Essential Manifolds for an Elliptic Problem in an Infinite Strip

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1 Introduction

In this paper we develop a new tool for the study of elliptic problems on a two–dimensional strip. In particular, we consider the semilinear Laplace equation

$$u_{xx} + u_{yy} - 2\alpha u_x - \beta u + f(x, y, u) = 0 \quad \text{on } \Omega = \mathbf{R} \times (0, \pi). \tag{1.1}$$

On the boundary we can prescribe homogeneous Dirichlet, Neumann, or periodic boundary conditions.

The general aim is to study all solutions of the problem which are bounded over all of Ω. Many local and global bifurcation methods are available, but most of them are restricted to special classes of functions, e.g., periodic in x. Here we show that *all bounded solutions* can be embedded into a finite–dimensional invariant manifold. According to [BM91] we call the set of all bounded solutions the *essential set* of (1.1); and a finite–dimensional invariant manifold containing the essential set is called an *essential manifold* for (1.1). Thus, the solutions in the essential set can be described by a reduced problem on the essential manifold, which is an ordinary differential equation.

The notation is in analogy to the *inertial manifold* for parabolic systems [FNST88, FST88, FST89] and damped hyperbolic systems [MS87, MS88]. However, essential manifolds are not exponentially attractive like the inertial manifolds. However, we will establish a generalization, the so–called *weak normal hyperbolicity.*

The methods are closely related to the center manifold approach for elliptic problems in cylindrical domains. In [Ki82] it was first shown how methods of the theory of dynamical systems are applicable to such elliptic problems. Out of this a whole theory of center manifolds for elliptic systems emerged. Bifurcation of small bounded solutions from the zero state can thus be studied by analyzing the reduced ODE on the finite–dimensional center manifold. In [Mi90] the exponential decay properties of general solutions close to the center manifold were established; one natural application of this weak normal hyperbolicity leads to the Saint–Venant principle in nonlinear elasticity. This dynamical approach was carried further by [BM91] and [CMS90], where a corresponding theory of “attractors” for semi–bounded solutions was developed.

Knowing that the bounded solutions in the essential set can be described by an ODE rather than a PDE leads to a lot of principal advantages. For instance, very often the existence of travelling waves in cylinders is considered [Ga86, CMV89, He89, BMPS91], where especially solutions are sought which connect two different types of states at infinity. For ODEs this just means a heteroclinic orbit, and many tools are available to study these. To do the same for the full elliptic problem we encounter the difficulty that the elliptic problem is infinite dimensional and is not of evolutionary type. But reducing the system to the finite–dimensional essential manifold one can study the system like an ODE. The case $\alpha = 0$ has applications in the theory of travelling waves in hyperbolic problems, e.g. for internal gravity waves [AT83, Ki82, BBT83, Mi86b]. Similarly, for the study of periodic orbits there are global topological methods [Fi91] which are up to now only valid in finite dimensions. Using the essential manifold for the elliptic problem we thus make available many ODE tools which are not available for PDEs. We discuss some applications in Section 6.

First results the existence of an essential manifold for (1.1) were obtained in [Mi91b], however only in the case $\alpha = 0$ and f independent of x. There, more general elliptic problems of higher order are treated, while the method developed here gives improved estimates for the dimension of the essential manifold. We remark, that our method works equally well for weakly coupled reaction diffusions systems, but for simplicity we demonstrate the method for one equation only.

The main restriction is that the domain has to be a strip rather than a cylinder $\mathbf{R} \times \Sigma$ where Σ is a smooth bounded domain in $\mathbf{R}^d$ with $d \geq 2$. It is the same restriction appearing in parabolic problems. It is intimately related to the so–called *gap condition* for the dominant linear operator of the problem. Writing (1.1) as a first order system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = L\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -f \end{pmatrix}, \quad \text{with } L = \begin{pmatrix} 0 & I \\ -\partial_y^2 + \beta & 2\alpha \end{pmatrix}, \tag{1.2}$$

we see that L has the eigenvalues $\sigma_{\pm n} = \alpha \pm \sqrt{\alpha^2 + \lambda_n}$, where λ_n are the eigenvalues of $-\partial_y^2 + \beta$ with the appropriate boundary conditions. Since $\lambda_n \approx n^2$ we see that the operator L satisfies the *gap condition of order* γ

$$\limsup_{n \to \pm\infty} |\sigma_n|^\gamma |\sigma_{n+1} - \sigma_n| = \infty. \tag{1.3}$$

with any $\gamma > 0$. For parabolic systems it is well–known that $\gamma \leq 0$ is needed, where γ has to be the smaller the more derivatives appear in the nonlinearity. For the problem at hand it is sufficient that L satisfies the gap condition of order $\gamma = 1/2$. This is due to the fact that the system is second order in x, implying that the Green's function has smoothing properties.

The plan of the paper is as follows: In Section 2 consider an abstract version of (1.1) and derive certain bounds for the essential set. We use an abstract version of the maximum principle due to [CMS90] as well as the classical one, both involve certain sign conditions for f for large u. In Section 3 we discuss the question of modifying the nonlinear part with the help of cut–off functions. Here it is important to do the cut–off such that the modified problem has still the same essential set as the original problem. Finally we derive explicit bounds on the Lipschitz constant of the modified nonlinearity. These will enable us to give bounds on the dimension of the essential manifold.

The construction of the essential manifold M_E is carried through in Section 4. The proof stays very close to the center manifold proof in [Ki82, Mi86a] and relies on the gap condition. We show that M_E is a C^1–manifold such that the reduced ODE on M_E is of the form

$$u_0 - 2\alpha u_{0x} - A_0 u_0 + F_{red}(x, u_0, u_{0x}) = 0,$$

where $u_0 \in E_0$, $F_{red} \in C^1(E_0 \times E_0, E_0)$, and E_0 is a finite–dimensional subspace of $L_2(0,\pi)$.

For specific examples, we give concrete estimates on the dimension of M_E: In the case $f(u) = \kappa^2 u - u^3$ with $\kappa > 3$ and $\beta = 1$ we find $\dim M_E \leq 574\kappa^{5/2}\sqrt{\log\kappa}$. For $f(u) = -u^3 + g(x,y)$ with $|g(x,y)|, |\partial_y g(x,y)| \leq \gamma$ we obtain $\dim M_E \leq C\gamma^{5/6}\log\gamma$ with some $C > 0$. And for $f(u) = \sigma \sin u$, the estimate $\dim M_E \leq C\sigma^2$ holds.

In Section 5 we work out the relevance of the essential manifold for the study of solution on the finite strip $\Omega_\ell = (-\ell,\ell) \times (0,\pi)$ when ℓ is large. First we show, again using growth and sign conditions at infinity, that there is a ball $\mathcal{B}_R \subset H^1(0,\pi) \times L_2(0,\pi)$ such that for every solution the distance of $(u(x), u_x(x))$ from $\mathcal{B}_R$ decays in an exponential way with the distance from both ends, $\ell - |x|$. In particular, the whole essential set is contained in $\mathcal{B}_R$. For large ℓ all solutions stay inside the ball $\mathcal{B}_{2R}$ for most values of x.

Inside this larger ball the solutions can now be very well approximated by solutions on the essential manifold. In analogy to the exponential attractivity of inertial manifolds we establish the property of *weak normal hyperbolicity* for essential manifolds (cf. [Mi90]). We show that there is a constant $C > 0$ such that for every solution $u : \Omega_\ell \to \mathbf{R}$ with $(u(x), u_x(x)) \in \mathcal{B}_{2R}$, $|x| \leq \ell$, there is a solution $\widetilde{u}$ on the essential manifold, such that the following two–sided exponential estimate holds:

$$\|u(x) - \widetilde{u}(x)\|_{H^1(0,\pi)} \leq C \max\{e^{-(\mu_N+\alpha)(\ell-1-x)}, e^{-(\mu_N-\alpha)(\ell-1+x)}\} \quad \text{for } |x| \leq \ell - 1,$$

where $\mu_N = \sqrt{\lambda_N + \alpha^2}$ and $2N + 1 = \dim M_E$. Note that this provides a very good approximation if ℓ is large and if we stay away from the ends $x = \pm\ell$ of the finite strip. Moreover, the decay rate can be made as large as we like, if the dimension of the essential manifold is increased. For solutions existing on the half–strip $\Omega_+ = (0,\infty) \times (0,\pi)$ we show exponential convergence towards a solution on the essential manifold, where the decay rate is given as $\mu_N - \alpha$.

In the last section we give three applications. First we treat the case $\alpha \neq 0$ and f independent of x, which is important when studying travelling waves (with speed α) for the parabolic problem $u_t = \Delta u + f(y,u)$. Using the Lyapunov function

$$V(u,u_x) = \int_0^\pi \left[\frac{1}{2}(u_x^2 - u_y^2 - \beta u^2) + g(x,u)\right] dy \quad \text{with } g(y,u) = \int_0^u f(y,v)\, dv$$

it is shown in [BMPS91] that all semi–bounded solution $u : \Omega_+ \to \mathbf{R}$ approach an equilibrium for $x \to \infty$. We give an alternative proof by reduction onto the essential manifold. Then the ODE results of [Au84] can be employed. As a consequence we find that all bounded solutions are heteroclinic.

In the case $\alpha = 0$ we use the reversibility property coming from the reflection $x \to -x$. As the reduced system is again reversible and C^1, we are able to apply the index theory of [Fi91]. For a simple example we establish the existence of periodic orbits for every period above a critical one.

Finally we use the function V in the case $\alpha = 0$ as a conserved quantity along any solution. In fact, V can be understood as a Hamiltonian function in the sense of [Mi91a]. We show that also the reduced system is a finite–dimensional Hamiltonian system on the essential manifold with a non–degenerate symplectic form. However, the low degree of smoothness of M_E renders the analysis more difficult and it is not clear how standard results from Hamiltonian systems theory can be applied in this context.

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2 Abstract formulation and a–priori estimates

The methods we will use are mainly functional analytical. We use the Hilbert spaces $H = L_2(0,\pi)$ and $E = H_B^1$, where H_B^1 is equal $H_D^1 = H_0^1(0,\pi)$, $H_N^1 = H^1(0,\pi)$, and $H_P^1 = \{\, u \in H^1(0,\pi) \;:\; u(0) = u(\pi)\,\}$ for Dirichlet, Neumann, and periodic boundary conditions, respectively.

The linear operator A is defined by $A : D(A) = H_B^2 \to H; u \mapsto -u_{yy} + \beta u$, where $H_D^2 = H^2(0,\pi) \cap H_D^1$, $H_N^2 = \{\, u \in H^2(0,\pi) \;:\; u'(0) = u'(\pi) = 0\,\}$, and $H_P^2 = \{\, u \in H^2(0,\pi) \;:\; u(0) = u(\pi),\; u'(0) = u'(\pi)\,\}$. We have then $E = D(A^{1/2})$. The parameter $\beta > 0$ is needed to ensure that the lowest eigenvalue of A is positive. (The boundary conditions $u(0) = u'(\pi) = 0$, or the other way round, could be treated similarly.) We use the notations

$$\langle u,v\rangle = \textstyle\int_0^\pi uv\, dy,\;\; \|u\|_0^2 = \langle u,u\rangle,\;\; \|u\|_1^2 = \langle Au,u\rangle,\;\; \|u\|_2 = \|Au\|,$$

for the scalar product in H, and the norms in H, E, and $D(A)$, respectively.

The function $f = f(x, y, u)$ is assumed to lie in $C^2_{b,unif}(\mathbf{R} \times [0, \pi] \times [-M, M], \mathbf{R})$ for each $M > 0$. In the case of Dirichlet boundary conditions we further impose $f(x, 0, 0) = f(x, \pi, 0) = 0$ for all $x \in \mathbf{R}$, implying that $f(x, \cdot, u(\cdot)) \in E = H^1_0$ for $u \in E$. (The case where this compatibility condition is not met can be handled with the methods in [Mi91b].) Then, the Nemitskii operator $F = F(x, u) : \mathbf{R} \times E \to E$ is well defined via $F(x, u)(y) = f(x, y, u(y))$, $y \in (0, \pi)$, since E embeds continuously into $C^0([0, \pi], \mathbf{R})$.

From the smoothness of f we easily see that every weak solution of (1.1) is a strong solution. Hence, further on we will not distinguish between different types of solutions. The problem can now be written in abstract form

$$u_{xx} - 2\alpha u_x - Au + F(x, u) = 0, \tag{2.1}$$

where a solution $u : J \subset \mathbf{R} \to H$ satisfies $u \in C^2(J, H) \cap C^1(J, E) \cap C^0(J, D(A))$.

We define the essential set $\mathcal{E}$ of (2.1) as a subset of the extended phase space $\mathbf{R}\times E\times H$. It is the union of all bounded solutions $u : \mathbf{R} \to H$:

$$\mathcal{E} = \{\, (x, u_0, u_1) \in \mathbf{R} \times E \times H \;:\; \exists \text{ bdd. soln. } u : u(x) = u_0,\ u_x(x) = u_1 \,\}.$$

The main assumption, we impose on F, is that there exist ε_0, ε_1, R_0, and R_1 that for $j = 0$ and $j = 1$ the estimate

$$\langle Au - F(x, u), A^j u\rangle \geq \varepsilon_j \ \text{ for all } x \in \mathbf{R} \text{ and } u \text{ with } \|u\|_j \geq R_j. \tag{2.2}$$

Using the abstract maximum principle of Calsina, Mora, and Solà–Morales [CMS90] we obtain the following a–priori estimates.

Theorem 2.1 (Abstract maximum principle)
Let $u : \mathbf{R} \to H$ be a bounded solution of (2.1) and assume that (2.2) holds for $j = 0$ or $j = 1$. Then, for all $x \in \mathbf{R}$ the estimate $\|u(x)\|_j \leq R_j$ holds.

Proof: For $\delta > 0$ we let $\rho_\delta(x) = \langle A^j u(x), u(x)\rangle / \cosh(\delta x)$ and $r_\delta = \sup_{x\in\mathbf{R}} \rho_\delta(x) \leq r_0 < \infty$. Differentiating with respect to x yields

$$\begin{aligned}
\rho'_\delta &= 2\langle A^j u, u_x\rangle / \cosh(\delta x) - \delta \rho_\delta \tanh(\delta x), \\
\rho''_\delta &= 2(\langle A^j u_x, u_x\rangle + \langle A^j u, u_{xx}\rangle) / \cosh(\delta x) - 2\delta \rho'_\delta \tanh(\delta x) \\
&\quad -\delta^2 (2\tanh^2(\delta x) + \cosh^{-2}(\delta x))\rho_\delta.
\end{aligned}$$

For $\delta > 0$, the supremum r_δ of ρ_δ is achieved at a finite x_δ: $\rho_\delta(x_\delta) = r_\delta$, $\rho'_\delta(x_\delta) = 0$, $\rho''_\delta(x_\delta) \leq 0$. Inserting eqn. (2.1) into the expression for $\rho''_\delta(x_\delta) \leq 0$ and multiplying with $\cosh(\delta x_\delta)$ results in

$$\begin{aligned}
0 \geq \rho''_\delta &= 2\langle A^j u_x, u_x\rangle + 2\langle A^j u, 2\alpha u_x + Au - F(x_\delta, u)\rangle \\
&\quad -\delta^2(2\tanh^2(\delta x_\delta) + \cosh^{-2}(\delta x_\delta))\langle A^j u, u\rangle \\
&\geq 2\langle A^j u, Au - F(x_\delta, u)\rangle - (2|\alpha|\delta + 3\delta^2) r_\delta.
\end{aligned} \tag{2.3}$$

Here we have used $2\langle A^j u, u_x\rangle = \cosh(\delta x)\rho_\delta' + \delta\tanh(\delta x)\rho_\delta \geq -\delta r_\delta$, if $\rho_\delta' = 0$. For all sufficiently small δ we have $(2|\alpha|\delta + 3\delta^2)r_\delta \leq \varepsilon_j/3$ (recall $r_\delta \leq r_0$); then, according to assumption (2.2), the last expression in (2.3) can only be negative if $R_j^2 \geq \|u(x_\delta)\|_j^2 = r_\delta$. Now, for all $x \in \mathbf{R}$,

$$\|u(x)\|_j^2 = \rho_0(x) = \rho_\delta(x)\cosh(\delta x) \leq r_\delta \cosh(\delta x) \leq R_j^2 \cosh(\delta x).$$

With $\delta \to 0$ the desired result follows. QED

We now relate the conditions $(2.2)_j$ to appropriate conditions for the scalar–valued function $f = f(x, y, u)$.

Lemma 2.2
a) *Assume that there are a, $b \geq 0$ such that $uf(x, y, u) \leq a^2 - bu^2$ for all $(x, y, u) \in \Omega \times \mathbf{R}$. Then, $\langle Au - F(x, u), u\rangle \geq (\lambda_1 + b)\|u\|_0^2 - \pi a^2$ for all $(x, u) \in \mathbf{R} \times E$.*

b) *Assume that f can be written as a sum of a bounded and a decreasing function depending only on (x, u):*

$$\begin{aligned} &f(x, y, u) = f_b(x, y, u) + f_d(x, u) \quad \text{with} \\ &|f_b(x, y, u)| \leq m,\ f_d(x, 0) = 0,\ \partial_u f_d(x, u) \leq 0, \end{aligned} \tag{2.4}$$

for all $(x, y, u) \in \Omega \times \mathbf{R}$. Then, for all $(x, u) \in \mathbf{R} \times D(A)$, the estimate

$$\langle Au - F(x, u), A^j u\rangle \geq \frac{\lambda_1}{2}(\|u\|_j^2 - \pi m^2 \lambda_1^{j-2}) \tag{2.5}$$

holds for $j = 0$ and $j = 1$.

Proof: Case a) follows simply from $\langle Au, u\rangle \geq \lambda_1\|u\|_0^2$ and integration of the inequality for uf.

In case b) we write $F = F_b + F_d$ according to the decomposition of f. Then, uf_d, $\partial_u f_d \leq 0$ implies $\langle F_d(x, u), A^j u\rangle \leq 0$. Thus, it suffices to establish (2.5) for F_b only. We find

$$\begin{aligned} 2\langle Au - F_b(x, u), A^j u\rangle &\geq 2\langle Au, A^j u\rangle - 2\sqrt{\pi} m\|A^j u\| \ \geq\ 2\lambda_1^{1-j}\|u\|_{2j}^2 - 2\sqrt{\pi} m\|u\|_{2j} \\ &\geq \lambda_1^{1-j}\|u\|_{2j}^2 - \pi m^2 \lambda_1^{j-1} \ \geq\ \lambda_1\left(\|u\|_j^2 - \pi m^2 \lambda_1^{j-2}\right). \end{aligned}$$

This proves the assertion. QED

In addition to these abstract maximum principles there is the classical one, which depends only on a sign condition for f.

Theorem 2.3 (Classical maximum principle)
Let $u : \Omega \to \mathbf{R}$ be a bounded solution of (1.1) *with one of the above–mentioned boundary conditions. Assume that there are ε_c, $R_c > 0$ such that*

$$-\mathit{sign}(u)(-\beta u + f(x,y,u)) \geq \varepsilon_c \quad \textit{for all } (x,y) \in \Omega \textit{ and } u \textit{ with } |u| \geq R_c. \tag{2.6}$$

Then, $|u(x,y)| \leq R_c$ for all $(x,y) \in \Omega$.

For the proof of this result we refer the reader to [Mi91b] where the case $\alpha = 0$ is treated. However, the generalization to $\alpha \neq 0$ can be done as in the proof of Thm. 2.1.

Note that $\beta > 0$ and (2.4) imply (2.6) for some R_c. Moreover, $|u(x,y)| \leq R_c$ implies $\|u(x,\cdot)\|_0 \leq \sqrt{\pi} R_c$. The same estimate follows in the abstract setting (Lemma 2.2a), if f satisfies the stronger estimate $uf(x,y,u) \leq \delta(R_c^2 - u^2)$ for some $\delta > 0$.

In general the bound R_0 will be much smaller than R_1. In that case the a–priori estimates can be improved considerably by using the Green's function K for the operator $u \mapsto u_{xx} - 2\alpha u_x - Au$.

Proposition 2.4
Let $u : \mathbf{R} \to E$ be a solution of (2.1) *with $\|u(x)\|_0 \leq \widetilde{R}_0$ and $\|F(x,u(x))\|_0 \leq S_0$ for all $x \in \mathbf{R}$. Let $S_1 = \max\{S_0, e\widetilde{R}_0\mu_1(\mu_1 - |\alpha|)\}$, where $\mu_1 = \sqrt{\lambda_1 + \alpha^2}$. Then,*

$$\|u(x)\|_1^2 + \alpha^2\|u(x)\|_0^2 \leq C(\widetilde{R}_0, S_0)^2 := e\frac{S_1\widetilde{R}_0\mu_1}{\mu_1 - |\alpha|}\log\frac{S_1}{\widetilde{R}_0\mu_1(\mu_1 - |\alpha|)}, \tag{2.7}$$

for all $x \in \mathbf{R}$.

Proof: We consider $G(x) = F(x,u(x))$ as a given function and estimate u as solution of $u_{xx} - 2\alpha u_x - Au + G = 0$. We let $H_B^s = D(A^{s/2})$ for $s \in [1,2]$, with the associated norm

$$|\!|u|\!|_s^2 = \|(A+\alpha^2)^{s/2}u\|_0^2 = \sum_{n=1}^{\infty}\mu_n^{2s}v_n^2, \quad \text{where } v_n = \int_0^\pi u(y)\varphi_n(y)\,dy,\ \mu_n = \sqrt{\lambda_n + \alpha^2}.$$

Note that $|\!|u(x)|\!|_1^2$ is the right–hand side in (2.7) and that $|\!|u|\!|_0 = \|u\|_0$. With the Green's function $K(t) = \frac{1}{2}e^{\alpha t}B^{-1}e^{-B|t|}$ we have $u(x,\cdot) = \int_{\mathbf{R}} K(x-\xi)G(\xi)\,d\xi$ and find the estimate

$$\|K(t)\|_{H\to H_B^s} = \sup_{n\in\mathbf{N}}\frac{\mu_n^s}{2\mu_n}e^{-\mu_n|t|+\alpha t} \leq \frac{1}{2}e^{\alpha t}\begin{cases}\mu_1^{s-1}e^{-\mu_1|t|} & \text{for } |t| \geq (s-1)/\mu_1,\\ \left(\frac{s-1}{e|t|}\right)^{s-1} & \text{for } |t| \leq (s-1)/\mu_1.\end{cases}$$

Integrating $\|K(t)\|_{H\to H_B^s}$ and using same elementary estimates yields

$$|\!|u(x,\cdot)|\!|_s \leq \int_{\mathbf{R}}\|K(x-\xi)\|_{H\to H_B^s}S_0\,d\xi \leq \frac{\mu_1^{s-1}S_0}{(2-s)(\mu_1 - |\alpha|)},$$

where we have used $s \in [1,2)$. Using Hölder's inequality we continue as follows:

$$\begin{aligned} \| u(x) \|_1^2 &= \sum \mu_n^2 v_n^2 = \sum v_n^{2(s-1)/s} \cdot \mu_n^2 v_n^{2/s} \\ &\le \left(\sum v_n^2\right)^{(s-1)/s} \left(\sum \mu_n^{2s} v_n^2\right)^{1/s} \\ &= \| u(x) \|_0^{2(s-1)/s} \| u(x) \|_s^{2/s} = \widetilde{R}_0^2 \left(\| u(x) \|_s / \widetilde{R}_0 \right)^{2/s}. \end{aligned}$$

Thus, for all $s \in [1,2)$ the estimate

$$\| u(x) \|_1 \le \widetilde{R}_0 \mu_1 \left(\frac{\rho}{2-s} \right)^{1/s}, \quad \text{where } \rho = \frac{S_0}{\widetilde{R}_0 \mu_1 (\mu_1 - |\alpha|)},$$

holds. For $s=1$ this gives the estimate $\|(A+\alpha^2)^{1/2} u(x)\| \le S_0/(\sqrt{\lambda_1+\alpha^2} - |\alpha|)$, which corresponds to the result of Theorem 2.1 when combined with Lemma 2.2b. However, taking s larger, we can diminish the influence of S_0. To prove the lemma it now suffices to show that it is possible to choose $s = s(\rho)$ such that $h(\rho,s) = (\rho/(2-s))^{1/s} \le (e\rho \log \rho)^{1/2}$. Therefore we replace S_0 by $S_1 \ge S_0$, such that $\rho \ge e$. Let $s = 2-2/r$ with $r = P + \log P + 1$, where $P = \log \rho \ge 1$, then the function $\widetilde{h}(\rho) = \log(h(\rho, s(\rho)))$ satisfies

$$\widetilde{h}(\rho) = \frac{1}{s}(P - \log(2-s)) = \frac{1}{2} \frac{P + \log P + 1}{P + \log P} (P + \log r - \log 2).$$

From $P \ge 1$ we find $P + \log P \ge P + \log(P + \log P + 1) - \log 2$ and thus $\widetilde{h}(\rho) \le (P + \log P + 1)/2$. Taking the exponential we find the desired result. QED

We now give three examples for nonlinearities, which fall under the above theory:

$$f_1(u) = \kappa^2 u - u^3, \quad f_2(x,y,u) = g(x,y) - u^3, \quad f_3(u) = \sigma \sin u, \tag{2.8}$$

where g is bounded by $\gamma = |g|_\infty = \sup\{|g(x,y)| : (x,y) \in \Omega\}$.

The classical maximum principle (Thm. 2.3) gives $|u| \le R_{cj}$ with $R_{c1} = \kappa$, $R_{c2} = \gamma^{1/3}$, and $R_{c3} = |\sigma|/\beta$ for $\beta > 0$. To apply Lemma 2.2a we use

$$u f_1(u) =\le \kappa^2(\kappa^2 - u^2), \; u f_2(x,y,u) \le \tfrac{3}{2}(\gamma - \gamma^{1/3} u^2), \; u(-\beta u + f_3(u)) \le \tfrac{\beta}{2}(\tfrac{\sigma^2}{\beta^2} - u^2).$$

Thus, $\|u(x,\cdot)\|_0 \le R_{0j} = \sqrt{\pi} R_{cj}$.

For Lemma 2.2b we need the decompositions $f_j = f_{bj} + f_{dj}$ with

$$f_{1b} = \begin{cases} f_1(u) & \text{for } |u| \le \kappa, \\ 0 & \text{for } |u| \ge \kappa; \end{cases} \quad f_{2b} = g(x,y), \quad f_{3b} = f_3.$$

Thus, the bound m takes the values $m_1 = 2\kappa^3/(3\sqrt{3})$, $m_2 = \gamma$, and $m_3 = |\sigma|$, respectively. Now, Lemma 2.2b and Theorem 2.1 give the estimates $R_{0j} \le \sqrt{\pi} m_j/\lambda_1$ and $R_{1j} \le \sqrt{\pi} m_j/\sqrt{\lambda_1}$. Obviously, for large κ or γ these estimates for R_{01} and R_{02} are much worse than those above. However, for f_3 the latter estimate is better, since $\lambda_1 \geq \beta$. In fact, in the case of Dirichlet boundary conditions we can allow $\beta \in (-1, 0]$ and still obtain $R_{03} \leq \sqrt{\pi}|\sigma/(1+\beta)$, whereas the classical maximum principle is no longer applicable.

The estimate for R_{1j}, $j = 1, 2$, can be improved by the help of Proposition 2.4, when the a–priori estimate $\|u(x, y)\|_0 \leq R_{0j}$ is employed. In this interval we easily find $|f_j(x, y, u)| \leq jm_j$. Thus, the theorem can be applied with $\widetilde{R}_{0j} = \sqrt{\pi}R_{cj}$ and $S_{0j} = \sqrt{\pi}jm_j$. For simplicity we assume $\lambda_1 = \mu_1 = 1$, $\alpha = 0$, and $S_0 \geq e\widetilde{R}_0$. Then, estimate (2.7) results in

$$R_{1j}^2 + \alpha^2 R_{0j}^2 \leq C_j^2 = \pi e j m_j R_{cj} \log(jm_j/R_{cj}).$$

As a consequence we have $C_1 \sim \kappa^2 \log\kappa$ and $C_2 \sim \gamma^{2/3}\log\gamma$. Roughly, this is an improvement against the previous value of R_1 by a factor of $R_1^{1/3}$. In the case of f_3 the bounds R_{03}, R_{13}, and $S_{03} = \sqrt{\pi}|\sigma|$ are all of the same order in σ. Hence, no improvement is to be expected from Prop. 2.4.

3 Modifications of the nonlinearity

In order to construct the essential manifold via the contraction mapping principle, we have to modify the nonlinearity $F(x, u) : \mathbf{R} \times E \to E$, such that its modification is uniformly Lipschitz continuous with respect to $u \in E$. The modification $\widetilde{F} = \widetilde{F}(x, u)$ will coincide with F on a cylinder $Z_r = \mathbf{R} \times B_E(r)$, where $B_E(r) = \{\, u \in E \,:\, \|u\|_1 \leq r \,\}$. On the one hand it is our aim to make the Lipschitz constant $\Theta = \mathrm{Lip}(\widetilde{F})$ as small as possible, since the dimension of the essential manifold will increase with Θ. On the other hand we want to guarantee that all bounded solutions of the original problem (2.1) are still solutions of the modified problem $(2.1)^\sim$, i.e., where F is replaced by $\widetilde{F}$. (We also use the notation $(2.1)^{e,i}$, if $\widetilde{F} = F_{e,i}$.) This means that the essential set $\mathcal{E} \subset \mathbf{R} \times E \times H$ of (2.1) is contained in the essential set $\widetilde{\mathcal{E}}$ of $(2.1)^\sim$. This property can be achieved when the critical radius r is taken larger than the a–priori bound R_1 obtained above. Of course, for any bounded solution of $(2.1)^\sim$ we can check a–posteriori whether it lies in $\mathcal{E}$, simply by testing for $\|u(x)\|_1 \leq r$ for all $x \in \mathbf{R}$.

In addition we can ask, that both essential sets are equal: $\mathcal{E} = \widetilde{\mathcal{E}}$. To guarantee this property we can proceed as follows. We have to find a modification $\widetilde{F}$ which still is emanable to some a–priori estimates. If $\|u(x)\|_1 \leq \widetilde{R}_1 \leq r$ can be shown for all bounded solutions, then all solutions in $\widetilde{\mathcal{E}}$ only experience the unmodified part of $\widetilde{F}$, and hence are solutions of the original problem. This implies the reverse inclusion $\widetilde{\mathcal{E}} \subset \mathcal{E}$.

A difficulty arises through the fact that, in general, a–priori estimates can be very sensitive to the modifications needed to ensure uniform Lipschitz continuity. There are two well–known modification methods, involving exterior and interior cut–off functions,

$$F_e(x, u) = \chi(\|u\|_1/r)F(x, u) \quad\text{and}\quad F_i(x, u) = F(x, \phi(\|u\|_1/r)u), \tag{3.1}$$

respectively. To make $\widetilde{F} = F_{e,i}$ Lipschitz continuous we may choose

$$\chi(t) = \begin{cases} \min\{1, 2-t\} & \text{for } t \in [0,2], \\ 0 & \text{for } t \geq 2; \end{cases} \qquad \phi(t) = \min\{1, 1/t\}. \tag{3.2}$$

Using C^1–mollifications of χ and ϕ we easily obtain $\widetilde{F} \in C^1(\mathbf{R} \times E, E)$.

Lemma 3.1
Let $\widetilde{F} = F_{e,i}$ be defined through (3.1) *and* (3.2). *Then, $\widetilde{F}$ is uniformly Lipschitz continuous with respect to $u \in E$ with*

$$\begin{aligned} \Theta_e = Lip(F_e) &= \sup_{(x,u)\in Z_{2r}} \left(\|D_u F(x,u)\|_{\mathcal{L}(E,E)} + \frac{1}{r}\|F(x,u)\|_1 \right), \\ \Theta_i = Lip(F_i) &= \sup_{(x,u)\in Z_r} \|D_u F(x,u)\|_{\mathcal{L}(E,E)}. \end{aligned}$$

The first result follows from $D_u F_e[h] = \chi D_u F[h] + \frac{1}{r}\chi' F \langle u, h\rangle_1 / \|u\|_1$. The second result is proved in [Mi91b, Lemma 2.3].

We remark that Θ_i is the optimal Lipschitz constant for given r. The interior cut–off can be used, when we are satisfied with a modification such that $\mathcal{E} \subset \widetilde{\mathcal{E}}$. From this Θ_i we can then derive optimal bounds for the dimension of an essential manifold containing the whole essential set $\mathcal{E}$. However, with respect to the a–priori estimates the interior cut–off has the disadvantage that the bounds from the classical maximum principle get destroyed. The abstract equation with nonlinearity F_i corresponds to the elliptic problem

$$u_{xx} - 2\alpha u_x - (u_{yy} - \beta u) + f(x, y, \phi(\|u(x,\cdot)\|_1/r)u) = 0.$$

Even if $f(x,y,u) < -\varepsilon_c$ for $u > R_c$ we can not conclude $u(x,y) \leq R_c$ for all $(x,y) \in \Omega$, since at a maximum of u we could have $u(x_0,y_0) > R_c$ but $\phi(\|u(x_0,\cdot)\|_1/r)u(x_0,y_0) < R_c$. Similarly the estimate part a) of Lemma 2.2 breaks done, whereas those of part b) are still valid.

In contrast to this the exterior cut–off leads to the elliptic problem

$$u_{xx} - 2\alpha u_x - (-u_{yy} + \beta u) + \chi(\|u\|_1/r) f(x,y,u) = 0,$$

which is still emanable to the classical maximum principle as well as all the abstract ones. For example condition (2.2) follows from

$$\langle Au - \chi F(x,u), A^j u\rangle = \chi\langle Au - F(x,u), A^j u\rangle + (1-\chi)\|u\|^2_{j+1}.$$

Since $0 \leq \chi \leq 1$, it is clear that this expression is bounded from below by some $\varepsilon^* > 0$ for $\|u\|_j \geq R_j$. From the a–priori estimates we obtain the following corollary.

Corollary 3.2
Assume that f satisfies (2.6). *Then every bounded solution u of* (2.1) *or the modified problem (2.1)^e satisfies the estimates*

$$\|u(x)\|_0 \leq \sqrt{\pi} R_c, \quad \|u(x)\|_1 \leq \sqrt{\pi} C(R_c, S_c),$$

where $S_0 = \sup\{\, |f(x,y,u)| \;:\; (x,y,u) \in \Omega \times [-R_c, R_c] \,\}$ *and* $C(\cdot,\cdot)$ *is defined in Prop. 2.4.*

From the examples at the end of the last section we see that $\|u\|_0$ and $\|u\|_1$ are typically of different order of magnitude. To take this into account properly we introduce an equivalent norm in E given by

$$|u|_\nu^2 = \|u\|_1^2 + \nu^2 \|u\|_0^2 = \int_0^\pi (u_y^2 + (\beta + \nu^2) u^2)\, dy.$$

We will dispose of ν later, according to the different bounds on f depending on the parameters. In particular we find

$$|u(x)|_\nu^2 \leq r^2(\nu, R_c, S_c) = \pi[\nu^2 R_c^2 + C^2(R_c, S_c)]. \tag{3.3}$$

for all $x \in \mathbf{R}$ and all bounded solutions.

Using this r in the definitions of $\Theta_{e,i}$ (where now $\|\cdot\|_1$ is replaced by $|\cdot|_\nu$) we encounter the fact, that a general u with $|u|_\nu \leq r$ explores the nonlinearity $f(x,\cdot)$ much further, namely up to r/ν (cf. Lemma 3.4 below), than all u appearing in $\mathcal{E}$. In order to keep $\|D_u F\|$ and $|F|_\nu$ small for all u with $|u|_\nu \leq r$ we change the scalar function f before using the cut–off. Since for all u appearing in $\mathcal{E}$ $|u(x,y)| \leq R_c$ we can modify f outside the u–interval $[-R_c, R_c]$ without changing the essential set, whenever for the modification $\widehat{f}$ the sign condition (2.6) still holds.

Lemma 3.3
Let f satisfy (2.6) *and* $f \in C^2_{b,unif}(\Omega \times [-R_c, R_c])$. *Then, there is a function* $\widehat{f} \in C^2_{b,unif}(\Omega \times \mathbf{R}, \mathbf{R})$ *such that*

$$\widehat{f}(x,y,u) = f(x,y,u) \quad \text{for } (x,y,u) \in \Omega \times [-R_c, R_c] \qquad \text{and } \widehat{f} \text{ satisfies (2.6).} \tag{3.4}$$

The result follows easily by letting $\widehat{f}(x,y,u) = \widetilde{\chi}(|u|/R_c)(f(x,y,u) + \varepsilon \text{sign}(u)) - \varepsilon \text{sign}(u)$, where $\widetilde{\chi} \in C^2$ is a cut–off function as given above and ε is sufficiently small.

However, in practice it might be better to construct $\widehat{f}$ in a different way, since the constants

$$d_{k,l} = \sup\{\, |\partial_u^k \partial_y^l \widehat{f}(x,y,u)| \;:\; (x,y,u) \in \Omega \times \mathbf{R} \,\}, \quad l = 0, 1,\ k + l \leq 2,$$

will determine the final estimate on the dimension of the essential manifold.

Altogether, we now define the modified abstract nonlinearity $\widetilde{F}$ as the abstract exterior cut–off of the function $\widehat{F}(x,u) = \widehat{f}(x,\cdot,u(\cdot))$, i.e., $\widetilde{F}(x,u) = \chi(|u|_\nu/r)\widehat{F}(x,u)$. To find upper bounds for the global Lipschitz constant $\Theta_e = \text{Lip}(\widetilde{F})$ in Lemma 3.1 we use

Lemma 3.4
a) For any $v \in H^1(0,\pi)$ we have, for all $y \in (0,\pi)$, the estimate

$$|v(y)|^2 \le \frac{1}{\nu\tanh(\nu\pi)}|v|_\nu^2. \tag{3.5}$$

b) For u, $v \in H^1(0,\pi)$ we have

$$|uv|_\nu^2 \le L(\nu)|u|_\nu^2|v|_\nu^2, \quad \textit{with } L(\nu) = \frac{4}{\nu\tanh(\nu\pi)} \tag{3.6}$$

Proof:
a) Estimate (3.5) follows from minimizing the $|\cdot|_\nu$–norm under the condition $v(\pi) = 1$. The minimum is achieved through the function $v(y) = \cosh(\nu y)/\cosh(\nu\pi)$.
b) By the definition of $|\cdot|_\nu$ we have

$$|uv|_\nu^2 = \int_0^\pi [\nu^2u^2v^2 + (u'v+uv')^2]\,dy \le 2\|u\|_\infty^2|v|_\nu^2 + 2\|v\|_\infty^2|u|_\nu^2,$$

where $\|u\|_\infty = \max\{\,|u(y)| \,:\, y\in(0,\ell)\,\}$. Employing part a) gives the result. QED

Choosing the functions $u = v = e^{-\nu y}$ we realize that the constant $L(\nu)$ is optimal for $\nu \to 0$ and ∞, up to a ν–independent factor.

To estimate the norm of the Fréchet derivative $D_uF \in \mathcal{L}(E,E)$ we use the fact that it is just a multiplication operator: $D_u\widehat{F}(u)[v](y) = \partial_u\widehat{f}(y,u(y))v(y)$. For $\nu \ge 1$ we have $\tanh(\pi\nu) \ge 4/5$; hence (3.6) implies

$$\begin{aligned}\tfrac{\nu}{5}\|D_u\widehat{F}(x,u)\|_{E\to E}^2 &\le |\partial_u\widehat{f}(x,\cdot,u(\cdot))|_\nu^2\\ &\le \textstyle\int_0^\pi \Big[(\nu\,\partial_u\widehat{f})^2 + (\partial_u\partial_y\widehat{f} + \partial_u^2\widehat{f}\,u_y)^2\Big]\,dy \le \pi\nu^2d_{1,0}^2 + (\sqrt{\pi}d_{1,1} + d_{2,0}\|u_y\|_0)^2.\end{aligned}$$

Similarly, we estimate $|\widehat{F}(x,u)|_\nu^2 \le \pi\nu^2d_{0,0}^2 + (\sqrt{\pi}d_{0,1} + d_{1,0}^2\|u_y\|)^2$. With $\|u_y\| \le |u|_\nu \le 2r$ we obtain

$$\widehat{\Theta}_e^2(\nu) \le \frac{10}{\nu}(\nu^2\pi d_{1,0}^2 + (\sqrt{\pi}d_{1,1} + 2d_{2,0}r)^2) + \frac{1}{r^2}(\nu^2\pi d_{0,0}^2 + (\sqrt{\pi}d_{0,1} + 2d_{1,0}r)^2),$$

where $r^2 = \nu^2R_0^2 + C(R_0,\sqrt{\pi}d_{0,0})$. This estimate can now be optimized by choosing $\nu \ge 1$ such that the right–hand side minimal.

We now return to the two examples introduced at the end of the last section $f_1 = \kappa^2u - u^3$, $f_2 = g(x,y) - u^3$, and $f_3 = \sigma\sin u$, where $|g(x,y)|, |\partial_y g(x,y)| \le \gamma$ is assumed. We have $R_{c1} = \kappa$, $R_{c2} = \gamma^{1/3}$, and $R_{c3} = \infty$ for general β. We choose the following modifications $\widehat{f}_j$:

$$\widehat{f}_1(u) = \begin{cases} \kappa^2u - u^3 & \text{for } u\in[0,\kappa],\\ \kappa(4\kappa-2u)^2/3 - \kappa^3/3 & \text{for } u\in[\kappa,\tfrac{4}{3}\kappa],\\ -\kappa^3/3 & \text{for } u \ge \tfrac{4}{3}\kappa\\ -\widehat{f}(-u) & \text{for } u \le 0;\end{cases}$$

$$\widehat{f}_2(x,y,u) = g(x,y) + \begin{cases} -u^3 & \text{for } u \in [0, \gamma^{1/3}, \\ \frac{3}{4}\gamma^{1/3}(3\gamma^{1/3} - 2u)^2 - \frac{7}{4}\gamma & \text{for } u \in [\gamma^{1/3}, \frac{3}{2}\gamma^{1/3}], \\ -\frac{7}{4}\gamma & \text{for } u \geq \frac{3}{2}\gamma^{1/3}, \\ \text{symmetric} & \text{for } u \leq 0; \end{cases}$$

and $\widehat{f}_3 = f_3$. The functions $\widehat{f}_j$ agree with the original ones inside of $(-R_{cj}, R_{cj})$ and are in $C^{1,Lip}_{b,unif}(\Omega \times \mathbf{R}, \mathbf{R})$. We obtain the following estimates for $d_{k,l}$:

	R_0	$d_{0,0}$	$d_{1,0}$	$d_{2,0}$	$d_{0,1}$	$d_{1,1}$
$f_1(u)$	$\sqrt{\pi}\kappa$	$2\kappa^3/(3\sqrt{3})$	$2\kappa^2$	6κ	0	0
$f_2(x,y,u)$	$\sqrt{\pi}\gamma^{1/3}$	$5\gamma/2$	$3\gamma^{2/3}$	$6\gamma^{1/3}$	γ	0
$f_3(u)$	$\|\sigma\|/\lambda_1$	$\|\sigma\|$	$\|\sigma\|$	$\|\sigma\|$	0	0

Now we can choose ν_j to our convenience. If let $\nu_1 = \kappa,\ \nu_2 = \gamma^{1/3}$, and $\nu_3 = 1$ we find after some elementary caluclations the asymptotic relations

$$\widehat{\Theta}_1 \sim \kappa^{5/2}\log\kappa,\ \widehat{\Theta}_2 \sim \gamma^{5/6}\log\gamma,\ \text{and } \widehat{\Theta}_3 \sim \sigma^2.$$

It should be noted that all the estimates above are explicit. Thus, it is possible to get the some concrete constants. We show this for the nonlinearity f_1. We also have to specify the eigenvalue λ_1 and the coefficient α; for simplicity let $\alpha = 0$ and $\lambda_1 = 1+\beta = 1$, which corresponds to Dirichlet conditions. Assuming $\kappa > 3$ we have

$$C^2(R_c, d_{0,0}) \leq \frac{2e}{3\sqrt{3}}\kappa^4 \log\frac{2\kappa^2}{3\sqrt{3}} \leq \frac{4e\kappa^4}{3\sqrt{3}}\log\kappa.$$

Since we will choose $\nu \geq \kappa$ we have

$$\begin{aligned} \widehat{\Theta}^2 &\leq \frac{10}{\nu}\Big(4\pi\nu^2\kappa^4 + 144\pi\kappa^2(\nu^2\kappa^2 + C^2)\Big) + \frac{\frac{4}{27}\pi\nu^2\kappa^6}{\pi(\nu^2\kappa^2 + C^2)} + 16\kappa^4 \\ &\leq 40\pi\kappa^2\Big(37\nu\kappa^2 + 36C^2/\nu\Big) + (16 + \frac{4}{27})\kappa^4. \end{aligned}$$

This expression is minimal for $\nu^2 = 36C^2/(37\kappa^2)$. With this choice we obtain

$$\widehat{\Theta} \leq 98.1\kappa^{5/2}\sqrt{\log\kappa}\,.$$

4 Global invariant manifolds

Now we are in a status where the modified abstract problem

$$u_{xx} - 2\alpha u_x - Au + \widetilde{F}(x,u) = 0, \tag{4.1}$$

is well prepared. The function $\widetilde{F} : \mathbf{R} \times E \to E$ is differentiable with respect to u such that the Fréchet derivative $D\widetilde{F}$ lies in $C^0_{b,unif}(\mathbf{R} \times E, \mathcal{L}(E))$. With

$$\Theta = \sup\{ \|D\widetilde{F}(x,u)\|_{E\to E} \; : \; x \in \mathbf{R}, \; u \in E \}$$

we denote the global Lipschitz constant of $\widetilde{F}$. The following theorem establishes the existence of a finite dimensional invariant manifold M_E in the extended phase space $\mathbf{R} \times E \times H$ for (4.1) containing all the bounded solutions, i.e., $\widetilde{\mathcal{E}} \subset M_E$. Since the essential set is contained in M_E we call it an *essential manifold of* (4.1).

Theorem 4.1
Let $\mu_N = (\lambda_N + \alpha^2)^{1/2}$ and assume that there is a positive integer N such that $\mu_N < \mu_{N+1}$ and

$$\Theta \cdot \min_{\mu_N < b < \mu_{N+1}} \left[\frac{1}{b^2 - \mu_N^2} + \frac{1}{\mu_{N+1}(\mu_{N+1} - b)} \right] < 1. \tag{4.2}$$

Then there exists a $(2N+1)$–dimensional manifold M_E which is invariant under the flow of (4.1) *and contains all bounded solutions of* (4.1). *Defining $P : E \to E \, (H \to H)$ to be the orthogonal projection onto span$\{\varphi_1, \ldots, \varphi_N\}$, the manifold M_E has the form*

$$M_E = \{ (x, u_0 + h(x, u_0, u_{0x}), u_{0x} + g(x, u_0, u_{0x})) \in \mathbf{R} \times E \times H : (x, u_0, u_{0x}) \in \mathbf{R} \times PE \times PH \},$$

where $(h,g) \in C^0_b(\mathbf{R} \times PE \times PH, (I-P)E \times (I-P)H)$ are continuously differentiable with respect to (u_0, u_{0x}). Moreover, every bounded solution $u_0 : (-\ell, \ell) \to PE$ of the reduced problem

$$u_{0xx} - 2\alpha u_x - A_0 u_0 = P\widetilde{F}(x, u_0 + h(x, u_0, u_{0x})) \tag{4.3}$$

yields via $u = u_0 + h(x, u_0, u_{0x})$ a solution of the full problem (4.1).

Remark: Note that the gap condition

$$\limsup_{N\to\infty} \lambda_{N+1} - \lambda_N = \infty, \tag{4.4}$$

is necessary and sufficient to be able to satisfy (4.2) for any Θ. The effect of the parameter α in the condition (4.2) is very small. For fixed N the minimum always decreases with growing $|\alpha|$, however going from $\alpha = 0$ to $|\alpha| = \infty$ we can only reduce the minimum by a factor of 1/2.

Proof of Theorem 4.1: Taking the integer N and the projection P as given in the theorem, we define $E_0 = PE$ and $E_1 = (I - P)E$ and similarly $H = H_0 \oplus H_1$. As P commutes with A, we can split (4.1) into

$$u_{0xx} - 2\alpha u_{0x} - A_0 u_0 = F_0(x, u_0 + u_1), \quad u_{1xx} - 2\alpha u_{1x} - A_1 u_1 = F_1(x, u_0 + u_1), \tag{4.5}$$

where $u_i \in E_i$, $A_0 = A|_{PD(A)}$, $A_1 = A|_{(I-P)D(A)}$, $F_0 = -P\widetilde{F}$, and $F_1 = (P-I)\widetilde{F}$. Using a shift in x–direction by τ we obtain for the bounded solutions $u(x) = u^\tau(x-\tau)$ of (4.5) the equivalent integral formulation

$$\begin{aligned} u^\tau(\cdot) &= S(\tau,\xi,\eta,u^\tau) = (S_0(\tau,\xi,\eta,u^\tau), S_1(\tau,u^\tau)) \\ &:= (K_0(\xi,\eta,F_0(\cdot+\tau,u^\tau)), K_1(F_1(\cdot+\tau,u^\tau))), \end{aligned} \tag{4.6}$$

where the linear operators K_i can be defined by using $B_i = (A_i + \alpha^2)^{1/2}$:

$$\begin{aligned} K_0(\xi,\eta,g_0)(x) &= e^{\alpha x}\Big(\cosh(B_0 x)\xi + B_0^{-1}\sinh(B_0 x)[\eta - \alpha\xi]\Big) + \\ &\quad + \int_0^x B_0^{-1}\sinh(B_0(x-t))e^{\alpha(x-t)}g_0(t)\,dt, \\ K_1(g_1)(x) &= \int_{\mathbf{R}} -\frac{1}{2}B_1^{-1}e^{-B_1|x-t|}e^{\alpha(x-t)}g_1(t)\,dt. \end{aligned}$$

Note that $\xi = u_0^\tau(0) = u_0(\tau)$ and $\eta = u_{0x}^\tau(0) = u_{0x}(\tau)$.

We now consider (4.6) as a fixed point equation on the space E^b of exponentially weighted continuous functions given by

$$E^b = \{\, u \in C(\mathbf{R}, E) \;:\; |u|_b < \infty \,\},\ |u|_b = \sup\{\, \|e^{-b|x|}e^{-\alpha x}u(x)\|_E \;:\; x \in \mathbf{R} \,\}.$$

We use here the notation $\|\cdot\|_E$ instead of $|\cdot|_\nu$ in order to avoid confusion with $|\cdot|_b$. Note the dependence of E^b and the associated norm on the parameter α. However, for notational convenience we desist from explicitly showing this dependence. The following estimate is then valid:

$$\begin{aligned} &|S(\tau,\xi,\eta,u) - S(\tau,\overline{\xi},\overline{\eta},\overline{u})|_b \\ &\le \alpha_N\|\xi-\overline{\xi}\|_E + \gamma_N\|\eta-\overline{\eta}\|_0 + \delta_{0N}|F_0(u)-F_0(\overline{u})|_b + \delta_{1N}|F_1(u)-F_1(\overline{u})|_b \\ &\le \alpha_N\|\xi-\overline{\xi}\|_E + \gamma_N\|\eta-\overline{\eta}\|_0 + (\delta_{0N}+\delta_{1N})\Theta|u-\overline{u}|_b, \end{aligned}$$

where

$$\begin{aligned} \alpha_N &= \sup\nolimits_{x\in\mathbf{R}} e^{-b|x|}\|\cosh(B_0 x) - \alpha B_0^{-1}\sinh(B_0 x)\|_{E\to E}, \\ \gamma_N &= \sup\nolimits_{x\in\mathbf{R}} e^{-b|x|}\|B_0^{-1}\sinh(B_0 x)\|_{H\to E}, \\ \delta_{0N} &= \sup\nolimits_{x\in\mathbf{R}} |e^{-b|x|}\textstyle\int_0^x \|B_0^{-1}\sinh(B_0(x-t))\|_{E\to E}e^{b|t|}\,dt| \\ &= \textstyle\int_0^\infty e^{-bt}\|B_0^{-1}\sinh(B_0 t)\|_{E\to E}dt \;=\; 1/(b^2-\mu_N^2), \\ \delta_{1N} &= \sup\nolimits_{x\in\mathbf{R}} e^{-b|x|}\textstyle\int_{\mathbf{R}} \|\frac{1}{2}B_1^{-1}e^{-B_1|x-t|}\|_{E\to E}e^{b|t|}\,dt \\ &= \textstyle\int_{\mathbf{R}} e^{b|t|}\|\frac{1}{2}B_1^{-1}e^{-B_1|t|}\|_{E\to E}dt \;=\; 1/(\mu_{N+1}(\mu_{N+1}-b)). \end{aligned} \tag{4.7}$$

Here we have used the fact that $B_i = (A_i+\alpha^2)^{1/2}$ is self–adjoint, hence, for any function Φ we have $\|\Phi(B_i)\|_{E\to E} = \sup|\Phi(\mu_n)|$ where the supremum is taken over $n \le N$ and $n \ge N+1$ for $i=0$ and $i=1$, respectively. In particular, there is now ν–dependence for the norms.

This shows that condition (4.2) guarantees that $S(\tau,\xi,\eta,\cdot) : E^b \to E^b$ is a contraction uniformly in (τ,ξ,η). Hence there is, for each (τ,ξ,η), a unique solution $u^\tau = U^\tau(\tau,\xi,\eta) \in E^b$ depending Lipschitz continuously on (ξ,η) and continuously on τ. Moreover, $S(\tau,\xi,\eta,U^\tau)$ considered as function from $\mathbf{R}$ into H is differentiable with derivative

$$\begin{aligned} u^\tau_{0x}(x) &= e^{\alpha x}\Big(\sinh(B_0x)[(B_0\xi - \alpha^2 B_0^{-1}) + \alpha B_0^{-1}\eta] + \cosh(B_0x)\eta\Big) + \\ &\quad + \int_0^x \Big[\cosh(B_0(x-t)) + \alpha B_0^{-1}\sinh(B_0(x-t))\Big] e^{\alpha(x-t)} F_0(\tau+t, u^\tau(t))\, dt, \\ u_{1x}(x)^\tau &= \int_{\mathbf{R}} \tfrac{1}{2}(\alpha B_1^{-1} - \operatorname{sign}(x-t)I) e^{-B_1|x-t|} e^{\alpha(x-t)} F_1(\tau+t, u^\tau(t))\, dt, \end{aligned} \tag{4.8}$$

and thus $U^\tau_x(\tau,\xi,\eta)$ lies in H^b with a Lipschitz continuous dependence on (ξ,η) (H^b is defined like E^b but with E replaced by H). Now we are able to define the function

$$(h,g) = \begin{cases} \mathbf{R}\times E_0 \times H_0 & \to \quad E_1 \times H_1, \\ (\tau,\xi,\eta) & \to \quad (U_1^\tau(\tau,\xi,\eta)(0), U^\tau_{1x}(\tau,\xi,\eta)(0)) \end{cases} \tag{4.9}$$

and the $(2N+1)$–dimensional manifold M_E as in the theorem.

To show that M_E is invariant we let $U(\tau,\xi,\eta)(\cdot) = U^\tau(\tau,\xi,\eta)(\cdot-\tau)$, $V=(U,U_x)$ and $V_i = (U_i, U_{ix}) \in E_i^b \times H_i^b$. The uniqueness of $U(\tau,\xi,\eta)$ implies $V(t, V_0(\tau,\xi,\eta)(t))(x) = V(\tau,\xi,\eta)(x)$ for all x,t,τ,ξ, and η. Whence, with $t=x$ we find

$$V_1(\tau,\xi,\eta)(x) = V_1(x, V_0(\tau,\xi,\eta)(x))(x) = V_1^x(x, V_0(\tau,\xi,\eta)(x))(0) = (h,g)(x, V_0(\tau,\xi,\eta)(x)).$$

This means that the whole solution $U(\tau,\xi,\eta)$ is contained in M_E.

Moreover, the last statement of the theorem is true, because any bounded solution $u_0 : (-\ell,\ell) \to E_0$ of the reduced equation (4.3) can be continued to all $x \in \mathbf{R}$. It then satisfies $u_0 = K_0(u_0(0), u_{0x}(0), F_0(\cdot+\tau, u_0 + h(\cdot, u_0, u_{0x})))$; and hence is equal to $U_0(\tau, u_0, u_{0x})$ by uniqueness.

It remains to be shown that (h,g) is in fact continuously differentiable. To this end let (ζ,κ) be any vector in $E_0\times H_0$ and differentiate the fixed point equation (4.6) formally in the direction (ζ,κ) to obtain

$$w = T(\zeta,\kappa,u,w) := (K_0(\zeta,\kappa, DF_0(u)[w]), K_1(DF_1(u)[w]))$$

where w is the place holder for the directional derivative $\frac{\partial u}{\partial(\zeta,\kappa)}$. For simplicity we do not express the dependence on x and τ explicitly.

Choosing $b^* \in (b, \mu_{N+1})$ sufficiently close to b, we easily see that $T(\zeta,\kappa,u,\cdot)$ is a contraction on E^b as well as on E^{b^*}. However, the fixed point is the same in both cases; let us denote the fixed point of $T(\zeta,\kappa,U(\xi,\eta),\cdot)$ by $W(\xi,\eta,\zeta,\kappa)$. Furthermore, $w\in E^b$ implies that the mapping $T(\zeta,\kappa,\cdot,w) : E^b \to E^{b^*}$ is continuous. This can be verified in the following way:

$$\begin{aligned} |T(\zeta,\kappa,u,w) - T(\zeta,\kappa,v,w)|_{b^*} &\le (\delta_{0N}+\delta_{1N})|D\widetilde{F}(u)[w] - D\widetilde{F}(v)[w]|_{b^*} \\ &\le (\delta_{0N}+\delta_{1N})|w|_b |D\widetilde{F}(u) - D\widetilde{F}(v)|_{b^*-b}. \end{aligned}$$

Using the uniform continuity of $D\widetilde{F}$, with ω being the modulus of continuity, yields

$$\begin{aligned} |D\widetilde{F}(u) - D\widetilde{F}(v)|_{b^*-b} &\leq \sup_{x\in\mathbf{R}} e^{-(b^*-b)|x|} \min\{\omega(|u-v|_b e^{b|x|}), \Theta\} \\ &\leq \max\{\omega(|u-v|_b e^{b\ell}), 2e^{-(b^*-b)\ell}\Theta\} \end{aligned}$$

for every $\ell > 0$. Taking $\ell = -\frac{1}{2b}\log|u-v|_b$ gives

$$|D\widetilde{F}(u) - D\widetilde{F}(v)|_{b^*-b} \leq \max\left\{\Theta|u-v|_b^{(b^*-b)/(2b)}, \omega(|u-v|_b^{1/2})\right\},$$

and hence $|T(\zeta,\kappa,u,w) - T(\zeta,\kappa,v,w)|_{b^*} \to 0$ for $|u-v|_b \to 0$.

Altogether this implies that the assumptions of the fiber contraction mapping theorem (cf. [Mi86a]) are fulfilled; thus the iteration sequence $(u_n, w_n) \in E^b \times E^{b^*}$, defined by $(u^{n+1}, w^{n+1}) = (S(\xi,\eta,u^n), T(\zeta,\kappa,u^n,w^n))$ and $(u^0,w^0) = (0,0)$, converges against the fixed point $(U(\xi,\eta), W(\xi,\eta,\zeta,\kappa))$. Moreover, by induction $\frac{\partial u^n}{\partial(\zeta,\kappa)} = w^n$ for all $n \in \mathbf{N}$; and hence $\frac{\partial U}{\partial(\zeta,\kappa)}(\xi,\eta) = W(\xi,\eta,\zeta,\kappa))$.

Since $W(\xi,\eta,\zeta,\kappa)$ is the unique fixed point of $T(\zeta,\kappa,U(\xi,\eta),\cdot) : E^{b^*} \to E^{b^*}$ and since this mapping is continuous in (ζ,κ,U), and thus in (ξ,η,ζ,κ), we also know that W depends continuously on (ξ,η,ζ,κ). As (ζ,κ) was arbitrary, we conclude $U = U(\xi,\eta) \in C^1(E_0 \times H_0, E^{b^*})$. Inserting this into (4.8) additionally gives $U_x(\xi,\eta) \in C^1(E_0 \times H_0, H^{b^*})$. Now from definition (4.9) the continuous differentiability with respect to (u_0, u_{0x}) of (h,g), and hence of M_E, follows. QED

Proposition 4.2

a) *Assume that there is an isomorphism $T : H \to H$ such that* (2.1) *is equivariant with respect to T (i.e., $ATu = TAu$ and $F(x,Tu) = TF(x,u)$ for all $u \in D(A)$). Then the essential manifold satisfies*

$$(h,g)(x, T_0u_0, T_0v_0) = (T_1h, T_1g)(x, u_0, v_0) \quad \text{for all } (x,u_0,v_0) \in \mathbf{R} \times E_0 \times H_0,$$

which means that the reduced problem is equivariant with respect to T_0. (Here T_j are the restrictions to E_j and H_j.)

b) *In the reversible case $\alpha = 0$ the essential manifold satisfies*

$$h(-x, u_0, -v_0) = h(x,u_0,v_0) \quad \text{and } g(-x,u_0,-v_0) = -g(x,u_0,v_0),$$

for all $(x,u_0,v_0) \in \mathbf{R} \times E_0 \times H_0$, which means that the reduced system is again reversible.

Proof: For part a) we first remark that T commutes with A, hence it commutes with the spectral projection P. Thus, $T_j : H_j \to H_j$ are well–defined isometries even in E_j. The equivariance implies that $TU(\tau,\xi,\eta)$ is a solution whenever $U(\tau,\xi,\eta)$ is one. From the uniqueness we conlude $TU(\tau,\xi,\eta) = U(\tau,T_0\xi,T_0\eta)$ which is the result.

For part b) we use that $U = U(x)$ is a solution whenever $\widetilde{U} = U(-x)$ is one. QED

Applications of part a) follow immediately if f does not depend in $y \in (0,\pi)$. Then, the reflection $Tu(y) = u(\pi - y)$ can be used. If f is odd in u, then $Tu = -u$ implies that also the reduced system is odd. In the case of periodic boundary conditions and no y–dependence there is a whole family of symmetries $T_\rho u(y) = u(y+\rho)$ where $\rho \in [0,\pi)$. Thus, the system has $O(2)$ symmetry.

Remark 4.3 The essential manifold is the graph of the C^1–function $\widetilde{h} = (h,g)$. An upper bound for the first derivative of $\widetilde{h}$ can be obtained as follows. We let $U = U^\tau(\tau,\xi,\eta)$ and $\overline{U} = U^\tau(\tau,\overline{\xi},\overline{\eta})$. From the fixed point equation we obtain

$$|U-\overline{U}|_b \leq \frac{\max\{\alpha_N,\gamma_N\}}{1-\Theta(\delta_{0N}+\delta_{1N})}(\|\xi-\overline{\xi}\|_E + \|\eta-\overline{\eta}\|_0)$$

Inserting this into the equation for u_1 we find

$$\begin{aligned}\|h(\tau,\xi,\eta)-h(\tau,\overline{\xi},\overline{\eta})\|_E &= \|U_1(0)-\overline{U}_1(0)\|_E \leq |U_1-\overline{U}_1|_b\\ &\leq |K_1[F_1(\cdot+\tau,U)-F_1(\cdot+\tau,\overline{U})]|_b\\ &\leq \delta_{1N}\Theta|U-\overline{U}|_b.\end{aligned}$$

A similar estimate holds for $g(\tau,\xi,\eta) \in H$ when the second equation in (4.8) is used together with the estimate $\|F_1\|_0 \leq \|F_1\|_E/\sqrt{\lambda_{N+1}}$. We find

$$\|D\widetilde{h}(u_0,v_0)\|_{E_0\times H_0\to E_1\times H_1} \leq L(N) = \frac{3\delta_{1N}\Theta(\alpha_N+\gamma_N)}{1-\Theta(\delta_{0N}+\delta_{1N})}$$

For fixed Θ and $N\to\infty$ we find $L(N) \sim 1/(\lambda_{N+1}-\lambda_N) \sim 1/N$.

The case of very large $|\alpha|$ and no x–dependence of f is treated in [CMS90] more carefully. Rescaling the x–variable we obtain

$$\varepsilon u_{xx} - u_x + u_{yy} + f(y,u) = 0, \qquad \text{with } \varepsilon = \frac{1}{4\alpha^2},$$

and are lead to a singularly perturbed parabolic problem. However, on the set of bounded solutions this limit behaves well; in particular, under additional assumptions on f, there are essential manifolds M_E^ε for each $\varepsilon \in (0,\varepsilon_0)$, which converge in the appropriate sense to the inertial manifold of the parabolic problem with $\varepsilon = 0$.

The dimension of the essential manifold can now be easily estimated using condition (4.2). The minimum over $b \in (\mu_N,\mu_{N+1})$ can be bounded from above by $(1+\sqrt{2})^2/(\lambda_{N+1}-\lambda_N)$. For all the boundary conditions the eigenvalues λ_N can be calculated explicitly and we find $\lambda_{N+1} - \lambda_N \geq 2N - 1$, where N should be odd in the case of periodic boundary conditions. Hence, the dimension of M_E is can be estimated by

$$\dim M_E = 2N + 1 \leq 2 + (1 + \sqrt{2})^2 \widehat{\Theta}.$$

This shows that the dimension grows linearly with the size of the Lipschitz constant $\widehat{\Theta}$. The estimates for $\widehat{\Theta}$ from the last section give now bounds on the dimension.

In the special example $f(u) = \kappa^2 u - u^3$ this leads to $\dim M_E \leq 574\kappa^{5/2}\sqrt{\log \kappa}$ for all $\kappa > 3$. For $\kappa > 10$ this is an considerable improvement over the estimate $35\kappa^4$ which was given in [Mi91b]. This number should also be compared with a lower estimate for the dimension, which is obtained by considering the dimension of a center manifold of the zero solution. Therefore we look at the linearized problem

$$u_{xx} - 2\alpha u_x - (-u_{yy} + \beta u) + \kappa^2 u = 0.$$

For $\alpha = 0$ the eigenvalues are $\pm\sqrt{\lambda_N - \kappa^2}$. From $\lambda_N \approx N^2$ we find that there are $2N_0 \approx 2[\kappa]$ eigenvalues on the imaginary axis. Thus, there is a $(2N_0 + 1)$–dimensional center manifold M_C which contains all *small* bounded solutions, cf. [Ki82, Mi86a]. Of course, this center manifold has to be contained in the essential manifold. From $M_C \subset M_E$ we conclude $\dim M_E \geq \dim M_C \geq 2(\kappa - 1)$.

5 Decay properties: weak normal hyperbolicity

In this section we are concerned with solutions existing only over a bounded x–interval, let us say $I_\ell = (-\ell, \ell)$, or over the half–line $[0, \infty)$. We always work in the abstract framework as introduced above. Under appropriate global assumptions on the nonlinearity we first show that every bounded solution decays exponentially with the distance from the ends as long as it is outside the ball where all the bounded solutions lie. We give an abstract and a classical version.

For the abstract global decay we make assumptions on the nonlinearity which are stronger than the assumptions (2.2) for the abstract maximum principle.

$$\exists \delta_j,\ R_j > 0 : \langle Au - F(x,u), A^j u\rangle \geq \frac{\delta_j}{2}(\|u\|_j^2 - R_j^2) \text{ for all } (x,u) \in I_\ell \times D(A), \tag{5.1}$$

where j is either 0 or 1. The condition (5.1) with $j = 1$ was used in [CMS90]. Sufficient conditions on the scalar function f, such that one of these conditions hold, are given in Lemma 2.2b.

Theorem 5.1
Let (5.1) *be satisfied, then for every bounded solution* $u : I_\ell \to E$ *of* (2.1) *the estimate*

$$\|u(x)\|_j^2 \leq R_j^2 + \coth(2\mu^j \ell)\left[e^{-(\alpha+\mu^j)(\ell - x)}\|u(\ell)\|_j^2 + e^{-(\mu^j - \alpha)(x+\ell)}\|u(-\ell)\|_j^2\right] \tag{5.2}$$

holds, where $\mu^j = \sqrt{\delta_j + \alpha^2} \geq |\alpha|$.

Proof: We define the bounded function $\rho(x) = \|u(x)\|_j^2 : I_\ell \to \mathbf{R}$. With $\rho' = 2\langle A^j u, u_x\rangle$ and $\rho'' = 2\langle A^j u, u_{xx}\rangle + 2\langle A^j u_x, u_x\rangle$ we obtain $\rho'' - 2\alpha\rho' = 2\|u_x\|^2 + 2\langle u, Au - F(x,u)\rangle$. Using (5.1) yields the differential inequality

$$\rho'' - 2\alpha\rho' - \delta_j\rho \geq -\delta_j R_j^2, \tag{5.3}$$

which is subject to the boundary conditions $\rho(\pm\ell) = \rho_\pm := \|u(\pm\ell)\|_j^2$. Since the differential operator on the left–hand side in (5.3) has a negative Green's function the solution $\widetilde{\rho}$ of the corresponding differential equation is an upper bound for ρ, i.e., $\rho(x) \leq \widetilde{\rho}(x)$ with

$$\widetilde{\rho}(x) \leq R_j^2 + \frac{1}{\sinh(2\mu^j\ell)}\left[(\rho_+ - R_j^2)e^{\alpha(x-\ell)}\sinh(\mu^j(x+\ell)) + (\rho_- - R_0^2)e^{\alpha(x+\ell)}\sinh(\mu^j(\ell-x))\right].$$

The desired result now follows from some elementary estimates. QED

This theorem shows that a solution with $|u(x)|_\nu \leq M$ can remain outside the $|\cdot|_\nu$–ball of radius $B = 2(\nu^2 R_0^2 + R_1^2)^{1/2}$ only on the intervals $[-\ell, -\ell+\gamma_-)$ and $(\ell-\gamma_+, \ell]$, where $\gamma_\pm = \log(2M/B)/(\sqrt{\delta+\alpha^2} \pm \alpha)$ with $\delta = \min\{\delta_0, \delta_1\}$. Note that $\gamma_\pm$ is bounded independently of ℓ. Thus, for large ℓ the solution will mostly lie inside the ball of radius B.

For the classical analogue we let $\Omega_\ell = I_\ell \times (0,\pi)$, and we assume

$$\exists \delta_c,\ R_c > 0 : \operatorname{sign}(u)[-\beta u + f(x,y,u)] \leq \delta_c(R_c - |u|) \text{ for all } (x,y,u) \in \Omega_\ell \times \mathbf{R}. \tag{5.4}$$

Integration over y yields immediately that also the abstract condition (5.1) with $j = 0, \delta_0 = \delta_c$, and $R_0 = \sqrt{\pi}R_c$ holds.

Theorem 5.2
Let (5.4) *be satisfied, then for every bounded solution* $u : \Omega_\ell \to \mathbf{R}$ *of* (1.1) *the estimate*

$$|u(x,y)| \leq R_c + \coth(2\mu^c\ell)\left[e^{-(\alpha+\mu^c)(\ell-x)}U_+ + e^{-(\mu^c-\alpha)(\ell+x)}U_-\right], \tag{5.5}$$

where $\mu^c = \sqrt{\delta_c + \alpha^2}$ *and* $U_\pm = \max\{\,|u(\pm\ell, y)| \;:\; y \in [0,\pi]\,\}$.

Proof: We establish the estimate from above ($u \leq R_c + \ldots$), then the estimate from below follows by replacing u by $-u$.

We define the x–dependent function v via

$$v_{xx} - 2\alpha v_x - \delta_c v = -\delta_c R_c, \quad v(\pm\ell) = U_\pm.$$

It is a lower bound for the right–hand side in (5.5); thus it suffices to show that v is a super–solution, i.e., $u(x,y) \leq v(x)$. Therefore let $w(x,y) = v(x) - u(x,y)$ which satisfies

$$\Delta w - 2\alpha w_x = \delta(u + w - R_c) - \beta u + f(x,y,u), \quad w(\pm\ell, y) = v(\pm\ell) - u(\pm\ell, y) \geq 0.$$

Additionally we have Dirichlet ($w(x,0) = w(x,\pi) = v(x) \geq 0$), Neumann, or periodic boundary conditions on the lateral sides.

Now assume that the minimum $m = \min\{\, w(x,y) \;:\; (x,y) \in \overline{\Omega_\ell}\,\}$ is negative. If m is attained at an interior point $(x_0,y_0) \in \Omega_\ell$, then $\Delta w(x_0,y_0) \geq 0$ and $w_x(x_0,y_0) = 0$. Moreover, $v(x_0) \geq 0$ implies $u(x_0,y_0) > 0$. Hence, condition (5.4) implies

$$0 \leq \Delta w - 2\alpha w_x = \delta_c(u + m - R_c) - \beta u + f(x_0,y_0,u) \leq \delta_c(u + m - R_c + R_c - u).$$

But this contradicts the assumption $m < 0$.

If $m < 0$ is attained on the boundary $\partial\Omega_\ell$ we know $|x_0| \leq \ell$, since $w(\pm\ell, y) \geq 0$. For Dirichlet boundary conditions $y_0 \in \{0,\pi\}$ is impossible. For Neumann conditions Hopf's maximum principle leads to a contraction, whereas in the case of periodicity any boundary point can be transformed into interior points by periodic continuation. Thus, we have shown $0 \leq m \leq w(x,y)$ for all $(x,y) \in \Omega_\ell$, which is the desired result. QED

We also show that the derivative u_x decays exponentially towards a finite ball in H. This implies that there is a cylinder $\mathbf{R} \times \mathcal{B}$ in the extended phase space $\mathbf{R} \times E \times H$, such that for every solution $u : I_\ell \to E$ the pair $(u(x), u_x(x))$ stays outside the bounded set $\mathcal{B}$ only close to the ends of I_ℓ. Particularly, the essential set lies in $\mathbf{R} \times \mathcal{B}$.

Lemma 5.3
Let $u : I_\ell \to E$ be a solution of (2.1) with $\|F(x,u(x))\|_0 \leq C_F$. Then u_x satisfies the following estimate:

$$\|u_x(x)\|_0 \leq 2\coth(2\mu_1\ell)\left[\frac{\mu_1 C_F}{\lambda_1} + e^{-(\mu_1+\alpha)(\ell-x)}\|u(\ell)\|_E + e^{-(\mu_1-\alpha)(\ell+x)}\|u(-\ell)\|_E\right] \quad (5.6)$$

for $x \in I_\ell$.

Proof: The function u has the explicit representation

$$u(x) = \textstyle\int_{-\ell}^{\ell} G_\ell(x,t)e^{\alpha(x-t)}F(t,u(t))\,dt +$$
$$(\sinh(2B\ell))^{-1}\left[\sinh(B(\ell{+}x))e^{\alpha(x-\ell)}u(\ell) + \sinh(B(\ell{-}x))e^{\alpha(x+\ell)}u(-\ell)\right],$$

where $B = (A+\alpha^2)^{1/2}$ and $G_\ell(x,t) = [B\sinh(2B\ell)]^{-1}\sinh(B(\ell+t))\sinh(B(\ell-x))$ for $-\ell \leq t \leq x \leq \ell$ and $G_\ell(x,t) = G_\ell(t,x)$ else. We have the estimates

$$\|G_\ell(x,t)\|_{H\to H} \leq \frac{\coth(2\mu_1\ell)}{2\mu_1}\,e^{-\mu_1|x-t|}, \quad \|\frac{\partial}{\partial x}G_\ell(x,t)\|_{H\to H} \leq \coth(2\mu_1\ell)e^{-\mu_1|x-t|}.$$

Differentiating the explicit expression for $u(x)$ and using these estimates along with standard estimates for the influence of the boundary values $u(\pm\ell)$, the result is obtained. QED

Further on we will restrict our view to that part of the solution which already lies in the ball $\mathcal{B} = \mathcal{B}_{R_1,R_d} = \{\, (u, u_x) \in E \times H \;:\; \|u\|_E \le R_1,\ \|u_x\|_0 \le R_d \,\}$. By decreasing ℓ we can assume that the whole solution $u : I_\ell \to E$ lies in $\mathcal{B}$. For such solutions we now show that they can be very well approximated by solutions on the essential manifold, namely in a way similar to estimate (5.2), however the exponential decay rate is now dependent on the dimension of the essential manifold, in particular we can replace μ^j by the much larger value $\mu_N = (\lambda_N + \alpha^2)^{1/2}$ if $\dim M_E = 2N + 1$. We call this property the *weak normal hyperbolicity* of the essential manifold. In [Mi90] this property was established for general center manifolds. Here, we generalize the theory to essential manifolds, giving an analogue to the exponential tracking property of inertial manifolds, see [FST89].

From now on it is sufficient to work with the modified problem (4.1):

$$u_{xx} - 2\alpha u_x - Au + \widetilde{F}(x, u) = 0,$$

where Θ is the global Lipschitz constant of $\widetilde{F}$ with respect to $u \in E$. The general idea is to continue a given $u : I_\ell \to E$ to a function $\overline{u} : \mathbf{R} \to E$ and to find a solution $\widetilde{u} \in M_E$ such that $\overline{u} - \widetilde{u}$ is small in the norm $|\cdot|_b$ of E^b.

Theorem 5.4
Let all the assumptions of Theorem 4.1 be satisfied. Then, for every R_d, $R_1 > 0$ there is a constant C such that:

a) *For all $\ell > 1$ and all solutions $u : I_\ell \to E$ with $(u(x), u_x(x)) \in \mathcal{B}_{R_1,R_d}$, $x \in I_\ell$, there is a solution $\widetilde{u} : \mathbf{R} \to E$ which lies in M_E and satisfies*

$$\|u(x) - \widetilde{u}(x)\|_E \le C \max\{e^{-(\mu_N+\alpha)(\ell-x)}, e^{-(\mu_N-\alpha)(\ell+x)}\} \quad \textit{for } |x| \le \ell - 1.$$

b) *For solutions $u : [0, \infty) \to E$ with $(u(x), u_x(x)) \in \mathcal{B}_{R_1,R_d}$, $x > 0$, there exists a solution $\widetilde{u}$ which lies on M_E and satisfies*

$$\|u(x) - \widetilde{u}(x)\|_E \le C e^{-(\mu_N-\alpha)x} \quad \textit{for } x > 1.$$

Proof: We first prove part **a)**. Therefore we use a continuation $\overline{u} : \mathbf{R} \to E$ of $u : I_\ell \to E$ to the whole real line:

$$\overline{u}(x) = \begin{cases} \phi(x)u(x) & \text{for } |x| \le 0, \\ 0 & \text{else;} \end{cases} \qquad \text{where } \phi(x) = \begin{cases} 1 & \text{for } |x| \le \ell - 1, \\ \cos^2 \frac{\pi}{2}(\ell - |x|) & \text{for } |x| \in [\ell-1, \ell], \\ 0 & \text{else.} \end{cases}$$

Inserting $\overline{u}$ into eqn. (4.1) we find $\overline{u}_{xx} - 2\alpha\overline{u}_x - A\overline{u} + \widetilde{F}(x, \overline{u}) = \overline{r}(x)$, where the residuum $\overline{r}$ is given by

$$\overline{r}(x) = \widetilde{F}(x, \phi u) - \phi\widetilde{F}(x, u) + (\phi'' - 2\alpha\phi')u + 2\phi' u_x.$$

Note that the residuum vanishes for $|x| \leq \ell - 1$ and satisfies $\|\overline{r}(x)\|_0 \leq C_1$ for some $C_1 = C_1(R_1, R_d)$ which is independent of u.

To compare $\overline{u}$ with a solution on the essential manifold M_E we recall the fixed point equation (4.6) which was used to find the solutions on M_E from their initial data $(\xi, \eta) = (\widetilde{u}_0(\tau), \widetilde{u}_{0x}(\tau)) \in E_0 \times H_0$. Here $E_0 = PE$ and $H_0 = PH$ where P is the projection onto the first N eigenfunction of A. Moreover, in the proof of Thm. 4.1 we have chosen the decay parameter $b \in (\mu_N, \mu_{N+1})$, such that (4.6) can be considered as a fixed point problem in the exponentially weighted space E^b.

We pick $\widetilde{u}$ such that $\delta u(x) = \overline{u}(x+\tau) - \widetilde{u}(x+\tau)$ satisfies $P(\delta u(0)) = P(\delta u_x(0)) = 0$ where $\tau = \alpha(\ell-1)/b$. Then δu is a solution of the equation

$$\delta u = K(0, 0, \widetilde{F}(\cdot + \tau, \overline{u}(\cdot + \tau)) - \widetilde{F}(\cdot + \tau, \overline{u}(\cdot + \tau) + \delta u)) - K(0, 0, \overline{r}(\cdot + \tau)), \tag{5.7}$$

where $K = (K_0, K_1) : E_0 \times H_0 \times E^b \to E^b$ is defined right after (4.6). Note that τ was chosen such that

$$|\overline{r}(\cdot + \tau)|_{H^b} = \sup\{ \|e^{-b|x| - \alpha x} \overline{r}(x + \tau)\|_0 \; : \; x \in \mathbf{R} \} \leq C_1 e^{-(\ell-1)(b^2 - \alpha^2)/b}$$

is minimal. A similar estimate holds for $|K(0, 0, \overline{r}(\cdot + \tau))|_b$, since $K(0, 0, \cdot)$ is a bounded operator from $H^b \to E^b$.

Because of our assumptions, the right–hand side of (5.7) defines a uniform contraction on E^b with Lipschitz constant $L < 1$. Thus, the unique fixed point can be estimated by $|\delta u|_b \leq L|\delta u|_b + |K(0, 0, \overline{r})|_b$. Recalling the shift with τ, we find

$$\begin{aligned} \|\overline{u}(x) - \widetilde{u}(x)\|_E &= \|\delta u(x - \tau)\|_E \leq |\delta u|_b e^{b|x-\tau| + \alpha(x-\tau)} \\ &\leq C_2 e^{b|x-\tau| + \alpha(x-\tau) - (\ell-1)(b^2-\alpha^2)/b} \\ &= C_3 \max\{e^{-(b-\alpha)(\ell+x)}, e^{-(b+\alpha)(\ell-x)}\}. \end{aligned}$$

Using $b > \mu_N$ and $\overline{u}(x) = u(x)$ for $|x| \leq \ell - 1$ the result is established.

For part **b)** we proceed analogously. We use an extension $\overline{u} : \mathbf{R} \to E$ such that $\overline{u}(x) = 0$ for $x < 0$. We obtain a residuum $\overline{r}$ with $\overline{r}(x) = 0$ for $x \geq 1$ and $\|\overline{r}(x)\|_0 \leq C_1$ else. However, when transferring the system into an integral equation we have to impose different behavior at infinity. We define

$$E^b_* = \{ u \in C^0(\mathbf{R}, E) \; : \; |u|^*_b < \infty \}, \quad |u|^*_b = \sup\{ \|e^{(b-\alpha)x} u(x)\|_E \; : \; x \in \mathbf{R} \}.$$

We are looking for $\delta u = \overline{u} - \widetilde{u} \in E^b_*$ solving a fixed point problem analogous to (5.7). Therefore we have to find the Green's functions K^*_0 and K^*_1 which are appropriate for the prescribed decay property. Since K_1 used above maps E^b_{1*} into itself, we only have to modify K_0. The unique decaying solution of $u_{0xx} - 2\alpha u_{0x} - Au_0 + g_0(x) = 0$ with $g_0 \in E^b_*$ is given by

$$u_0(x) = (K^*_0 g_0)(x) = \int_x^\infty B_0^{-1} \sinh(B_0(x - t)) e^{\alpha(x-t)} g_0(t)\, dt.$$

Thus, it is no longer allowed to prescribe initial data, since the asymptotic behavior is fixed. As in Thm. 4.1 we obtain

$$\|K_0^*\|_{E^b_{0,*}\to E^b_{0,*}} + \|K_1\|_{E^b_{1,*}\to E^b_{1,*}} \le \frac{1}{\mu_N(b-\mu_N)} + \frac{1}{\mu_{N+1}^2 - b^2}.$$

Additionally, the mapping $F^* : \delta u \to \widetilde{F}(\cdot,\overline{u}) - \widetilde{F}(\cdot,\overline{u}+\delta u)$ is a Lipschitz continuous mapping from E^b_* into itself with Lipschitz constant Θ.

Together, we see that δu is the unique solution of the fixed point equation $\delta u = K^*(F^*(\delta u) - \overline{r})$ and hence satisfies $|\delta u|_{b*} \le C_4$. Now, the result follows as above. QED

6 Various applications

In this last section we want to give three applications for the theory of essential manifolds. We restrict ourselfs to the case of $f = f(x, y, u)$ being independent of x. Then the system can be thought of as a dynamical system in the phase space $E \times H$. Especially, the essential set $\mathcal{E}$ is the invariant subset containing all bounded solutions.

Further on we only consider the case $\alpha \le 0$, the case $\alpha > 0$ can be handled by changing x into $-x$. We define the function

$$V(u, u_x) = \int_0^\pi \left[\frac{1}{2}(u_x^2 - u_y^2 - \beta u^2) + g(x,u)\right] dy, \quad \text{where } g(y,u) = \int_0^u f(y,v)\,dv.$$

Differentiating with respect to x shows that V is nonincreasing along solutions of (2.1), viz., $\frac{d}{dx}V = 2\alpha\|u_x\|_0^2$. In particular, for $\alpha = 0$ the function V is constant along solutions. Below we will exhibit that V can be interpreted as a Hamiltonian function.

6.1 Asymptotic convergence

For $\alpha < 0$ this Lyapunov function was used in [BMPS91] in order to show that all (semi–) bounded solutions $u : [0,\infty) \to E$ actually converge to a simple steady state for $x \to \infty$. The proof developed there deals with the full infinite dimensional problem. We now give a simpler proof by using the essential manifold and its weak normal hyperbolicity.

We first use Theorem 5.4b) which says that every semi–bounded solution u is exponentially approximated by a solution $\widetilde{u}$ on the essential manifold. Thus, the ω–limit sets of both solutions are equal. To prove the asymptotic convergence it is hence sufficient to consider solutions on the essential manifold M_E. The flow on M_E is described by the ordinary differential equation

$$u_{0xx} + 2|\alpha|u_{0x} - A_0u_0 + F_0(u_0 + h(u_0, u_{0x})) = 0. \tag{6.1}$$

Restricting the Lyapunov function V to M_E we find the Lyapunov function $V_0(u_0, u_{0x}) = V(u_0 + h(u_0, u_{0x}), u_{0x} + g(u_0, u_{0x}))$. For such gradient–like systems it is well–known that the ω–limit set of every bounded semi–orbit is a compact, connected union of equilibria, since V_0 decreases strictly along every non–constant solution.

The equilibria of our problem are given as $(u, u_x) = (v, 0)$ where v solves the ordinary differential equation

$$v_{yy} - \beta v + f(y, v) = 0 \quad \text{and boundary conditions.} \tag{6.2}$$

In [BMPS91] it is shown that, in the case of Dirichlet or Neumann boundary conditions, the only possible compact and connected sets of equilibria are either a single point or a C^1–curve in E. Since the linearization of (6.2) at each point in the curve is a Sturm–Liouville operator with simple eigenvalues we see that the curve is normally hyperbolic. Note that if $\widehat{A}w = -w_{yy} + \beta w - \partial_u f(y, v(y))w$ has eigenvalues $\widehat{\sigma}_n$, $n \in \mathbf{N}$, then the equilibrium $(v, 0)$ of (2.1) has the eigenvalues $\widehat{\lambda}_n^\pm = -\alpha \pm \sqrt{\widehat{\sigma}_n + \alpha^2} \notin i\mathbf{R} \setminus \{0\}$.

Since the sets of equilibria are the same for the full system and for the reduced system on the essential manifold, we conclude that the ω–limit set of any semi–bounded orbit of (6.1) is either a single equilibrium or a normally hyperbolic closed curve of equilibria. Since we are now in a finite dimensional setting we can use the results of Aulbach [Au84], which imply convergence against one single equilibrium.

Thus, we have proved that, in the case of Dirichlet or Neumann boundary conditions, every semi–bounded solution converges to a single equilibrium. This implies trivially that every bounded solution is heteroclinic, i.e., it converges for $x \to \infty$ and for $x \to -\infty$ to an equilibrium. A similar result should hold for periodic boundary conditions and f independent of y, however there connected sets of equilibria could be two–dimensional surfaces due to the rotational symmetry $y \to y + \phi \,(\mathrm{mod}\, \pi)$ with $\phi \in [0, \pi)$, if f is independent of y, cf. [CMV89] and Prop. 4.2a).

6.2 Periodic solutions in the reversible case

The case $\alpha = 0$ is special, since the system is reversible, i.e., the system remains unchanged when x is changed in to $-x$. For every solution $u = u(x)$ we know that also $u(-x)$ is a solution. Defining the reflection $\mathcal{R} : E \times H \to E \times H; (u, u_x) \to (u, -u_x)$ we see that the essential set satisfies $\mathcal{R}\mathcal{E} = \mathcal{E}$. For reversible ordinary differential equations (with C^1–smoothness) Fiedler [Fi91] constructed an index which allows to study global bifurcation of periodic orbits. The simplest result can be described as follows:

Assume that the set of all bounded solutions is bounded and that all stationary points are hyperbolic except for one, let us say S_0. The linearization at S_0 should have one pair of purely imaginary eigenvalues $\pm i\rho$ while all the other eigenvalues are off the imaginary axis. Then, the system has a periodic orbit for each period $T > 2\pi/\rho$.

Further conclusions on the minimal period can be obtained also.

We show that this result can be applied to our elliptic problem. We note that the associated reduced problem on the essential manifold is reversible due to Prop. 4.2b. For simplicity we consider the nonlinearity $f(u) = \kappa^2 u - u^3$ and assume Dirichlet boundary conditions $u(0) = u(\pi) = 0$. The set of equilibria can be found by studying (6.2), and it is well–known, that for $\kappa \leq 1$ the only equilibrium in $u \equiv 0$. For $\kappa \in (M, M+1)$, $M \in \mathbf{N}$, there are $2M + 1$ equilibria with $j = 0, \ldots, M$ distinct pairs of purely imaginary eigenvalues, respectively. Especially, for $\kappa \in (1, 2)$ there are two non–trivial hyperbolic equilibria and $u = 0$ has the single eigenvalue pair $\pm i\kappa$ on the imaginary axis. Now, we can restrict the whole system onto an essential manifold M_E, such that the essential set $\mathcal{E}$ remains unchanged. The three equilibria for the reduced problem are still the same including their first $2N = \dim M_E$ eigenvalues. Thus, the index theory can be applied to the reduced system (6.1) with $\alpha = 0$.

As a conclusion, for the elliptic problem $\Delta u + \kappa^2 u - u^3 = 0$ with $\kappa \in (1, 2)$ we have established the existence of a periodic solution for every period $T > 2\pi/\kappa$. Of course, we do not know whether the period is minimal, and therefore the solutions with periods T and $kT, k \in \mathbf{N}$, could be identical. For $0 < \kappa - 1 \ll 1$ this result follows from a local analysis on the center manifold, see [Ki82]. The associated center manifold is two–dimensional and the origin is enclosed by a family of periodic orbits which limit with period going to infinity in a pair of heteroclinic solutions, which join the two bifurcated saddle points.

In [BBT83] comparable global results are obtained by using the variational methods and symmetrization. Using nodal–like properties for the elliptic problem Healey & Kielhöfer [HK91] obtianed global bifurcation branches with fixed periods for parameter dependent systems.

6.3 The Hamiltonian structure

Finally we want to comment on the Hamiltonian nature of the problem in the case $\alpha = 0$. Therefore we let $w = u_x$ and $H(u, w) = V(u, w)$, then the equations can be rewritten as

$$u_x = \frac{\partial}{\partial w} H(u, w) = w, \quad w_x = -\frac{\partial}{\partial u} H(u, w) = Au - F(u).$$

Here the derivative with respect to u has to be understood as a variational derivative involving partial integration with respect to y and using the boundary conditions. A general theory of Hamiltonian structures for second order elliptic variational problems is given in [Mi91a]. There it is shown that the reduction of such a problem to a finite dimensional center manifold leads in a natural way to a reduced Hamiltonian system.

A similar property is asked for essential manifolds in [BM91]. Here we want to indicate the arising problems when reducing the Hamiltonian structure to the essential manifold M_E. We recall that, in general, it is only possible to construct M_E as a C^1–manifold, in particular M_E can be written as a graph of the C^1–function $\widetilde{h} = (h, g)$ over the finite–

dimensional base space $E_0 \times H_0$. We introduce the short hand notation $z = (u, w) \in Z = E \times H$ and $z_0 = (u_0, w_0) \in Z_0 = E_0 \times H_0$. Then, $M_E = \{ z_0 + \widetilde{h}(z_0) \in Z \ : \ z_0 \in Z_0 \}$.

The symplectic structure for the full space $Z = E \times H$ is the canonical one

$$\omega_{can}(\delta z, \delta\widetilde{z}) = \textstyle\int_0^\pi (\delta u \delta\widetilde{w} - \delta\widetilde{u}\delta w)\, dy = \langle \delta u, \delta\widetilde{w}\rangle - \langle \delta\widetilde{u}, \delta w\rangle = \langle\langle \delta z, \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)\delta\widetilde{z}\rangle\rangle,$$

where $\delta z, \delta\widetilde{z} \in E \times H$. The Hamiltonian equations can now be written as $\frac{d}{dx}z = X_H(z)$, where the vector field X_H is defined via $\omega_{can}(X_H(z), \delta\widetilde{z}) = DH(z)[\delta\widetilde{z}]$ for all $\delta\widetilde{z} \in E \times H$.

The induced structure ω_E on the essential manifold is obtained by restricting ω_{can} to the tangent space of M_E. Taking $z_0 \in E_0 \times H_0$ as coordinates in M_E we find the vector tangent at M_E and associated to δz_0 as $\delta z = z_0 + D\widetilde{h}(z_0)[\delta z_0]$. Thus, the reduced structure reads

$$\omega_{E\, z_0}(\delta z_0, \delta\widetilde{z}_0)) = \langle\langle \delta z_0, \Omega_E(z_0)\delta\widetilde{z}_0\rangle\rangle \ \text{ with } \Omega_E = \Omega_{0\, can} + D\widetilde{h}(z_0))^*\Omega_{1\, can}D\widetilde{h}(z_0). \tag{6.3}$$

Here * means the adjoint operator and $\Omega_{j\, can}$ is the restriction of Ω_{can} to $E_j \times H_j$.

For Hamiltonian systems one needs that ω_E is a symplectic structure, which means that it is bilinear, skew–symmetric, closed, and nondegenerate. The first two properties are immediate. The closedness is usually expressed by the vanishing exterior derivative $\mathbf{d}\omega_E = 0$. Since in our case M_E is only a C^1–manifold, the exterior derivative is not defined. Thus, we have the weaker condition that $\int_{S_0} \omega_E = 0$ for each closed two–dimensional surface $S_0 \subset Z_0$. (For smooth manifolds this condition is equivalent to $\mathbf{d}\omega_E = 0$ by Stokes' theorem for differential forms.) Here the weak closedness follows from the fact that ω_E is the restriction of the closed two–form ω_{can} and that $\int_{S_0} \omega_E = \int_S \omega_{can} = 0$ where $S = \{ z_0 + \widetilde{h}(z_0) \in Z \ : \ z_0 \in S_0 \}$.

The non–degeneracy of ω_E means that the matrix $\Omega_E(z_0) \in \mathcal{L}(Z_0, Z_0)$ is invertible for all z_0. To establish this property we use the fact that the Lipschitz constants of the function $\widetilde{h} = (h, g)$ can be made small. In Remark 4.3 we have shown $\|D\widetilde{h}\|_{Z_0 \to Z_1} \le L(N) \sim 1/(\lambda_{N+1} - \lambda_N)$.

Proposition 6.1
If N is chosen such that $\lambda_N^{1/2} L(N)^2 < 1$, then ω_E is non–degenerate.

Proof: The matrix Ω_E can be written as $\Omega_E = \Omega_{0\, can}[I - R]$ with $R = \Omega_{0\, can}D\widetilde{h}^*\Omega_{1\, can}D\widetilde{h}$. Here $\Omega_{j\, can} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ satisfy

$$\|\Omega_{0\, can}\|_{Z_0 \to Z_0} = \lambda_N^{1/2}, \quad \|\Omega_{1\, can}\|_{Y_1 \to Y_1} = 1.$$

where $Y_1 = H_1 \times H_1$. This implies the estimate

$$\|R\|_{Z_0 \to Z_0} \le \|D\widetilde{h}\|_{Z_0 \to Y_1} \|\Omega_{1\, can}\|_{Y_1 \to Y_1} \|D\widetilde{h}^*\|_{Y_1 \to Z_0} \|\Omega_{0\, can}\|_{Z_0 \to Z_0} \le \lambda_N^{1/2} L(N)^2.$$

According to our assumption the norm of R is less than 1; and hence Ω_E is invertible by the Neumann series $\Omega_E^{-1} = -(\sum_0^\infty R^k)\Omega_{0\, can}$. QED

Note that the gap condition $\lambda_N^{-1/4}(\lambda_{N+1}-\lambda_N) \to \infty$ is sufficient to guarantee the existence of an N needed in the above proposition. This condition is a little stronger than (4.4) which was used for the existence proof for the essential manifold. Since $\lambda_N \approx N^2$ we have $\lambda_N^{-1/4}(\lambda_{N+1} - \lambda_N) \approx N^{1/2}$ which is sufficient.

Using the restricted Hamiltonian $H_E(z_0) = H(z_0 + \widetilde{h}(z_0))$, which is a C^1–function, and ω_E we can define uniquely the induced Hamitonian vector field X_{H_E} via $\omega_{E\,z_0}(X_{H_E}(z_0), \delta\widetilde{z}_0) = DH_E(z_0)[\delta\widetilde{z}_0]$ for all $\delta\widetilde{z}_0$. From this expression we obtain only that X_{H_E} is a continuous vector field. However, the restriction procedure has the nice property that X_{H_E} is simply the restriction of the vector field of the full problem, X_H (cf. [Mi91a, Thm.4.1]). Hence, in the coordinates $z_0 \in Z_0$ we have

$$\frac{d}{dx}\begin{pmatrix} u_0 \\ w_0 \end{pmatrix} = X_{H_E}(u_0, w_0) = \begin{pmatrix} w_0 \\ A_0 u_0 + F_0(u_0 + h(u_0, w_0)) \end{pmatrix},$$

which is a C^1–vector field.

This vector field defines a local C^1–flow Φ_x^0 in the ball $\mathcal{B} \times H_0 \subset E_0 \times H_0$ where the essential manifold was constructed. (We have to work with the unmodified problem, since the multiplication of the nonlinearity with a cut–off function destroys energy conservation.) Obviously, H_E is constant along solutions $z_0(x) = \Phi_x^0(z_0(0))$, i.e. $H_E \circ \Phi_x^0 = H_E$. Another important property of Hamiltonian systems is that the symplectic structure is conserved.

Proposition 6.2
For all z_0 and all x such that $\Phi_x^0(z_0)$ is defined we have

$$\omega_{E\,z_0}(\delta\widehat{z}_0, \delta\widetilde{z}_0) = \omega_{E\,\Phi_x^0(z_0)}(D\Phi_x^0(z_0)\delta\widehat{z}_0, D\Phi_x^0(z_0)\delta\widetilde{z}_0) \ \ \textit{for all } \delta\widehat{z}_0, \delta\widetilde{z}_0 \in Z_0.$$

Proof: We call the right–hand side $\rho(x)$. The assertion is true for $x = 0$, hence it suffices to show $\rho'(x) = 0$.

We note that $z(x) = \Phi_x^0(z_0) + \widetilde{h}(\Phi_x^0(z_0)) \in E,\ x \in I$, defines a solution of the full problem. Moreover, $\widehat{z}(x) = [I + D\widetilde{h}(\Phi_x^0(z_0))]D\Phi_x^0(z_0)\delta\widehat{z}_0$ and the similarly defined $\widetilde{z}(x)$ are solutions of the first variational equation around the solution z, i.e., $\widehat{z}_{xx} - A\widehat{z} + DF(z(x))[\widehat{z}] = 0$. Since ω_E is the restriction of ω_{can} we have $\rho(x) = \omega_{can}(\widehat{z}(x), \widetilde{z}(x))$. Hence,

$$\begin{aligned} \rho'(x) = \tfrac{d}{dx}(\langle \widehat{u}(x), \widetilde{w}(x)\rangle - \langle \widehat{w}(x), \widetilde{u}(x)\rangle) &= \langle \widehat{u}_x, \widetilde{u}_x\rangle + \langle \widehat{u}, \widetilde{u}_{xx}\rangle - \langle \widehat{u}_x, \widetilde{u}_x\rangle - \langle \widehat{u}_{xx}, \widetilde{u}_x\rangle \\ &= \langle \widehat{u}, (A - DF(u))\widetilde{u}\rangle - \langle (A - DF(u))\widehat{u}, \widetilde{u}\rangle \ = \ 0. \end{aligned}$$

The last equality holds since A and DF are symmetric. QED

These results show that the reduced problem can be viewed as a Hamiltonian system. Yet, the low degree of differentiability leads to complications. Many results in Hamiltonian systems theory rely heavily on smoothness properties of the systems. In particular, the bifurcation results for periodic orbits indicated in [BM91] would need more regularity. However, as demonstrated in the proposition above it seems possible to circumvent this difficulty by doing the calculus (the differentiations) in the full problem and then interpret the result for the reduced finite dimensional system.

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