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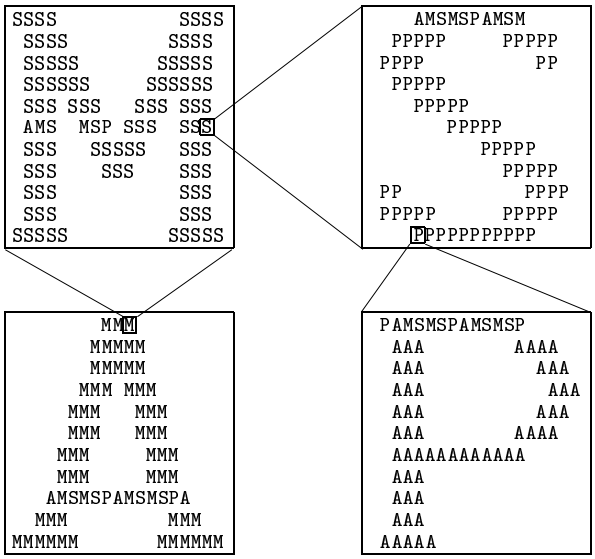

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A rate-independent model for inelastic behavior of shape-memory alloys

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Abstract. We formulate a model describing rate-independent hysteretic response of shape-memory alloys under slow external forcing. Under natural assumptions we prove that this model has solution. The microstructure is treated on a "mesoscopic" level, described by volume fractions of particular phases in terms of Young measures. The whole formulation is based on energetic functionals for energy storage and energy dissipation. The latter is built into the model by a dissipation distance between different values of these volume fractions.

Key Words. Martensitic transformation, Young measures, energetic formulation, dissipation distance.

AMS Subject Classification: 74N30, 49S05.

1. Introduction, mesoscopic models

In the last decades the research in *shape-memory alloys* (SMA) received enormous attention in engineering, theoretical and experimental material science, and in mathematics, too. SMAs are characterized by the ability to transform, with a relatively small activation energy, between the different variants of a less symmetrical phase of a crystal (i.e., tetragonal, trigonal, orthorhombic, or monoclinic) or even to transform between crystallographic phases with different symmetries (typically cubic/tetragonal, or cubic/monoclinic, etc.). This process is called martensitic or martensitic/austenitic *phase transformation* (PT). In mathematics it is common to call each of the different variants of a crystal configuration a *phase* (e.g., a tetragonal martensite consists of 3 phases, contrary to what is understood in physics as a single phase with 3 variants).

Modeling of such "smart" materials challenged a rise of variety of mathematical models. Disregarding atomistic-level models, the mathematical models based on *continuum mechanics* can be classified according to a description of the microstructure in SMA to

micro-, meso- and macro-scopic, depending on whether they use the full microscopic deformation, or only the macroscopic deformation together with locally averaged *volume fractions*, or only the macroscopic deformation itself.

The general drawback of the calculations on the microscopic level is that they can basically model only domains of scales far below millimeters, and predict macroscopic response only on the assumption that this domain is representative. In other words, the microstructure created during the loading process should be rather homogeneous and not much chaotic. This assumption is quite realistic in laboratory experiments with single crystals but often it is not satisfied in engineering workpieces which may have complex geometries, are loaded in non-homogeneous ways or are made from commercially produced SMAs which are rather *polycrystals*. Therefore, it is desirable to develop models on the mesoscopic level, which is exactly the focus of this work. For microscopic models we refer to Remark 1.2.

The essential problem of such a multiscale modeling is to keep suitable information of the microscopic structures such that their influence on the macroscale can be described effectively and such that an evolution law for this model can be found without resolving the microstructure in full detail. We will use (gradient-) Young measures for this purpose. They characterize the distribution of the microscopic gradients and phases, but they neglect the micropatterns. However, since gradient Young measures have encoded the knowledge about geometric compatibility of different gradients they carry a lot of information on the possible micropattern, which often can be reconstructed in a post-processing step from the gradient Young measures.

There are many models for the hysteretic behavior of SMA using volume fractions. Starting from Frémond [Fre87, FrMi96, Fre02] they were further developed in [CFV90, CoSp92] and [HNZ90] and many other works, cf. [BrSp96] or [Rou00] for further references. Models of this sort have also been developed by Müller et al. [Mül79, MüSe01, MüWi80]. However, these models are very phenomenological and do not model correctly the microscopic interplay and geometrical compatibilities between different phases (or phase variants). Additionally these models do not have the potential for treating fully nonlinear elasticity and the full menagerie of phase(variant)s occuring in more complex situations, e.g. 4, 7, or 13 different ones, or sometimes even much more. Moreover, all these models are rate-dependent and it is not clear whether hysteretic behavior remains in the rate-independent limit. Later, the dissipation in Frémond's model has been augmented by a rate-independent term in [Fre02, Sect.13.13]. Another model was proposed by James [Jam95] but again does not involve a rate-independent rule for volume-fraction evolution. A rate-independent model based on multiple configuration concept has been proposed by Rajagopal and Srinivasa [RaSr95, RaSr97] though implicitly it addresses rather polycrystalic materials in a highly phenomenological way. Another model using rate-independent plasticity-like dissipation was used in [TaAn03].

A different type of models was developed in [MiTh99, MiTh01, MTL02, GMH02, The02]. This class of models is based on energy minimization of a functional depending simply on the macroscopic deformation and the macroscopic volume fractions of the phases. The new constitutive quantity needed for this model is the effective macroscopic

energy which depends only on the macroscopic deformation gradient and the volume fractions of the phases. There, this energy is called *mixture function* or *free energy of mixing* and it encodes the crystallographic information of geometric (in)compatibilities of different microscopic deformation gradients. Moreover, the model uses a dissipation functional which amounts in postulating that the dissipated energy is the product of a material constant, depending on the two involved phases, and the volume of the part of the body which has undergone the PT. The difficulty with this class of models is that it is based on the free energy of mixing which, in principle, can be obtained by a quasiconvexification but explicit formulae are only available in very special cases [GMH02, Mie00].

Another model, describing volume fractions (denoted by λ) in terms of Young measures (denoted by ν) was proposed and computationally tested in [Rou00, Rou02]. The effective stored energy V is evaluated automatically in this model but the definite disadvantage of this model is that it is formulated only for scalar, anti-plane-like deformation only. Also, the mathematical formulation using finitely-additive measures (because the time derivative is explicitly involved) and giving results only generically with respect to the final time horizon is not completely elegant.

Our goal is, by joining the ideas of the two latter models, to develop a mesoscopic model which covers vectorial situations and does not need explicit knowledge of the free energy of mixing. In fact, this combination of rate-independent mesoscopic volume-fraction evolution with the usage of Young-measure relaxation to evaluate the effective energy has been outlined already in [Mie02] for general rate-independent material models displaying elastic and inelastic effects including damage models, elastoplasticity, micromagnetism and SMAs. Here, we show that this framework allows for a rigorous analysis of some models for SMAs.

Our configuration at time t, let us denote it by $q(t)$, will thus include, beside the macroscopic displacement u, also ν and λ. As proposed in [MiTh99], for a prescribed dissipation (pseudo)potential R and given time-dependent loading F, the solution q will then be characterized by *stability* (1.1a) and the *energy inequality* (1.1b):

$$\text{(1.1a)} \qquad \forall\, t \ \ \forall \tilde{q}: \quad G\big(t,q(t)\big) \le G(t,\tilde{q}) + R\big(q(t),\tilde{q}\big),$$

$$\text{(1.1b)} \qquad \forall\, t \ge s: \quad G\big(t,q(t)\big) + \mathrm{Var}(q;s,t) \le G\big(s,q(s)\big) - \int_s^t \frac{\partial F}{\partial \theta}\big(\theta,q(\theta)\big)\,\mathrm{d}\theta,$$

where $G(t,q) = V(q) - F(t,q)$ is the Gibbs stored energy and $\mathrm{Var}(q;s,t)$ is the total variation of the process q over the time interval $[s,t]$ with respect to R, i.e. the dissipated energy; cf. (2.31)–(2.33) for a detailed definition. It was shown in [MiTh99, MiTh01, MTL02] that, in qualified cases (covering our model in the dimension $n > 1$), the two inequalities (1.1) can be written in the form of a doubly nonlinear problem

$$\text{(1.2)} \qquad \partial R\Big(\frac{\mathrm{d}q}{\mathrm{d}t}\Big) + \partial V(q) \ni F$$

where ∂ denotes the subdifferential. While (1.2) is a well–accepted model in rate–dependent cases, where $R(v) \ge c\|v\|^p$ for some $p > 1$ (see [CoVi90]), it is shown in [MiTh01], that the rate-independent case, where $R(\alpha v) = \alpha R(v)$ and hence $p = 1$, is better treated in the weaker energetic formulation (1.1). In [Rou02] the doubly nonlinear

problem (1.2) was applied to a scalar model for SMA, cf. (4.6) below. The major advantage of our energetic formulation is that it allows for more general nonconvexities, since no derivatives for the solutions nor of the involved functionals are needed.

After giving a detailed derivation and specifying the form of the Gibbs energy G and the dissipation distance R in Section 2, we provide the analysis of the model in Section 3. Using a time discretization we obtain an incremental problem which is a minimization problem at each time step, which corresponds to a fully implicit (backward) Euler method. For the limit of step size going to 0 we show that the approximate solutions have limit points which provide solutions of the time-continuous problem (1.1) and (1.1b), see Theorem 3.4.

Remark 1.1. *(Young-measure implementation.)* Of course, modeling of SMA evolution is a very ambitious goal in connection with an extremely complicated reality. This is reflected, in particular, by involving the set of so-called gradient Young measures whose characterization cannot be made explicit. In fact, only an implicit characterization by all quasiconvex functions has been invented by Kinderlehrer and Pedregal [KiPe91, KiPe94], cf. (3.27) below. Therefore, one must make some simplification to implement the model, presumably using eventually only an inner approximation of this mysterious set e.g. by so-called laminates up to a certain order. For 2D calculations with first-order laminates but not rate-independent model see [RoKr98]. For steady-state models but higher-order laminates see also [Ara01, AFO02, Kru98, Rou97]. Similar constructions were employed for finite–strain elastoplasticity in [ORS00, MiLa01]. Yet, this issue will not be addressed in this paper.

Remark 1.2. *(Microscopic models.)* There is also a variety of models on the microscopic level. They originated basically from Falk's model [Fal80]; we refer to [BrSp96] or also [Rou00] for further references. These models are usually either rate dependent or the possible rate-independent hysteretic response cannot be set up independently of the stored energy, cf. [RaRo03] for a thorough discussion. A model that does not inherit these discrepancies has been proposed in [Rou00] and further analyzed and developed in [AGR02, PlRo02], showing ambitions to model real 3D laboratory experiments with *single crystal* SMAs. Another model that plays with rate-independent hysteretic mechanism independent of the stored energy has recently been proposed by Krejčí et al. [KSS01] (even for nonisothermal situations) but analyzed only in dimension one and not accompanied with any calculations.

2. The model

The difficult question in efficient modeling of multiscale inelastic processes like those in SMAs is how to describe the usually very complicated, multiscale configurations in keeping only the relevant information. Moreover, one has to model the, to some extend rather mutually independent, mechanisms describing how the material stores energy and how it dissipates energy. Beside modeling of material properties, a macroscopic geometry as the shape of the specimen and its external loading as well as the orientation of the crystalic material must be dealt with, as usual.

2.1. **Stored energy, relaxation.** Let $\Omega \subset \mathbb{R}^n$ be a reference configuration of the (undeformed) body, n refer to the dimension ($n = 3$ is therefore the physically interested case, though we will not exclude symmetrical cases formulated for $n < 3$). On a microscopic (but still continuum-mechanical) level, the configuration is standardly described by the *displacement* $u : \Omega \to \mathbb{R}^n$ so that the deformed body occupies the region $\{x + u(x);\ x \in \Omega\} \subset \mathbb{R}^n$.

The energy storage mechanism is described by the *specific stored energy* $\varphi : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R}$,

$$\varphi(x, \nabla u) = \hat\varphi\Big((\mathrm{I}+\nabla u)\mathfrak{Q}(x)\Big) \quad \text{where } \hat\varphi : \mathbb{R}^{n\times n} \to \mathbb{R} \text{ and } \mathfrak{Q} : \Omega \to \mathrm{SO}(n). \tag{2.1}$$

Here $\mathrm{I} \in \mathbb{R}^{n\times n}$ is the identity matrix and $\mathrm{SO}(n) = \{Q \in \mathbb{R}^{n\times n}\,;\, Q^\top Q = \mathrm{I} = QQ^\top,\ \det Q = 1\}$ stands for the special orthogonal group, i.e., orientation-preserving rotations. The function $\hat\varphi$ describes phenomenologically how the material stores energy while the function $\mathfrak{Q} : \Omega \to \mathrm{SO}(n)$ determines the orientation of the reference (austenitic) crystal lattice. Thus, $\mathfrak{Q}$ is constant over Ω in case of a single crystal or piecewise constant in case of a polycrystal, the regions of constancy of $\mathfrak{Q}$ are interpreted as grains.

The standard requirement of *frame-indifference* means that $\hat\varphi$ in fact depends only on the (right) Cauchy-Green stretch tensor $C := (\mathrm{I}+\nabla u)^\top(\mathrm{I}+\nabla u)$.

For analytical reasons, let us assume that the potential φ has p-growth as well as the respective coercivity:

$$\exists\, c_1 \ge c_0 > 0\ \forall (a.a.)\, x\in\Omega\ \forall\, A\in\mathbb{R}^{n\times n}\colon \quad c_0(|A|^p-1) \le \varphi(x,A) \le c_1(1+|A|^p) \tag{2.2}$$

with some $p \ge 2n/(n+1)$; this value just guarantees that the traces on Γ of functions from $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ are in $\mathrm{L}^2(\Gamma;\mathbb{R}^n)$. The restriction (2.2) to φ does not cover true nonlinear elasticity where interpenetration of material is forbidden by assuming $\varphi(x, A) = \infty$ for $\det A \le 0$ and $\varphi(x, A) \to \infty$ for $\det A \searrow 0$. However, the deformation regime, where SMAs are used, is usually not so large that these extreme values matter.

Of course, likewise in lab experiments or engineering applications, we must somehow fix the body by boundary conditions to exclude rigid-body motions. Let us consider a linear elastic support on the boundary Γ through a non-negative coefficient α, assuming

$$\alpha\in\mathrm{L}^\infty(\Gamma;\mathbb{R}^n), \quad \forall\, i = 1, ..., n: \quad \alpha_i \ge 0 \ \text{ and } \ \mathrm{meas}_{n-1}\{\alpha_i > 0\} > 0. \tag{2.3}$$

The coercivity (2.2) in ∇u and the support condition (2.3) guarantee that, through a Poincaré-type inequality, the overall (Helmholtz-type) stored energy is coercive in the sense that, for some $\varepsilon > 0$, it holds

$$\forall u\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n): \quad \int_\Omega \varphi^{\mathrm{qc}}(x,\nabla u)\,\mathrm{d}x + \int_\Gamma (\alpha\cdot u)^2\,\mathrm{d}S \ge \varepsilon\|u\|^p_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)} - \frac{1}{\varepsilon} \tag{2.4}$$

where $\varphi^{\mathrm{qc}}(x,\cdot)$ is the quasiconvex envelope of $\varphi(x,\cdot)$, i.e. the supremum of all quasiconvex minorants of $\varphi(x,\cdot)$. Recall that a function $v : \mathbb{R}^{n\times n} \to \mathbb{R}$ is called *quasiconvex* if $v(A) = \min_{u\in\mathrm{W}^{1,\infty}_0(\Omega;\mathbb{R}^n)} \int_\Omega v(A+\nabla u(x))\,\mathrm{d}x/\mathrm{vol}(\Omega)$ for all $A \in \mathbb{R}^{n\times n}$.

To drive the system, we need to impose a time-dependent loading, e.g., through a surface traction $f : \Gamma \to \mathbb{R}^n$, although a volume-force loading might be considered as well. Hence, this loading will contribute to the stored energy by the boundary integral $\int_\Gamma f(x)\cdot u(x)\,\mathrm{d}S$. Here, we assume $f \in \mathrm{L}^\gamma(\Gamma;\mathbb{R}^n)$ with $\gamma \ge (np-p)/(np-n)$ if $p < n$

or simply $\gamma = 1$ if $p \geq n$. to guarantee that $f{\cdot}u|_\Gamma \in L^1(\Gamma)$ if $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$; further qualification of f will be given below, cf. (2.20).

The assumption (2.2) allows one to prove rigorously that any minimizing sequence $\{u_k\}_{k\in\mathbb{N}}$ of the functional $u \mapsto \int_\Omega \varphi(x,\nabla u)\,\mathrm{d}x + \int_\Gamma (\alpha{\cdot}u)^2 - f{\cdot}u\,\mathrm{d}S$ does not concentrate energy in the sense that $\{|\nabla u_k|^p;\ k\in\mathbb{N}\}$ as well as $\{\varphi(\cdot,\nabla u_k);\ k\in\mathbb{N}\}$ are relatively weakly compact subsets of $\mathrm{L}^1(\Omega)$. This suggests that a correct relaxation for this minimization problem can be based on the so-called gradient L^p-Young measures

$$\mathcal{G}^p(\Omega;\mathbb{R}^{n\times n}) := \Big\{ \nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})\,;\ \exists\, u\in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n):\quad \mathrm{Id}\bullet\nu = \nabla u, \\ (1\otimes v)\bullet\nu \geq v(\nabla u)\quad \forall\, 0\leq v\in C^{(p)}(\mathbb{R}^{n\times n}) \text{ quasiconvex}\Big\}, \tag{2.5}$$

where $C^{(p)}(\mathbb{R}^{n\times n}) := \{v:\mathbb{R}^{n\times n}\to\mathbb{R} \text{ continuous, } \sup_{A\in\mathbb{R}^{n\times n}} v(A)/(1+|A|^p) < +\infty\}$ and the so-called L^p-Young measures are defined as

$$\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n}) := \Big\{\nu\in \mathrm{L}^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n}))\,;\ \int_\Omega\int_{\mathbb{R}^{n\times n}} |A|^p \nu_x(\mathrm{d}A)\,\mathrm{d}x < +\infty, \\ \nu_x \text{ is a probability measure on } \mathbb{R}^{n\times n} \text{ for a.a. } x\in\Omega\Big\}, \tag{2.6}$$

where $\nu_x := \nu(x)$. Here, $\mathrm{L}^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n})) \cong \mathrm{L}^1(\Omega;C_0(\mathbb{R}^{n\times n}))^*$ is the Banach space of weakly measurable functions from Ω to the set of Radon measures $\mathrm{rca}(\mathbb{R}^{n\times n}) \cong C_0(\mathbb{R}^{n\times n})^*$ on $\mathbb{R}^{n\times n}$. The "product" $\bullet$ is defined as the contraction over the measure on $\mathbb{R}^{n\times n}$ but not over $x\in\Omega$:

$$[h\bullet\nu](x) := \langle \nu_x, h(x,\cdot)\rangle = \int_{\mathbb{R}^{n\times n}} h(x,A)\nu_x(\mathrm{d}A). \tag{2.7}$$

In (2.5), we used it for the matrix- and scalar-valued integrands $\mathrm{Id}\colon \Omega\times\mathbb{R}^{n\times n}\to\mathbb{R}^{n\times n} : (x,A)\mapsto A$ and $(1\otimes v):\Omega\times\mathbb{R}^{n\times n}\to\mathbb{R} : (x,A)\mapsto v(A)$, respectively. The important property is that $\mathcal{G}^p(\Omega;\mathbb{R}^{n\times n})$ contains precisely those Young measures ν for which there is a sequence $\{u_k\}_{k\in\mathbb{N}}$ bounded in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ such that $\delta_{\nabla u_k} \rightharpoonup \nu$ weakly*, with δ_A denoting the Dirac measure supported at $A\in\mathbb{R}^{n\times n}$. This means, for all $h\in \mathrm{L}^1(\Omega;C_0(\mathbb{R}^{n\times n}))$, it holds $\langle\delta_{\nabla u_k},h\rangle \to \langle\nu,h\rangle$ or, in other words, $\int_\Omega h(x,\nabla u_k(x))\,\mathrm{d}x \to \int_\Omega h\bullet\nu\,\mathrm{d}x$. In fact, the above brief statements summarize (some) deep results of intensive research, especially from [BaJa88, KiPe91, KiPe94, Mül99] and many others, cf. also [Rou97, Ch. 6] for some details.

The original potential can be weak*-continuously extended in terms of Young measures and the correct relaxed problem then looks as

$$\begin{cases} \text{Minimize} & \displaystyle\int_\Omega \varphi\bullet\nu\,\mathrm{d}x + \int_\Gamma \Big(\big(\alpha{\cdot}u\big)^2 - f(t,\cdot){\cdot}u\Big)\,\mathrm{d}S \\ \text{subject to} & \nabla u = \mathrm{Id}\bullet\nu, \\ & (u,\nu)\in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n){\times}\mathcal{G}^p(\Omega;\mathbb{R}^{n\times n}). \end{cases} \tag{2.8}$$

Of course, $\varphi\bullet\nu$ is meant as defined by (2.7).

Remark 2.1. Steady-state configurations have been studied intensively during the past decades by minimization of the stored energy; cf. Ball and James [BaJa88, BaJa92] or Müller [Mül99], see also [Rou97, Chap.6] and references therein. This assumes implicitly

non-dissipative PT which does not require any activation energy. This viewpoint is useful to realize how the "mesoscopic" description of the microstructure should look like. In particular, it led to identification of the structure of $\mathcal{G}^p(\Omega;\mathbb{R}^{n\times n})$ from (2.5) although the set of all quasiconvex test functions v is not explicitly known so the definition (2.5) is still quite vague. Anyhow, it outlines some structure of gradient Young measures that can be exploited in computer implementation of the problem, as mentioned in Remark 1.1.

Though thousands of SMA materials have been examined both theoretically and experimentally, concrete forms of $\hat\varphi$ can be found in the literature only very exceptionally. Anyhow, some examples does exist:

Example 2.2. One of such case is the cubic to tetragonal transformation of In-20.7 at% Tl alloy for which Ericksen and James [Eri86] (see also, e.g., [CoLu89, Lus96]) used the potential $\hat\varphi$ (dependent on temperature θ) in the form

$$\begin{aligned}\hat\varphi(F) \;=\;& \tfrac{a(\theta)}{6}\left[\left(\tfrac{3C_{11}}{\operatorname{tr}C}-1\right)^2+\left(\tfrac{3C_{22}}{\operatorname{tr}C}-1\right)^2+\left(\tfrac{3C_{33}}{\operatorname{tr}C}-1\right)^2\right]\\ &+\tfrac{b}{2}\left(\tfrac{3C_{11}}{\operatorname{tr}C}-1\right)\left(\tfrac{3C_{22}}{\operatorname{tr}C}-1\right)\left(\tfrac{3C_{33}}{\operatorname{tr}C}-1\right)\\ &+\tfrac{c}{36}\left[\left(\tfrac{3C_{11}}{\operatorname{tr}C}-1\right)^2+\left(\tfrac{3C_{22}}{\operatorname{tr}C}-1\right)^2+\left(\tfrac{3C_{33}}{\operatorname{tr}C}-1\right)^2\right]^2\\ &+\tfrac{d}{2}\left(C_{12}^2+C_{13}^2+C_{23}^2+C_{21}^2+C_{31}^2+C_{32}^2\right)+e(\operatorname{tr}C-3)^2,\end{aligned}\tag{2.9}$$

where again $C=F^\top F$, θ is a given fixed temperature, and the phenomenological coefficients take the value $a(\theta)=0.38+(1.22\times10^{-3})(\theta-\theta_T)$, $b=-29.23$, $c=562.13$, $d=3.26$, $e=5.25$, (all in GPa) and $\theta_T=70$°C. This potential is obviously *frame indifferent* in the sense $\hat\varphi(RF)=\hat\varphi(F)$ for all $R\in\mathrm{SO}(3)$ and fits with the assumption (2.2) for $p=4$ because the first three terms in (2.9) are, in fact, homogeneous of degree 0 (and hence bounded). The fourth term controls the non-diagonal part of $F^\top F$, and the last term itself controls its trace. Hence, for bounded $\hat\varphi(F)$, the whole matrix F is bounded since $\operatorname{tr}C=\operatorname{tr}(F^\top F)=\|F\|^2$. On the other hand, it does not fit with our assumption of continuity because $\hat\varphi$ from (2.9) is not continuous at $F=0$ but this potential is anyhow reliable only around its 4 wells (of the form $\mathrm{SO}(3)U_\alpha$ with $U_1=\mathrm{I}:=\mathrm{diag}(1,1,1)$ the identity matrix and $U_{2,\dots,4}=\mathrm{diag}(\eta_1,\eta_1,\eta_2)$ up to permutations with η's numbers close to 1, cf. [Eri86] for details) so that the reader can equally imagine $\hat\varphi$ modified far from these wells, in particular around 0, so that $\hat\varphi$ will indeed be continuous.

Example 2.3. Another type of potentials with more explicit reference to measured data and more general use can be constructed as follows. We consider that the material can occur in L stress-free configurations that are determined by *distortion matrices* U_α, $\alpha=1,...,L$. One can imagine $U_1=\mathrm{I}$ corresponding to the cubic parent austenite in the stress-free configuration taken as the reference one, while the others U_α are related with particular martensitic variants. E.g., for a *tetragonal* martensite, U_α, $\alpha=2,...,L=4$ are as in Example 2.2 with $\eta_1=a/a_0$ and $\eta_2=c/a_0$ determined from *lattice parameters* that are usually known with 4-digit accuracy; here, a_0 is the size of the cubic cell of the austenite while $(a\times a\times c)$ is related with prism of the martensitic cell. In case of *orthorhombic* martensite, all sides of this prism differ from each other, which gives rise to 6 diagonal matrices and then $L=7$. A *monoclinic* martensite creates even 12 (i.e. $L=13$) nondiagonal matrices. To ensure frame-indifference, the *free energy* of particular phase(variant)s is considered as a function of *Green's strain tensor* $\varepsilon^\alpha=\frac12(F^\top F-U_\alpha^\top U_\alpha)$.

In the simplest case, one can consider a quadratic function of the form

$$\hat\phi_\alpha(F)=\sum_{i,j,k,l=1}^{n}\varepsilon^\alpha_{ij}\mathbb{C}^\alpha_{ijkl}\varepsilon^\alpha_{kl}-c_\alpha, \tag{2.10}$$

where $\mathbb{C}^\alpha=\{\mathbb{C}^\alpha_{ijkl}\}$ is the 4th-order tensor of elastic moduli satisfying the usual symmetry relations depending also on symmetry of the specific phase(variant) α, while c_α is some offset (depending on a temperature consider fixed, however). Let us remark that sometimes the quadratic form in (2.10) involves rather the tensor $(U_\alpha^\top)^{-1}F^\top FU_\alpha^{-1}-\mathrm{I}$ instead of ε^α, which is however equivalent to (2.10) provided the elastic tensor $\mathbb{C}^\alpha$ is transformed accordingly. The overall stored energy is assembled as

$$\hat\varphi(F):=\min_{\alpha=1,\dots,L}\hat\phi_\alpha(F)\qquad\text{or}\qquad\hat\phi(F):=-k_{\mathrm{B}}\theta\ln\left(\sum_{\alpha=1}^{L}\mathrm{e}^{-\hat\phi_\alpha(F)/k_{\mathrm{B}}\theta}\right) \tag{2.11}$$

where k_{B} is the Boltzmann constant (related per unit volume) and θ the considered (fixed) temperature. Both options exhibit the same *multi-well* character, the latter option being backed up by statistical physics while the former one being computationally simpler and keeping the wells precisely at the orbits $\mathrm{SO}(3)U_\alpha$. Assuming that all $\mathbb{C}^\alpha$ are positive definite, both options fit with the assumption (2.2) for $p=4$ as well as with the continuity assumption. The former option in (2.11) has been used in [AFO02] for CoAlNi undergoing cubic/orthorhombic PT and in [AGNRS03] for NiMnGa with cubic/tetragonal PT. However, the data required for this potential are available for many other alloys; yet (up to few exceptions) the elastic tensors $\mathbb{C}^\alpha$ for the martensites are not known and have then to be taken from the austenite as a certain approximation.

2.2. **Dissipation.** When loaded by a sufficiently large force, the PTs may be activated. On a microscopic level, it means that surfaces between microscopic domains of relatively homogeneous deformation ∇u near some well of $\varphi(\cdot,x)$ may start to move, changing thus the mesoscopic volume fractions described by the measure ν_x. This process usually needs some energy to be activated, which is intimately connected with a dissipation of energy. As there is usually no markable change of structure (the martensitic PT is usually perfectly reversible and no structural changes are observed sometimes even after 10^7 cycles), this energy is obviously dissipated to heat. Anyhow, we assume a sufficiently slow loading processes that allow us to assume that all produced heat can be transferred out of Ω and we may neglect temperature changes.

The amount of dissipated energy is essentially influenced by defects in the atomic grid as various impurities and dislocations. These 1- or 0-dimensional defects essentially do not influence the energy $\hat\varphi$ stored in the n-dimensional bulk, $n\geq 2$, which says, in other words, that the elastic and inelastic effects are to a large extent independent to each other; this is well known from classical plasticity as well as ferromagnetism, and is equally valid for martensitic PT.

As outlined above, the dissipation mechanisms are determined on the atomistic level and it seems that the only efficient way to incorporate them in higher-level model is a phenomenology. Our, probably quite simplified standpoint is that the amount of dissipated energy within the PT from one phase to the other can be described by a single, phenomenologically given number (of the dimension $\mathrm{J/m^3}$=Pa). Hence, we need to identify

the particular phases and particular PTs according to the stretch tensor (and according, in case of a polycrystalic alloy, to a current point x). Inspired by [MiTh99, MTL02, GMH02] and considering L phases, we define a continuous mapping $\mathfrak{L} : \Omega \times \mathbb{R}^{n\times n} \to \triangle_L$ where $\triangle_L := \{\, \zeta \in \mathbb{R}^L \,;\ \zeta_i \geq 0,\ i = 1, ..., L,\ \sum_{i=1}^L \zeta_i = 1 \,\}$ is a simplex with L vertices. Like (2.1), we assume

$$\mathfrak{L}(x, A) = \hat{\mathfrak{L}}\Big((\mathrm{I}+A)\mathfrak{Q}(x)\Big),\ \text{with } \hat{\mathfrak{L}} : \mathbb{R}^{n\times n} \to \triangle_L \text{ and } \mathfrak{Q} \text{ from (2.1)}. \tag{2.12}$$

Again, $\hat{\mathfrak{L}}$ is related with the material itself and thus is expected to be frame indifferent. We have in mind that the components $\{\hat{\mathfrak{L}}_1, ..., \hat{\mathfrak{L}}_L\}$ of $\hat{\mathfrak{L}} = (\hat{\mathfrak{L}}_1, \ldots, \hat{\mathfrak{L}}_L)^\top$ form a *partition of unity* on $\mathbb{R}^{n\times n}$ such that $\mathfrak{L}_i(F)$ is equal 1 if F is in the i-th phase, i.e. F is in a neighborhood of i-th well $\mathrm{SO}(n)U_i$ of $\hat{\varphi}$. Of course, $\hat{\mathfrak{L}}(F)$ in the (relative) interior of $\triangle_L$ indicates F in the spinodal region where no definite phase is specified. Hence λ plays the rôle of what is often called a vector of *order parameters* or a vector-valued *internal variable*, cf. [Mie03].

In terms of the mesoscopic microstructure described by the Young measure ν, the "mesoscopic" order parameter is naturally defined as

$$\lambda = \mathfrak{L} \bullet \nu : \Omega \to \Delta_L \subset \mathbb{R}^L, \tag{2.13}$$

where the bilinear mapping "$\bullet$" is defined as in (2.7) with $\mathbb{R}^L$-valued functions. Thus, $\lambda_i(x)$ may be interpreted as the mesoscopic volume fraction of the i-th phase and $\nu_x \in \mathrm{rca}(\mathbb{R}^{n\times n})$ gives the distribution of the displacement gradient on the microscopic level.

From the viewpoint of rigorous analysis, it appears advantageous to consider the mesoscopic order parameter λ as a part of the *generalized configuration* of the system which is thus a triple $q := (u, \nu, \lambda)$. In view of both (2.8) and (2.13), we define the set Q of admissible configurations as

$$Q := \big\{(u, \nu, \lambda) \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^n) \times \mathcal{G}^p(\Omega; \mathbb{R}^{n\times n}) \times \mathrm{L}^\infty(\Omega; \mathbb{R}^L) : \nabla u = \mathrm{Id} \bullet \nu,\ \lambda = \mathfrak{L} \bullet \nu\big\}. \tag{2.14}$$

For a phenomenological description of the dissipation mechanism one usually postulates a (pseudo)*potential for the dissipational forces* as a function of the rate of λ (i.e., the dissipational forces are obtained as derivative of the potential with respect to the rates of the internal variables). Here, we want to avoid specifying what the derivative means if q ranges the manifold Q and, instead, we define the *dissipation distance* $R(q_1, q_2) \equiv R(u_1, \nu_1, \lambda_1, u_2, \nu_2, \lambda_2)$ on Q by means of a metric $\varrho : \Delta_L \times \Delta_L \to [0, \infty)$ by

$$R(q_1, q_2) \equiv R(u_1, \nu_1, \lambda_1, u_2, \nu_2, \lambda_2) = \int_\Omega \varrho(\lambda_1(x), \lambda_2(x))\,\mathrm{d}x. \tag{2.15}$$

For simplicity we let $\varrho(\lambda_1, \lambda_2) = |\lambda_1 - \lambda_2|_L$ for a norm $|\cdot|_L$ on $\mathbb{R}^L$. The important properties is that the "dissipation semi-distance" R on Q satisfies the triangle inequality, i.e.

$$\forall q_1, q_2, q_3 \in Q : \qquad R(q_1, q_3) \leq R(q_1, q_2) + R(q_2, q_3). \tag{2.16}$$

For certain SMAs it might be important to allow for an unsymmetric metric ϱ, i.e., $\varrho(\lambda_1, \lambda_2) \neq \varrho(\lambda_2, \lambda_1)$ is admissible. For the subsequent such an unsymmetry is perfectly allowed.

For the analysis below, we will need to consider rather a certain *regularization* of the stored energy V which controls the spatial smoothness of λ. For this, like in [Fre02, p.364], we will augment V by a higher-order term

$$V(u,\nu,\lambda) := \int_\Omega \Big(\varphi(\nu) + \rho|\nabla\lambda|^2\Big)\,\mathrm{d}x + \int_\Gamma (\alpha\cdot u)^2\,\mathrm{d}S, \qquad \rho>0 \text{ fixed.} \tag{2.17}$$

For a certain interpretation and justification of this regularizing term see Remark 2.6 below.

Remark 2.4. *(More general dissipation potentials.)* In the case (2.15), R is also symmetric but we will not exploit this property. In fact, $R(0,\cdot)$ is the Minkowski functional of the set $\{\lambda\in \mathrm{L}^\infty(\Omega;\mathbb{R}^L);\ \forall(\text{a.a.})x\in\Omega:\ |\lambda(x)|_L\le 1\}$ but one can equally imagine another bounded, convex set in $\mathrm{L}^\infty(\Omega;\mathbb{R}^L)$ with a nonempty interior. Its surface can be interpreted as an *activation threshold* for the PTs. If this set is not balanced with respect to 0, ϱ would not be symmetric, which would reflect that a PT from one phase to another one may need a different activation energy than the reverse transformation. See [Mie03] for more general dissipation distances when $\triangle_L$ is replaced by a manifold.

Remark 2.5. The λ's indicating particular phases have also been used in [AGR02] but without the natural requirement $\sum_{i=1}^L \lambda_i(\cdot,x)=1$ so that they cannot be interpreted directly as volume fractions. Besides, λ's indicating particular PTs rather than the phases themselves have been proposed in [Rou02].

Remark 2.6. *(A justification of the regularization.)* Inspired by the so-called Ericksen-Timoshenko beam, cf. [ReTr00, RoTr97], the ρ-term in (2.17) can be recognized as asymptotically resulting from the stored energy occuring in (2.4) if augmented as follows:

$$\begin{aligned} V_{\varepsilon_1,\varepsilon_2}(u) := \int_\Omega \Big(&\varphi(x,\nabla u) + \varepsilon_1|\nabla^2 u|^2 \\ &+ \frac{1}{\varepsilon_2}|\lambda - \mathfrak{L}(x,\nabla u)|^2 + \rho|\nabla\lambda|^2\Big)\,\mathrm{d}x + \int_\Gamma (\alpha\cdot u)^2\,\mathrm{d}S \end{aligned} \tag{2.18}$$

with $\varepsilon_1>0$ a small parameter related with a bending rigidity of the material, ε_2 a small parameter allowing a (presumably small) deviation of the "macroscopic" order parameter λ from the "microscopic" order parameter $\mathfrak{L}(\nabla u)$, and with $\rho>0$ a (small) regularization parameter preventing large spatial variations of volume fractions (measured in terms of the order parameters $\mathfrak{L}$'s) but still admitting sharp interfaces in ∇u. For the special case $n=L=1$ and $\mathfrak{L}:(x,A)\mapsto A$, the stored energy (2.18) indeed coincides with Ericksen-Timoshenko beam, $\sqrt{\varepsilon_2\rho}$ being identified as an internal length scale in [ReTr00, RoTr97]. Thus, (2.18) can be viewed as the generalization of the Ericksen-Timoshenko model. Passing $\varepsilon_1\to 0$, we arrive at the relaxed potential $V_{\varepsilon_2}(u,\nu,\lambda) := \int_\Omega \big(\varphi(\nu) + \frac{1}{\varepsilon_2}|\lambda - \mathfrak{L}(\nu)|^2 + \rho|\nabla\lambda|^2\big)\,\mathrm{d}x + \int_\Gamma(\alpha\cdot u)^2\,\mathrm{d}S$ and, passing furthermore $\varepsilon_2\to 0$ to push the difference $\lambda-\mathfrak{L}(\nu)$ penalized in $\mathrm{L}^2(\Omega;\mathbb{R}^L)$-norm to zero, we eventually obtain (2.17) together with the constraint $\lambda=\mathfrak{L}(\nu)$ involved in (2.14).

2.3. **Solution processes and their energetics.** Considering the external load f to vary within a fixed time interval, say $[0,T]$, we can naturally expect the response $q(t)$ to vary with $t\in[0,T]$. The delicate point is to define the solution process in such a way which would meet the desired energetics and is still amenable to a rigorous analysis.

As now $f:[0,T]\times\Gamma\to\mathbb{R}$, we shall write $F:[0,T]\times Q\to\mathbb{R}$ defined by

$$F(t,q)\equiv F(t,u,\nu,\lambda):=\big\langle f(t,\cdot),u\big\rangle=\int_\Gamma f(t,x)\cdot u(x)\,\mathrm{d}S; \tag{2.19}$$

as usual, we identified $f(t,\cdot)$ with a linear functional on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$.

We assume that F is smooth as a function of time; namely in terms of f:

$$f\in\mathrm{W}^{1,1}(0,T;\mathrm{L}^\gamma(\Gamma;\mathbb{R}^n)); \tag{2.20}$$

recall that $\gamma\ge(np-p)/(np-n)$ for $p<n$ or simply $\gamma=1$ for $p\ge n$. Then, naturally, we will write

$$\frac{\partial F}{\partial t}(t,q):=\int_\Gamma\frac{\partial f}{\partial t}(t,x)\cdot u(x)\,\mathrm{d}S. \tag{2.21}$$

Furthermore, let us abbreviate the Gibbs stored energy by

$$G(t,q):=V(q)-F(t,q). \tag{2.22}$$

The modeling of rate-independent processes is based on the assumption that the scale of the macroscopic processes enforced by the external loading varying with the (process) time t, is much slower than any internal time scale due to internal viscous effects. Hence, for each time the system is in a relaxed and hence stable state. Of course, stability here must include the rate-independent frictional-type forces modeled via the dissipation semi-distance (2.15). At time t, we define the *stability set*, i.e. the set of stable states, by

$$S(t):=\big\{q\in Q;\quad\forall\tilde q\in Q:\quad G(t,q)\le G(t,\tilde q)+R(q,\tilde q)\big\}. \tag{2.23}$$

We impose a global stability: we consider a state q stable if and only if $q\in S(t)$, i.e. the gain of Gibbs' energy $G(t,q)-G(t,\tilde q)$ at any other state $\tilde q$ is not larger than the dissipation $R(q,\tilde q)$; cf. [MTL02] for a discussion.

We will denote by $S:[0,T]\rightrightarrows Q$ the set-valued mapping $t\mapsto S(t)$. The following structural assertion is essential to the theory, cf. also [MiTh01].

Proposition 2.7. *Let (2.2), (2.3) and (2.20) hold. Consider a sequence (t_k,q_k) with $t_k\to t$, $q_k\to q$ weakly* in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\times\mathrm{L}^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n}))\times\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)$ and $q_k\in S(t_k)$. Then,*

(i) *$q\in S(t)$ (i.e., the graph of the set-valued map S is (sequentially) weakly* closed)*

(ii) *$G(t_k,q_k)\to G(t,q)$ for $k\to\infty$.*

Proof. Using $\nu_k\to\nu$ weakly* in $\mathrm{L}^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n}))$ and $\liminf_{|A|\to\infty}\varphi(x,A)\ge0$ (due to (2.2)), we can claim that $\liminf_{k\to\infty}\int_\Omega\varphi\bullet\nu_k\,\mathrm{d}x\ge\int_\Omega\varphi\bullet\nu\,\mathrm{d}x$. (In fact, by arguments we use later, it even holds $\lim_{k\to\infty}\int_\Omega\varphi\bullet\nu_k\,\mathrm{d}x=\int_\Omega\varphi\bullet\nu\,\mathrm{d}x$.) Using further the weak lower semicontinuity of the quadratic terms in V, we have

$$\begin{aligned}\liminf_{k\to\infty}V(q_k)\ &=\ \liminf_{k\to\infty}\int_\Omega\varphi\bullet\nu_k+|\nabla\lambda_k|^2\,\mathrm{d}x+\int_\Gamma(\alpha\cdot u_k)^2\,\mathrm{d}S\\ &\ge\ \int_\Omega\varphi\bullet\nu+|\nabla\lambda|^2\,\mathrm{d}x+\int_\Gamma(\alpha\cdot u)^2\,\mathrm{d}S=V(q).\end{aligned} \tag{2.24}$$

By (2.20) we have $f\in C\big([0,T];\mathrm{L}^\gamma(\Gamma;\mathbb{R}^n)\big)$, hence $F(t_k,q_k)=\int_\Gamma f(t_k,x)\cdot u_k(x)\,\mathrm{d}S\to\int_\Gamma f(t,x)\cdot u(x)\,\mathrm{d}S=F(t,q)$. Altogether, we have obtain the lower semicontinuity

$$G(t,q)\le\liminf_{k\to\infty}G(t_k,q_k) \tag{2.25}$$

Second, we have also $\lim_{k\to\infty} R(q_k,\tilde q) = R(q,\tilde q)$ for any $\tilde q\in Q$ because, by (2.15) and by the triangle inequality (2.16), we have

$$\big|R(q_k,\tilde q)-R(q,\tilde q)\big| \le R(q_k,q) = \int_\Omega \varrho(\lambda_k,\lambda)\,\mathrm{d}x = \|\lambda_k-\lambda\|_{\mathrm{L}^1(\Omega;\mathbb{R}^L)} \to 0 \tag{2.26}$$

if $\mathbb{R}^L$ is equipped with the norm $|\cdot|_L$; of course, in (2.26) we used the compact embedding $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)\subset\mathrm{L}^1(\Omega;\mathbb{R}^L)$.

Thus, starting from $q_k\in S(t_k)$ and using both (2.25) and (2.26), we have

$$G(t,q) \le \liminf_{k\to\infty} G(t_k,q_k) \le \lim_{k\to\infty} G(t_k,\tilde q)+R(q_k,\tilde q) = G(t,\tilde q)+R(q,\tilde q). \tag{2.27}$$

Inserting $\tilde q = q$ we obtain (ii) since all "$\le$" must be equalities, which implies that the "lim inf" is a limit (by using a contradiction argument).

Since $\tilde q\in Q$ was arbitrary in (2.27), to conclude that $q\in S(t)$ it remains to show that $q\in Q$. For this, we will prove an a-priori estimate for $|\cdot|^p\bullet\nu_k$. Using (2.2), we find

$$\begin{aligned}\int_\Omega\int_{\mathbb{R}^{n\times n}} |A|^p[\nu_k]_x(\mathrm{d}A)\,\mathrm{d}x &\le \mathrm{meas}(\Omega)\\ &\quad + c_0^{-1}\int_\Omega\int_{\mathbb{R}^{n\times n}} \varphi(x,A)[\nu_k]_x(\mathrm{d}A)\,\mathrm{d}x \le \mathrm{meas}(\Omega)+c_0^{-1}V(q_k),\end{aligned} \tag{2.28}$$

From the assumption $q_k\in S(t_k)$ by using the definition (2.23) with $\tilde q=(0,\{\delta_0\},\tilde\lambda)\in Q$ (hence inevitably $\tilde\lambda(x)=\mathfrak{L}(x,0)$), we obtain the estimate

$$\begin{aligned}V(q_k) &\le G(t_k,q_k)+F(t_k,q_k)\\ &\le \int_\Omega \varphi(x,0)+|\lambda_k(x)-\mathfrak{L}(x,0)|_L\,\mathrm{d}x + \int_\Gamma f(t_k,x)\cdot u_k(x)\,\mathrm{d}S.\end{aligned} \tag{2.29}$$

Realizing that $\lambda_k\to\lambda$ in $\mathrm{L}^1(\Omega;\mathbb{R}^L)$ and, again due to (2.20), $f(t_k,\cdot)\to f(t,\cdot)$ in $\mathrm{L}^\gamma(\Gamma;\mathbb{R}^n)$, from (2.28) and (2.29) we can see that

$$\sup_{k\in\mathbb{N}}\int_\Omega\int_{\mathbb{R}^{n\times n}} |A|^p[\nu_k]_x(\mathrm{d}A)\,\mathrm{d}x<+\infty. \tag{2.30}$$

As $\lambda_k=\mathfrak{L}\bullet\nu_k$ and $\nabla u_k=\mathrm{Id}\bullet\nu_k$, by [Bal89] we find in the limit $\lambda=\mathfrak{L}\bullet\nu$ and $\nabla=\mathrm{Id}\bullet\nu$ because these relations are linear and thus conserved under weak* limits; here we used the estimate (2.30) and that the growth of both $\mathrm{Id}(x,\cdot)$ and $\mathfrak{L}(x,\cdot)$ is less than p-power. In view of the definition (2.14) of Q, we have still to show $\nu\in\mathcal{G}^p(\Omega;\mathbb{R}^{n\times n})$. Based on (2.30) and on an explicit construction of gradient Young measures with "nonconcentrating energy" [Kri94, FMP98], this requires a rather fine technique and is outlined in the Appendix in Lemma 4.3. □

Definition 2.8. We say that a *process* $q:[0,T]\to Q:t\mapsto q(t)$ is *stable* if $q(t)\in S(t)$ for all $t\in[0,T]$ (for S see (2.23)), i.e.

$$\forall\, t\in[0,T]\ \ \forall\tilde q\in Q:\qquad G\big(t,q(t)\big)\le G(t,\tilde q)+R\big(q(t),\tilde q\big). \tag{2.31}$$

Furthermore, we say that the process $q : [0,T] \to Q$ satisfies the *energy inequality* if, for all $s,t \in [0,T]$, $s \le t$, it holds

$$\underbrace{G\big(t,q(t)\big)}_{\substack{\text{Gibbs's energy}\\ \text{at time } t}} + \underbrace{\mathrm{Var}(q;s,t)}_{\substack{\text{dissipated}\\ \text{energy}}} \le \underbrace{G\big(s,q(s)\big)}_{\substack{\text{Gibbs' ener-}\\ \text{gy at time } s}} \underbrace{-\int_s^t \frac{\partial F}{\partial\theta}\big(\theta,q(\theta)\big)\,\mathrm{d}\theta}_{\substack{\text{reduced work}\\ \text{of external load}}}, \tag{2.32}$$

where the total variation of the process q over the time interval $[s,t]$ (with respect to the semidistance R) is defined standardly, without using explicitly any time derivative, as

$$\mathrm{Var}(q;s,t) := \sup \sum_{i=1}^{I} R\big(q(t_{i-1}),q(t_i)\big), \tag{2.33}$$

where the supremum is taken over all $I \in \mathbb{N}$ and over all partitions of $[s,t]$ in the form $s = t_0 < t_1 < ... < t_{I-1} < t_I = t$. In particular, (2.32) requires the mapping $[0,T] \to \mathbb{R} : t \mapsto \frac{\partial}{\partial t}F(t,q(t))$ measurable.

We will address an initial-value problem, hence we should consider a prescribed initial condition q_0 satisfying

$$q_0 \in Q \quad \text{and} \quad G(0,q_0) \le G(0,\tilde q) + R(q_0,\tilde q) \qquad \forall \tilde q \in Q. \tag{2.34}$$

In view of (2.23), this assumption simply says that $q_0 \in S(0)$, i.e. the initial state is stable. Note that it implies, in particular, that $V(q_0) < +\infty$; cf. also Remark 3.2 below.

Therefore, the quantities we play with are the stored energy V, the dissipative (semi-) distance R, the external loading F, and the initial condition q_0. In view of this, we will speak about a problem (V,R,F,q_0) and define the notion of its solution like in [Mie03, MiTh01, MTL02, The02]. Let us agree to understand the space $\mathrm{BV}([0,T];Z)$ of functions with bounded variations as containing measurable functions defined *everywhere* on $[0,T]$, and normed by $\|z\|_{\mathrm{BV}([0,T];Z)} := \|z\|_{\mathrm{L}^1(0,T;Z)} + \mathrm{Var}(z;0,T)$ with "Var" defined like (2.33) but with $\|\cdot - \cdot\|_Z$ instead of $R(\cdot,\cdot)$.

Definition 2.9. The process $q : [0,T] \to Q : t \mapsto q(t)$ will be considered as a *solution* to the problem (V,R,F,q_0) if

1) $u \in \mathrm{L}^\infty(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n))$ and $\lambda \in \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^L))$,
2) $q(t) \in Q$ for all $t \in [0,T]$,
3) it is stable in the sense (2.31), and
4) it satisfies the energy inequality (2.32) for all $s,t \in [0,T]$ with $s \le t$,
5) $q(0) = q_0$.

Remark 2.10. By the "reduced work" in (2.32) we mean (up to a sign) the usual work, i.e. $\int_s^t \big\langle f, \frac{\mathrm{d}u}{\mathrm{d}\theta}\big\rangle\,\mathrm{d}\theta$, but reduced by $F(s,q(s)) - F(t,q(t))$ which is just the difference between the Gibbs' and Helmholtz' energy at times s and t.

3. Existence via time discretization

We will prove the existence of a response q with the above mentioned properties rather constructively by a semi-discretization in time, using the fully implicit (backward) Euler scheme; also, this is often called Rothe's method. This also suggests implementable

numerical approach after further spatial discretization like in [AFO02, RoKr98]. For simplicity, let us consider an equi-distant partition of the time interval $[0,T]$ with a time step $\tau>0$, assuming T/τ integer, and assume $\tau\to0$ in such a way that the equi-distant partitions will be nested; for example, the reader can think about a sequence of time steps $\tau=2^{-k}T$ for $k\in\mathbb{N}$.

Then we put $q_\tau^0=q_0$, a given initial condition, and, for $k=1,...,T/\tau$ we define q_τ^k recursively as a solution of the *time-incremental minimization problem*

$$\left\{\begin{array}{l}\text{Minimize}\quad V(q)+R(q_\tau^{k-1},q)-F(k\tau,q)\\ \text{subject to}\ \ q\equiv(u,\nu,\lambda)\in Q\ ,\end{array}\right. \tag{3.1}$$

where Q is from (2.14), V is from (2.17), R from (2.15), and F from (2.19). If a solution (i.e. a *global* minimizer) to (3.1) is not unique, we just take an arbitrary one for q_τ^k. Then we define the piecewise constant interpolation $q_\tau\in\mathrm{L}^\infty\big(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\times\mathrm{L}^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n}))\times\mathrm{L}^\infty(\Omega;\mathbb{R}^L)\big)$ so that $q_\tau|_{((k-1)\tau,k\tau]}=q_\tau^k$ for $k=1,...,T/\tau$. Likewise, F_τ will denote the piecewise constant interpolation so that $F_\tau(t,q):=F_\tau(k\tau,q)$ for any $q\in Q$ and $t\in((k-1)\tau,k\tau]$. Moreover, we define naturally the piece-wise constant approximation of Gibbs' enegry by

$$G_\tau(t,q):=V(q)-F_\tau(t,q). \tag{3.2}$$

Proposition 3.1. *Let (2.2), (2.3), (2.20), and (2.34) hold. Then q_τ "constructed" recursively by (3.1) does exist and this q_τ is stable in the sense of Definition 2.8 with G_τ taken instead of G, i.e.*

$$\forall\tilde q\in Q:\qquad G_\tau\big(t,q_\tau(t)\big)\le G_\tau(t,\tilde q)+R\big(q_\tau(t),\tilde q\big) \tag{3.3}$$

for all $t\in[0,T]$, and satisfies the two-side approximate energy inequality

$$\begin{aligned}-\int_{t_1}^{t_2}\frac{\partial F}{\partial t}\big(t,q_\tau(t)\big)\,\mathrm{d}t\ \le\ G\big(t_2,q_\tau(t_2)\big)+\mathrm{Var}(q_\tau;t_1,t_2)\\ -G\big(t_1,q_\tau(t_1)\big)\le-\int_{t_1}^{t_2}\frac{\partial F}{\partial t}\big(t,q_\tau(t-\tau)\big)\,\mathrm{d}t,\end{aligned} \tag{3.4}$$

for all $t_1\le t_2$ from the set $\{k\tau\}_{k=0}^{T/\tau}$, where naturally $q_\tau(t):=q_0$ for $t<0$. Also, the following a-priori estimates hold:

$$\|u_\tau\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n))}\le C_1, \tag{3.5}$$

$$\|(1\otimes|\cdot|^p)\bullet\nu_\tau\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))}\le C_2, \tag{3.6}$$

$$\|\lambda_\tau\|_{\mathrm{L}^\infty\left(0,T;\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^L)\right)\cap\mathrm{BV}\left([0,T];\mathrm{L}^1(\Omega;\mathbb{R}^L)\right)}\le C_3, \tag{3.7}$$

$$\|\mathfrak{G}_\tau\|_{\mathrm{BV}([0,T])}\le C_4\quad\textit{where}\ \mathfrak{G}_\tau(t):=G_\tau\big(t,q_\tau(t)\big). \tag{3.8}$$

Remark 3.2. If the initial condition $q_0\in Q$ has just finite energy, i.e. $V(q_0)<+\infty$, instead of being stable as in (2.34), we would get only the second estimate in (3.4) while the first one could obviously break down for $t_1=0$.

Proof. The existence of a solution to (3.1) is demonstrated in Lemma 4.2 in Appendix below.

Let us abbreviate $F_\tau^k(q) := F(k\tau, q)$ and $G_\tau^k(q) := G(k\tau, q)$. As in [MTL02, Thm.3.4], by using successively (3.1) and (2.16), we derive stability from the minimization property of q_τ^k in (3.1) and the triangle inequality:

$$\begin{aligned} V(q_\tau^k) &\le V(\tilde q) + R(q_\tau^{k-1}, \tilde q) - R(q_\tau^{k-1}, q_\tau^k) - F_\tau^k(\tilde q) + F_\tau^k(q_\tau^k) \\ &\le V(\tilde q) + R(q_\tau^k, \tilde q) - F_\tau^k(\tilde q) + F_\tau^k(q_\tau^k) \end{aligned} \tag{3.9}$$

for any $k = 1, ..., T/\tau$. In view of the definition of q_τ and F_τ, it just means that (3.3) holds for all $t \in (0, T]$. For $t = 0$, stability is assumed in (2.34).

Moreover, consider $0 \le k_1 \le k_2 \le T/\tau$, and $t_1 = k_1\tau$, $t_2 = k_2\tau$. As in [MTL02, Formula (2.13)], we can test the first inequality in (3.9) by $\tilde q = q_\tau^{k-1} \in Q$, and sum it for all $k = k_1, ..., k_2$. After a small re-arrangement, it gives

$$\begin{aligned} G_\tau^{k_2}(q_\tau^{k_2}) - G_\tau^{k_1}(q_\tau^{k_1}) + \sum_{k=k_1}^{k_2} R(q_\tau^k, q_\tau^{k-1}) &\le \sum_{k=k_1}^{k_2-1} F_\tau^k(q_\tau^k) - F_\tau^{k+1}(q_\tau^k) \\ &= -\int_{t_1}^{t_2} \frac{\partial F_\tau}{\partial t}\big(t, q_\tau(t-\tau)\big)\,\mathrm{d}t. \end{aligned} \tag{3.10}$$

As q_τ is piecewise constant and jumps only at $t = \tau, 2\tau, ...$, we have a simple explicit formula for its variation, namely

$$\mathrm{Var}(q_\tau; t_1, t_2) = \sum_{k=k_1}^{k_2} R(q_\tau^k, q_\tau^{k-1}). \tag{3.11}$$

In particular, from (3.10) we get the second inequality in the two-side estimate (3.4).

Using (3.9) to express the stability of q_τ^{k-1} with respect to q_τ^k, we find

$$V(q_\tau^k) - F_\tau^k(q_\tau^k) + R(q_\tau^{k-1}, q_\tau^k) - V(q_\tau^{k-1}) - F_\tau^k(q_\tau^{k-1}) \ge F_\tau^{k-1}(q_\tau^k) - F_\tau^k(q_\tau^k). \tag{3.12}$$

Note that, for $k = 1$, we use the assumption (2.34). Summing up (3.12) for $k = k_1, ..., k_2$, we obtain

$$\begin{aligned} V(q_\tau^{k_2}) &- F_\tau^{k_2}(q_\tau^{k_2}) - V(q_\tau^{k_1}) - F_\tau^{k_1}(q_\tau^{k_1}) + \sum_{k=k_1}^{k_2} R(q_\tau^{k-1}, q_\tau^k) \\ &\ge \sum_{k=k_1}^{k_2-1} F_\tau^k(q_\tau^{k+1}) - F_\tau^{k+1}(q_\tau^{k+1}) = \int_{t_1}^{t_2} -\frac{\partial F_\tau}{\partial t}\big(t, q_\tau(t)\big)\,\mathrm{d}t\ , \end{aligned} \tag{3.13}$$

which gives the first inequality in (3.4).

From the coercivity (2.4), for any $q = (u, \nu, \lambda) \in Q$, we obtain the estimate

$$\begin{aligned} \varepsilon\|u\|^p_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)} &\le \int_\Omega \varphi^{\mathrm{qc}}(x, \nabla u)\,\mathrm{d}x + \int_\Gamma (\alpha\cdot u)^2\,\mathrm{d}S + \varepsilon^{-1} \\ &= \int_\Omega \varphi\bullet\nu\,\mathrm{d}x + \int_\Gamma (\alpha\cdot u)^2\,\mathrm{d}S + \varepsilon^{-1} \\ &\le \int_\Omega \varphi\bullet\nu + \rho|\nabla\lambda|^2\,\mathrm{d}x + \int_\Gamma (\alpha\cdot u)^2\,\mathrm{d}S + \varepsilon^{-1} = V(q) + \varepsilon^{-1}. \end{aligned} \tag{3.14}$$

Using (3.14) for $q=q_\tau^k$ and also (3.10) for $k_1=0$ and written in u-component, we obtain

$$\varepsilon\|u_\tau^{k_2}\|^p_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)} \le \int_\Gamma \left(f_\tau^{k_2}\cdot u_\tau^{k_2} + \sum_{k=0}^{k_2-1}(f_\tau^k - f_\tau^{k+1})\cdot u_\tau^k\right)\mathrm{d}S + C_0 \tag{3.15}$$

with a constant C_0 depending on q_0 and on $\|f(0)\|_{\mathrm{L}^\gamma(\Gamma;\mathbb{R}^n)}$ only. Using the short-hand $U_\tau = \max_{k=1,\dots,T/\tau}\|u_\tau^k\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)}$, we find

$$\varepsilon U_\tau^p \le \Big(\max_{t\in[0,T]}\|f(t,\cdot)\|_{\mathrm{L}^\gamma(\Gamma;\mathbb{R}^n)} + \int_0^T \Big\|\frac{\partial f}{\partial t}(t,\cdot)\Big\|_{\mathrm{L}^\gamma(\Gamma;\mathbb{R}^n)}\mathrm{d}t\Big)U_\tau + C_0, \tag{3.16}$$

which gives (3.5) when using (2.20), i.e. $f\in \mathrm{W}^{1,1}(0,T;\mathrm{L}^\gamma(\Gamma;\mathbb{R}^n))$. Now we can see that the right-hand side of (3.10) written for $k_1=0$ is a-priori bounded independently of k_2, hence $G_\tau^{k_2}(q_\tau^{k_2})$ is a-priori bounded, and thus $V(q_\tau^{k_2})$, too. Using (2.28), we arrive at (3.6). Improving (3.14), cf. (2.17), to get even $\varepsilon\|u\|^p_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)} + \varrho\|\lambda\|^2_{\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)} \le V(q)+\varepsilon^{-1}$, we obtain the L^∞-part of (3.7); realize that always $\lambda_\tau(t,x)\in\Delta_L$ which is bounded in $\mathbb{R}^L$. Furthermore, the R-term in (3.10) gives the remaining BV-estimate in (3.7) if (3.11) is taken into account.

Finally, using (3.10) for $k_2=k$ and $k_1=k-1$ and (3.12), we find

$$\begin{aligned}|G_\tau^k(q_\tau^k) - G_\tau^{k-1}(q_\tau^{k-1})| &\le R(q_\tau^k,q_\tau^{k-1})\\ &\quad+\max\left(|F_\tau^k(q_\tau^{k-1})-F_\tau^{k-1}(q_\tau^{k-1})|\,,\,|F_\tau^k(q_\tau^k)-F_\tau^{k-1}(q_\tau^k)|\right)\end{aligned} \tag{3.17}$$

for $k=1,\dots,T/\tau$. Due to the already proved estimate (3.5), the BV-estimate in (3.7) and due to assumption (2.20) of F, the right-hand side of (3.17) is boundedly summable, which eventually gives (3.8). □

Now, we want to use and modify the results from [MiTh01, MTL02] to investigate the convergence of the processes $q_\tau(\cdot)$ for $\tau\to 0$. For this, it seems inevitable to require that the dissipative mechanism described by $\mathfrak{L}$ is sufficiently strong (with respect to the loading regime F and the geometry of Ω) in the sense that

$$\forall t\in[0,T]\ \ \forall q_i=(u_i,\nu_i,\lambda_i)\in S(t):\quad \lambda_1=\tilde\lambda_2 \ \ \Rightarrow\ \ u_1=\tilde u_2. \tag{3.18}$$

This can equally be formulated as

$$\begin{aligned}\forall t\in[0,T]\ \ \forall q=(u,\nu,\lambda)\in S(t)\ \ \forall\tilde q=(\tilde u,\tilde\nu,\lambda)\in Q:\\ \tilde u\neq u\ \ \Rightarrow\ \ G(t,\tilde q)>G(t,q).\end{aligned} \tag{3.19}$$

Note that (3.18) thus requires uniqueness of the u-component of the minimizer of the relaxed problem (2.8) augmented by the (compatible) constraint $\mathfrak{L}\bullet\nu=\lambda$. Realizing the coercivity of R from (2.15), it prevents variation of macroscopic displacement u without changing the stored energy and without dissipating any energy. This seems a reasonable requirement in real experiments with (or applications of) SMAs if one counts, of course, fixing the body by the elastic support on Γ to prevent rigid-body motions. In particular, the condition (3.18) *excludes buckling-type response* on a given load regime.

Based on Proposition 2.7, we obtain the following auxiliary result.

Proposition 3.3. *Let (3.18) hold and let $q_k=(u_k,\nu_k,\lambda_k)\in S(t_k)$ with $t_k\to t$ and $\lambda_k\to\lambda$ weakly in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)$. Then, $u_k\to u$ weakly in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$.*

Proof. Using (3.14) written for $q = q_k$ and (2.28), we can see that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$. Hence, up to a subsequence, we can assume $u_k \to u$ and $\nu_k \to \nu$ weakly* in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ and in $\mathrm{L}^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n}))$, respectively. By Proposition 2.7, $q = (u,\nu,\lambda) \in S(t)$, and by (3.18) this limit u (contrary, in general, to ν!) is determined uniquely. Hence even the whole sequence $\{u_k\}_{k\in\mathbb{N}}$ converges to it. □

Let us recall the concept of a net from general topology. A set Ξ is called *directed* by an ordering "$\preceq$" if, for any $\xi_1,\xi_2 \in \Xi$, there is $\xi_3 \in \Xi$ such that both $\xi_1 \preceq \xi_3$ and $\xi_2 \preceq \xi_3$. A subset A of a directed set Ξ is called cofinal if for any $\xi_1 \in \Xi$ there is $\xi_2 \in A$ such that $\xi_1 \preceq \xi_2$. Having a directed set Ξ and another set X, we say that $\{x_\xi\}_{\xi\in\Xi}$ is a *net* in X if there is a mapping $\Xi \to X : \xi \mapsto x_\xi$. If X is a topological space, we write $x = \lim_{\xi\in\Xi} x_\xi$ if, for any neighbourhood N of x there is $\xi_0 \in \Xi$ such that $x_\xi \in N$ whenever $\xi_0 \preceq \xi$, and then we say that the net $\{x_\xi\}_{\xi\in\Xi}$ *converges* to x (in the so-called Moore-Smith sense). Having another net $\{\tilde x_{\tilde\xi}\}_{\tilde\xi\in\tilde\Xi}$ in X, we say that this net is *finer* than the net $\{x_\xi\}_{\xi\in\Xi}$ if there is a mapping $j : \tilde\Xi \to \Xi$ such that, for any $\tilde\xi \in \tilde\Xi$, it holds $\tilde x_{\tilde\xi} = x_{j(\tilde\xi)}$ and moreover, for any $\xi \in \Xi$ there is $\tilde\xi \in \tilde\Xi$ large enough so that $j(\tilde\xi_1) \succeq \xi$ whenever $\tilde\xi_1 \succeq \tilde\xi$. For example, every non-decreasing mapping $j : \tilde\Xi \to \Xi$ such that $j(\tilde\Xi)$ is cofinal in Ξ produces a finer net by putting $\tilde x_{\tilde\xi} = x_{j(\tilde\xi)}$. Obviously, a finer net may have an index set of strictly greater cardinality than the original net. *Compact* sets are characterized by the property that every net posseses a finer net that converges. We use $\Xi \subset \mathbb{N}$ (ordered standardly, hence a net indexed by Ξ is called a sequence) and $\tilde\Xi \subset \{\text{finite subsets of } [0,T]\}$ ordered by inclusion. Note that $\tilde\Xi$ is indeed directed by this way.

Theorem 3.4. [Convergence and existence]
Let the assumptions of Proposition 3.1 and together with (3.18) be valid. Then there are a net $\{q_{\tau_\xi}\}_{\xi\in\Xi}$, finer than the sequence $\{q_\tau\}_{\tau=T/2^k,\ k\in\mathbb{N}}$ and such that $\lim_{\xi\in\Xi}\tau_\xi = 0$, and a process $q : [0,T] \to \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\times\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})\times\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L) : t \mapsto q(t) = \big(u(t),\nu(t),\lambda(t)\big)$ such that:

(i) $\mathrm{w}\text{-}\lim_{\xi\in\Xi} u_{\tau_\xi}(t) = u(t)$ *(weak convergence in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$) for all $t \in [0,T]$, and $u \in \mathrm{L}^\infty(0,T,\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n))$*

(ii) $\mathrm{w}^*\text{-}\lim_{\xi\in\Xi} \nu_{\tau_\xi}(t) = \nu(t)$ *(weak* convergence in $\mathrm{L}^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n}))$ for all $t \in [0,T]$,*

(iii) $\mathrm{w}^*\text{-}\lim_{\xi\in\Xi} \lambda_{\tau_\xi}(t) = \lambda(t)$ *(weak* convergence in $\mathrm{L}^\infty(\Omega;\mathbb{R}^L) \cap \mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)$) for all $t \in [0,T]$, and $\lambda \in \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^L))$,*

(iv) $\lim_{\xi\in\Xi} G_{\tau_\xi}(t,q_{\tau_\xi}(t)) = G(t,q(t))$ *for all $t \in [0,T]$.*

Moreover, every $q : [0,T] \to Q$ obtained as such a limit is a solution process according to Definition 2.9, in particular u and λ are measurable, $\nu(t) \in \mathcal{G}^p(\Omega;\mathbb{R}^{n\times n})$, $\nabla u(t) = \mathrm{Id}\bullet\nu(t)$, $\lambda(t) = \mathfrak{L}\bullet\nu(t)$ for all $t \in [0,T]$, and the energy equality holds for all $s,t \in [0,T]$ with $s < t$:

$$G(t,q(t)) + \mathrm{Var}(q,[s,t]) = G(s,q(s)) - \int_s^t \frac{\partial F}{\partial t}(\theta,q(\theta))\,\mathrm{d}\theta. \tag{3.20}$$

Proof. For clarity, let us divide it into six steps.

Step 1: The point (iii) *sequentially.* By the a-priori estimates (3.5) and (3.7) and by a slight modification of Helly's theorem [MTL02, Cor.2.8], we can select a subsequence, indexed for simplicity again by τ, such that $u_\tau \to u$ weakly* in $\mathrm{L}^\infty(0,T,\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n))$ and $\lambda_\tau(t) \to \lambda(t)$ weakly in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)$ for any $t \in [0,T]$, and this limit λ is measurable and belongs to

$\mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^L))$. The a-priori estimate of $\lambda_\tau(t)$ in $L^\infty(\Omega;\mathbb{R}^L)\cap \mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)$, cf. (3.7), then gives the mode of convergence claimed in the point (iii). Besides, we can assume that we selected the subsequence in such a way that also the approximate Gibbs energy $\mathfrak{G}_\tau$, see Proposition 3.1, converges pointwise to some $\mathfrak{G}\in \mathrm{BV}([0,T])$.

Step 2: The point (i) *sequentially.* Fixing t, we already know that $\lambda_\tau(t)\to\lambda(t)$ weakly in $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)$. By (2.31) and in view of the definition of G_τ, we can write $q_\tau(t)\in S(\vartheta(t,\tau))$ for some $\vartheta(t,\tau)\in[t,T]$ such that $\lim_{\tau\to0}\vartheta(t,\tau)=t$; in fact, $\vartheta(t,\tau)$ is $\min_{k\in\mathbb{N}\cup\{0\}}\{k\tau\ge t\}$. Then we can use readily Proposition 3.3 to get the claim (i) for the subsequence selected already in Step 1.

Step 3: The point (ii). The set $\mathrm{L}^\infty_{\mathrm{w}}\big(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n})\big)\cong \mathrm{L}^1\big(\Omega;C_0(\mathbb{R}^{n\times n})\big)^*$ endowed by the weak* topology is a metrizable compact set. Then $X:=\mathrm{L}^\infty_{\mathrm{w}}\big(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n})\big)^{[0,T]}$ endowed by the standard product topology is, by the famous Tikhonov's theorem, compact (although not metrizable since the index set $[0,T]$ is uncountable). Considering $\nu_\tau\equiv\{\nu_\tau(t)\}_{t\in[0,T]}$ as an element of X, there is a finer net, denoted by $\{q_{\tau_\xi}\}_{\xi\in\Xi}$, such that $\{\nu_{\tau_\xi}\}_{\xi\in\Xi}$ converges in X. Let $\nu\equiv\{\nu(t)\}_{t\in[0,T]}$ denotes its limit. Thus, we obtain the convergence claimed at the point (ii), while the others at the points (i) and (iii) are preserved for this finer net, too. Hence, altogether, $\mathrm{w}^*\text{-}\lim_{\xi\in\Xi} q_{\tau_\xi}(t)=q(t)$.

Step 4: $q(t)\in S(t)$ for all t and the point (iv). Let us fix t. When endowed with the weak* topology, bounded sets in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\times\mathrm{L}^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n}))\times\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)$ are metrizable and therefore we can, for a moment, consider a sequence $\{q_{\tau_{\xi_k}}(t)\}_{k\in\mathbb{N}}$, not necessarily finer than $\{q_{\tau_\xi}(t)\}_{\xi\in\Xi}$, that anyhow converges to $q(t)$ weakly*. Using $q_\tau(t)\in S(\vartheta(t,\tau))$ with $\vartheta(\cdot,\cdot)$ as in Step 2 and Proposition 2.7, we can see that $q(t)\in S(t)$. In particular, we thus proved $\lambda(t)=\mathfrak{L}\bullet\nu(t)$, $\nabla u(t)=\mathrm{Id}\bullet\nu(t)$, and $\nu(t)\in\mathcal{G}^p(\Omega;\mathbb{R}^{n\times n})$. Realizing $G_\tau(t,q_\tau(t))\to G(t,q(t))$, like in (2.27), we find $G(t,q(t))\le\mathfrak{G}(t)\le G(t,\tilde q)+R(q(t),\tilde q)$. As we just proved $q(t)\in Q$, we can choose $\tilde q=q(t)$, which results to $G(t,q(t))=\mathfrak{G}(t)$. Hence the point (iv) has been proved, too.

Step 5: The energy (in)equality (2.32) almost everywhere. One can pass to the limit in (3.4) considered with $t_1=0$ and $t_2=t$ with t being some grid-point belonging to some partition of $[0,T]$ so that (3.4) is at our disposal for each finer partition (for the limit passage, we will therefore consider only those partitions, i.e. with τ small enough with respect to these t). Note that the set of such t's is dense in $[0,T]$. We use $\lim_{\xi\in\Xi}G_{\tau_\xi}(t,q_{\tau_\xi}(t))=G(t,q(t))$ proved in Step 4., and similarly for s in place of t. From the pointwise converge of $\lambda_{\tau_\xi}(\cdot)$ and from the definition (2.33) of $\mathrm{Var}(\,\cdot\,;0,t)$, we find $\liminf_{\xi\in\Xi}\mathrm{Var}(q_{\tau_\xi};0,t)\ge\mathrm{Var}(q;0,t)$. Moreover,

$$\begin{aligned}(3.21)\qquad \lim_{\xi\in\Xi}\int_0^t F_{\tau_\xi}\big(\theta,q_{\tau_\xi}(\theta)\big)\,\mathrm{d}\theta &= \lim_{\xi\in\Xi}\int_{[0,t]\times\Gamma} f_{\tau_\xi}(\theta,x)u_{\tau_\xi}(\theta,x)\,\mathrm{d}S\,\mathrm{d}\theta\\ &= \int_{[0,t]\times\Gamma} f(\theta,x)u(\theta,x)\,\mathrm{d}S\,\mathrm{d}\theta=\int_0^t F\big(\theta,q(\theta)\big)\,\mathrm{d}\theta\end{aligned}$$

with f_τ denoting naturally the piecewise constant approximation of f inducing the approximation F_τ. By (2.20), $\|f-f_\tau\|_{\mathrm{L}^1(0,T;\mathrm{L}^\gamma(\Gamma;\mathbb{R}^n))}=\mathcal{O}(\tau)$ giving (3.21), since, by the already proved point (i) and by the continuity of the trace operator $u\mapsto u|_\Gamma:\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\to\mathrm{L}^{\gamma/(\gamma-1)}(\Gamma;\mathbb{R}^n))$, we also have $u_\tau|_{[0,T]\times\Gamma}\to u|_{[0,T]\times\Gamma}$ weakly* in $\mathrm{L}^\infty(0,T;\mathrm{L}^{\gamma/(\gamma-1)}(\Gamma;\mathbb{R}^n))$.

Hence, the reduced work of the external forces $\int_0^t \frac{\partial}{\partial t}F(\theta, q_\tau(\theta-\tau))\,\mathrm{d}\theta$, occuring on the right-hand side of (3.4) converges to the same limit as $\int_0^t \frac{\partial}{\partial t}F(\theta, q_\tau(\theta))\,\mathrm{d}\theta$ occuring on the left-hand side of (3.4), i.e. to $\int_0^t \frac{\partial}{\partial t}F(\theta, q(\theta))\,\mathrm{d}\theta$ by similar argument as already used for (3.21); here we need that the shifted $u_\tau(\cdot-\tau)$ has the same weak* limit as u_τ, which can quite easily be proved by testing it by functions of type $\chi_{[k\tau,l\tau]}(t)g$ which forms a dense subset in $L^1([0,T]\times\Omega;\mathbb{R}^n)$. This holds for any finer net, too. Then, we can pass to the limit in both inequalities in (3.4) with $t_1=0$ and $t_2=t$, proving thus

$$\mathfrak{m}(t) := G\big(t,q(t)\big) - G\big(0,q(0)\big) + \mathrm{Var}(q;0,t) - \int_0^t \frac{\partial F}{\partial\theta}\big(\theta,q(\theta)\big)\,\mathrm{d}\theta = 0 \tag{3.22}$$

at each t of the form $k\tau\in[0,T]$, $k=1,...,T/\tau$, τ from the considered sequence of time steps. The (only countable) set of such t's is dense in $[0,T]$ and thus (3.22) must hold also at each $t\in[0,T]$ at which all functions involved in (3.22) are continuous. Since all functions have a bounded variations, they are continuous with the exception of at most countable number of points. Hence, (3.22) holds everywhere on $[0,T]$ with the only exception of at most countable number of points.

Step 6: The energy (in)equality (2.32) everywhere. As λ is a BV-mapping, it possesses limits from the left and from the limits at each $t\in[0,T]$, in particular at a point ϑ where some function involved in (3.22) is not continuous. Denote $\lambda^-(\vartheta) := \lim_{t\nearrow\vartheta}\lambda(t)$ and $\lambda^+(\vartheta) := \lim_{t\searrow\vartheta}\lambda(t)$. By Proposition 3.3, there are also weak limits $u^-(\vartheta) :=$w*-$\lim_{t\nearrow\vartheta}u(t)$ and $u^+(\vartheta) :=$w*-$\lim_{t\searrow\vartheta}u(t)$. By a-priori estimate (3.6) we can assume (for a subsequence of t's) that also the limits of ν's do exists, so that altogether we have $q^+(\vartheta) =$w*-$\lim_{t\searrow\vartheta}q(t)$ and $q^-(\vartheta) =$w*-$\lim_{t\nearrow\vartheta}q(t)$. Furthermore, put $\mathfrak{G}^-(\vartheta) := \lim_{t\nearrow\vartheta}G(t,q(t))$ and $\mathfrak{G}^+(\vartheta) := \lim_{t\searrow\vartheta}G(t,q(t))$; in fact, these limits exists even without selecting a subsequence as the Gibbs energy has a bounded variation. From Proposition 3.3(ii) we know

$$\mathfrak{G}^+(\vartheta) = G\big(\vartheta,q^+(\vartheta)\big) \quad\text{and}\quad \mathfrak{G}^-(\vartheta) = G\big(\vartheta,q^-(\vartheta)\big). \tag{3.23}$$

As we have proved $q(\vartheta)\in S(\vartheta)$ in Step 4, putting $\tilde q := q^+(\vartheta)$ into (2.31) written, of course, for $t=\vartheta$, we obtain

$$G\big(\vartheta,q(\vartheta)\big) \le G\big(\vartheta,q^+(\vartheta)\big) + R\big(q(\vartheta),q^+(\vartheta)\big). \tag{3.24}$$

Likewise, by Proposition 2.7 also $q^-(\vartheta)\in S(\vartheta)$ and thus, together with (3.24),

$$\begin{aligned}\mathfrak{G}^-(\vartheta) &= G\big(\vartheta,q^-(\vartheta)\big) \le G\big(\vartheta,q(\vartheta)\big) + R\big(q^-(\vartheta),q(\vartheta)\big)\\ &\le \mathfrak{G}^+(\vartheta) + R\big(q^-(\vartheta),q(\vartheta)\big) + R\big(q(\vartheta),q^+(\vartheta)\big).\end{aligned} \tag{3.25}$$

By definition (2.33), we have $\mathrm{Var}(q;s,t) = \mathrm{Var}(q;s,\vartheta) + \mathrm{Var}(q;\vartheta,t)$ for $s<\vartheta<t$. Moreover, $\lim_{s\nearrow\vartheta}\mathrm{Var}(q;s,\vartheta) = R(q^-(\vartheta),q(\vartheta))$ and $\lim_{t\searrow\vartheta}\mathrm{Var}(q;\vartheta,t) = R(q(\vartheta),q^+(\vartheta))$. Passing to the limit in (3.22) and using (3.23) we obtain

$$\mathfrak{G}^+(\vartheta) - \mathfrak{G}^-(\vartheta) + R\big(q^-(\vartheta),q(\vartheta)\big) + R\big(q(\vartheta),q^+(\vartheta)\big) = 0, \tag{3.26}$$

which shows that (3.25) and hence (3.24) are in fact equalities. For $\vartheta>0$ we find

$$\mathfrak{m}(\vartheta) - \lim_{s\nearrow\vartheta}\mathfrak{m}(t) = G\big(\vartheta,q(\vartheta)\big) - \mathfrak{G}^-(\vartheta) + R\big(q^-(\vartheta),q(\vartheta)\big) = 0.$$

Hence $\mathfrak{m}(s,\cdot)$ is proved to be continuous from the left. Similarly, we find

$$\lim_{t\searrow\vartheta}\mathfrak{m}(t)-\mathfrak{m}(\vartheta)=\mathfrak{G}^+(\vartheta)-G(\vartheta,q(\vartheta))+R(q(\vartheta),q^+(\vartheta))=0,$$

which is continuity from the right. Together with (3.22) we conclude $\mathfrak{m}(t)=0$ for all $t\in[0,T]$. However, the claimed energy equality (3.20) is equivalent to $\mathfrak{m}(t)=\mathfrak{m}(s)$ and thus is established. □

Remark 3.5. *(Sequential approach.)* One might be tempted to formulate more conventionally (and explicitly) Theorem 3.4 in terms of sequences and to obtain measurability of ν in time, too. This attempt, however, brings the following trouble. From a weak* convergence $\nu_\tau\to\nu\equiv\{\nu_{t,x}\}_{(t,x)\in[0,T]\times\Omega}$ in $\mathrm{L}^\infty_{\mathrm{w}}([0,T]\times\Omega;\mathrm{rca}(\mathbb{R}^{n\times n}))$, one cannot deduce convergence of $\nu_\tau(t)$ for a.a. t. To show that $\nu(t)\in\mathcal{G}^p(\Omega;\mathbb{R}^{n\times n})$, by Kinderlehrer and Pedregal's results [KiPe91, KiPe94], one must ultimately show Jensen's inequality

$$\forall v{\in}C^{(p)}(\mathbb{R}^{n\times n})\text{ quasiconvex, nonnegative}\qquad\int_{\mathbb{R}^{n\times n}}v(A)\nu_{t,x}(\mathrm{d}A)\ge v(\mathrm{Id}\bullet\nu_{t,x})=v\big(\nabla u(t,x)\big),\tag{3.27}$$

for a.a. $(t,x)\in[0,T]\times\Omega$, cf. the definition (2.5). As $\nu_\tau(t)\in\mathcal{G}^p(\Omega;\mathbb{R}^{n\times n})$ for all $t{\in}[0,T]$, (3.27) holds with ν_τ and u_τ in place of ν and u, respectively. In particular, for all $O\subset\Omega$ measurable, it holds

$$\int_O\int_{\mathbb{R}^{n\times n}}v(A)\big[\nu_\tau\big]_{t,x}(\mathrm{d}A)\,\mathrm{d}x\ge\int_O v(\nabla u_\tau(t,x))\,\mathrm{d}x\tag{3.28}$$

for all $v{\in}C^{(p)}(\mathbb{R}^{n\times n})$ quasiconvex and positive and all $t\in[0,T]$. Our goal now is to pass to the limit with $\tau\searrow0$ but we can rely only on the weak* convergence of $\nu_\tau\to\nu$ if restricted on a fix interval $[t_1,t_2]$. For a.a. $t\in[0,T]$, we know that, by Theorem 3.4(i), $u_\tau(t)\to u(t)$ weakly. As $v\in C^{(p)}(\mathbb{R}^{n\times n})$ is quasiconvex and nonnegative, one has weak lower semicontinuity of $u\mapsto\int_O v(\nabla u)\,\mathrm{d}x$ at disposal [KiPe92], i.e.

$$\forall t\in[0,T]:\qquad\liminf_{\tau\to0}\int_O v\big(\nabla u_\tau(t,x)\big)\,\mathrm{d}x\ge\int_O v\big(\nabla u(t,x)\big)\,\mathrm{d}x.\tag{3.29}$$

Then, by (3.28) and (3.29) and by Fatou's lemma, for all $0\le t_1<t_2\le T$ one can estimate

$$\begin{aligned}\lim_{\tau\to0}\int_{t_1}^{t_2}\int_O\int_{\mathbb{R}^{n\times n}}v(A)\big[\nu_\tau\big]_{t,x}(\mathrm{d}A)\,\mathrm{d}x\,\mathrm{d}t&\ge\liminf_{\tau\to0}\int_{t_1}^{t_2}\int_O v\big(\nabla u_\tau(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t\\&\ge\int_{t_1}^{t_2}\bigg(\liminf_{\tau\to0}\int_O v\big(\nabla u_\tau(t,x)\big)\,\mathrm{d}x\bigg)\,\mathrm{d}t\ge\int_{t_1}^{t_2}\int_O v\big(\nabla u(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.30}$$

However, then one would still need to estimate the left-hand side in (3.30) from above by $\int_{t_1}^{t_2}\int_O\int_{\mathbb{R}^{n\times n}}v(A)\nu_{t,x}(\mathrm{d}A)\,\mathrm{d}x\,\mathrm{d}t$ and then (3.27) follows because O, t_1 and t_2 can vary arbitrarily. Unfortunately, while just the opposite estimate and the sought estimate for v with growth less than p are obvious, the proof for a general $v\in C^{(p)}(\mathbb{R}^{n\times n})$ would require a uniform (in t, at least in an integrable sense) nonconcentration of the energy of a sequence attaining $\nu(t)$. This seems difficult.

On the other hand, the possible gain of measurability of ν in time would bring a loss of the (Moore-Smith) convergence of $\nu_\tau(t)$ for all t.

4. Appendix

Following and modifying [Rou97], we will present a suitable locally compact extension of $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$. This gives a certain general viewpoint on the problem itself which can be extended continuously on this extension and offers some fine results needed.

For this, let us take $H \subset \mathrm{Car}^p(\Omega;\mathbb{R}^{n\times n})$ a suitable normed, separable, linear space of some Carathéodory integrands $h : \Omega\times\mathbb{R}^{n\times n} \to \mathbb{R}$ with p-growth. More in detail, $h : \Omega\times\mathbb{R}^{n\times n} \to \mathbb{R}$ is called a Carathéodory integrand if $h(x,\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous for a.a. $x \in \Omega$ and $h(\cdot, A) : \Omega \to \mathbb{R}$ is measurable for all $A \in \mathbb{R}^{n\times n}$ and

$$\mathrm{Car}^p(\Omega;\mathbb{R}^{n\times n}) := \Big\{h : \Omega\times\mathbb{R}^{n\times n} \to \mathbb{R};\ \text{Carathéodory},\ \exists a\in \mathrm{L}^1(\Omega)\ \exists b\in\mathbb{R} :\ |h(x,A)| \le a(x) + b|A|^p\Big\}, \tag{4.1}$$

which can be normed naturally by putting $\|h\| := \inf \|a\|_{\mathrm{L}^1(\Omega)} + b$ where the infimum is taken over all (a,b) satisfying, for the integrand h in question, the estimate in (4.1). E.g., the space

$$H = \Big\{(x,A) \mapsto a_1\phi(x,A) + a_2|A|^p + g_1(x)\cdot\lambda(x,A) + g_2(x){:}A + h;\ a_1,a_2\in\mathbb{R},\ g_1\in\mathrm{L}^1(\Omega;\mathbb{R}^L),\ g_2\in\mathrm{L}^{p/(p-1)}(\Omega;\mathbb{R}^{n\times n}),\ h\in\mathrm{L}^1\big(\Omega;C_0(\mathbb{R}^{n\times n})\big)\Big\} \tag{4.2}$$

is separable in $\mathrm{Car}^p(\Omega;\mathbb{R}^{n\times n})$, contains all possible nonlinearities which can appear in our problem (3.1) or which we need for its analysis, in particular all test-functions for Young measures, i.e. $\mathrm{L}^1\big(\Omega;C_0(\mathbb{R}^{n\times n})\big)$. The notation in (4.2) is standard: the dot means scalar product of vectors while $g{:}A$ means $\sum_{i,j=1}^n h_{ij}(x)A_{ij}$.

Then we define the embedding $j_H : \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n) \to \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\times H^*\times\mathrm{L}^\infty(\Omega;\mathbb{R}^L)$ by putting $j_H(u) = (u, i_H(\nabla u), \mathfrak{L}(\nabla u))$ where $i_H : \mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}) \to H^*$ is defined as $h \mapsto \int_\Omega h(x,\nabla u(x))\,\mathrm{d}x$. Taking into account the assumed coercivity of at least one integrand from H, here e.g. φ or $1\otimes|\cdot|^p$, the set of "generalized Young functionals" $Y_H^p(\Omega;\mathbb{R}^{n\times n})$, defined as the weak* closure of $i_H(\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n}))$, endowed with the weak* topology of H^*, is a *convex, locally sequentially compact envelope* of the Lebesgue space $\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})$; cf. [Rou97]. Moreover, i_H is a (norm,weak*)-*homeomorphical embedding*.

Furthermore, we set

$$\bar{Q} := \text{w*-cl}\, j_H\big(\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\big) = \Big\{(u,\eta,\lambda)\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\times H^*\times\mathrm{L}^\infty(\Omega;\mathbb{R}^l);\ \eta\in G_H^p(\Omega;\mathbb{R}^{n\times n}),\ \mathrm{Id}\bullet\eta = \nabla u\ ,\mathfrak{L}\bullet\eta = \lambda\Big\} \tag{4.3}$$

where the set of "gradient generalized Young functionals" is defined via $G_H^p(\Omega;\mathbb{R}^{n\times n}) := \{\eta \in Y_H^p(\Omega;\mathbb{R}^{n\times n});\ \eta = \text{w*-}\lim_{k\to\infty} i_H(\nabla u_k),\ u_k \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)\}$, the "product" $\mathrm{Id}\bullet\eta \in \mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})$ is defined by $\langle\mathrm{Id}\bullet\eta, g\rangle := \langle\eta, g{:}\mathrm{Id}\rangle$ for any $g \in \mathrm{L}^{p/(p-1)}(\Omega;\mathbb{R}^{n\times n})$, and $\mathfrak{L}\bullet\eta \in \mathrm{L}^\infty(\Omega;\mathbb{R}^L)$ is defined analogously. The extension $\bar{Q}$ forms a locally sequentially compact envelope of the Sobolev space $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$. However, $\bar{Q}$ is not convex for $n \ge 2$, which is the case of our interest. As H from (4.2) contains $\mathrm{L}^1(\Omega;C_0(\mathbb{R}^{n\times n}))$, we can consider Q as a subset of $\bar{Q}$ because every L^p-Young measure ν determines a functional $\eta \in Y_H^p(\Omega;\mathbb{R}^{n\times n})$

by

$$\langle \eta, h\rangle = \int_\Omega \int_{\mathbb{R}^{n\times n}} h(x,A)\nu_x(\mathrm{d}A)\,\mathrm{d}x. \tag{4.4}$$

Conversely, if η is *p-nonconcentrating* in the sense that $\eta = \text{w*-}\lim_{k\to\infty} i_H(y_k)$ for some sequence $\{y_k\}_{k\in\mathbb{N}} \subset \mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})$ such that $\{|y_k|^p\}_{k\in\mathbb{N}}$ is relatively weakly compact in $\mathrm{L}^1(\Omega)$, then η admits only one $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$ that determines it by this way; cf. [Rou97] for more details.

Lemma 4.1. (Kristensen [Kri94], Fonseca, Müller, Pedregal [FMP98], reformulated.) *Any $\bar q = (u,\eta,\lambda) \in \bar Q$ admits a p-nonconcentrating modification $q = (u,\mathring\eta,\lambda) \in \bar Q$ in the sense that $\mathring\eta$ is p-nonconcentrating and $\langle \eta, h\rangle = \langle \mathring\eta, h\rangle$ for any $h \in H$ with growth less than power p, i.e. $\lim_{|A|\to\infty} \|h(\cdot,A)\|_{\mathrm{L}^\infty(\Omega)}/(1+|A|^p) = 0$.*

This important assertion is based on a modification on a set with small Lebesgue measure (in fact, converging to 0) of a sequence $\{u_k\}_{k\in\mathbb{N}} \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$ whose gradients attain $\eta \in G^p_H(\Omega;\mathbb{R}^{n\times n})$ such that $\{|\nabla u_k|^p\}_{k\in\mathbb{N}}$ is relatively weakly compact in $\mathrm{L}^1(\Omega)$; this was done in [Kri94, Theorem 3.10] and later, independently and more explicitly, in [FMP98, Lemma 1.2]. This modified sequence just attains $\mathring\eta$ and, as Id and $\mathfrak{L}$ have certainly a growth strictly less than power p, it holds $\mathrm{Id}\bullet\eta = \mathrm{Id}\bullet\mathring\eta$ and $\mathfrak{L}\bullet\eta = \mathfrak{L}\bullet\mathring\eta$.

The problem (3.1) can be continuously extended onto $\bar Q$ by

$$\begin{cases} \text{Minimize} \quad \bar V(\bar q) + R(q_\tau^{k-1},\bar q) - F(k\tau,\bar q) \\ \text{subject to } \ \bar q \equiv (u,\eta,\lambda) \in \bar Q\ , \end{cases} \tag{4.5}$$

with $\bar V(u,\eta) = \langle \eta, 1\otimes\varphi\rangle + \rho\|\nabla\lambda\|^2_{\mathrm{L}^2(\Omega;\mathbb{R}^{n\times L})} + \|\alpha\cdot u\|^2_{\mathrm{L}^2(\Gamma;\mathbb{R}^n)}$, while R and F are as before because these functionals act only on λ's and u's, respectively.

Lemma 4.2. *The extended problem (4.5) has a solution $\bar q_\tau^k = (u_\tau^k,\eta_\tau^k,\lambda_\tau^k)$ and every η_τ^k obtained by this way has an L^p-Young measure representation $\nu_\tau^k \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$ in the sense of (4.4) with η_τ^k and ν_τ^k instead of η and ν, respectively, and $q_\tau^k = (u_\tau^k,\nu_\tau^k,\lambda_\tau^k)$ solves (3.1).*

Sketch of the proof. Existence of a solution to (4.5), let us denote it by $\bar q_\tau^k = (u_\tau^k,\eta_\tau^k,\lambda_\tau^k) \in \bar Q$, follows by standard arguments from the coercivity and weak* lower semicontinuity of the functional $\bar V$ minimized on the locally compact set $\{q\in\bar Q;\ \mathfrak{L}\bullet\eta \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)\}$ equipped with the topology projectively induced from the weak* topology of $\bar Q$ and, via $\eta \mapsto \mathfrak{L}\bullet\eta$, by the weak topology on $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^L)$. Here, $\mathfrak{L}\bullet\eta$ is naturally defined by the formula $\int_\Omega (\mathfrak{L}\bullet\eta)g\,\mathrm{d}x := \langle \eta, g\cdot\mathfrak{L}\rangle$ for all $g \in \mathrm{L}^1(\Omega;\mathbb{R}^L)$. Considering the p-nonconcentrating modification $\mathring\eta_\tau^k$ of η_τ^k from Lemma 4.1, and assuming that η_τ^k itself is not p-nonconcentrating, by the coercivity (2.2) we get $\langle \eta_\tau^k,\varphi\rangle > \langle \mathring\eta_\tau^k,\varphi\rangle$ (see [Rou97, Lemma 4.2.3(ii)]), hence $(u_\tau^k,\mathring\eta_\tau^k,\lambda_\tau^k)$ is an admissible state for (4.5) yielding a lower energy than $\bar q_\tau^k$, a contradiction. Hence $\eta_\tau^k = \mathring\eta_\tau^k$, so η_τ^k itself is p-nonconcentrating, and therefore has an (even unique) L^p-Young measure representation ν_τ^k. As $Q \subset \bar Q$, $q = (u_\tau^k,\nu_\tau^k,\lambda_\tau^k)$ solves (3.1). □

Lemma 4.3. *Assume the situation in Proposition 2.7, in particular $q_k \to q$ weakly*, $q_k \in S(t_k)$, $t_k \to t$. Then $\nu \in \mathcal{G}^p(\Omega;\mathbb{R}^{n\times n})$.*

Sketch of the proof. Let us consider q_k embedded into $\bar{Q}$ by the formula (4.4) with ν_k instead of ν. By estimate (2.30), we can select a subsequence such that $q_k \to \bar{q} = (u,\eta,\lambda) \in \bar{Q}$ weakly*. Passing to the limit in $q_k \in S(t_k)$, like (2.27) we obtain $\bar{V}(\bar{q}) - F(t,\bar{q}) \le \bar{V}(\tilde{q}) + R(\bar{q},\tilde{q}) - F(t,\tilde{q})$ for any $\tilde{q} \in Q$, and thus by continuity even for any $\tilde{q} \in \bar{Q}$. This is rather trivial due to the definition (2.14) of $\bar{Q}$ by closure. Fixing λ, we can therefore see that, in particular, (u,η) minimizes the functional $(\tilde{u},\tilde{\eta}) \mapsto \bar{V}(\tilde{u},\tilde{\eta},\lambda) - \int_\Gamma f(t,\cdot)\tilde{u}\,\mathrm{d}S$ over $\{(\tilde{u},\tilde{\eta});\ (\tilde{u},\tilde{\eta},\lambda) \in \bar{Q}\}$. Now, again by Lemma 4.1 and by the contradiction argument as in the proof of Lemma 4.2, we can see that η must be p-nonconcentrating, and therefore has an (even unique) L^p-Young measure representation ν. By Lemma 4.1, this ν can be generated by gradients. □

Remark 4.4. *(Doubly nonlinear inclusion.)* Let us mention a relation with an inclusion of the type (1.2) in the scalar case $n = 1$. To have a link to [Rou02], we formulate it in terms of a "reduced" state $\mathfrak{q} = (u,\eta)$; our original "full" state is $\bar{q} = (\mathfrak{q}, \Lambda\mathfrak{q})$ with $\Lambda\mathfrak{q} := \mathfrak{L} \bullet \eta$. Thus $\Lambda : \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n) \times H^* \to \mathrm{L}^\infty(\Omega;\mathbb{R}^L)$ is the linear operator which is adjoint to $\Lambda' : \mathrm{L}^1(\Omega;\mathbb{R}^L) \to \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)^* \times H : g \mapsto (0, g\otimes\lambda)$. Using the notation $\mathfrak{R}(\lambda) = \int_\Omega |\lambda|_L \,\mathrm{d}x$ (hence $R(\bar{q}_1,\bar{q}_2) \equiv R(u_1,\eta_1,\lambda_1,u_2,\eta_2,\lambda_2) = \mathfrak{R}(\lambda_1-\lambda_2)$), $\mathfrak{V}(\mathfrak{q}) := \bar{V}(\mathfrak{q},\Lambda\mathfrak{q})$, $\mathfrak{Q} := \big\{(u,\eta) \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n) \times H^*;\ \eta \in G^p_H(\Omega;\mathbb{R}^{n\times n}),\ \mathrm{Id} \bullet \eta = \nabla u\big\}$, and $\langle \mathfrak{F}(t), \mathfrak{q}\rangle := F(t,q)$, we can formulate the doubly nonlinear inclusion as in [Rou02]:

$$\Lambda^* \partial\mathfrak{R}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Lambda\mathfrak{q}\Big) + \partial[\mathfrak{V} + \delta_{\mathfrak{Q}}](\mathfrak{q}) \ni \mathfrak{F}(t), \tag{4.6}$$

where ∂ denotes the subdifferential and $\delta_{\mathfrak{Q}}$ denotes the indicator function of $\mathfrak{Q}$ so that $\partial\delta_{\mathfrak{Q}}(\mathfrak{q})$ is just the normal cone to $\mathfrak{Q}$ at $\mathfrak{q}$.

In [Rou02], $\mathfrak{q} \in \mathrm{L}^\infty_{\mathrm{w}}(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n) \times H^*)$ was called a weak solution to (4.6) if there exist $\omega \in \mathrm{L}^\infty([0,T]\times\Omega;\mathbb{R}^L)$, $z \in \mathrm{L}^\infty_{\mathrm{w}}(0,T;B^*)$, and $\mu \in \mathrm{vba}([0,T]\times\Omega;\mathbb{R}^L) \cong \mathrm{L}^\infty([0,T]\times\Omega;\mathbb{R}^L)^*$, also the limits $\lim_{t\searrow 0} \mathfrak{G}(t) = \mathfrak{G}(0)$ and $\lim_{t\nearrow T} \mathfrak{G}(t) = \mathfrak{G}(T)$ with the Gibbs energy $\mathfrak{G}(t) = \mathfrak{V}(\mathfrak{q}(t)) - \langle \mathfrak{F}(t), \mathfrak{q}(t)\rangle$ do exist, $\Lambda'\omega + z = \mathfrak{F}$ with $\mu|_{C([0,T]\times\Omega;\mathbb{R}^L)} = \Lambda\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{q}$ in the sense of distributions, and

$$\int_0^T \left(\langle \xi, v\rangle - \langle \omega, v\rangle - \Big\langle \frac{\mathrm{d}\mathfrak{F}}{\mathrm{d}t}, \mathfrak{q}\Big\rangle \right) \mathrm{d}t - \langle \mu, \xi\rangle \ge \mathfrak{G}(T) - \mathfrak{G}(0) \tag{4.7}$$

for all $v \in \mathrm{L}^1([0,T]\times\Omega;\mathbb{R}^L)$ and $\xi \in \mathrm{L}^\infty([0,T]\times\Omega;\mathbb{R}^L)$ such that $\xi(t) \in \partial\mathfrak{R}\big(v(t)\big)$ for a.a. $t \in [0,T]$, and

$$\int_0^T \langle z - \xi, \mathfrak{q} - v\rangle \,\mathrm{d}t \ge 0 \tag{4.8}$$

for all $v \in \mathrm{L}^1(0,T;\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n) \times H^*)$ and $\xi \in \mathrm{L}^\infty_{\mathrm{w}}(0,T;(\mathrm{W}^{1,p}(\Omega;\mathbb{R}^n) \times H^*)^*)$ such that $\xi(t) \in \partial[\mathfrak{V} + \delta_{\mathfrak{Q}}](v(t))$ for a.a. $t \in [0,T]$. In [Rou02], such weak solution has been proved to exists generically with respect to T, i.e. for a.a. $T > 0$, and it has been shown (see [Rou02, Prop.1]) that this weak solution indeed solves (4.6) in a classical sense at least if $\mathfrak{G}$, $\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{F}, \mathfrak{q}\rangle$, and μ are absolutely continuous (as functions or as a measure, respectively). In [Rou02, Proof of Lemma 3] it has further been shown (even for multidimensional but scalar problems) that, for a.a. $t \in [0,T]$, $\mathfrak{q}(t)$ minimizes the functional $\tilde{\mathfrak{q}} \mapsto \mathfrak{V}(\tilde{\mathfrak{q}}) - \langle \mathfrak{F}(t), \tilde{\mathfrak{q}}\rangle + \|\Lambda\tilde{\mathfrak{q}} - L(t)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^L)}$ on $\mathfrak{Q}$ with $L(t) = \Lambda\mathfrak{q}(t)$ considered fixed, which is just the stability condition (2.31). Moreover, the energy inequality (2.32) follows from (4.7)–(4.8)

at least under the above mentioned absolute-continuity assumption; cf. [Rou02, Formula (3.11)].

The converse relation, i.e. how (2.31)–(2.32) leads to the doubly-nonlinear inclusion of the type (4.6) has, in a general context, been investigated in [MTL02, Theorem 2.3].

Remark 4.5. *(Maximum-dissipation principle.)* The dissipation mechanism through the convex, homogeneous potential $\mathfrak{R}$ is intimately related with Hill's maximum-dissipation principle [Hil48]. In fact, (4.6) can be written as the system of two inclusions: $\Lambda^*\omega+\partial[\mathfrak{V}+\delta_{\mathfrak{Q}}](\mathfrak{q}) \ni \mathfrak{F}$ and $\omega \in \partial\mathfrak{R}(\frac{\mathrm{d}}{\mathrm{d}t}\Lambda\mathfrak{q})$. Denoting $\lambda := \Lambda\mathfrak{q}$, the latter relation means equivalently that $\langle \omega - z, \frac{\mathrm{d}}{\mathrm{d}t}\lambda - v\rangle \geq 0$ for all pairs (z, v) such that $z \in \partial\mathfrak{R}(v)$. In particular, for $v = 0$ one gets the statement about maximum dissipation:

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\lambda, \omega \right\rangle = \max_{z\in\partial\mathfrak{R}(0)} \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\lambda, z \right\rangle . \tag{4.9}$$

This says that, for the considered volume-fraction rate $\frac{\mathrm{d}}{\mathrm{d}t}\lambda$, the driving energies ω makes the dissipation caused by the PTs maximal among all other admissible driving energies, i.e. those from $\partial\mathfrak{R}(0)$. In plasticity theory, this maximum-dissipation principle can alternatively be expressed as a normality in the sense that the rate of plastic deformation belongs to the cone of outward normals to the elasticity domain. Here, this would result in the observation that the rate $\frac{\mathrm{d}}{\mathrm{d}t}\lambda$ of PTs belongs to the normal cone of the "*elasticity domain*" $\partial\mathfrak{R}(0)$ at the point ω. In particular, (4.9) says that $\frac{\mathrm{d}}{\mathrm{d}t}\lambda = 0$ (i.e. the volume fractions do not change) if ω is inside $\partial\mathfrak{R}(0)$ (i.e. there in not enough stress to activate PT). Also recall that $\partial\mathfrak{R}$ is maximal responsive in the sense of [ERR90].

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