from the considered sequence of time steps. The (only countable) set of such t's is dense in $[0,T]$ and thus (3.23) must hold also at each $t\in[0,T]$ at which all functions involved in (3.23) are continuous. Those functions have, however, a bounded variations and are thus continuous with the exception of at most countable number of points. Hence (3.23) holds everywhere on $[0,T]$ with the only exception of at most countable number of points.

Step 4: The energy (in)equality (2.24) everywhere. As ζ is a BV-mapping, it possesses limits from the left and from the limits at each $t\in[0,T]$, in particular at a point ϑ where some function involved in (3.23) is not continuous. Denote $\zeta^-(\vartheta):=$w*-$\lim_{t\nearrow\vartheta}\zeta(t)$ and $\zeta^+(\vartheta):=$w*-$\lim_{t\searrow\vartheta}\zeta(t)$. By (3.2), $w(\cdot)$ is (even absolutely) continuous everywhere, and as $u(t)$ depends (norm×weak,norm)-continuously on $(w(t),\zeta(t))$, there are also limits $u^-(\vartheta):=\lim_{t\nearrow\vartheta}u(t)$ and $u^+(\vartheta):=\lim_{t\searrow\vartheta}u(t)$. So that altogether we have $q^+(\vartheta)=$w*-$\lim_{t\searrow\vartheta}q(t)$ and $q^-(\vartheta)=$w*-$\lim_{t\nearrow\vartheta}q(t)$. Furthermore, put $\mathfrak{G}^-(\vartheta):=\lim_{t\nearrow\vartheta}G(t,q(t))$ and $\mathfrak{G}^+(\vartheta):=\lim_{t\searrow\vartheta}G(t,q(t))$; these limits exists as the Gibbs energy has a bounded variation. From Lemma 3.2, we know

$$\mathfrak{G}^+(\vartheta) = G\big(\vartheta,q^+(\vartheta)\big) \quad\text{and}\quad \mathfrak{G}^-(\vartheta) = G\big(\vartheta,q^-(\vartheta)\big). \tag{3.24}$$

As we have proved $q(\vartheta)\in S(\vartheta)$ in Step 2, putting $\tilde q := q^+(\vartheta)$ into (2.16) written, of course, for $t=\vartheta$, we obtain

$$G\big(\vartheta,q(\vartheta)\big) \le G\big(\vartheta,q^+(\vartheta)\big) + R\big(q^+(\vartheta)-q(\vartheta)\big). \tag{3.25}$$

Likewise, by Proposition 3.2 also $q^-(\vartheta)\in S(\vartheta)$ and thus, together with (3.25),

$$\begin{aligned}\mathfrak{G}^-(\vartheta) &= G\big(\vartheta,q^-(\vartheta)\big) \le G\big(\vartheta,q(\vartheta)\big) + R\big(q(\vartheta)-q^-(\vartheta)\big)\\ &\le \mathfrak{G}^+(\vartheta) + R\big(q(\vartheta)-q^-(\vartheta)\big) + R\big(q^+(\vartheta)-q(\vartheta)\big).\end{aligned} \tag{3.26}$$

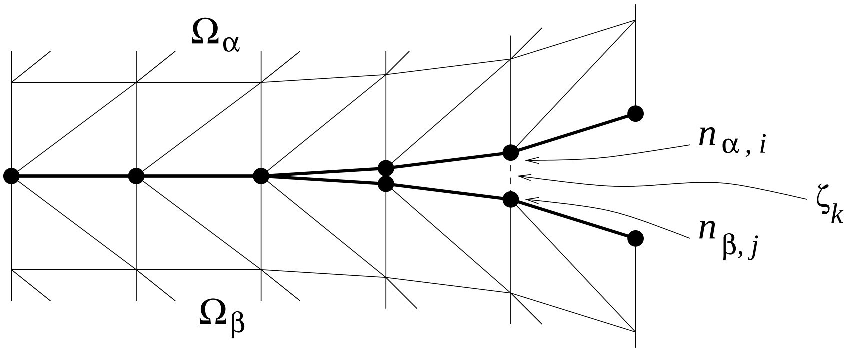

Figure 2. *Detail of the finite element mesh on the interface boundary.*

By definition (2.25), we have $\mathrm{Var}_R(q;s,t) = \mathrm{Var}_R(q;s,\vartheta) + \mathrm{Var}_R(q;\vartheta,t)$ for $s < \vartheta < t$. Moreover, $\lim_{s\nearrow\vartheta} \mathrm{Var}_R(q;s,\vartheta) = R(q(\vartheta) - q^-(\vartheta))$ and $\lim_{t\searrow\vartheta} \mathrm{Var}_R(q;\vartheta,t) = R(q^+(\vartheta) - q(\vartheta))$. Passing to the limit in (3.23) and using (3.24) we obtain

$$\mathfrak{G}^+(\vartheta) - \mathfrak{G}^-(\vartheta) + R\big(q(\vartheta) - q^-(\vartheta)\big) + R\big(q^+(\vartheta) - q(\vartheta)\big) = 0, \tag{3.27}$$

which shows that (3.26) and hence (3.25) are in fact equalities. For $\vartheta > 0$ we find

$$\mathfrak{m}(\vartheta) - \lim_{s\nearrow\vartheta} \mathfrak{m}(t) = G\big(\vartheta, q(\vartheta)\big) - \mathfrak{G}^-(\vartheta) + R\big(q(\vartheta - q^-(\vartheta)\big) = 0.$$

Hence $\mathfrak{m}$ is proved to be continuous from the left. Similarly, we find

$$\lim_{t\searrow\vartheta} \mathfrak{m}(t) - \mathfrak{m}(\vartheta) = \mathfrak{G}^+(\vartheta) - G(\vartheta, q(\vartheta)) + R(q^+(\vartheta) - q(\vartheta)) = 0,$$

which proves the continuity from the right. Together with (3.23) we conclude $\mathfrak{m}(t) = 0$ for all $t \in [0,T]$. This proves $\mathfrak{m}(t) - \mathfrak{m}(s) = 0$ which is the claimed equality in (2.24). □

4. Numerical approximation and model examples

The minimization problem (3.1), obtained by semi-discretization in the time variable, is further discretized in the space variable by the standard finite element method. In particular, we use linear triangular elements. Vectors associated with the discretized variables will be denoted by boldface letters. We use the same mesh for both variables, u and ζ. In order to allow for the separation (delamination) of the elastic bodies, each joint boundary $\Gamma_{\alpha\beta}$ is discretized by pairs of nodes, say $(n_{\alpha,i}, n_{\beta,j})$, with the same position. The notation $n_{\alpha,i}$ means the i-th node from the set of nodes discretizing Ω_α. Denote by $I_{\alpha\beta}$ the set of triple indices (i,j,k). The first two indices refer to the above nodes; the third index k relates to the component of the vector $\boldsymbol{\zeta}$ corresponding to node $(n_{\alpha,i}, n_{\beta,j})$; see Figure 2.

Let A_α be the stiffness matrices of the elastic bodies Ω_α, $\alpha = 1, \dots, m$. The discretized elastic stored energy becomes

$$\mathbf{V}(\mathbf{u}, \boldsymbol{\zeta}) = \sum_{\alpha=1}^{m} \Bigg(\mathbf{u}_\alpha^T A_\alpha \mathbf{u}_\alpha + \sum_{\beta=1}^{\alpha-1} \sum_{(i,j,k)\in I_{\alpha\beta}} \omega_k \boldsymbol{\zeta}_k \, (\mathbf{u}_{\alpha,i} - \mathbf{u}_{\beta,j})^\top b_{\alpha\beta}(n_{\alpha,i})(\mathbf{u}_{\alpha,i} - \mathbf{u}_{\beta,j}) \Bigg)$$

where ω_i are integration weights and ν_{ij} is the outer normal to $\Gamma_{\alpha\beta}$ at the i-th node from Ω_α. Similarly, the discretized dissipation potential is

$$\mathbf{R}(\boldsymbol{\zeta}) = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{\alpha-1} \sum_{(i,j,k)\in I_{\alpha\beta}} -\omega_k d_{\alpha\beta}(n_{\alpha,i}) \boldsymbol{\zeta}_k .$$

The case $\mathbf{R}(\boldsymbol{\zeta}) = +\infty$ will be implemented through a corresponding linear constraint. Finally, let L be a rectangular matrix selecting the boundary components from the whole vector u.

The discrete version of the optimization problem (3.1) is then

$$(4.1) \qquad \begin{cases} \text{Minimize} & \mathbf{V}(\mathbf{u}, \boldsymbol{\zeta}) + \mathbf{R}(\boldsymbol{\zeta} - \boldsymbol{\zeta}^{\kappa-1}) \\ \text{subject to} & L\mathbf{u} \ge \mathbf{w}^\kappa, \\ & (\mathbf{u}_{\alpha,i} - \mathbf{u}_{\beta,j}) \cdot \nu_{ij} \ge 0, \quad (i,j) \in I_{\alpha\beta}, \ \alpha, \beta = 1, \dots, m, \\ & \boldsymbol{\zeta}^{\kappa-1} \ge \boldsymbol{\zeta} \ge 0 \quad \text{componentwise.} \end{cases}$$

Here the index $\kappa-1$ refers to the previous time step. The irreversibility of the dissipation process is guaranteed by the left-hand side of the box constraint.

As we are only interested in the components of the displacement vector lying on the boundary Γ, we can eliminate all components corresponding to the interior nodes. This reduces the number of variables in (4.1) to the number of interface boundary nodes times five (two times two components of the displacement vector plus the components of $\boldsymbol{\zeta}$) plus the number of boundary nodes with prescribed non-zero Dirichlet condition (loaded nodes) times two.

Problem (4.1) is a medium-size nonconvex optimization problem that has to be solved repeatedly. Being nonconvex, the problem has several local minima from which only one is "physical". The "non-physical" minima refer to situations when the delamination parameter $\boldsymbol{\zeta}$ is smaller than it really should be, i.e. the delamination occurs sooner than in reality. The only tool by which we can (try to) control the minima is the choice of the initial point. At the beginning of the time iteration process, when no load is applied, we initiate $\boldsymbol{\zeta} \equiv 1$, meaning non-delaminated state. Then, in each time step, we initiate $\boldsymbol{\zeta}$ by its upper bound, i.e., $\boldsymbol{\zeta}^{\kappa-1}$. Note that the mathematical model is quite different, since it supposes a global stability condition, where the system is able to find any global minimizer, see the discussion below.

Obviously, the selection of the optimization code was a critical issue for getting realistic solutions efficiently. The code of our choice was SNOPT [12] that proved to be a robust and reliable solver for medium and large-scale problems. We implemented problem (4.1) in MATLAB and used SNOPT version available in the toolbox TOMLAB[1] and callable from MATLAB.

Example 4.1. Figure 3 shows the geometry and the boundary conditions. By the Dirichlet condition applied at the right-hand end we try to split apart the two elastic bodies. The prescribed displacement is a linear function of time.

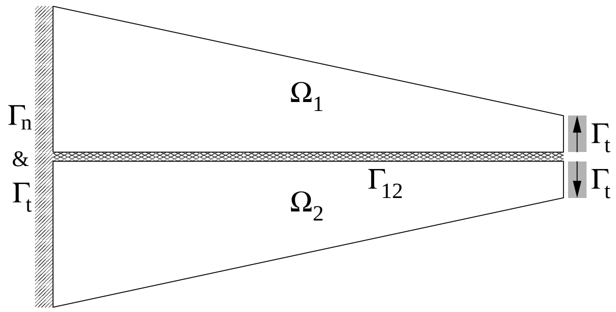

FIGURE 3. *Geometry and boundary conditions for Example 4.1.*

Figure 4 presents the deformed body after 150 and 600 time units, along with the energy balance as a function of time. From top to bottom, Line 1 shows the right-hand and the left-hand sides of (3.6) which are almost identical in the figure, Line 2 denotes the computed stored energy $V(q_\tau(T)) + \mathrm{Var}_R(q_\tau; 0, T)$ (see again (3.6)), Line 3 is the energy stored in the elastic bodies, Line 4 is the dissipated energy $\mathrm{Var}_R(q_\tau; 0, T)$, and Line 5 the energy stored in the adhesive on the interface boundary. We can nicely see the jumps in the energies whenever a new node is delaminated. After some steps, the computed energy tends outside the bounds, i.e. Lines 1 and 2 are distinct, contrary to what one expects in view of the two-sided energy estimate (3.6), which is valid even for discrete times. This is most likely due to two reasons. First, we had to approximate the integrals in (3.6) by keeping the reaction forces F constant over the time intervals $((k-1)\tau, k\tau)$, i.e., we used $F(w_\tau(\theta), \zeta_\tau(\theta))$ instead of $F(w(\theta), \zeta_\tau(\theta))$. This problem could only be avoided by "sub-discretization" of each interval and calculating the reaction forces in each substep, which is a very time-consuming task. Second, our numerical code may not always find the global minimum in the noncovex problem (4.1). Getting stuck in local minima will easily violate (3.6), as we show in the trivial finite-dimensional Example 4.3.

To have a better impression of the delamination process, we plot in Figure 5 the (level lines of the) local energy density of the adhesive $b_{\alpha\beta}\zeta(t,x)(u_\alpha(t,x) - u_\beta(t,x))^2$ as a function of time and the "horizontal" space variable x. Also in Figure 5 we present a 3D view on the function $-\zeta(t,x)$. We notice a remarkable fact that $\zeta(t,x)$ only attains the values 0 or 1 and never any intermediate value. This fact is also seen in the following Example 4.3.

Remark 4.2. *(Clapeyron's principle* [3].*)* Figure 4 (left/down) shows a remarkable fact: at the beginning of the loading process, the work of external loading is equally distributed

[1] `http://www.tomlab.biz/`

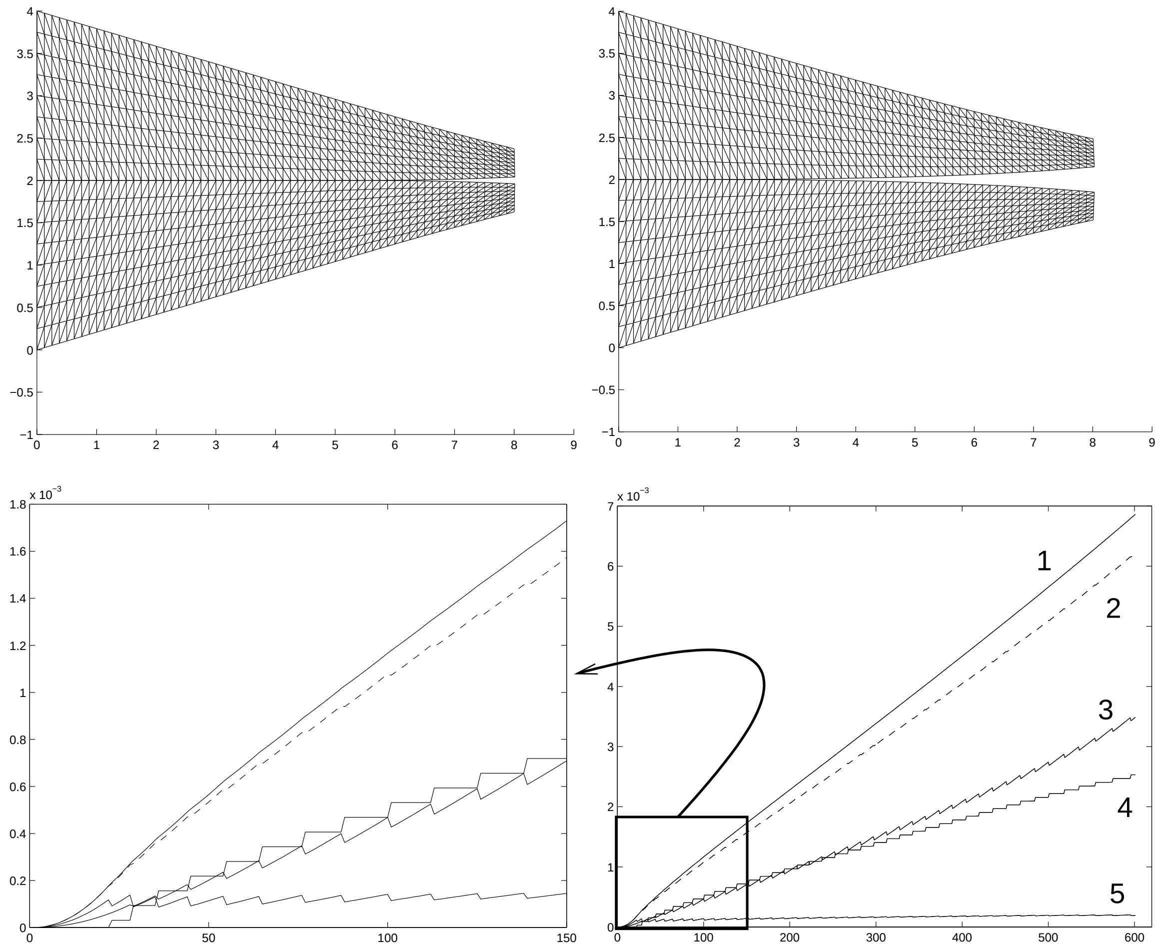

Figure 4. *Example 4.1: delamination after 150 (left) and 600 time steps (right) and evolution of energetic in time:*
1: work of external loading calculated as $\int_0^t \langle F, \mathrm{d}w/\mathrm{d}t\rangle$,
2: this work but calculated as $V(q(t)) + \mathrm{Var}_R(q; 0, t) - V(q_0)$,
3: stored elastic energy in all domains Ω_α,
4: energy $\mathrm{Var}_R(q_\tau; 0, t)$ *dissipated in the adhesive,*
5: stored elastic energy in the adhesive.
Hence, curve 2 is the sum of 3, 4 and 5.

to the stored energy and to the dissipated energy. This corresponds to Clapeyron's principle of linear elasticity for slowly loaded bodies with viscous damping, see [11] for a modern treatment. For further loading, the delamination process is close to be finished and the external work turns rather to the stored energy, and this is why the curves 3 and 4 start to diverge from each other on Figure 4 (right/down).

Example 4.3. We consider a very simplistic finite-dimensional model for delamination which displays the difference between using a local versus a global stability condition.

We consider the system of three elastic strings as displayed in Figure 6. All strings are assumed to be linearly elastic with identical constants $e > 0$. The right upper string

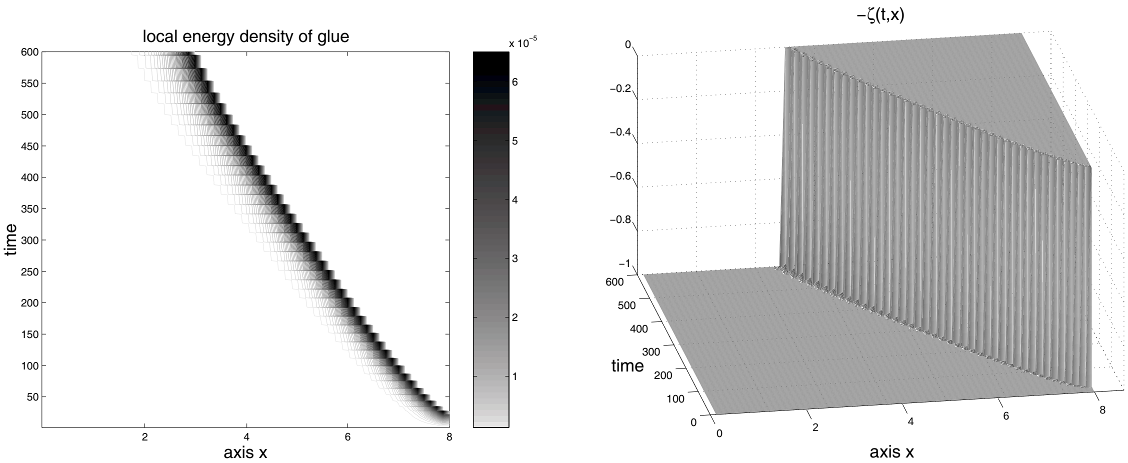

FIGURE 5. *Example 4.1: evolution of local energy density of the adhesive and values of* $\zeta(t,x)$.

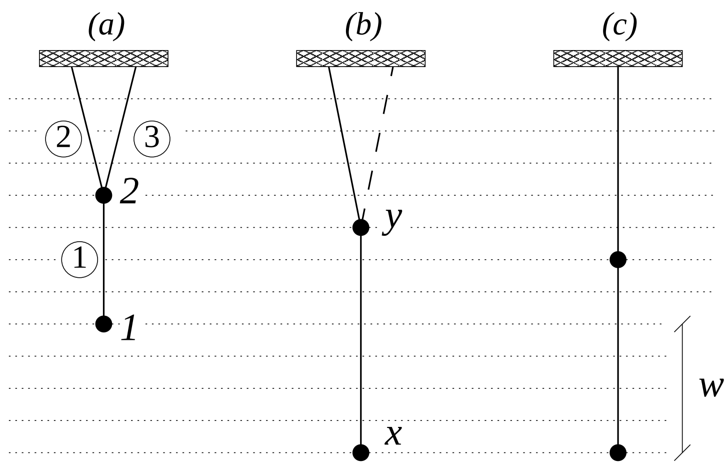

FIGURE 6. Three elastic strings.

(number 3) may break partially break with a percentage $\zeta \in [0,1]$ of intact material. The energy dissipated in breaking will be $d(\zeta_{\rm old}{-}\zeta_{\rm new})$ with a positive material constant d. The system is driven by the Dirichlet condition $x(t) = w(t)$ and the only remaining elastic freedom is the position $y(t)$ which will depend on ζ.

The stored energy is given by $V(t,y,\zeta) = \frac{e}{2}((w(t){-}y)^2 + y^2 + \zeta y^2)$ and the dissipation is

$$R(\zeta{-}\tilde\zeta) = d(\tilde\zeta{-}\zeta) \text{ for } \tilde\zeta \geq \zeta \text{ and } +\infty \text{ else.}$$

The unique elastic equilibrium is given by $y = Y(w,\zeta) = w/(2{+}\zeta)$ and the minimal energy (cf. Lemma 2.3) reads

$$\mathfrak{v}(w,\zeta) = \frac{e\,w^2}{2}\,\frac{1+\zeta}{2+\zeta}.$$

A particular feature of this model is that $\mathfrak{v}(w,\cdot)$ is strictly concave on $[0,1]$. Hence, the initial state $\zeta = 1$ is global stable in the sense of (2.16) if and only if

$$\mathfrak{v}(w,1) \le \mathfrak{v}(w,0) + d,$$

which is equivalent to $w \le w_{\rm crit}^{\rm glob} = \sqrt{12d/e}$. Hence, if w increases through this value, then ζ jumps from 1 down to 0 and the energy balance is kept.

However, if we only ask for local stability of $\zeta = 1$ (as is done in numerics and probably in nature) the criterion is

$$0 \ge \frac{\partial}{\partial\zeta}\mathfrak{v}(w,1) - d$$

which gives the condition $w \le w_{\rm crit}^{\rm loc} = \sqrt{18d/e}$. Hence, local stability holds longer than the global one. If w increases through this value we again find a jump from 1 down to 0, however, now the dissipation is just d whereas the energy release in the springs is larger:

$$\mathfrak{v}(w_{\rm crit}^{\rm loc},1) - \mathfrak{v}(w_{\rm crit}^{\rm loc},0) = \frac{e\,(w_{\rm crit}^{\rm loc})^2}{12} = \frac{3}{2}d > d.$$

Example 4.4. In this example we try to model a real-world problem of crash of a vehicle. The model is sketched in Figure 7. The crash element, drafted in grey, is a laminated elastic body. By delamination, the element absorbs part of the crash energy, thus preventing the passengers' injuries. Again, the crash element is modeled by two elastic and one adhesive layer. Figure 7 also shows the geometry and the boundary conditions.

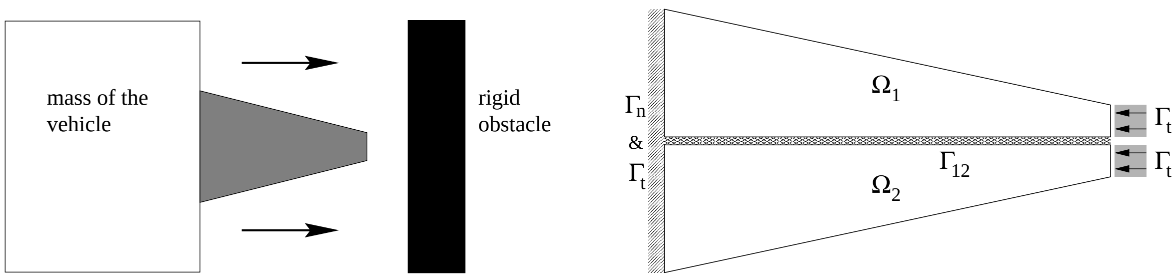

FIGURE 7. *Motivation, geometry and boundary conditions for Example 4.4.*

The deformation of the body after 100 time steps with $\tau = 0.0625$ (starting from $t_0 = 10$) is shown in Figure 8. In this case, the inner segment was delaminated (almost) at once.

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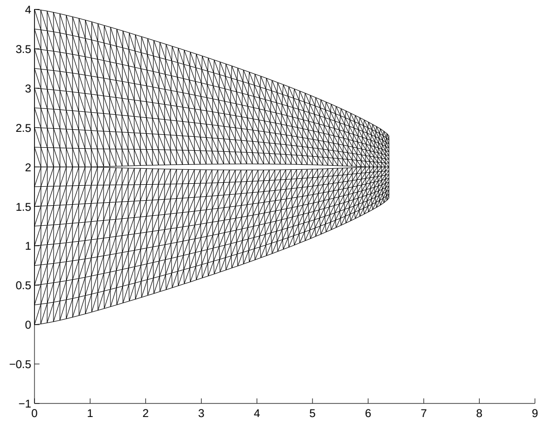

Figure 8. *Delamination after 100 time steps in Example 4.4.*

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