RATE-INDEPENDENT EVOLUTIONS AND MATERIAL MODELING
(Special Section of EQUADIFF 2007) Eds.: T.Roubíček, U.Stefanelli.
Pubblicazione IMATI-CNR, 29PV10/27/0, Pavia, 2010, pp.23–32, ISSN 1772-8964

On an evolutionary model for complete damage based on energies and stresses

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Abstract

A recent model [BMR09] allows for complete damage, such that the deformation is not well-defined. The evolution can be described in terms of energy densities and stresses. We introduce the notion of *generalized energetic solution* and show how the existence theory can be generalized to convex, but non-quadratic elastic energies. We also discuss Γ-convergence from partial to complete damage.

Keywords: generalized energetic solution, rate independent energetic system, complete damage, Gamma convergence.

1 Introduction

There is a rich literature [Ort85, FrM93, DPO94, FrN96, DMT01, MaA01, HaS03] on rate-independent mechanical models for damage in brittle materials, and recently several mathematical approaches [FrM98, FKS99, FrG06] were developed, in particular the abstract theory of rate-independent processes [MiT99, MiT04, Mie05] proved very helpful as it allows one to employ the machinery of incremental minimization.

Here we want to contribute to the models discussed in [MiR06, BMR09, MRZ10]. Let $u : \Omega \to \mathbb{R}^d$ be the displacement and $z : \Omega \to [0, 1]$ the damage variable, then the rate-independent system is given by the triple $(\mathcal{F}\times\mathcal{Z}, \mathcal{E}, \mathcal{D})$, where $u \in \mathcal{F}$, $z \in \mathcal{Z}$. The energy-storage functional has the form

$$\mathcal{E}_\delta(t, u, z) = \int_\Omega W(x, \mathbf{e}(u_\mathrm{D}(t){+}u)(x), z(x)) + \delta|\mathbf{e}(u_\mathrm{D}(t){+}u)|^p \,\mathrm{d}x + \mathcal{G}(z),$$

and the dissipation is $\mathcal{D}(z, \widehat{z}) = \int_\Omega D(x, z(x), \widetilde{z}(x))\,\mathrm{d}x$. For $\delta > 0$ existence of energetic solutions (u_δ, z_δ) is known [MiR06] for general W. The limit passage for $\delta \to 0$ in the sense of Γ-limits works under the assumption that $e \mapsto W(x, e, z)$ is quadratic [BMR09, MRZ10]. The difficulty is that W is not coercive, hence in the limit $\delta \to 0$ we are not able to control u_δ, and convergence should only be valid for z_δ. The task is to define a limit equation in terms of z. In particular one needs a replacement of the power of the external forces giving the limit of $\partial_t \mathcal{E}_\delta(u_\delta(t), z_\delta(t))$.

Here we discuss the changes needed to generalize from a quadratic $W(x, \cdot, z)$ to arbitrary strictly convex potentials with p growth from above. The main idea is to use a reduced functional $\mathcal{I}_\delta(t, z)$ avoiding the usage of u; however, to keep control over stresses one introduces an auxiliary functional $\mathcal{V}_\delta : \mathrm{L}^p(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \times \mathcal{Z} \to \mathbb{R}$ such that

$$\mathcal{I}_\delta(t, z) = \min\left\{\, \mathcal{E}_\delta(t, \widetilde{u}, z) \mid u \in \mathcal{F} \,\right\} = \mathcal{V}_\delta(\mathbf{e}(u_\mathrm{D}(t)), z) + \mathcal{G}(z),$$

and $\mathrm{D}_e \mathcal{V}_\delta(\mathbf{e}(u_\mathrm{D}(t)), z(t)) \in \mathrm{L}^{p/(p-1)}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ gives the equilibrium stress.

In $(\mathcal{Z}, \mathcal{I}_\delta, \mathcal{D})$ it is possible to pass to the Γ-limit for $\delta \to 0$ with respect to the weak convergence in $\mathcal{Z} \subset \mathrm{W}^{1,r}(\Omega)$. However, the Γ-limit $\mathfrak{I}(t, \cdot)$ loses in general differentiability in t, since we are not able to show that the Γ-limit $\mathfrak{V}(e, \cdot)$ of $\mathcal{V}_\delta(e, \cdot)$ remains differentiable in e. Nevertheless, the convexity of $\mathfrak{V}(\cdot, z)$ allows us to characterize the Clarke differential using the left and right partial derivative in t:

$$\partial_t^{\mathrm{Cl}} \mathfrak{I}(t, z) = \Big[\partial_t^- \mathfrak{I}(t, z), \partial_t^+ \mathfrak{I}(t, z)\Big],$$

where $\partial_t^\pm \mathfrak{I}(t, z) = \pm \sup\big\{\, \langle \pm\sigma, \mathbf{e}(\dot u_\mathrm{D}(t))\rangle \mid \sigma \in \partial_e^{\mathrm{sub}} \mathfrak{V}(\mathbf{e}(u_\mathrm{D}(t)), z) \,\big\}$.

We generalize the notion of energetic solutions [Mie05] to *generalized energetic solutions* by keeping stability (S) and replacing the energy balance by

$$\mathfrak{I}(t, z(t)) + \mathrm{Diss}_\mathcal{D}(z, [0, t]) = \mathfrak{I}(0, z(0)) + \int_0^t p(\tau)\,\mathrm{d}\tau,$$

where p has to satisfy $p(\tau) \in \partial_\tau^{\mathrm{Cl}} \mathfrak{I}(\tau, z(\tau))$ a.e. in $[0, T]$, see Definition **4**.3. Theorem **4**.4 establishes existence of such generalized energetic solutions to $(\mathcal{Z}; \mathfrak{I}, \mathcal{D})$. Moreover, assuming that a certain conjecture holds, we show that a subsequence $(z_{\delta_j})_{j\in\mathbb{N}}$ converges to weak energetic solution for $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$.

2 Setup of the model

The body $\Omega \subset \mathbb{R}^d$ is described by a bounded Lipschitz domain. The state of the system is described by the displacement $\widetilde{u} : \Omega \to \mathbb{R}^d$ and the scalar damage variable $z : \Omega \to [0, 1]$, where $z = 1$ denotes no damage and $z = 0$ means that the maximal damage has been reached (all microscopic breakable structures are broken). The displacement $\widetilde{u}$ will satisfy time-dependent Dirichlet boundary conditions on $\Gamma_\mathrm{D} \subset \partial\Omega$ via $u_\mathrm{D} \in \mathrm{C}^1([0, T], \mathrm{W}^{1,p}(\Omega))$ in the form

$$\widetilde{u}(t) = u_\mathrm{D}(t) + u(t) \quad \text{with } u(t) \in \mathcal{F} = \left\{\, v \in \mathrm{W}^{1,p}(\Omega) \mid v|_{\Gamma_\mathrm{D}} \equiv 0 \,\right\}.$$

We also use the infinitesimal strain tensor $\mathbf{e}(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^\mathsf{T}\big)$ and set

$$\mathbf{e}_\mathrm{D}(t) = \mathbf{e}(u_\mathrm{D}(t)) \text{ and } \dot{\mathbf{e}}_\mathrm{D}(t) = \mathbf{e}(\dot u_\mathrm{D}(t)) \text{ where } \dot{} = \partial_t.$$

The stored energy of the system is given via the functional

$$\mathcal{E}(t,u,z) = \int_\Omega W(x, \mathbf{e}_{\mathrm{D}}(t,x){+}\mathbf{e}(u)(x), z(x))\,\mathrm{d}x + \mathcal{G}(z)$$
$$\text{with } \mathcal{G}(z) = \int_\Omega b(z(x)) + \kappa(x)|\nabla z(x)|^r\,\mathrm{d}x,$$

where $\kappa \in \mathrm{L}^\infty(\Omega)$ and $\kappa(x) \geq c_0$ a.e. Thus, the suitable space for the deformation states is $\mathcal{Z} = \{\, z \in \mathrm{W}^{1,r}(\Omega) \mid 0 \leq z \leq 1 \,\}$. The additional term b is intended to model cohesive effects and should satisfy $b'(z) \leq 0$, i.e., if the stresses in the material are released then the damage may heal ($\dot{z} > 0$) by using up some energy.

The stored energy density $W : \Omega \times \mathbf{E}_d \times [0,1] \to \mathbb{R}$, where $\mathbf{E}_d = \mathbb{R}^{d\times d}_{\mathrm{sym}}$, is a Carathéordory function satisfying

$$\forall\,(x,z) \in \Omega : \quad W(x,\cdot,z) \in \mathrm{C}^1(\mathbf{E}_d), \tag{1a}$$
$$\exists\, C>0\;\forall\,(x,e,z) : \quad W(x,e,z) \leq C|e|^p + C, \tag{1b}$$
$$\forall\,(x,z) : \quad e \mapsto W(x,e,z) \text{ is strictly convex}, \tag{1c}$$
$$\forall\,(x,e) : \quad z \mapsto W(x,e,z) \text{ is nondecreasing}, \tag{1d}$$
$$\exists\, c_1,c_2\;\forall\,(x,e,z) : \; |\partial_e W(x,e,z)| \leq c_1(W(x,e,z){+}c_2)^{1-1/p}. \tag{1e}$$

Condition (1d) means that the material becomes weaker if damage increases, and (1e) is called "stress control", since it allows us to control the size of the stresses in terms of the energy alone, uniformly in (x,z). A typical function W has the form

$$W(x,e,z) = W_0(x,e) + a(z)W_1(x,e),$$

where W_0 and W_1 are smooth and convex, W_0 may be non-coercive while W_1 is coercive, $a(z) \geq cz^\alpha$ and $a'(z) \geq 0$.

Finally we describe the dissipation functional $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ via

$$\mathcal{D}(z_0,z_1) = \int_\Omega D(x, z_0(x), z_1(x))\,\mathrm{d}x,$$

where, for each x, D satisfies the triangle inequality and the coercivity $D(x,z,\widetilde{z}) \geq C|z{-}\widetilde{z}|$. The typical choice is $D(x,z,\widetilde{z}) = \delta_+(z{-}\widetilde{z})$ for $\widetilde{z} \leq z$ and $\delta_-(\widetilde{z}{-}z)$ for $z \leq \widetilde{z}$, where $\delta_+ \in (0,\infty)$ and $\delta_- \in (0,\infty]$. Here $\delta_- = \infty$ forbids healing, which can only take place if $\delta_- + b'(z) < 0$ for some z.

With these functionals we define notion of energetic solution [MiT99, MiT04] (see also the surveys [Mie05, MiR09]) for the rate-independent energetic system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, where $\mathcal{Q} = \mathcal{F}\times\mathcal{Z}$. A mapping $q = (u,z) : [0,T] \to \mathcal{Q}$ is called **energetic solution** if for all $t \in [0,T]$ we have **stability (S)** and **energy balance (E)**:

$$\begin{array}{ll}\textbf{(S)} & \forall\,\widetilde{q} = (\widetilde{u},\widetilde{z}) \in \mathcal{Q} : \quad \mathcal{E}(t,q(t)) \leq \mathcal{E}(t,\widetilde{q}) + \mathcal{D}(z(t),\widetilde{z});\\ \textbf{(E)} & \mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{D}}(z,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_\tau \mathcal{E}(\tau,q(\tau))\,\mathrm{d}\tau.\end{array} \tag{2}$$

Here $\mathrm{Diss}_{\mathcal{D}}(z,[r,s])$ is defined to be the supremum of $\sum_1^N \mathcal{D}(z(t_{j-1}, z(t_j))$ over all finite partitions $r \leq t_0 < t_1 \cdots t_N \leq s$. Here we use that for each q the power of the external forces $\partial_t\mathcal{E}(t,q)$ is well defined by using (1e), and (E) implicitly assumes that $t \mapsto \partial_t\mathcal{E}(t,q(t))$ is measurable.

For non-coercive problems, where u is no longer well-defined, we will see that it is the main problem how to define this partial derivative $\partial_t\mathcal{E}(t,q)$. Thus, it is an open problem whether under the above assumption a general existence result holds. However, the coercive case was solved under more general assumptions including unilateral constraints and volume forces [MiR06]. To make this theory applicable we strengthen the lower bound in (1b) to make it coercive in e for all $z \in [0,1]$.

Theorem 2.1. *If the above assumption hold with $p > 1$ and $r > d$ and if W additionally satisfies*

$$\exists\, C, c > 0\ \forall\, (x,e,z): \quad c|e|^p - C \leq W(x,e,z),$$

then for all stable initial states $q_0 \in \mathcal{Q}$ (i.e., (S) holds at $t = 0$ with $q(0)$ replaced by q_0) there exists an energetic solution $q : [0,T] \to \mathcal{Q}$ with $q(0) = q_0$ and $q \in \mathrm{L}^\infty([0,T], \mathrm{W}^{1,p}(\Omega)\times\mathrm{W}^{1,r}(\Omega)$ and $z \in \mathrm{B}([0,T], \mathrm{W}^{1,r}(\Omega))$.

3 Reformulation based on stress and energy

The approach for solving non-coercive problems was indicated already in [MiR06] and finally solved in [BMR09] under the additional assumption that W is quadratic: $W(x,e,z) = \frac{z}{2}e{:}\mathbb{C}{:}e$; however more general quadratic forms $\frac{1}{2}e{:}\mathbb{C}(z){:}e + g(z){:}e + \gamma(z)$ would work equally well. The main idea is to approximate the non-coercive case with a coercive one by setting

$$W_\delta(x,e,z) = W(x,e,z) + \delta(1+|e|^2)^{p/2}. \tag{3}$$

Then for each $\delta > 0$ there is a solution $q_\delta = (u_\delta, z_\delta)$ of the rate-independent energetic system $(\mathcal{Q}, \mathcal{E}_\delta, \mathcal{D})$. Moreover, using the stress control (1e) it is not difficult to show that there exists $C > 0$ such that for all $\delta \in (0,1)$ and all $t \in [0,T]$ we have $\mathcal{E}_\delta(t, q_\delta(t)) + \mathrm{Diss}_\mathcal{D}(z_\delta, [0,t]) \leq C$.

Now, using the theory of Γ-convergence of rate-independent energetic systems [MRS08] it is then possible to pass to the limit in the reduced system, where the displacement u is minimized out. The latter step is essential, since it is not to be expected that u_δ or $\mathbf{e}(u_\delta)$ converges in any reasonably sense. In regions where $z = 0$ holds we may have $W(x,e,0) = 0$ for a large and possibly unbounded set of strains $e \in \mathbf{E}_d$ due to the missing coercivity.

To define the reduced problem we use the strict convexity (1c) to find that $\mathcal{E}_\delta(t,\cdot,z)$ has a unique minimizer $u = U_\delta(t,z) \in \mathcal{F}$. With this we have

$$\mathcal{I}_\delta(t,z) = \int_\Omega W_\delta(x, \mathbf{e}_\mathrm{D}(t) + \mathbf{e}(U_\delta(t,z)), z)\,\mathrm{d}x + \mathcal{G}(z).$$

A classical argument [KnM08, KMZ08] shows that $\partial_t\mathcal{I}_\delta(t,z) = \partial_t\mathcal{E}_\delta(t, U_\delta(t,z), z)$.

While the limit of the energy $\mathcal{I}_\delta(t, z_\delta)$ along energetic solutions q_δ can be understood in the sense of Γ-limits, it is nontrivial to control the power

$$\begin{aligned}&\partial_t \mathcal{I}_\delta(t, z_\delta) = \textstyle\int_\Omega \sigma_\delta(t){:}\dot{\mathbf{e}}_{\mathrm{D}}(t)\,\mathrm{d}x \text{ with}\\ &\sigma_\delta(t,x) = \partial_e W(x, \mathbf{e}_{\mathrm{D}}(t,x){+}\mathbf{e}(u_\delta(t))(x), z_\delta(t,x)).\end{aligned}$$

The main observation is that the stress-control assumption (1e) and the usual energy a priori estimates provide bounds for σ_δ in $\mathrm{L}^{p/(p-1)}(\Omega, \mathbf{E}_d)$ that are independent of $\delta > 0$.

The essential idea to make the limit tractable is to introduce an auxiliary functional in which it is possible to keep control over the Γ-limit. Denote by $\mathbb{E} = \mathrm{L}^p(\Omega; \mathbf{E}_d)$ the strain space, and for $(e, z) \in \mathbb{E} \times \mathcal{Z}$ let

$$\begin{aligned}&\mathcal{H}_\delta(e, z) = \mathcal{V}_\delta(e, z) + \mathcal{G}(z) \text{ with}\\ &\mathcal{V}_\delta(e, z) = \min\left\{ \textstyle\int_\Omega W_\delta(x, e{+}\mathbf{e}(u), z)\,\mathrm{d}x \mid u \in \mathcal{F} \right\}.\end{aligned} \tag{4}$$

In fact, the functional $\mathcal{V}_\delta$ should not be considered as a functional on $\mathbb{E}$ but rather on $\mathbb{B} = \{ u|_{\partial\Omega} \mid u \in \mathcal{F} \}$, since all the other information is minimized out. Moreover, for fixed $z \in \mathcal{Z}$, the mapping $e \mapsto \mathcal{V}_\delta(e, z)$ is convex and differentiable with

$$\mathrm{D}_e \mathcal{V}_\delta(e, z) = \partial_e W(x, e{+}\mathbf{e}(V(e, z)), z) \in \mathbb{E}^* = \mathrm{L}^{p/(p-1)}(\Omega; \mathbf{E}_d),$$

where $V(e, z) \in \mathcal{F}$ is the unique minimizer in (4). This shows that $\sigma = \mathrm{D}_e \mathcal{V}_\delta(e, z)$ is in fact an equilibrium stress, and thus satisfies $\operatorname{div} \sigma = 0$ in Ω and $\sigma\,\nu = 0$ on $\partial\Omega\backslash\Gamma_{\mathrm{D}}$.

The importance of the functional $\mathcal{V}_\delta$ is that on the one hand it is possible to do the Γ-limit for $\delta \to 0$ and keep some of the main features and that on the other hand, by construction the reduced functional $\mathcal{I}_\delta$ and its partial derivative with respect to t can be easily expressed:

$$\mathcal{I}_\delta(t, z) = \mathcal{V}_\delta(\mathbf{e}_{\mathrm{D}}(t), z){+}\mathcal{G}(z) \text{ and } \partial_t \mathcal{I}_\delta(t, z) = \langle \mathrm{D}_e \mathcal{V}_\delta(\mathbf{e}_{\mathrm{D}}(t), z), \dot{\mathbf{e}}_{\mathrm{D}}(t)\rangle.$$

Thus, we have found a way to express the energies in terms of the damage alone and we still have control over the equilibrium stresses $\mathrm{D}_e \mathcal{V}_\delta(\mathbf{e}_{\mathrm{D}}(t), z)$ that are needed to control the power generated by the boundary data $u_{\mathrm{D}}(t)$.

4 Existence for the complete-damage problem

A functional $\mathfrak{I}(t, \cdot) : \mathcal{Z} \to \mathbb{R}$ is called the Γ-limit of $(\mathcal{I}_\delta(t, \cdot))_\delta$ if

$$\begin{aligned}&(\Gamma1) \quad z_\delta \rightharpoonup z \text{ in } \mathcal{Z} \quad \Longrightarrow \quad \mathfrak{I}(t, z) \le \liminf_{\delta\to 0} \mathcal{I}_\delta(t, z_\delta),\\ &(\Gamma2) \quad \forall\, z \in \mathcal{Z}\ \exists\, (z_\delta)_\delta :\ z_\delta \rightharpoonup z \text{ in } \mathcal{Z} \text{ and } \mathcal{I}_\delta(t, z_\delta) \to \mathfrak{I}(t, z).\end{aligned}$$

We note that Γ-convergence is quite different from pointwise convergence, see Example **4**.2. Moreover, while each $\mathcal{I}_\delta$ was strongly continuous, this is not true for $\mathfrak{I}(t, \cdot)$; only the important weak lower semicontinuity is maintained (as for all Γ-limits).

The main difficulty is to control the temporal smoothness of $\mathfrak{I}$, or more precisely to show that the following implication holds

$$\left.\begin{array}{c} z_\delta \rightharpoonup z \text{ in } \mathcal{Z} \\ \mathcal{I}_\delta(t,z_\delta) \to \mathfrak{I}(t,z) \end{array}\right\} \implies \partial_t \mathcal{I}_\delta(t,z_\delta) \to \partial_t \mathfrak{I}(t,z),$$

cf. condition (2.9) in [MRS08]. To provide this result we use the functional $\mathcal{V}_\delta$, since its Γ-limit can be studied more easily. The following result is a direct generalization of [BMR09, Prop. 2.10].

Proposition 4.1. *Let* (1) *hold with* $p>1$ *and* $r>d$. *On* $\mathbb{E}\times\mathcal{Z}$ *define*

$$\mathfrak{V}(e,z) = \lim_{\varepsilon\to 0^+}\Big(\lim_{\delta\to 0^+} \mathcal{V}_\delta\big(e,\max\{z-\varepsilon,0\}\big)\Big).$$

Then, $\mathfrak{V}$ *satisfies*

$$\exists\, C>0\ \forall\,(e,z)\in\mathbb{E}\times\mathcal{Z}: \quad -C \le \mathfrak{V}(e,t) \le C + C\|e\|_{\mathbb{E}}^p, \tag{5a}$$

$$\forall\, z\in\mathcal{Z}: \quad \mathfrak{V}(\cdot,z) \text{ is convex on } \mathbb{E}, \tag{5b}$$

$$\text{if } W(x,\cdot,z) \text{ is quadratic, then } \mathfrak{V}(\cdot,z) \text{ is quadratic.} \tag{5c}$$

Moreover, we have $\mathfrak{I}(t,z) = \mathfrak{V}(\mathbf{e}_D(t),z) + \mathcal{G}(z)$.

The proof relies on the compact embedding of $\mathrm{W}^{1,r}(\Omega)$ into $\mathrm{C}^0(\overline{\Omega})$ and uses essentially the monotonicity properties of the mapping $(\varepsilon,\delta)\mapsto \mathcal{V}_\delta\big(e,\max\{z-\varepsilon,0\}\big)$: it is non-increasing in ε because of (1d) and it is nondecreasing in δ because of the definition of W_δ in (3). Thus, the limit $\mathfrak{V}(e,z)$ always exists as a pointwise limit in δ and then in ε. Moreover, for each fixed z the convexity in e is preserved by pointwise convergence. The following example, which is inspired by [BoV88, Ex. 3] and further discussed in [BMR09], shows that in general $\mathfrak{V}$ is strictly smaller than $\mathcal{V}_0(e,z) = \lim_{\delta\to 0^+}\mathcal{V}_\delta(e,z)$.

Example 4.2. *Consider* $\Omega = \,]{-1},1[$ *and the energy*

$$\mathcal{I}_\delta(t,z) = \int_\Omega \frac{\delta+z}{2}(\mathbf{e}_D(t)+u')^2\,\mathrm{d}x + \mathcal{G}(z).$$

Then, $\mathcal{V}_\delta(e,z) = \big(\int_\Omega e\ \mathrm{d}x\big)^2/\int_\Omega \frac{2}{\delta+z}\ \mathrm{d}x$. *Clearly, the pointwise limit* $\mathcal{V}_0$ *is obtained by letting* $\delta=0$. *However, the* Γ*-limit* $\mathfrak{V}(e,\cdot)$ *in* $\mathrm{W}^{1,r}(\Omega)$ *satisfies*

$$\mathfrak{V}(e,z) = \mathcal{V}_0(e,z) \text{ for } \min z>0 \quad \text{and } \mathfrak{V}(e,z)=0 \text{ for } \min z = 0.$$

For $\alpha\in\,](r-1)/r,1[$ *we let* $z_\alpha(x) = |x|^\alpha$, *then* $z_\alpha\in\mathcal{Z}$ *and* $0=\mathfrak{V}(e,z) < \mathcal{V}_0(e,z) = (1-\alpha)\big(\int_\Omega e\,\mathrm{d}x\big)^2/4$.

The formula for $\mathfrak{I}$ allows us to study the question whether the power exists. For this, we use that convex functions have one-sided Gateaux derivatives in all points:

$$\begin{aligned} \delta_e\mathfrak{V}(e,z;\widehat{e}) &= \lim_{h\to 0^+}\frac{1}{h}\Big(\mathfrak{V}(e+h\widehat{e},z)-\mathfrak{V}(e,z)\Big) \\ &= \sup\big\{\,\langle\sigma,\widehat{e}\rangle \mid \sigma\in\partial_e^{\mathrm{sub}}\mathfrak{V}(e,z)\,\big\}, \end{aligned} \tag{6}$$

where $\partial_e^{\rm sub}\mathfrak{V}(e,z) \subset \mathbb{E}^*$ denotes the subdifferential of the convex function $\mathfrak{V}(\cdot,z)$. Using $\mathbf{e}_{\rm D} \in {\rm C}^1([0,T];\mathbb{E})$ we find that the left and right partial derivatives $\partial_t^{\pm}\mathfrak{I}(t,z) = \lim_{h\to 0^+} \frac{\pm 1}{h}\big(\mathfrak{I}(t{\pm}h,z) - \mathfrak{I}(t,z)\big)$ with respect to t of $\mathfrak{I}$ exists. We have the relations

$$\partial_t^-\mathfrak{I}(t,z) = -\delta_e\mathfrak{V}(t,\mathbf{e}_{\rm D}(t);-\dot{\mathbf{e}}_{\rm D}(t)) \le \delta_e\mathfrak{V}(t,\mathbf{e}_{\rm D}(t);\dot{\mathbf{e}}_{\rm D}(t)) = \partial_t^+\mathfrak{I}(t,z).$$

Definition 4.3. *Let $z:[0,T]\to\mathcal{Z}$ satisfy (S) in* (2) *for all $t\in[0,T]$. Then, z is called a* generalized energetic solution *of the rate-independent energetic system $(\mathcal{Z},\mathfrak{I},\mathcal{D})$, if there exists $p:[0,T]\to\mathbb{R}$ such that $p(\tau)\in\partial_\tau^{Cl}\mathfrak{I}(\tau,z(\tau))$ a.e. in $[0,T]$ and for all $t\in[0,T]$ we have*

$$\mathfrak{I}(t,z(t)) + {\rm Diss}_{\mathcal{D}}(z,[0,t]) = \mathfrak{I}(0,z(0)) + \int_0^t p(\tau)\,{\rm d}\tau. \tag{7}$$

Now a slight generalization of the abstract existence theory for rate-independent systems gives the following. Note that we construct weak energetic solutions for $(\mathcal{Z},\mathfrak{I},\mathcal{D})$ directly directly, without reference to the solutions z_δ for $(\mathcal{Z},\mathfrak{I}_\delta,\mathcal{D})$.

Theorem 4.4. *For all stable $z^0\in\mathcal{Z}$ there exists a generalized energetic solution for $(\mathcal{Z},\mathfrak{I},\mathcal{D})$.*

Proof. The existence theory follows the usual steps in the abstract theory for rate-independent processes [Mie05, FrM06] via incremental minimization, uniform a priori estimates and Helly's selection principle. This part and the proof of the stability of the limit process work as in [BMR09].

For the upper energy estimate we obtain, by setting $A(t) = \mathfrak{I}(t,z(t)) + {\rm Diss}_{\mathcal{D}}(z,[0,t])$,

$$A(s) - A(r) \le \int_r^s p^{\max}(t)\,{\rm d}t \text{ with } p^{\max}(t) = \max\partial_t^{\rm Cl}\mathfrak{I}(t,z(t)).$$

With a slight generalization of [Mie05, Prop. 5.7] we see that stability of the limit process z implies the lower bound $A(s)-A(r) \ge \int_r^s p^{\min}(t)\,{\rm d}t$ with $p^{\min}(t) = \min\partial_t^{\rm Cl}\mathfrak{I}(t,z(t))$.

Thus, we conclude that A is absolutely continuous and satisfies $p^{\min}(t) \le A'(t) \le p^{\max}(t)$. Hence, setting $p(t) = A'(t)$ the proof is complete. ☐

In the following example we show that the notion of generlaized energetic solution, which involves the weakened energy balance (7) with the Clarke differential, is really necessary in cases where the one-sided partial derivatives satisfy $\partial_t^-\mathfrak{I}(t,z) < \partial_t^+\mathfrak{I}(t,z)$ at some points. In particular, it is not possible to make an a priori choice like $p(t) = \max\{\partial_t^{\rm Cl}\mathfrak{I}(t,z(t))\}$, which worked in [KZM10, MiR09], since there $\partial_t^-\mathfrak{I}(t,z) \ge \partial_t^+\mathfrak{I}(t,z)$ holds.

Example 4.5. *This example has a smooth energy $\mathfrak{I}_\delta$ such that $\partial_t\mathfrak{I}_\delta$ exists, while in the limit $\mathfrak{I}$ is only Lipschitz in t. We let $\mathcal{Z}=\mathbb{R}$ and $\mathcal{D}(z,\widetilde{z}) = |\widetilde{z}{-}z|$. The energy functional reads*

$$\mathfrak{I}_\delta(t,z) = H_\delta\big(z{-}\alpha(t)\big) \quad \textit{and} \quad \mathfrak{I}(t,z) = 2|z{-}\alpha(t)|,$$

where $\alpha \in \mathrm{C}^1([0,T])$ *is given and* $H_\delta(u) = 2u^2/\sqrt{\delta^2{+}u^2}$. *For the partial derivatives with respect to time we have*

$$\partial_t \mathcal{I}_\delta(t,z) = -H_\delta'(z{-}\alpha(t))\dot\alpha(t) \text{ and } \partial_t^{Cl}\mathfrak{I}(t,z) = -2\,\mathrm{Sign}(z{-}\alpha(t))|\dot\alpha(t)|.$$

Since $\mathcal{I}_\delta(t,\cdot)$ *is smooth and strictly convex, the energetic solutions for* $(\mathbb{R}, \mathcal{I}_\delta, \mathcal{D})$ *are exactly the solutions of the doubly nonlinear equation [MiT04]*

$$0 \in \mathrm{Sign}(\dot z(t)) + H_\delta'(z(t){-}\alpha(t)).$$

For $\delta > 0$ *the system is smooth, while for* $\delta = 0$ *we have* $H_0(u) = 2|u|$ *and set* $\mathfrak{I}(t,z) = H_0(z{-}\alpha(t))$.

Consider the special case $\alpha(t) = t$ *and* $z_\delta(0) = 0$. *If* β_δ *is the unique solution of* $H_\delta'(\beta_\delta) = 1$, *then the unique energetic solution is* $z_\delta(t) = \max\{0, t{-}\beta_\delta\}$. *Using* $0 < \beta_\delta \to 0$ *we find the limit solution* $z(t) = t = \lim_{\delta\to 0} z_\delta(t)$. *It is a generalized energetic solution in the sense of Definition* **4**.*3 by using* $p(t) = 1 \in [-2,2] = \partial_t^{Cl}\mathfrak{I}(t,t)$.

5 Γ-convergence for $\delta \to 0$

Here we discuss the Γ-limit for the solutions z_δ of the rate-independent energetic system $(\mathcal{Z}, \mathcal{I}_\delta, \mathcal{D})$. First note that the a priori estimates give the boundedness of the family $(z_\delta)_\delta$ in $\mathrm{BV}([0,T], \mathrm{L}^1(\Omega)) \cap \mathrm{L}^\infty([0,T], \mathrm{W}^{1,r}(\Omega))$, and hence Helly's selection principle allows us to extract a subsequence $(z_{\delta_k})_{k\in\mathbb{N}}$ which converges pointwise on $[0,T]$ to a limit $z : [0,T] \to \mathcal{Z}$ satisfying the same bound, i.e., $z_\delta(t) \rightharpoonup z(t)$ in $\mathcal{Z}$.

To conclude that z is a generalized energetic solution for $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$ it is sufficient to check two compatibility conditions, namely *conditioned continuous convergence of the power*, cf. [MRS08, (2.9)], and *conditioned upper semicontinuity of stable sets*, cf. [MRS08, (2.11)]. The latter condition is purely static and it is not difficult to generalize it to the present case. As in [BMR09] we obtain the energy convergence $\mathcal{I}_\delta(t, z_\delta(t)) \to \mathfrak{I}(t, z(t))$, which in turn implies strong convergence $\|z_\delta(t){-}z(t)\|_{\mathrm{W}^{1,r}} \to 0$.

The conditional continuous convergence of the power would be satisfied if the following conjecture would be true.

Conjecture 5.1. *Assume that* z_{δ_j} *is stable for* $(\mathcal{Z}, \mathcal{I}_{\delta_j}, \mathcal{D})$ *at time* t, $z_{\delta_j} \rightharpoonup z$, $\mathcal{I}_{\delta_j}(t, z_{\delta_j}) \to \mathfrak{I}(t,z)$, *and* $\sigma_{\delta_j} = \mathrm{D}_e \mathcal{V}_{\delta_j}(\mathbf{e}_D(t), z_{\delta_j}) \rightharpoonup \sigma_*$ *in* $\mathbb{E}^*$, *then* $\sigma_* \in \partial_e^{sub}\mathfrak{V}(\mathbf{e}_D(t), z)$.

The conjecture holds [BMR09] under the assumption that $W(x,e,z)$ is quadratic in e. The relevant consequence is obtained via (6):

$$\partial_t^- \mathfrak{I}(t,z) \le \liminf_{\delta\to 0} \partial_t \mathcal{I}_\delta(t, z_\delta) \le \limsup_{\delta\to 0} \partial_t \mathcal{I}_\delta(t, z_\delta) \le \partial_t^+ \mathfrak{I}(t,z). \tag{8}$$

Combining this estimate with the abstract Γ-convergence for rate-independent systems [MRS08] and the existence theory for complete damage [BMR09] it is possible to obtain the following convergence result.

Theorem 5.2. *Assume that the (yet unproved) estimate* (8) *holds. If* $(z_\delta)_{\delta\in(0,1)}$ *is a family of solutions to* $(\mathcal{Z}, \mathcal{I}_\delta, \mathcal{D})$ *satisfying*

$$z_\delta(0) \rightharpoonup z^0 \text{ in } \mathrm{W}^{1,r}(\Omega) \quad \text{and} \quad \mathcal{I}_\delta(0, z_\delta(0)) \to \mathfrak{I}(0, z^0),$$

then there exist a subsequence $(z_{\delta_j})_{j\in\mathbb{N}}$ *and a generalized energetic solution* $z : [0,T] \to \mathcal{Z}$ *for* $(\mathcal{Z}, \mathfrak{I}, \mathcal{D})$ *with* $z(0) = z^0$ *such that for all* $t \in [0,T]$

$$\begin{gathered} z_{\delta_j}(t) \to z(t) \text{ in } \mathrm{W}^{1,r}(\Omega), \quad \mathcal{I}_\delta(t, z_\delta(t)) \to \mathfrak{I}(t, z(t)), \\ \mathrm{Diss}_{\mathcal{D}}(z_\delta, [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(z, [0,t]). \end{gathered}$$

Acknowledgements

This research was partially supported by DFG Research Center MATHEON, subproject C18: *Analysis and numerics of multidimensional models for elastic phase transformations in shape-memory alloys.* The author is grateful to Tomáš Roubíček and Ulisse Stefanelli for helpful discussions.

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