Multiscale modeling via evolutionary Γ-convergence

Alexander Mielke

Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin

Institut für Mathematik, Humboldt-Universität zu Berlin

www.wias-berlin.de/people/mielke/

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Coworkers on various parts of the work:

Aida Timofte 2007 — homogenization for plasticity

Tomas Roubicek, Ulisse Stefanelli 2008-today — rate-independent evolutionary Γ-convergence

Marita Thomas, Tomas Roubicek — damage, delamination

Matthias Liero 2011 — elastoplastic plate theory

Riccarda Rossi, Giuseppe Savaré 2009-today — general gradient systems

Lev Tuskinovsky 2012 — wiggly energies as origin of plasticity

Jan Maas, M. Liero (2012-today) — chemical reaction-diffusion systems

Mark Peletier, Michiel Renger (2013) — large-deviations principle

Partial support via

erc ERC Ad-Grant **AnaMultiScale**

"*Analysis of multiscale systems driven by functionals*"

Overview

Aim of these lectures:

- Evolutionary systems with multiple scales
  $0 < \varepsilon = 1/n \ll 1$ small parameter
- Describe mathematical methods for limit passage $\varepsilon \to 0$

Restriction:

- only generalized gradient systems
- only very simple applications
- proofs only for the simpler results

1. Introduction

General evolutionary equations

Multiscale limit corresponds to interchanging to limits, namely

"$\lim_{\varepsilon\to 0}$" and "$u^\varepsilon(t) = u_0^\varepsilon + \int_0^t \ldots \mathrm{d}s$"

	microsc. system		macrosc. system
	$\dot u_\varepsilon = \boldsymbol{G}_\varepsilon(u_\varepsilon)$		$\dot y = \boldsymbol{G}_0(y)$
initial state	u_ε^0	$\xrightarrow[\boldsymbol{M}_\varepsilon]{\text{upscaling}}$	y^0
time evolution	$\downarrow$		$\downarrow$
	$u_\varepsilon(t) = \boldsymbol{S}_\varepsilon(t, u_\varepsilon^0)$	$\xrightarrow[\boldsymbol{M}_\varepsilon]{\text{upscaling}}$	$y(t) = \boldsymbol{S}_0(t, y^0)$

Mathematical task: Prove $\lim_{\varepsilon\to 0} \boldsymbol{M}_\varepsilon \circ \boldsymbol{S}_\varepsilon(t,\cdot) = \boldsymbol{S}_0(t, \lim_{\varepsilon\to 0} \boldsymbol{M}_\varepsilon(\cdot))$

Gradient flows = evolution driven by gradient systems $(\boldsymbol{X}, \mathcal{E}, \mathbb{G})$

$u \in \boldsymbol{X}$ = state space (closed convex subset of a reflexive Banach space)

$\mathcal{E} : \boldsymbol{X} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ energy functional

Three equivalent formulations:

- Riemannian case $\mathbb{G}(u) : \mathrm{T}_u\boldsymbol{X} \to \mathrm{T}^*_u\boldsymbol{X}$ metric tensor)

$$\boxed{\mathbb{G}(u)\dot{u} = -\mathrm{D}\mathcal{E}(u)}$$ = force balance (viscous f. = restoring f.)

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- Rate equation with Onsager operator $\mathbb{K}(u) : \mathrm{T}_u^*\boldsymbol{X} \to \mathrm{T}_u\boldsymbol{X}$

  $$\boxed{\dot{u} = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u)}$$ where $\mathbb{K}(u) = \mathbb{K}(u)^* \geq 0 \quad (\mathbb{K} = \mathbb{G}^{-1})$

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- Energetic balance using dissipation potentials (see later)

$\mathcal{R}(u, \dot{u}) := \frac{1}{2}\langle \mathbb{G}(u)\dot{u}, \dot{u}\rangle$ and $\mathcal{R}^*(u, \xi) := \frac{1}{2}\langle \xi, \mathbb{G}(u)^{-1}\xi\rangle$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) = -\mathcal{R}(u, \dot{u}) - \mathcal{R}^*(u, \mathrm{D}\mathcal{E}(u(t))) \in \mathbb{R}}$$

Energy-Dissipation Balance (EDB)

Generalized gradient systems $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$

$\mathcal{R}(u,\dot u)$ dissipation potential $\rightsquigarrow$ $\partial_{\dot u}\mathcal{R}(u,\dot u)$ = dissipative force

$\mathcal{R}(u,\cdot) : \mathrm{T}_u\boldsymbol{X} \to [0,\infty]$ convex, lsc, $\mathcal{R}(u,0)=0$.

$$0 \in \partial_{\dot u}\mathcal{R}(u,\dot u) + \mathrm{D}\mathcal{E}(u) - \ell(t)$$

Classical gradient system $\mathcal{R}(u,v) = \frac{1}{2}\langle \mathbb{G}(u)v, v\rangle$

More general $\mathcal{R}(u,v) = \|\mathbb{A}(u)v\|_B + \frac{1}{2}\|\mathbb{V}(u)v\|_H^2 + \frac{1}{p}\|\mathbb{M}(u)\|_Z^p$

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In **multiscale modeling** on is interested in

Γ-convergence for families of gradient systems $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$

- homogenization
- dimension reductions (plates, interfaces, ...)
- singular perturbations
- modulation equations

Our working definition for this course:

Definition (Γ-convergence of geneneralized gradient systems = evolutionary Γ-convergence)

We write $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$ if and only if

$$\left. \begin{array}{c} u^\varepsilon : [0,T] \to \boldsymbol{X} \\ \text{solves } (\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \\ u^\varepsilon(0) \rightharpoonup u_0, \\ \mathcal{E}_\varepsilon(u^\varepsilon(0)) \to \mathcal{E}_0(u_0) \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \exists\, u \text{ sln. of } (\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0) \text{ with } u(0) = u_0 \\ \text{and a subsequence } \varepsilon_k \to 0 : \\ \forall\, t \in [0,T] : \ u^{\varepsilon_k}(t) \rightharpoonup u(t) \\ \mathcal{E}_\varepsilon(u^{\varepsilon_k}(t)) \to \mathcal{E}_0(u(t)) \end{array} \right.$$

Aim: Find conditions of $(\mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \rightsquigarrow (\mathcal{E}_0, \mathcal{R}_0)$ to guarantee evolutionary Γ-convergence.

Overview

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Inhomogenous diffusion equations

$$\gamma_\varepsilon(x)\dot u(t,x) = \mathrm{div}\,\big(A_\varepsilon(x)\nabla u) - f_\varepsilon(x,u(t,x)), \qquad t>0,\ x\in\Omega$$

(& suitable BC)

L^2-type gradient system $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathbb{G}_\varepsilon)$ with $\boldsymbol{X}=\mathrm{L}^2(\Omega)$

$(\mathbb{G}_\varepsilon v)(x) = \gamma_\varepsilon(x)v(x) \quad\Rightarrow\quad \mathcal{R}_\varepsilon(v) = \int_\Omega \frac12 \gamma_\varepsilon(x)v(x)^2\,\mathrm{d}x$

$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac12 \nabla u\cdot A_\varepsilon(x)\nabla u + F_\varepsilon(x,u(x))\,\mathrm{d}x \qquad F_\varepsilon(x,u) = \int_0^u f\_eps(x,w)\,\mathrm{d}w$

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- Homogenization $\gamma_\varepsilon(x) = g(x,\frac{x}{\varepsilon})$ and $A_\varepsilon(x,\frac{x}{\varepsilon})$

  Aim: $\mathcal{R}_{\mathsf{eff}}(v) = \int_\Omega \frac{\gamma_{\mathsf{eff}}}{2} v^2\,\mathrm{d}x$ and $\mathcal{E}_{\mathsf{eff}}(u) = \int_\Omega \frac12 \nabla u\cdot A_{\mathsf{eff}}\nabla u + F_{\mathsf{eff}}(u)\,\mathrm{d}x$

2. Motivating examples

Inhomogenous diffusion equations

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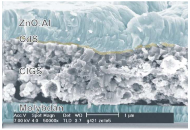
- Dimension reduction (modeling of active interfaces)

$$x\in\,]-1,1[\,\subset\mathbb{R}^1 \qquad A_\varepsilon(x) = \begin{cases}\alpha & \text{for } |x|>\varepsilon/2,\\ \beta\varepsilon & \text{for } |x|<\varepsilon/2\end{cases}$$

$$\mathcal{E}_{\text{eff}}(u) = \int_{-1}^0 \frac{\alpha}{2}u_x^2\,\mathrm{d}x + \underbrace{\frac{\beta}{2}\big(u(0^-)-u(0^+)\big)^2}_{\text{gives interface conditions}} + \int_0^1 \frac{\alpha}{2}u_x^2\,\mathrm{d}x$$



Amplitude equations

Swift-Hohenberg equation for weakly unstable systems

$$\dot{u} = -\frac{1}{\varepsilon^2}(1+\varepsilon^2\Delta)^2 u + Ru - u^3$$

Typical solutions behave highly oscillatory in space:

$u(t,x) = \mathrm{Re}\left(A(t,x)\mathrm{e}^{-\mathrm{i}k\cdot x/\varepsilon}\right)$ with $|k| \approx 1$

Expected amplitude/enveloppe equation
(cf. Eckhaus 1965, first proofs ≈ 1990)

$$\dot{A} = c_0\Delta A + RA - c_1|A|^2 A$$

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Vortex equations (Sandier-Serfaty 2004)

$\mathcal{E}_\varepsilon(\psi) = \int_\Omega \frac{1}{2}|\nabla\psi|^2 + \frac{1}{\varepsilon^2}(1-|\psi|^2)^2 \,\mathrm{d}x$

$\mathcal{R}_\varepsilon(\dot{\psi}) = \frac{1}{2\log(1/\varepsilon)} \int_\Omega |\dot{\psi}|^2 \,\mathrm{d}x$

$\rightsquigarrow$ ODE for vortex positions

Chemical reaction systems with detailed balance

E.g. $\quad 2B+C \overset{k_1}{\rightleftharpoons} 2A \qquad A+2C \overset{1/\varepsilon}{\rightleftharpoons} B \qquad A+B+C \overset{k_3}{\rightleftharpoons} 3C$

Fast reaction versus slow reactions $k_1,\ k_3 = O(1)$

$$\begin{pmatrix}\dot c_A\\ \dot c_B\\ \dot c_C\end{pmatrix} = k_1(c_B^2c_C - c_A^2)\begin{pmatrix}2\\-2\\-1\end{pmatrix} + \frac{1}{\varepsilon}(c_Ac_C^2-c_B)\begin{pmatrix}-1\\1\\-2\end{pmatrix} + k_3(c_Ac_Bc_C-c_C^3)\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

Energy = relative entropy $\mathcal{E}(\boldsymbol{c}) = \sum_{i=A,B,C}\lambda_{\mathrm{Bz}}(c_i) \qquad \lambda_{\mathrm{Bz}}(z) = z\log z - z + 1$

$$\dot{\boldsymbol{c}} = -\mathbb{K}(\boldsymbol{c})\mathrm{D}\mathcal{E}(\boldsymbol{c}) \text{ with } \mathbb{K}_\varepsilon(\boldsymbol{c}) = \mathbb{K}_{1,3}(\boldsymbol{c}) + \frac{1}{\varepsilon}\frac{c_Ac_C^2-c_B}{\log(c_Ac_C^2/c_B)}\begin{pmatrix}1&-1&2\\-1&1&-2\\2&-2&4\end{pmatrix}$$

Gradient system $([0,\infty[^3,\mathcal{E},\mathbb{K})$: $\mathcal{E}$ indep. of ε but $\mathbb{K}_\varepsilon(\boldsymbol{c})$

Overview

$\boldsymbol{X}$ reflexive Banach space and functionals $\mathcal{J}_\varepsilon : \boldsymbol{X} \to \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$

Definition (Weak/strong Γ and Mosco convergence)

Weak Γ-convergence: $\mathcal{J}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{J}$ if (G1w) and (G2w) hold:

(G1w) $u_\varepsilon \rightharpoonup u \implies \mathcal{J}(u) \leq \liminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon(u_\varepsilon)$ (liminf estimate)

(G2w) $\forall \widehat{u}\ \exists (\widehat{u}_\varepsilon)_\varepsilon$: $\widehat{u}_\varepsilon \rightharpoonup \widehat{u}$ and $\mathcal{J}(\widehat{u}) = \lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\widehat{u}_\varepsilon)$ (ex. recovery seq.)

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Strong Γ-convergence $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}$ if (G1s) and (G2s) hold:

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Mosco convergence $\mathcal{J}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{J}$ if $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}$ and $\mathcal{J}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{J}$ hold
(or (G1w) and (G2s))

The (primal) dissipation potentials $\mathcal{R}(u,\dot u)$ is always **convex in $\dot u$**.

The dual dissipation potential $\mathcal{R}^*$ is always **convex in ξ**.

$$\mathcal{R}^*(u,\xi) := \sup\{\, \langle \xi, v\rangle - \mathcal{R}(u,v) \,|\, v \in \boldsymbol{X} \,\}$$

Theorem (Attouch 1984)

Let $\boldsymbol{X}$ be a reflexive Banach space and assume that all $\mathcal{F}_n : \boldsymbol{X} \to \mathbb{R}_\infty$ are proper, convex, equicoercive and that $(\mathcal{F}_n^*)^*$. Then,

$$\mathcal{F}_n \xrightarrow{\Gamma} \mathcal{F} \qquad \Longleftrightarrow \qquad \mathcal{F}_n^* \xrightarrow{\Gamma} \mathcal{F}^*\,.$$

In particular, we have $\mathcal{F}_n \xrightarrow{\mathsf{M}} \mathcal{F} \quad \Leftrightarrow \quad \mathcal{F}_n^* \xrightarrow{\mathsf{M}} \mathcal{F}^*$.

Easy to remember via the well-known convergence result of linear functional analysis:

$v_n \rightharpoonup v$ and $\xi_n \to \xi$ implies $\langle \xi_n, v_n\rangle \to \langle \xi, v\rangle$

Overview

$\boldsymbol{X} = \mathbb{R}^2$

$\mathcal{E}_\varepsilon(u) = \frac{1}{2}u_1^2 + \frac{1}{2\varepsilon^2}(u_2 - \varepsilon u_1)^2 = \frac{1}{2}u \cdot A_\varepsilon u$ with $A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$

$\mathcal{R}_\varepsilon(\dot u) = \frac{1}{2}\dot u_1^2 + \frac{1}{2\varepsilon^\beta}\dot u_2^2 = \frac{1}{2}\dot u \cdot G_\varepsilon \dot u$ with $G_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon^\beta \end{pmatrix}$

ODE $G_\varepsilon \dot u_\varepsilon = -A_\varepsilon u_\varepsilon$ with $u_\varepsilon(0) = \binom{1}{0}$

Explicit solutions can be calculated. We find, for all $t \geq 0$,

$\beta \in [0,2[\,:\quad u_\varepsilon(t) \to \binom{\mathrm{e}^{-t}}{0}$ as $\varepsilon \to 0$

$\beta = 2:\quad u_\varepsilon(t) \to \binom{w(t)}{0}$ as $\varepsilon \to 0$

$\beta > 2:\quad u_\varepsilon(t) \to \binom{\mathrm{e}^{-2t}}{0}$ as $\varepsilon \to 0$

where $w(t) = \frac{1}{2\sqrt{5}}\big((\sqrt{5}+1)\mathrm{e}^{-\mu_1 t} + (\sqrt{5}-1)\mathrm{e}^{-\mu_2 t}\big)$ with $\mu_{1,2} = (3 \pm \sqrt{5})/2$

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2}u_1^2 + \frac{1}{2\varepsilon^2}(u_2 - \varepsilon u_1)^2 = \frac{1}{2}u \cdot A_\varepsilon u \qquad \text{with } A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$$

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What are the limits of the functionals?

$$\mathcal{E}_\varepsilon(u) = \tfrac{1}{2}u_1^2 + \tfrac{1}{2\varepsilon^2}(u_2-\varepsilon u_1)^2 = \tfrac{1}{2}u\cdot A_\varepsilon u \qquad \text{with } A_\varepsilon = \begin{pmatrix} 2 & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon^2 \end{pmatrix}$$

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What are the limits of the functionals?

$$\mathcal{E}_\varepsilon \xrightarrow{\text{pointwise}} \mathcal{E}_{\text{pw}} : u \mapsto \begin{cases} (\frac{1}{2}+\frac{1}{2})u_1^2 & \text{for } u_2=0, \\ \infty & \text{otherwise} \end{cases}$$

$$\mathcal{E}_\varepsilon \xrightarrow{\text{M}} \mathcal{E} : u \mapsto \begin{cases} \frac{1}{2}u_1^2 & \text{for } u_2=0, \\ \infty & \text{otherwise} \end{cases} \qquad \mathcal{R}_\varepsilon \xrightarrow{\text{M}} \mathcal{R} : v \mapsto \begin{cases} \frac{1}{2}v_1^2 & \text{for } v_2=0, \\ \infty & \text{otherwise} \end{cases}$$

$(\mathbb{R}^2, \mathcal{E}, \mathcal{R})$ gives $u(t) = \binom{e^{-t}}{0}$ and $(\mathbb{R}^2, \mathcal{E}_{\text{pw}}, \mathcal{R})$ gives $u(t) = \binom{e^{-2t}}{0}$.

$\boxed{\beta < 2}$ $(\mathbb{R}^2, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}^2, \mathcal{E}, \mathcal{R})$

$\boxed{\beta = 2}$ **no** evolutionary Γ convergence

$\boxed{\beta > 2}$ $(\mathbb{R}^2, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}^2, \mathcal{E}_{\text{pw}}, \mathcal{R})$

Overview

We consider **one-dimensional homogenization** of a parabolic equation on $x \in \Omega =]0,\ell[$ for $t > 0$:

$$c(\tfrac{x}{\varepsilon})\dot{u}(t,x) = \big(a(\tfrac{x}{\varepsilon})u_x(t,x)\big)_x - b(\tfrac{x}{\varepsilon})u(t,x) \qquad u_x(t,0) = 0 = u_x(t,\ell)$$

where $a, b, c \in \mathrm{L}^\infty(\mathbb{R})$ are 1-periodic and are $\geq c_0 > 0$.

Family of gradient system $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ with

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega a(\tfrac{x}{\varepsilon})u_x(x)^2 + b(\tfrac{x}{\varepsilon})u(x)^2 \,\mathrm{d}x, \qquad \mathcal{R}_\varepsilon(v) = \frac{1}{2}\int_\Omega c(\tfrac{x}{\varepsilon})v(x)^2 \,\mathrm{d}x$$

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Aim: Find $\mathcal{E}_{\text{eff}}$ and $\mathcal{R}_{\text{eff}}$ in the form

$$\mathcal{E}_{\text{eff}}(u) = \tfrac{1}{2}\int_\Omega a_{\text{eff}}\, u_x(x)^2 + b_{\text{eff}}\, u^2 \,\mathrm{d}x, \qquad \mathcal{R}_{\text{eff}}(v) = \tfrac{1}{2}\int_\Omega c_{\text{eff}}\, v^2 \,\mathrm{d}x$$

$a_{\text{eff}} = ?$

$b_{\text{eff}} = ?$

$c_{\text{eff}} = ?$

Quadratic functionals:

$$\Psi_\varepsilon(v) = \frac{1}{2}\int_\Omega v(x)\cdot G_\varepsilon(x)v(x)\,\mathrm{d}x \iff \Psi^*_\varepsilon(x) = \frac{1}{2}\int_\Omega \xi(x)\cdot G_\varepsilon(x)^{-1}\xi(x)\,\mathrm{d}x$$

Lemma (One-dimensional homogenization)
Let $G_\varepsilon(x) = \mathbb{G}(x/\varepsilon)$ with $0 < c_0 \leq \mathbb{G}(y) \leq C_1$ and $\mathbb{G}$ 1-periodic.
In $\mathrm{L}^2(]x_1, x_2[)$ we have

weak-Γ: $\Psi_\varepsilon \xrightharpoonup{\Gamma} \Psi_{\mathsf{harm}} : v \mapsto \frac{1}{2}\int_{x_1}^{x_2} v\cdot G_{\mathsf{harm}} v\,\mathrm{d}x$

strong-Γ: $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_{\mathsf{arith}} : v \mapsto \frac{1}{2}\int_{x_1}^{x_2} v\cdot G_{\mathsf{arithm}} v\,\mathrm{d}x$

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$\Psi_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{x_1}^{x_2} v_\varepsilon\cdot G_\varepsilon v_\varepsilon\,\mathrm{d}x =$

$$\frac{1}{2}\int_{x_1}^{x_2} \underbrace{(G_\varepsilon v_\varepsilon - G_{\mathsf{ha}}v)\cdot G_\varepsilon^{-1}(G_\varepsilon v_\varepsilon - G_{\mathsf{ha}}v)}_{\ge 0} + 2\underbrace{G_\varepsilon v_\varepsilon\cdot G_\varepsilon^{-1}}_{=v_\varepsilon \rightharpoonup v} G_{\mathsf{ha}}v - G_{\mathsf{ha}}v\cdot \underbrace{G_\varepsilon^{-1}}_{\stackrel{*}{\rightharpoonup} G_{\mathsf{ha}}^{-1}} G_{\mathsf{ha}}v\,\mathrm{d}x$$

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Hence, $\liminf_{\varepsilon\to 0}\Psi_\varepsilon(v_\varepsilon) \geq \frac{1}{2}\int_{x_1}^{x_2} \quad 0 \quad + 2v\cdot G_{\mathsf{ha}}v \quad - \quad v\cdot G_{\mathsf{ha}}v \quad \mathrm{d}x \;=\; \Psi_{\mathsf{harm}}(v)$

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Given $\widehat{v}$ choose the recovery sequence $\widehat{v}_\varepsilon = G_\varepsilon^{-1}G_{\mathsf{ha}}\widehat{v} \rightharpoonup \widehat{v}$ and first term = 0.
Hence, $\Psi_\varepsilon(\widehat{v}_\varepsilon) = \int_{x_1}^{x_2} \quad 0 \quad + G_{\mathsf{ha}}\widehat{v}\cdot G_\varepsilon^{-1} G_{\mathsf{ha}}\widehat{v}\,\mathrm{d}x \;\to\; \Psi_{\mathsf{harm}}(\widehat{v})$

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Proof of strong-Γ is much simpler:
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Result is compatible with Attouch's theorem: $\Psi_\varepsilon \xrightharpoonup{\Gamma} \Psi_{\mathsf{harm}} \iff \Psi^*_\varepsilon \xrightarrow{\Gamma} \Psi^*_{\mathsf{harm}}$

For this, simply use $\mathsf{arith}(\mathbb{G}^{-1}) = \mathsf{harm}(\mathbb{G})^{-1}$.

One-dimensional homogenization for parabolic equation on $x \in \Omega =]0, \ell[$:

$$c(\tfrac{x}{\varepsilon})\dot u(t,x) = \big(a(\tfrac{x}{\varepsilon})u_x(t,x)\big)_x - b(\tfrac{x}{\varepsilon})u(t,x) \qquad u_x(t,0) = 0 = u_x(t,\ell)$$

where $a, b, c \in \mathrm{L}^\infty(\mathbb{R})$ are 1-periodic and are $\geq c_0 > 0$.

Gradient system $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \Psi_\varepsilon)$ with

$$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega a(\tfrac{x}{\varepsilon})u_x(x)^2 + b(\tfrac{x}{\varepsilon})u(x)^2\,\mathrm{d}x, \qquad \Psi_\varepsilon(v) = \frac{1}{2}\int_\Omega c(\tfrac{x}{\varepsilon})v(x)^2\,\mathrm{d}x$$

- $\Psi_\varepsilon \xrightharpoonup{\Gamma} \Psi_{\mathsf{harm}}$ or $\Psi_\varepsilon \xrightarrow{\Gamma} \Psi_{\mathsf{arith}}$ in the **dynamic space** $\mathrm{L}^2(\Omega)$
- Analogously the energy satisfies in the **energy space** $\mathrm{H}^1(\Omega) \Subset \mathrm{L}^2(\Omega)$

$$\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_{\mathsf{ha}} : u \mapsto \frac{1}{2}\int_\Omega a_{\mathsf{harm}} u_x^2 + b_{\mathsf{arith}} u^2\,\mathrm{d}x \qquad \text{(weakly in } \mathrm{H}^1(\Omega))$$

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We will use later: $\mathcal{E}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}_\text{ha}$ (Mosco in $\mathrm{L}^2(\Omega)$)

$\rightsquigarrow$ expected limit eqn $c_\text{eff}\,u_t = a_\text{harm}u_{xx} - b_\text{arith}u$ with $c_\text{eff} \in \{c_\text{harm}, c_\text{arith}\}$

Overview

Overview

Legendre-Fenchel theory for a reflexive Banach space

$\Psi : \boldsymbol{X} \to \mathbb{R}_\infty$ proper, convex, lower semicontinuous

Legendre transform $\Psi^* = \mathcal{L}\Psi : \boldsymbol{X}^* \to \mathbb{R}_\infty$ with

$\Psi^*(\xi) := \sup\{ \langle \xi, v\rangle - \Psi(v) \mid v \in \boldsymbol{X} \}$

Basic properties:

- $\mathcal{L}(\mathcal{L}\Psi) = \Psi$ or $\Psi^{**} = \Psi$
- Young-Fenchel estimate: $\forall v \in \boldsymbol{X}\ \forall \xi \in \boldsymbol{X}^* :\ \Psi(v) + \Psi^*(\xi) \geq \langle \xi, v\rangle$
- $\Psi(v) = \frac{1}{2}\langle Gv, v\rangle \implies \Psi^*(\xi) = \frac{1}{2}\langle \xi, G^{-1}\xi\rangle$
- $\Psi(v) = \frac{1}{p}\|v\|_{\boldsymbol{X}}^p \implies \Psi^*(\xi) = \frac{1}{p^*}\|\xi\|_{\boldsymbol{X}^*}^{p^*}$ for $1 < p < \infty$, $p^* = p/(p-1)$

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Subdifferential of convex Ψ

$\partial\Psi(v) = \{ \eta \in \boldsymbol{X}^* \mid \forall\, w \in \boldsymbol{X} : \; \Psi(w) \geq \Psi(v) + \langle \eta, w{-}v\rangle \} \subset \boldsymbol{X}^*$

If $\Psi \in \mathrm{C}^1(\boldsymbol{X};\mathbb{R})$ and convex, then $\partial\Psi(v) = \{\mathrm{D}\Psi(v)\}$.

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Theorem (Fenchel equivalence)

(i) $\xi \in \partial\Psi(v) \iff$ *(ii)* $v \in \partial\Psi^*(\xi) \iff$ *(iii)* $\Psi(v) + \Psi^*(\xi) \leq \langle \xi, v \rangle$

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Generalized gradient system $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$

Energy funct. $\mathcal{E} : [0,T]\times\boldsymbol{X} \to \mathbb{R}_\infty$, dissipation pot. $\mathcal{R}(u,\cdot) : \boldsymbol{X} \to [0,\infty]$

(i) $0 \in \partial_{\dot u}\mathcal{R}(u(t),\dot u(t)) + \mathrm{D}\mathcal{E}(t,u(t)) \ \in \boldsymbol{X}^*$ for a.a. $t \in [0,T]$

force balance in $\boldsymbol{X}^*$ — Biot's equation 1954

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force balance in $\boldsymbol{X}^*$ — Biot's equation 1954

Dual dissipation potential $\mathcal{R}^*(u,\xi) = \mathcal{L}(\mathcal{R}(u,\cdot))(\xi)$

(ii) $\dot u(t) \in \partial_\xi\mathcal{R}^*(u(t), -\mathrm{D}\mathcal{E}(t,u(t))) \;\in \boldsymbol{X}$ for a.a. $t \in [0,T]$

rate equation in $\boldsymbol{X}$ — Onsager's equation 1931

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(iii) $\mathcal{R}(u(t),\dot u(t)) + \mathcal{R}^*(u(t), -\mathrm{D}\mathcal{E}(t,u(t))) \le \langle -\mathrm{D}\mathcal{E}(t,u(t)), \dot u(t)\rangle$

power balance in $\mathbb{R}$ (equivalent to equality by Young-Fenchel)

De Giorgi's (Ψ, Ψ^*) formulation 1980

3. Energy-dissipation formulations

(i) $0 \in \partial_{\dot u}\mathcal{R}(u,\dot u) + \mathrm{D}\mathcal{E}(t,u)$ (ii) $\dot u \in \partial_\xi \mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(t,u))$

Theorem (Energy-Dissipation Estimate)

Assume that $\mathcal{E}$ solves the ***chain rule on $\boldsymbol{X}$****, then $u \in \mathrm{W}^{1,1}([0,T];\boldsymbol{X})$ solves (i) or (ii) if and only if* ***(EDE)*** *holds:*

(EDE)
$$\mathcal{E}(T,u(T)) + \int_0^T \mathcal{R}(u(t),\dot u(t)) + \mathcal{R}^*(u(t),-\mathrm{D}\mathcal{E}(t,u(t)))\,\mathrm{d}t \leq \mathcal{E}(0,u(0)) + \int_0^T \partial_s\mathcal{E}(s,u(s))\,\mathrm{d}s$$

Final energy + dissipated energy = initial energy + external work

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Proof: $\int_0^T -\langle \mathrm{D}\mathcal{E}(t,u),\dot u\rangle\,\mathrm{d}t \overset{\text{YF}}{\le} \int_0^T \mathcal{R}(u,\dot u) + \mathcal{R}^*(u,-\mathrm{D}\mathcal{E})\,\mathrm{d}t$

$$\overset{\text{(EDE)}}{\le} \mathcal{E}(0,u(0)) + \int_0^T \partial_t\mathcal{E}\,\mathrm{d}t - \mathcal{E}(T,u(T)) \overset{\text{Ch.Rule}}{=} \int_0^T -\langle \mathrm{D}\mathcal{E}(t,u),\dot u\rangle\,\mathrm{d}t$$

$\Rightarrow$ all estimates are equalities $\Rightarrow$ Young-Fenchel estimate is equality a.e. QED

(i) $0 \in \partial_{\dot u}\mathcal{R}(u,\dot u) + \mathrm{D}\mathcal{E}(t,u)$ (ii) $\dot u \in \partial_\xi \mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(t,u))$

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Fundamental and more general tool **Chain-Rule Estimate (CR)**

$\mathcal{E} : \boldsymbol{X} \to \mathbb{R}_\infty$ satisfies **CRE**, if

$$\left.\begin{array}{c} u \in \mathrm{W}^{1,p}([0,T];\boldsymbol{X}),\ \xi \in \mathrm{L}^{p'}([0,T];\boldsymbol{X}^*) \\ \xi(t) \in \partial\mathcal{E}(u(t)) \end{array}\right\} \implies \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) \geq \langle \xi(t), \dot u(t)\rangle$$

(e.g. always true for lsc and convex $\mathcal{E}(\cdot)$)

Overview

$0 \in \partial_{\dot u} \mathcal{R}_\varepsilon(u, \dot u) + \mathrm{D}\mathcal{E}_\varepsilon(u) \overset{\text{Fenchel}}{\Longleftrightarrow}$ (EDE) = Energy-Dissipation Estimate

(EDE) $\mathcal{E}_\varepsilon(u^\varepsilon(t)) + \int_0^T \mathcal{R}_\varepsilon(u^\varepsilon, \dot u^\varepsilon) + \mathcal{R}^*_\varepsilon(u^\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\,\mathrm{d}t \le \mathcal{E}_\varepsilon(u^\varepsilon(0))$

Evolutionary Γ convergence based on (EDE)

- Sandier-Serfaty'04 (general approach)
- here: special case of M-Rossi-Savare'12 (CVPDE) $\mathcal{R}(u, v) = \Psi(v)$

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Theorem (Mosco convergence implies evolutionary Γ-convergence)

$\boldsymbol{X}$ *reflexive,* $\exists\, c, C, \lambda_\mathrm{c} > 0,\ p > 1$ *such that* $\mathcal{E}_\varepsilon(\cdot) + \lambda_\mathrm{c}\|\cdot\|^2_{\boldsymbol{X}}$ *is convex,*
$\Psi_\varepsilon(v) \ge c\|v\|^p_{\boldsymbol{X}} - C,\ \Psi^*_\varepsilon(\xi) \ge c\|\xi\|^p_{\boldsymbol{X}^*} - C,\ \mathcal{E}_\varepsilon(u) \ge c\|u\|_{\boldsymbol{Z}} - C$ *with* $\boldsymbol{Z} \Subset \boldsymbol{X}$

$$\Big(\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0 \ \&\ \Psi_\varepsilon \xrightarrow{M} \Psi_0 \text{ in } \boldsymbol{X}\Big) \quad \Longrightarrow \quad (\boldsymbol{X}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{evol} (\boldsymbol{X}, \mathcal{E}_0, \Psi_0)$$

Compatibility: $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ and $\Psi_\varepsilon \xrightarrow{\mathrm{M}} \Psi_0$ in SAME topology $\boldsymbol{X}$

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Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

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Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

ODE model on $\boldsymbol{X} = \mathbb{R}^2$

We always have $\mathcal{E}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}$ and $\mathcal{R}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{R}$.

$\mathcal{R}(v) = \frac{1}{2}(v_1^2 + v_2^2/\varepsilon^\beta)$ and $\mathcal{R}^*(\xi) = \frac{1}{2}(\xi_1^2 + \varepsilon^\beta \xi_2^2)$

Theorem is applicable for $\beta = 0$.

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Application to our two simple problems (coercivity of Ψ defines $\boldsymbol{X}$)

Homogenization: $c\|v\|^2_{\mathrm{L}^2} \leq \Psi_\varepsilon(v) \leq C\|v\|^2_{\mathrm{L}^2} \quad \Longrightarrow \quad \boldsymbol{X} = \mathrm{L}^2(0,\ell)$.

$\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_0^\ell a_\varepsilon u_x^2 + b_\varepsilon u^2 \,\mathrm{d}x$: $\quad \mathcal{E}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}_0$ in $\boldsymbol{X} = \mathrm{L}^2(0,\ell)$ $\quad \oplus$

$\Psi_\varepsilon(v) = \frac{1}{2}\int_0^\ell c(x/\varepsilon) v(x)^2 \,\mathrm{d}x$: $\quad \Psi_{\text{weak}} \lneqq \Psi_{\text{strong}}$ $\quad \ominus$

Theorem is applicable in the case $c_\varepsilon = c_* = c^* =$const.

$$c u_t = (a_\varepsilon u_x)_x - b_\varepsilon u \quad \xrightarrow{\text{evol}} \quad c u_t = (a_* u_x)_x - b^* u$$

Sketch of proof of theorem: u_ε are solutions of (i) = (EDB)$_\varepsilon$:

$$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \Psi_\varepsilon(\dot u_\varepsilon) + \Psi_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t = \mathcal{E}_\varepsilon(u_\varepsilon(0)) \text{ where } -\xi_\varepsilon(t) \in \mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon(t))$$

- Uniform coercivity of $\mathcal{E}_\varepsilon$, Ψ_ε. and Ψ_ε^* yield uniform a priori bounds

$$\|u_\varepsilon\|_{\mathrm{L}^\infty([0,T];\boldsymbol{Z})} + \|u_\varepsilon\|_{\mathrm{W}^{1,p}([0,T];\boldsymbol{X})} + \|\xi_\varepsilon\|_{\mathrm{L}^p([0,T];\boldsymbol{X}^*)} \le C$$

- We find convergent subsequences

$$u_\varepsilon(t) \to u(t) \text{ in } \boldsymbol{X},\ u_\varepsilon \rightharpoonup u \text{ in } \mathrm{W}^{1,p}([0,T];\boldsymbol{X}),\ \xi_\varepsilon \rightharpoonup \xi \text{ in } \mathrm{L}^p([0,T];\boldsymbol{X}^*)$$

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- Lower semicontinuity of the dissipation (use $\Psi_\varepsilon \xrightarrow{\mathsf{M}} \Psi \iff \Psi^*_\varepsilon \xrightarrow{\mathsf{M}} \Psi^*$)

Ioffe's lsc result: $\int_0^T \Psi(\dot u)+\Psi^*(\xi)\,\mathrm{d}t \leq \liminf_{\varepsilon\to 0} \int_0^T \Psi_\varepsilon(\dot u_\varepsilon)+\Psi^*_\varepsilon(\xi_\varepsilon)\,\mathrm{d}t$

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- Strong-weak closedness of $\mathrm{D}\mathcal{E}$ if $\mathcal{E}_\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}$ & $\mathcal{E}$ λ_c-convex (cf. Attouch'84)

$$u_\varepsilon \to u \text{ in } \boldsymbol{X} \text{ and } \xi_\varepsilon \rightharpoonup \xi \text{ in } \boldsymbol{X}^* \;\Rightarrow\; \xi \in \mathrm{D}\mathcal{E}(\mathrm{u})$$

- Passing to $\varepsilon \to 0$ in (EDE)$_\varepsilon$ we obtain

$$\mathcal{E}(u(T)) + \int_0^T \Psi(\dot u)+\Psi^*(\xi)\,\mathrm{d}t \leq \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(0)) \overset{\text{ass}}{=} \mathcal{E}(u(0))$$ QED

Main tools is Strong-Weak Closedness of the graph of $(\partial\mathcal{E}_\varepsilon)_{\varepsilon\in]0,1[}$

(SWC) $\quad u_\varepsilon \to u$ in $\boldsymbol{X}$ and $\xi_\varepsilon \rightharpoonup \xi$ in $\boldsymbol{X}^*$ $\;\Rightarrow\;$ $\xi \in \mathrm{D}\mathcal{E}(\mathrm{u})$

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This is a consequence of Mosco convergence and convexity!

Theorem (Convexity and $\xrightarrow{\text{M}}$ imply (SWC), cf. Attouch 1983)

If all $\mathcal{E}_\varepsilon$ are lsc and convex, then $\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$ implies (SWC).

Proof: Assume $u_\varepsilon \to u$, $\xi_\varepsilon \rightharpoonup \xi$, and $\mathcal{E}_\varepsilon(u_\varepsilon) \to e_*$

Then convexity gives (Conv)$_\varepsilon$ $\quad \mathcal{E}_\varepsilon(w) \geq \mathcal{E}_\varepsilon(u_\varepsilon) + \langle \xi_\varepsilon, w{-}u_\varepsilon\rangle$

For given $\widehat{u}$ the M-convergence gives a rec. seq. $\widehat{u}_\varepsilon$ with $\widehat{u}_\varepsilon \to \widehat{u}$, $\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \to \mathcal{E}_0(\widehat{u})$

Hence, setting $w = \widehat{u}_\varepsilon$ in (Conv)$_\varepsilon$ gives $\underbrace{\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon)}_{\to\mathcal{E}_0(\widehat{u})} \geq \underbrace{\mathcal{E}_\varepsilon(u_\varepsilon)}_{\to e_*} + \langle \underbrace{\xi_\varepsilon}_{\rightharpoonup\xi}, \underbrace{\widehat{u}_\varepsilon{-}u_\varepsilon}_{\to\widehat{u}-u}\rangle$

Taking the limit $\varepsilon \to 0$ we obtain the relation $\mathcal{E}_0(\widehat{u}) \geq e_* + \langle \xi, \widehat{u}{-}u\rangle$

Choose $\widehat{u} = u$ we see that $\mathcal{E}_0(u) \geq e_*$ but M-liminf gives $e_* \geq \mathcal{E}_0(u)$. Thus, $e_* = \mathcal{E}_0(u)$ and we conclude $\xi \in \partial\mathcal{E}_0(u)$ as desired. □

(EDE) $\mathcal{E}_\varepsilon(u_\varepsilon(t)) + \int_0^T \Psi_\varepsilon(\dot u_\varepsilon) + \Psi^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\,\mathrm{d}t \leq \mathcal{E}_\varepsilon(u_\varepsilon(0))$

The **Sandier-Serfaty [2004]** approach is more general.

They do assume

neither Strong-Weak Closedness of $(\partial\mathcal{E}_\varepsilon)_{\varepsilon\in[0,1]}$

nor the Mosco convergence of $\Psi_\varepsilon \xrightarrow{\mathsf{M}} \Psi$

Instead they assume

(i) $v_\varepsilon \rightharpoonup v \implies \liminf_{\varepsilon\to} \Psi_\varepsilon(v_\varepsilon) \geq \Psi_0(v)$ (w-Γ-liminf)

(ii) $u_\varepsilon \to u \implies \liminf_{\varepsilon\to} \Psi^*_\varepsilon(\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)) \geq \Psi^*_0(\mathrm{D}\mathcal{E}_0(u))$ (dual w-Γ-liminf)

Clearly, (SWC) & $\Psi_\varepsilon \xrightarrow{\mathsf{M}} \Psi$ imply (i) and (ii) but not vice-versa.

Overview

(EDE) $\mathcal{E}_\varepsilon(u^\varepsilon(t)) + \int_0^T \mathcal{R}_\varepsilon(u^\varepsilon, \dot{u}^\varepsilon) + \mathcal{R}_\varepsilon^*(u^\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\,\mathrm{d}t \leq \mathcal{E}_\varepsilon(u^\varepsilon(0))$

EDE is quite flexible
- general $\mathcal{R}_\varepsilon(u,\cdot)$
- λ_c-conv. of $\mathcal{E}_\varepsilon$ not needed
- convergence of individual terms not needed

It suffices to find $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$ and $\mathcal{M}$ such that

- $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$
- Chain rule holds for $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$

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- $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$
- Chain rule holds for $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$
- $\int_0^T \mathcal{M}(u,\dot{u})\,\mathrm{d}t \leq \liminf_\varepsilon \int_0^T \mathcal{R}_\varepsilon(u^\varepsilon, \dot{u}^\varepsilon) + \mathcal{R}_\varepsilon^*(u^\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\,\mathrm{d}t$

  (a) $\mathcal{M}(u,v) \geq -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle$ and
  (b) $\mathcal{M}(u,v) = -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle \implies$

$$\mathcal{R}_0(u,v) + \mathcal{R}_0^*(u, -\mathrm{D}\mathcal{E}_0(u)) = -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle$$

Remark:
$\mathcal{M}(u,v) \geq \mathcal{R}_0(u,v) + \mathcal{R}_0^*(u, -\mathrm{D}\mathcal{E}_0(u))$ is suffic. for (a,b) but not necessary!

Even, passage from quadratic $\mathcal{R}_\varepsilon(v) = r_\varepsilon \|v\|_H^{\mathbf{2}}$
to 1-homogeneous $\mathcal{R}_0(v) = r_0 \|v\|_{\boldsymbol{X}}^{\mathbf{1}}$ is possible!

To illustrate the method in homogenization for ODE with $a(y+1) = a(y)$

$\dot{u}_\varepsilon(t,x) = -a(x/\varepsilon)u_\varepsilon(t,x), \quad t>0,\ x\in\,]0,\ell[$ Tartar 1990

Is there an effective equation of the form $\dot{u} = -a_{\mathsf{eff}}\,u$??

Tartar: **NO!** Memory term needed: $\dot{u}(t,x) = -a^0 u(t,x) + \int_0^t K_a(x,t{-}s)u(s,x)\,\mathrm{d}s$

Today: YES, there is a local evolutionary Γ limit

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For the L^2-gradient structure $(\mathrm{L}^2(0,\ell), \widehat{\mathcal{E}}_\varepsilon, \frac{1}{2}\|\cdot\|_2^2)$ with
$\widehat{\mathcal{E}}_\varepsilon(u) = \int_0^\ell \frac{a(x/\varepsilon)}{2}u^2\,\mathrm{d}x$ it does not work (wrong topology)!

Take a **nontrivial gradient structure** $(\boldsymbol{M}_+(0,\ell), \mathcal{E}_\varepsilon, \mathcal{R}_\mathrm{H})$

- state space = nonnegative Radon measures in $\boldsymbol{M}(0,\ell) = \mathrm{C}_0^0(0,\ell)^*$
- linear energy functional $\mathcal{E}_\varepsilon(u) = \int_0^\ell a(x/\varepsilon)\,\mathrm{d}u(x)$
- Hellinger dissipation potential $\mathcal{R}_\mathrm{H}^*(u,\xi) = \frac{1}{2}\int_0^\ell \xi(x)^2\,\mathrm{d}u(x)$ (indep. of ε)

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Theorem (Pisa 27.11.2013): Let $\mathcal{E}_0(u) = a_{\text{eff}}\int_0^\ell \mathrm{d}u$ with $a_{\text{eff}} = \min a$, then **(A)** $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ and **(B)** $(\boldsymbol{M}_+(0,\ell), \mathcal{E}_\varepsilon, \mathcal{R}_{\mathrm{H}}) \xrightarrow{\text{evol}} (\boldsymbol{M}_+(0,\ell), \mathcal{E}_0, \mathcal{R}_{\mathrm{H}})$

Sketch of proof:

ad (A): Using $a_{\text{eff}} = \min a \le a(x/\varepsilon)$ we have $\mathcal{E}_\varepsilon(u) \ge \mathcal{E}_0(u)$

Since $\mathcal{E}_0$ is weak* continuous, $u_\varepsilon \overset{*}{\rightharpoonup} u$ implies $\liminf \mathcal{E}_\varepsilon(u_\varepsilon) \ge \mathcal{E}_0(u)$

For the recovery sequence assume w.l.o.g. $a(0) = \min a$.
Given $\widehat{u}$ consider $\widehat{u}_\varepsilon = \sum_{k=0}^{\ell/\varepsilon} \nu_{k,\varepsilon} \delta_{\varepsilon k}(x)$ with $\nu_{k,\varepsilon} = \int_{\varepsilon k}^{\varepsilon k+\varepsilon} \mathrm{d}\widehat{u}$.

By construction $\widehat{u}_\varepsilon \overset{*}{\rightharpoonup} \widehat{u}$ and $\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) = \mathcal{E}_0(\widehat{u}_\varepsilon) = \mathcal{E}_0(\widehat{u})$

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ad (B): Consider the (EDE) for solutions u_ε:

$$\int_0^\ell a_\varepsilon u_\varepsilon(T)\,\mathrm{d}x + \int_0^T \int_0^\ell \frac{\dot{u}_\varepsilon^2}{2u_\varepsilon} + \frac{u_\varepsilon}{2} a_\varepsilon^2 \,\mathrm{d}x\ \mathrm{d}t \leq \int_0^\ell a_\varepsilon u_\varepsilon(0)\,\mathrm{d}x$$

We estimate from below via $a_\varepsilon^2 \geq a_{\mathsf{eff}}^2$, use $u_\varepsilon \overset{*}{\rightharpoonup} u$ and

the convexity of $(u,v) \mapsto \frac{v^2}{2u}$ to obtain

$$\int_0^\ell a_{\mathsf{eff}}\, u(T)\,\mathrm{d}x + \int_0^T \int_0^\ell \frac{\dot{u}^2}{2u} + \frac{u}{2} a_{\mathsf{eff}}^2 \,\mathrm{d}x\ \mathrm{d}t \leq \lim_{\varepsilon\to 0} \int_0^\ell a_\varepsilon u_\varepsilon(0)\,\mathrm{d}x = \int_0^\ell a_{\mathsf{eff}}\, u(0)\,\mathrm{d}x$$

Thus, u is a solution of (EDE) for $(\boldsymbol{M}_+(0,\ell), \mathcal{E}_0, \mathcal{R}_{\mathrm{H}})$. $\square$

Overview

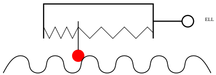
Aim: Derive dry friction as evol. Γ-limit of viscous friction

$(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}, \mathcal{E}_0, \Psi_0)$

where $\Psi_\varepsilon(v) = \frac{\varepsilon^\alpha}{2} v^2$ (quadratic)
and $\Psi_0(v) = \rho|v|$ (one-homogeneous)

Here $\mathcal{E}_\varepsilon(t, \cdot)$ is a **wiggly energy landscape**

James '96, Puglisi&Truskinovsky '02,'05



Prandtl Gedankenmodell 1928

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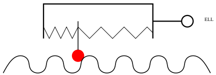

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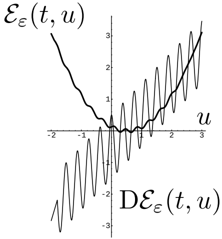


Driven gradient system $(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon)$

$$\mathcal{E}_\varepsilon(t,u) = \underbrace{\tfrac{1}{2}u^2 - \ell(t)u}_{\text{macroscopic part}} + \underbrace{\varepsilon\rho\cos(u/\varepsilon)}_{\text{wiggly part}}$$

$$\boxed{\varepsilon^\alpha \dot{u} = -\mathrm{D}_u \mathcal{E}_\varepsilon(t,u) = -\big(u - \ell(t)\big) + \rho \sin(u/\varepsilon)}$$

3. Energy-dissipation formulations

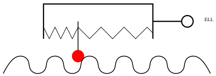
Aim: Derive dry friction as evol. Γ-limit of viscous friction

$(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}, \mathcal{E}_0, \Psi_0)$

where $\Psi_\varepsilon(v) = \frac{\varepsilon^\alpha}{2} v^2$ (quadratic)
and $\Psi_0(v) = \rho|v|$ (one-homogeneous)

Here $\mathcal{E}_\varepsilon(t,\cdot)$ is a **wiggly energy landscape**

James '96, Puglisi&Truskinovsky '02,'05

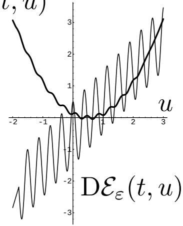


Prandtl Gedankenmodell 1928

Driven gradient system $(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon)$

$$\mathcal{E}_\varepsilon(t,u) = \underbrace{\tfrac{1}{2}u^2 - \ell(t)u}_{\text{macroscopic part}} + \underbrace{\varepsilon\rho\cos(u/\varepsilon)}_{\text{wiggly part}}$$

$$\varepsilon^\alpha \dot{u} = -\mathrm{D}_u\mathcal{E}_\varepsilon(t,u) = -\big(u-\ell(t)\big) + \rho\sin(u/\varepsilon)$$

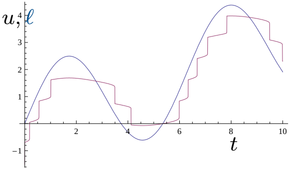


$\mathcal{E}_\varepsilon(t,u) \xrightarrow{\text{pw}} \mathcal{E}_0(t,u) = \frac{1}{2}u^2 - \ell(t)u + 0$ and $\Psi_\varepsilon \to \Psi_0 \equiv 0$

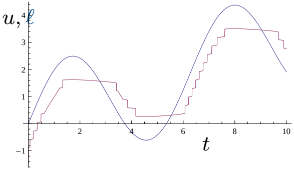
However, $u = \lim u^\varepsilon$ **does not solve** $0 = -\mathrm{D}_u\mathcal{E}_0(t,u(t))$ **!!**

Simulation: $\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u - \varepsilon\cos(u/\varepsilon)$,

$\ell(t) = 2\sin t + 0.3\,t$, $q(0) = -1.0$, $\varepsilon^\alpha = 10^{-3}$



$\varepsilon = 0.125$



$\varepsilon = 0.05$

Simulation: $\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u - \varepsilon\cos(u/\varepsilon)$,
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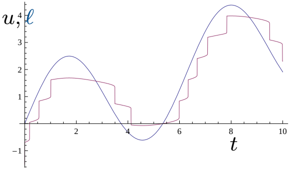

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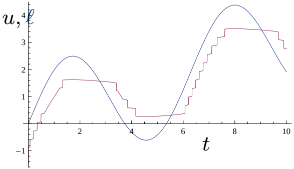

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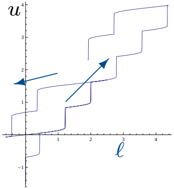

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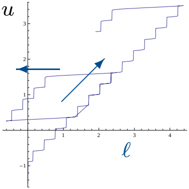

$\varepsilon = 0.05$

For $\varepsilon \to 0$ (vanishing wiggles and vanishing viscosity):
Convergence to a rate-independent hysteresis operator

$$\mathcal{E}_\varepsilon(t,u) = \tfrac{1}{2}u^2 - \ell(t)u + \varepsilon\rho\cos(u/\varepsilon), \quad \Psi_\varepsilon(v) = \tfrac{\varepsilon^\alpha}{2}v^2, \quad \Psi^*_\varepsilon(\xi) = \tfrac{1}{2\varepsilon^\alpha}\xi^2$$

Theorem (M'11 Cont. Mech. Thermodyn. / Puglisi-Truskinovsky'05)

$(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}, \mathcal{E}_0, \Psi_0)$ *where* $\mathcal{E}_0(u) = \frac{1}{2}u^2 - \ell(t)u$ *and* $\Psi_0(v) = \rho|v|$

Use (EDE) $\mathcal{E}_\varepsilon(T, u_\varepsilon(T)) + \mathbb{J}_\varepsilon(u_\varepsilon) = \mathcal{E}_\varepsilon(u_\varepsilon(0))$ with

$$\mathbb{J}_\varepsilon(u) = \int_0^T \Psi_\varepsilon(\dot u) + \Psi^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(t,u))\,\mathrm{d}t \geq \int_0^T |\dot u||\mathrm{D}\mathcal{E}_\varepsilon(t,u)| + \tfrac{1}{\varepsilon^{\alpha/2}}\mathrm{D}\mathcal{E}_\varepsilon(t,u)^2\,\mathrm{d}t$$

$$\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u + \varepsilon\rho\cos(u/\varepsilon), \qquad \Psi_\varepsilon(v) = \frac{\varepsilon^\alpha}{2}v^2, \qquad \Psi^*_\varepsilon(\xi) = \frac{1}{2\varepsilon^\alpha}\xi^2$$

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Proposition: $u^\varepsilon \rightsquigarrow u^0 \implies \liminf_{\varepsilon\to 0} \mathbb{J}_\varepsilon(u^\varepsilon) \geq \int_0^T \mathcal{M}(u^0, \dot u^0, t)\,\mathrm{d}t$ with

$\mathcal{M}(u,v,t) = |v|K(\ell(t)-u) + \chi_{[-\rho,\rho]}(\ell(t)-u)$ and $K(\xi) = \frac{1}{2\pi}\int_0^{2\pi} |\xi+\rho\cos y|\,\mathrm{d}y$

$K(\xi) = |\xi|$ for $|\xi| \geq \rho$ and $K(\xi) \gneqq |\xi|$ for $|\xi| < \rho \implies$

$\mathcal{M}(u,v,t) \geq |v|\,|\ell(t)-u| \geq -v\,\mathrm{D}\mathcal{E}_0(t,u) \quad \implies \quad ..... \quad \implies \quad \Psi_0(v) = \rho|v|$

Overview

1. Introduction

2. Motivating examples

3. Energy-dissipation formulations

4. Evolutionary variational inequality (EVE)

5. Rate-independent systems (RIS)

Overview

The **Evolutionary Variational Estimate (EVE)** is **derivative free**, so we can use Γ-convergence for $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ more directly.

Simplest case $\boxed{\mathbb{V}\dot{u} = -\mathbb{L}u}$ with Hilbert space $\boldsymbol{X}$,

energy $\mathcal{E}(u) = \frac{1}{2}\langle \mathbb{L}u, u\rangle \geq 0$ and viscous dissipation $\Psi(v) = \frac{1}{2}\langle \mathbb{V}v, v\rangle$

Theorem (Benilan'72 Hilbert space; [AGS05] general metric spaces)
We have **(i)** $\Leftrightarrow$ **(ii)** $\Leftrightarrow$ **(iii)=(EDB)** $\Leftrightarrow$ **(EDE)** $\Leftrightarrow$ **(EVI)**

with **(EVI)** $\begin{cases} \forall\, 0 \leq s < t \leq T\ \forall\, w \in \boldsymbol{X} : \\ \Psi(u(t){-}w) - \Psi(u(s){-}w) \leq (t{-}s)\big(\mathcal{E}(w){-}\mathcal{E}(u(t))\big) \end{cases}$

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"$\Rightarrow$"

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(u{-}w) = \langle \mathbb{V}\dot{u}, u - w\rangle \overset{\text{(i)}}{=} -\langle \mathbb{L}u, u{-}w\rangle$$
$$= \frac{1}{2}\langle \mathbb{L}w, w\rangle - \frac{1}{2}\langle \mathbb{L}u, u\rangle - \frac{1}{2}\langle \mathbb{L}(u{-}w), u{-}w\rangle \leq \mathcal{E}(w) - \mathcal{E}(u) - 0$$

Integration over time gives
$\Psi(u(t){-}w) - \Psi(u(s){-}w) = \int_s^t \mathcal{E}(w){-}\mathcal{E}(u(\tau))\,\mathrm{d}\tau \leq (t{-}s)\big(\mathcal{E}(w){-}\mathcal{E}(u(t))\big)$.

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"$\Leftarrow$" Rearrangement of quadratic expressions gives

(EVI) $\Leftrightarrow$ $\frac{1}{2}\langle \mathbb{V}(u(t){-}u(s)), u(t){+}u(s){-}2w\rangle \leq \frac{t-s}{2}\langle \mathbb{L}(u(t){+}w), w{-}u(t)\rangle$

Now set $s = t - h$, divide by h and let $h \to 0_+$, then we find

$$\langle \mathbb{V}\dot{u}(t), u(t){-}w\rangle \leq \frac{1}{2}\langle \mathbb{L}(u(t){+}w), w{-}u(t)\rangle$$

Setting $w = u(t){-}\delta\widehat{v}$, dividing by δ and letting $\delta \to 0_+$ gives

$\langle \mathbb{V}\dot{u}(t), \widehat{v}\rangle \leq -\langle \mathbb{L}u(t), \widehat{v}\rangle$ for all $\widehat{v}$ $\Rightarrow$ (i). $\square$

4. Evolutionary variational inequality (EVE)

$$\Psi_\varepsilon(u(t){-}w) - \Psi_\varepsilon(u(s){-}w) \leq (t{-}s)\big(\mathcal{E}_\varepsilon(w){-}\mathcal{E}_\varepsilon(u(t))\big)$$

Homogenization $\boxed{c_\varepsilon \dot u = (a_\varepsilon u_x)_x - b_\varepsilon u}$ with $a_\varepsilon = a(x/\varepsilon); b_\varepsilon(x) = \cdots$

(EVI)$_\varepsilon$ with $\Psi_\varepsilon(v) = \frac{1}{2}\int_\Omega c_\varepsilon(x) v(x)^2 \,\mathrm{d}x$ and $\mathcal{E}_\varepsilon(u) = \frac{1}{2}\int_\Omega a_\varepsilon u_x^2 + b_\varepsilon u^2 \,\mathrm{d}x$

$$\Psi_\varepsilon(u(t)-w) - \Psi_\varepsilon(u(s)-w) \leq (t-s)\big(\mathcal{E}_\varepsilon(w)-\mathcal{E}_\varepsilon(u(t))\big)$$

Homogenization $\boxed{c_\varepsilon \dot{u} = (a_\varepsilon u_x)_x - b_\varepsilon u}$ with $a_\varepsilon = a(x/\varepsilon); b_\varepsilon(x) = \cdots$

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We know $u_\varepsilon \rightharpoonup u \in \mathrm{H}^1([0,T],\mathrm{L}^2(\Omega))$ and $u_\varepsilon(t) \rightharpoonup u(t)$ in $\mathrm{H}^1(\Omega)$.

Hence, fixing $s<t$ and $\widehat{w}$ we choose $\widehat{w}_\varepsilon \rightharpoonup \widehat{w}$ with $\mathcal{E}_\varepsilon(w_\varepsilon) \to \mathcal{E}_0(\widehat{w})$

Using $\mathrm{H}^1(\Omega) \Subset \mathrm{L}^2(\Omega)$ and $\mathcal{E}_0(u(t)) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(t))$ gives

$$\Psi_\varepsilon(\underbrace{u_\varepsilon(t)-\widehat{w}_\varepsilon}_{\underset{\mathrm{L}^2}{\to} u(t)-\widehat{w}}) - \Psi_\varepsilon(\underbrace{u_\varepsilon(s)-\widehat{w}_\varepsilon}_{\underset{\mathrm{L}^2}{\to} u(s)-\widehat{w}}) \leq (t-s)\Big(\underbrace{\mathcal{E}_\varepsilon(\widehat{w}_\varepsilon)}_{\to\mathcal{E}_0(\widehat{w})}-\underbrace{\mathcal{E}_\varepsilon(u_\varepsilon(t))}_{\liminf\ \text{suff.}}\Big)$$

$$\Psi_\varepsilon(u(t)-w) - \Psi_\varepsilon(u(s)-w) \leq (t-s)\big(\mathcal{E}_\varepsilon(w)-\mathcal{E}_\varepsilon(u(t))\big)$$

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We find (EVI)$_0$ $\Psi_0(u(t)-\widehat{w}) - \Psi_0(u(s)-\widehat{w}) \leq (t-s)\big(\mathcal{E}_0(\widehat{w}) - \mathcal{E}_0(u(t))\big)$

with $\Psi_0(v) = \frac{1}{2}\int_\Omega c_{\text{arith}} v^2\,\mathrm{d}x$ and $\mathcal{E}_0(u) = \frac{1}{2}\int_\Omega a_{\text{harm}} u_x^2 + b_{\text{arith}} u^2\,\mathrm{d}x$

Effective equation $c_{\text{arith}}\dot{u} = a_{\text{harm}} u_{xx} - b_{\text{arith}} u$

Overview

Ambrosio-Gigli-Savaré'05, Daneri-Savaré'08'10

Gradient system $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$ with **quadratic** $\mathcal{R}(u,v) = \frac{1}{2}\langle G(u)v, v\rangle$

- **Geodesic distance** $d_{\mathcal{R}} : \boldsymbol{X} \times \boldsymbol{X} \to [0,\infty]$ defined via
$$d_{\mathcal{R}}(u_0,u_1)^2 = \inf\{ \int_0^1 2\mathcal{R}(\widetilde{u},\dot{\widetilde{u}})\,\mathrm{d}s \mid u_0 \overset{\widetilde{u}}{\rightsquigarrow} u_1 \}$$
- $\widetilde{u} : [s_0,s_1] \to \boldsymbol{X}$ is called a **geodesic curve** in $(\boldsymbol{X}, d_{\mathcal{R}})$
if $d_{\mathcal{R}}(\widetilde{u}(r),\widetilde{u}(t)) = |t-r| d_{\mathcal{R}}(\widetilde{u}(s_0),\widetilde{u}(s_1))$ for all $r,t \in [s_0,s_1]$

Ambrosio-Gigli-Savaré'05, Daneri-Savaré'08'10

Gradient system $(\boldsymbol{X},\mathcal{E},\mathcal{R})$ with **quadratic** $\mathcal{R}(u,v)=\frac{1}{2}\langle G(u)v,v\rangle$

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- $\widetilde{u}:[s_0,s_1]\to\boldsymbol{X}$ is called a **geodesic curve** in $(\boldsymbol{X},d_\mathcal{R})$ if $d_\mathcal{R}(\widetilde{u}(r),\widetilde{u}(t))=|t{-}r|d_\mathcal{R}(\widetilde{u}(s_0),\widetilde{u}(s_1))$ for all $r,t\in[s_0,s_1]$
- $\mathcal{E}:\boldsymbol{X}\to\mathbb{R}_\infty$ is called **geodesically λ-convex** on $(\boldsymbol{X},d_\mathcal{R})$ if
$s\mapsto\mathcal{E}(\widetilde{u}(s))-\lambda\frac{d_\mathcal{R}(\widetilde{u}(s_0),\widetilde{u}(s))^2}{2}$ is convex on $[s_0,s_1]$ for all geod. $\widetilde{u}$

Trivial but useful and important case:

$G(u)=G_0=$const. $\Longrightarrow$ $d_\mathcal{R}(u_0,u_1)=\|u_1{-}u_0\|_{G_0}$ with $\|w\|^2_{G_0}=\langle G_0w,w\rangle$

Then, $\mathcal{E}$ geod. λ-convex on $(\boldsymbol{X},d_{G_0})$ $\iff$ $\mathrm{D}^2\mathcal{E}\geq\lambda G_0$

Truely derivative-free reformulation of gradient system

(i) $0 \in G(u)\dot u + \mathrm{D}\mathcal{E}(u)$ (ii) $\dot u = -\nabla_{\mathbb{G}}\mathcal{E}(u) = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u)$ (iii) ....

(EDE) $\mathcal{E}(u(T)) + \int_0^T \mathcal{R}(u,\dot u) + \mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(u))\,\mathrm{d}t \leq \mathcal{E}(u(0))$

Theorem [AGS'05] (Benilan'72: case $d = d_{G_0}$)

If $(\boldsymbol{X}, \mathcal{E}, G)$ is geodesically λ-convex, then

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (EDE) $\Leftrightarrow$ **(EVI)**$_\lambda$ $\Leftrightarrow$ **(EVI')**$_\lambda$

where

(EVI)$_\lambda$ $\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} d_G(u(t),w)^2 + \frac{\lambda}{2} d_G(u(t),w)^2 + \mathcal{E}(u(t)) \leq \mathcal{E}(w)$ for $t > 0$, $w \in \boldsymbol{X}$

(EVI')$_\lambda$ $\frac{\mathrm{e}^{\lambda\tau}}{2} d_G(u(t+\tau),w)^2 - \frac{1}{2} d_G(u(t),w)^2 \leq \frac{\mathrm{e}^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}(w) - \mathcal{E}(u(t+\tau))\big)$ for $t, \tau > 0$, $w \in \boldsymbol{X}$

Truely derivative-free reformulation of gradient system

(i) $0 \in G(u)\dot{u} + \mathrm{D}\mathcal{E}(u)$ (ii) $\dot{u} = -\nabla_{\mathbb{G}}\mathcal{E}(u) = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u)$ (iii) ....

(EDE) $\mathcal{E}(u(T)) + \int_0^T \mathcal{R}(u,\dot{u}) + \mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(u))\,\mathrm{d}t \le \mathcal{E}(u(0))$

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(EVI')$_\lambda$ $\frac{\mathrm{e}^{\lambda\tau}}{2} d_G(u(t{+}\tau),w)^2 - \frac{1}{2} d_G(u(t),w)^2 \le \frac{\mathrm{e}^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}(w) - \mathcal{E}(u(t{+}\tau))\big)$ for $t,\tau > 0$, $w \in \boldsymbol{X}$

Exercise:

(a) Prove (EDE) $\Leftrightarrow$ (EVI)$_\lambda$ (b) Prove (EVI)$_\lambda$ $\Leftrightarrow$ (EVI'$_\lambda$

Truely derivative-free reformulation of gradient system

(i) $0 \in G(u)\dot u + \mathrm{D}\mathcal{E}(u)$ (ii) $\dot u = -\nabla_{\mathbb{G}}\mathcal{E}(u) = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u)$ (iii) ....

(EDE) $\mathcal{E}(u(T)) + \int_0^T \mathcal{R}(u,\dot u) + \mathcal{R}^*(u,-\mathrm{D}\mathcal{E}(u))\,\mathrm{d}t \leq \mathcal{E}(u(0))$

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(EVI)$_\lambda$ $\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} d_G(u(t),w)^2 + \frac{\lambda}{2} d_G(u(t),w)^2 + \mathcal{E}(u(t)) \leq \mathcal{E}(w)$ for $t>0$, $w \in \boldsymbol{X}$

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⊕ no derivatives of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ appear $\rightsquigarrow$ ideal for Γ-convergence

⊕ no time derivative $\dot u$ is involved

$$(\text{EVI'})_\lambda \quad \frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau),w)^2 - \frac{1}{2} d_\varepsilon(u(t),w)^2 \ \le\ \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$$

Theorem (Savaré'11 (personal communication))

If $(\boldsymbol{X},\mathcal{E}_\varepsilon,d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$, and $d_\varepsilon \xrightarrow{\text{cont}} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X},\mathcal{E}_\varepsilon,d_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X},\mathcal{E},d)$.

(Convergence of the whole sequence u^ε to u, since solutions are unique.)

(EVI')$_\lambda$ $\frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau), w)^2 - \frac{1}{2} d_\varepsilon(u(t), w)^2 \le \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$

Theorem (Savaré'11 (personal communication))

If $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}$, and $d_\varepsilon \xrightharpoonup{\text{cont}} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}, d)$.

(Convergence of the whole sequence u^ε to u, since solutions are unique.)

The relatively strong assumption $d_\varepsilon \xrightharpoonup{\text{cont}} d$ in $\boldsymbol{X}$ means
$u_\varepsilon \rightharpoonup u$ & $w_\varepsilon \rightharpoonup w$ in $\boldsymbol{X}$ $\quad\Longrightarrow\quad d_\varepsilon(u_\varepsilon, w_\varepsilon) \to d(u, w)$

This can be weakened to
Gromov-Hausdorff convergence $(\boldsymbol{X}, d_\varepsilon) \xrightarrow{\text{GH}} (\boldsymbol{X}, d)$.

4. Evolutionary variational inequality (EVE)

(EVI')$_\lambda$ $\frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau),w)^2 - \frac{1}{2} d_\varepsilon(u(t),w)^2 \;\leq\; \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$

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(Convergence of the whole sequence u^ε to u, since solutions are unique.)

Sketch of proof: u_ε solves (EVI')$_\lambda$ for $(\boldsymbol{X},\mathcal{E}_\varepsilon,d_\varepsilon)$

- ε-uniform bounds from (EVI')$_\lambda$ $\Longrightarrow$ $u_{\varepsilon_k}(t) \rightharpoonup u(t)$ for all $t \in [0,T]$
- Pass to the limit in (EVI')$_\lambda$ using
  recovery sequence $w_\varepsilon \rightharpoonup w$ with $\mathcal{E}_\varepsilon(w_\varepsilon) \to \mathcal{E}(w)$
  $\Rightarrow$ $d_\varepsilon(u_\varepsilon(t{+}\tau),w_\varepsilon) \to d(u(t{+}\tau),w)$ and $d_\varepsilon(u_\varepsilon(t),w_\varepsilon) \to d(u(t),w)$
  $\Rightarrow$ $\mathcal{E}(u(t{+}\tau)) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(t{+}\tau))$ by Γ-liminf estimate
- Hence, $u:[0,T]\to\boldsymbol{X}$ satisfies (EVI')$_\lambda$ for $(\boldsymbol{X},\mathcal{E},d)$ QED

Overview

Theorem (Savaré'11 (personal communication))

If $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$, and $d_\varepsilon \xrightarrow{cont} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon) \xrightarrow{evol} (\boldsymbol{X}, \mathcal{E}, d)$.

Nonlinear parabolic PDE

$c(x/\varepsilon)u_t = \big(a(x/\varepsilon)u_x\big)_x - f(x/\varepsilon, u)$ for $t > 0,\ x \in \Omega =]0, \ell[$ & Neum. BC

$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{a_\varepsilon}{2} u_x^2 + F(x/\varepsilon, u)\,\mathrm{d}x$ on $\boldsymbol{X} = \mathrm{H}^1(0, \ell)$

where $F(y+1, u) = F(y, u) \geq \rho_0 u^2 - C$ for $C, \rho_0 > 0$
and $f(y, u) = \partial_u F(y, u)$, $\partial_u^2 F(y, u) \geq \lambda_0$

$\mathcal{E}_\varepsilon \xrightarrow[\mathrm{H}^1]{\Gamma} \mathcal{E}_0 : u \mapsto \int_\Omega \frac{a_*}{2} u_x^2 + F^*(u)\mathrm{d}x$ $\qquad$ ($a_* = \big(\int_0^1 \frac{\mathrm{d}y}{a(y)}\big)^{-1},\ F^*(u) = \int_0^1 F(y, u)\mathrm{d}y$)

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$\Psi_\varepsilon(v) = \int_\Omega \frac{c_\varepsilon}{2} v^2\,\mathrm{d}x$: $\quad \Psi_\varepsilon \xrightharpoonup[\mathrm{H}^1]{\text{cont}} \Psi : v \mapsto \int_0^\ell \frac{c^*}{2} v^2\,\mathrm{d}x$ in $\boldsymbol{X} = \mathrm{H}^1(0,\ell)$ weakly

Hence, $(\mathrm{H}^1(\Omega), \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathrm{H}^1(\Omega), \mathcal{E}, \Psi) \,\widehat{=}\,$ $c^* u_t = a_* u_{xx} - f^*(u)$

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Second application of (EVI) to homogenization:

Linear PDE $c_\varepsilon u_t = (a_\varepsilon u_x)_x$ for $t > 0$ and $x \in \Omega =]0,\ell[$ & Neum. BC

$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{c_\varepsilon}{2} u^2 \, \mathrm{d}x$ and $\mathbb{K}_\varepsilon \xi = -\frac{1}{c_\varepsilon}\big(a_\varepsilon (\frac{\xi}{c_\varepsilon})_x\big)_x$ **(Exerc. 5)**

$u_t = \dot{u} = -\mathbb{K}_\varepsilon \mathrm{D}\mathcal{E}_\varepsilon(u) = -\mathbb{K}_\varepsilon(c_\varepsilon u) = \frac{1}{c_\varepsilon}\big(a_\varepsilon u_x\big)_x$ correct equations!

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Correct Banach space is $\boldsymbol{X} = \mathrm{L}^2(\Omega)$, because $\mathcal{E}_\varepsilon$ are ε-unif. coercive

$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E} : u \mapsto \int_\Omega \frac{c_*}{2} u^2 \,\mathrm{d}x$ in $\mathrm{L}^2(\Omega)$ weakly (recall $c_* \lneqq c^* =$ good value)

$\Psi_\varepsilon^*(\xi) = \frac{1}{2} \int_\Omega a_\varepsilon (\xi/c_\varepsilon)_x^2 \,\mathrm{d}x$ and $\Psi_\varepsilon(v) = \frac{1}{2} \int_\Omega \frac{1}{a_\varepsilon}\Big(\int_0^x c_\varepsilon v \,\mathrm{d}y\Big)^2 \mathrm{d}x$

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Recall $d_\varepsilon(u_0,u_1) = (2\Psi_\varepsilon(u_1-u_0))^{1/2}$

Fortunately, we do NOT have $\Psi_\varepsilon \xrightarrow{\text{cont}} \Psi$ in $\mathrm{L}^2(\Omega)$ weakly! $\oplus$

Theorem is not applicable!

(EVI')$_\lambda$ $\frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau),w)^2 - \frac{1}{2} d_\varepsilon(u(t),w)^2 \;\leq\; \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$

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Reformulate in new variable $w = c_\varepsilon u$:

$\dot w = \big(a_\varepsilon(\frac{w}{c_\varepsilon})_x\big)_x = -\widehat{\mathbb{K}}_\varepsilon \mathrm{D}\widehat{\mathcal{E}}_\varepsilon(w)$ with $\widehat{\mathcal{E}}_\varepsilon(w) = \int_\Omega \frac{w^2}{2c_\varepsilon}\,\mathrm{d}x$ in $\boldsymbol{X} = \mathrm{L}^2(\Omega)$

Now $\widehat{\mathcal{E}}_\varepsilon \xrightarrow{\Gamma} \widehat{\mathcal{E}} : w \mapsto \int_\Omega \big(\frac{1}{2c}\big)_* w^2 \,\mathrm{d}x = \int_\Omega \frac{w^2}{2c^*}\,\mathrm{d}x$ in $\mathrm{L}^2(\Omega)$ weakly!

and $\widehat{\Psi}_\varepsilon(\dot w) = \frac{1}{2}\int_\Omega \frac{1}{a_\varepsilon}\big(\int_0^x \dot w \,\mathrm{d}y\big)^2 \mathrm{d}x$

$\Longrightarrow\; \widehat{\Psi}_\varepsilon \xrightarrow{\text{cont}} \widehat{\Psi} : \dot w \mapsto \int_\Omega \underbrace{\big(\frac{1}{a}\big)^*}_{=1/a_*} \underbrace{\big(\int_0^x \dot w\,\mathrm{d}y\big)}_{\in \mathrm{H}^1}{}^2 \,\mathrm{d}x$ in $\mathrm{L}^2(\Omega)$ weakly!

(EVI')$_\lambda$ $\frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau),w)^2 - \frac12 d_\varepsilon(u(t),w)^2 \leq \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$

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$\Longrightarrow$ $\widehat{\Psi}_\varepsilon \xrightarrow{\text{cont}} \widehat{\Psi} : \dot w \mapsto \int_\Omega \underbrace{\big(\tfrac{1}{a}\big)^*}\,\underbrace{\big(\int_0^x \dot w \, dy\big)}^2 dx$ in $L^2(\Omega)$ weakly!

Theorem applies: $(L^2(\Omega), \widehat{\mathcal{E}}_\varepsilon, \widehat{\Psi}_\varepsilon) \xrightarrow{\text{evol}} (L^2(\Omega), \widehat{\mathcal{E}}, \widehat{\Psi})$ $\dot w = (a_*(w/c^*))_x$

Overview

1. Introduction

2. Motivating examples

3. Energy-dissipation formulations

4. Evolutionary variational inequality (EVE)

5. Rate-independent systems (RIS)

Overview

$(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$ is a **Rate-Independent System (RIS)**

if the dissipation is 1-homogeneous: $\mathcal{R}(u, \gamma\dot{u}) = \gamma^1 \mathcal{R}(u, \dot{u})$

$\rightsquigarrow$ the friction forces $\partial_{\dot{u}}\mathcal{R}(u, \lambda\dot{u}) = \lambda^0 \partial\mathcal{R}(u, \dot{u})$ are independent of the size but not the direction of the rate $\dot{u}$

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For solutions $u \in \mathrm{W}^{1,1}([0,T]; \boldsymbol{X})$ we still have the three equivalent formulations

(i) $0 \in \partial_{\dot{u}}\mathcal{R}(u, \dot{u}) + \mathrm{D}\mathcal{E}(t, u) \ \in \boldsymbol{X}^*$ force balance

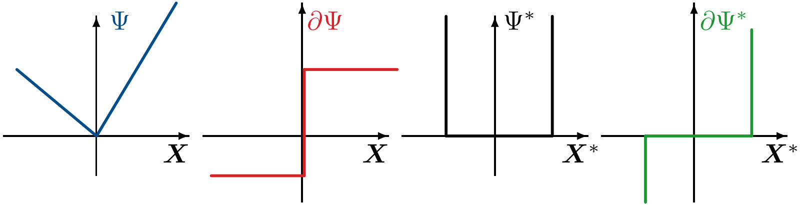
(ii) $\dot{u} \in \partial_{\xi}\mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(t, u)) \ \in \boldsymbol{X}$ flow law

(iii) $\mathcal{R}(u, \dot{u}) + \partial_{\xi}\mathcal{R}^*(u, -\mathrm{D}\mathcal{E}(t, u)) = -\langle \mathrm{D}\mathcal{E}(t, u), \dot{u}\rangle$ power balance

Special form of the subdifferential for 1-homogeneous Ψ:

Lemma. $\Psi : \boldsymbol{X} \to [0,\infty]$ convex, lsc, 1-homogeneous, then

$$\xi \in \partial\Psi(v) \implies \begin{cases} \xi \in K^* := \partial\Psi(0) \\ \langle \xi, v \rangle = \Psi(v) \end{cases} \qquad \Psi^*(\xi) = \begin{cases} 0 & \text{if } \xi \in K^*, \\ \infty & \text{else.} \end{cases}$$



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(i) $\boxed{0 \in \partial_{\dot u}\mathcal{R}(u,\dot u) + \mathrm{D}\mathcal{E}(t,u)}$ (or (ii), (iii)) can be reformulated:

$$\begin{cases} \textbf{(S)}_{\text{loc}}\text{:} \ \ 0 \in \partial_v \mathcal{R}(u,{\color{red}0}) + \mathrm{D}\mathcal{E}(t,u) & \text{local stability} \quad \text{(purely static!)} \\ \textbf{(E)}_{\text{loc}}\text{:} \ \ 0 = \mathcal{R}(u,\dot u) + \langle \mathrm{D}\mathcal{E}(t,u), \dot u\rangle & \text{power balance} \quad \text{(only scalar!)} \end{cases}$$

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Intregating $\textbf{(E)}_{\text{loc}}$ gives equivalent (note $\mathcal{R}^* = 0 \;\Leftrightarrow\; (\text{S})_{\text{loc}}$):

$$\textbf{(E)} \quad \underbrace{\mathcal{E}(T, u(T))}_{\text{final energy}} + \underbrace{\int_0^T \mathcal{R}(u, \dot u)\,\mathrm{d}t}_{\text{dissipated energy}} = \underbrace{\mathcal{E}(0, u(0))}_{\text{initial energy}} + \underbrace{\int_0^T \partial_t \mathcal{E}(t, u(t))\,\mathrm{d}t}_{\text{work of external forces}}$$

(S)$_{\text{loc}}$ $0 \in \partial_v \mathcal{R}(u,0) + \mathrm{D}\mathcal{E}(t,u)$ **(E)** $\mathcal{E}(T,u(T)) + \int_0^T \mathcal{R}\,\mathrm{d}t = \mathcal{E}(0,u(0)) + \int_0^T \partial_t \mathcal{E}\,\mathrm{d}t$

Since the dissipation $\int_0^T \mathcal{R}(u(t),\dot u(t))\,\mathrm{d}t$ controls the BV-norm only, solutions u will not be absolutely continuous $\rightsquigarrow$ jumps $u(t^-) \neq u(t^+)$!

Definition (Energetic solutions for RIS)

A function $u : [0,T] \to \boldsymbol{X}$ is called **energetic solution** for $(\boldsymbol{X}, \mathcal{E}, \mathcal{D})$ if for all $t \in [0,T]$ we have **stability (S)** and **energy balance (E)**:

(S) $\mathcal{E}(t,u(t)) \leq \mathcal{E}(t,w) + \mathcal{D}(u(t),w)$ for all $w \in \boldsymbol{X}$ (global stability)

(E) $\mathcal{E}(t,u(t)) + \mathrm{Diss}_{\mathcal{D}}(u,[0,t]) = \mathcal{E}(0,u(0)) + \int_0^t \partial_s \mathcal{E}(s,u(s))\,\mathrm{d}s.$

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Same definition as **quasistatic evolution** of SISSA (Dal Maso et al.)

- $\mathcal{D} : \boldsymbol{X} \times \boldsymbol{X} \to [0,\infty]$ distance induced by $\mathcal{R}$
- $\mathrm{Diss}_{\mathcal{D}}(u,[0,t]) = \sup \big\{ \sum_{j=1}^N \mathcal{D}(u(t_{j-1}),u(t_j)) \,\big|\, \text{all partitions} \big\}$

(S)$_{\text{loc}}$ $0 \in \partial_v \mathcal{R}(u,0) + \mathrm{D}\mathcal{E}(t,u)$ **(E)** $\mathcal{E}(T,u(T)) + \int_0^T \mathcal{R}\,\mathrm{d}t = \mathcal{E}(0,u(0)) + \int_0^T \partial_t \mathcal{E}\,\mathrm{d}t$

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$\oplus$ again solution concept has no derivative $\dot u$

$\oplus$ only functionals $\mathcal{E}$ and $\mathcal{D}$ are used (no derivatives $\mathrm{D}\mathcal{E}$ or $\partial_v \mathcal{R}(u,v)$)

$\rightsquigarrow$ natural Γ-convergence theory

Typical case:

$\boldsymbol{X}$ Banach space

$\mathcal{R}(u,v) = \Psi(v) \quad \Longrightarrow \quad \mathcal{D}(u_0,u_1) = \Psi(u_1{-}u_0)$

Theorem (Equivalence for convex energies)

If $(\boldsymbol{X},\mathcal{E},\mathcal{R})$ as above and

$\mathcal{E}(t,\cdot)$ convex. Then,

(S) & (E) $\quad \Longleftrightarrow \quad$ (S)$_{loc}$ & (E).

Overview

(S)$_\varepsilon$ $\mathcal{E}_\varepsilon(t,u(t)) \leq \mathcal{E}_\varepsilon(t,w) + \mathcal{D}_\varepsilon(u(t),w)$ for all $w \in \boldsymbol{X}$

(E)$_\varepsilon$ $\mathcal{E}_\varepsilon(t,u(t)) + \mathrm{Diss}_{\mathcal{D}}(u,[0,t]) = \mathcal{E}_\varepsilon(0,u(0)) + \int_0^t \partial_s \mathcal{E}_\varepsilon(s,u(s))\,\mathrm{d}s.$

First a simple result.

Theorem (Evolutionary Γ-convergence for RIS, M-Roubicek-Stefanelli'08)

Assume that $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)$ *satisfies* $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ *and* $\mathcal{D}_\varepsilon \xrightarrow{\text{cont}} \mathcal{D}_0$*,*

then $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X},\mathcal{E}_0,\mathcal{D}_0)$ *in the sense of energetic solutions.*

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then $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X},\mathcal{E}_0,\mathcal{D}_0)$ *in the sense of energetic solutions.*

Sketch of proof:

- (S) Closedness of the sets of stable states:

To show: $\Big(u^\varepsilon \rightharpoonup u$ and u^ε satisfies (S)$_\varepsilon\Big) \implies u$ satisfies (S)$_0$

$$\mathcal{E}_0(u) \overset{\Gamma}{=} \liminf_{\varepsilon\to0} \mathcal{E}_\varepsilon(t,u^\varepsilon) \overset{(\mathsf{S})_\varepsilon}{\leq} \liminf_{\varepsilon\to0} \mathcal{E}_\varepsilon(w^\varepsilon) + \mathcal{D}_\varepsilon(u^\varepsilon,w^\varepsilon) \overset{\text{rec}}{=} \mathcal{E}_0(w) + \mathcal{D}_0(u,w)$$

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First a simple result.

Theorem (Evolutionary Γ-convergence for RIS, M-Roubicek-Stefanelli'08)

Assume that $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon)$ satisfies $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ and $\mathcal{D}_\varepsilon \xrightarrow{cont} \mathcal{D}_0$,

then $(\boldsymbol{X},\mathcal{E}_\varepsilon,\mathcal{D}_\varepsilon) \xrightarrow{evol} (\boldsymbol{X},\mathcal{E}_0,\mathcal{D}_0)$ in the sense of energetic solutions.

Sketch of proof:

- (S) Closedness of the sets of stable states:
- (E)$_0$ "$\leq$" follows by simple liminf-estimates from (E)$_\varepsilon$

  "$\geq$" is a consequence of (S)$_0$ (cf. chain rule for global slope)

Generally, independent Mosco convergence is not enough:

$$\left.\begin{array}{c}\mathcal{E}^\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}^0 \\ \mathcal{R}^\varepsilon \xrightarrow{\mathsf{M}} \mathcal{R}^0\end{array}\right\} \quad \not\Longrightarrow \quad (\boldsymbol{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon) \xrightarrow{\mathsf{evol}} (\boldsymbol{Q}, \mathcal{E}^0, \mathcal{R}^0)$$

Example in $\mathcal{Q} = \mathbb{R}^2$ [MRS'08]:

$\mathcal{E}^\varepsilon(t,q) = \frac{1}{2}q_1^2 + \frac{1}{2}(q_1 - \frac{q_2}{\varepsilon})^2 - tq_1, \quad \mathcal{R}^\varepsilon(v) = |v_1| + |v_2|/\varepsilon^2$ for $\varepsilon > 0$.

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Example in $\mathcal{Q}=\mathbb{R}^2$ [MRS'08]:

$\mathcal{E}^\varepsilon(t,q)=\frac12 q_1^2+\frac12(q_1-\frac{q_2}{\varepsilon})^2-tq_1, \quad \mathcal{R}^\varepsilon(v)=|v_1|+|v_2|/\varepsilon^2$ for $\varepsilon>0$.

Then $\mathcal{E}^\varepsilon \xrightarrow{\mathsf{M}} \mathcal{E}^0$ and $\mathcal{R}^\varepsilon \xrightarrow{\mathsf{M}} \mathcal{R}^0$ (also Mosco convergence) with

$$\mathcal{E}^0(t,q)=\begin{cases}\frac12 q_1^2-tq_1 & \text{for } q_2=0,\\ \infty & \text{for } q_2\neq 0;\end{cases} \quad \text{and } \mathcal{R}^0(v)=\begin{cases}|v_1| & \text{for } v_2=0,\\ \infty & \text{for } v_2\neq 0.\end{cases}$$

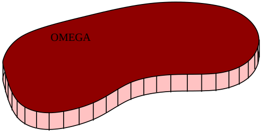
For the unique solutions with $q^\varepsilon(0)=0$ we find

$q^0(t)=\binom{\max\{t-1,0\}}{0} \neq \lim_{\varepsilon\to 0} q^\varepsilon(t)=\binom{\max\{0,t/2-1\}}{0}$.

Overview

Aim: Derive a **plate model** for elastoplastic materials

Start from linearized 3D elastoplasticity as RIS on a thin plate-like domain $\Omega_\varepsilon = \omega \times]-\varepsilon, \varepsilon[$



$u_\varepsilon^{3D} : \Omega_\varepsilon \to \mathbb{R}^3$ displacement

$p_\varepsilon^{3D} : \Omega_\varepsilon \to \mathbb{R}^{3\times 3}_{\mathrm{dev}}$ plastic strain

$$(u_\varepsilon^{3D}, p_\varepsilon^{3D}) \xrightarrow{\text{rescaling}} (u^\varepsilon, p^\varepsilon) \xrightarrow{\varepsilon\to 0} (u^0, p^0)$$

Find evolution law for the limit $(u^0, p^0) : [0, T] \to \boldsymbol{Q}$.

Motivation: Guenther/Krejčí/Sprekels: ZAMM'08.
Small strain oscillations of an elastoplastic Kirchhoff plate.

- **Scaling of the domain**: $y = S_\varepsilon x$ with $S_\varepsilon = \mathrm{diag}(1, 1, 1/\varepsilon)$

  $\Omega_\varepsilon = \underbrace{\omega \times \,]-\varepsilon, \varepsilon[}_{\text{thin plate}} \;\ni x \;\mapsto\; y \in \underbrace{\omega \times \,]-1, 1[}_{\text{fixed domain}} =: \Omega$

- **Plate-like scaling of displacement and plastic strains:**

  $u_\varepsilon^{\mathrm{3D}}(x) = \varepsilon S_\varepsilon u^\varepsilon(S_\varepsilon x)$ and $p_\varepsilon^{\mathrm{3D}}(x) = \varepsilon p^\varepsilon(S_\varepsilon x)$

$\rightsquigarrow$ in-plane displacements $u_1^{\mathrm{3D}}, u_2^{\mathrm{3D}}$ are one order smaller ($O(\varepsilon^1)$) than out-of-plane component u_3^{3D} ($O(1)$).

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- Linearized strain tensor satisfies $\mathrm{e}(u_\varepsilon^{\mathrm{3D}})(x) = \varepsilon S_\varepsilon \mathrm{E}(u^\varepsilon)(S_\varepsilon x) S_\varepsilon$

$$\mathrm{e}(u_\varepsilon^{\mathrm{3D}}) = \varepsilon^1 \begin{pmatrix} E_{11} & E_{12} & \frac{1}{\varepsilon}E_{13} \\ E_{12} & E_{22} & \frac{1}{\varepsilon}E_{23} \\ \frac{1}{\varepsilon}E_{13} & \frac{1}{\varepsilon}E_{23} & \frac{1}{\varepsilon^2}E_{13} \end{pmatrix} \text{ with } \mathrm{E}(u)(y) = \tfrac{1}{2}(\nabla_y u + \nabla_y u)^{\mathsf{T}})$$

- Scaled state $q^\varepsilon = (u^\varepsilon, p^\varepsilon) \in \boldsymbol{Q} := \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3) \times \mathrm{L}^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})$

$\rightsquigarrow$ **fixed state space Q**

Scaled functionals (note $\mathrm{d}x = \varepsilon\,\mathrm{d}y$)

- Energy functional

$$\mathcal{E}^\varepsilon(t,q^\varepsilon) = \frac{1}{\varepsilon^3}\mathcal{E}^{3\mathrm{D}}_\varepsilon(t,q^{3\mathrm{D}}_\varepsilon) = \int_\Omega W_\varepsilon(\mathrm{E}(u)(y),p(y))\,\mathrm{d}y - \langle \ell(t),u\rangle$$

where $W_\varepsilon(E,p) = W(S_\varepsilon E S_\varepsilon, p)$

- Dissipation potential

$$\mathcal{R}^\varepsilon(\dot p^\varepsilon) = \frac{1}{\varepsilon^3}\mathcal{R}^{3\mathrm{D}}_\varepsilon(p^{3\mathrm{D}}_\varepsilon) = \int_\Omega \frac{\sigma^{3\mathrm{D}}_{\textsf{yield}}(\varepsilon)}{\varepsilon}|\dot p(y)|\,\mathrm{d}y.$$

Hence, we must choose $\sigma^{3\mathrm{D}}_{\mathsf{yield}}(\varepsilon) = \varepsilon^1 \sigma^*_{\mathsf{yield}}$.

(This corresponds to the fact that the yield stress needs to be of the same order as the typical strains in $\mathrm{e}(u^{3\mathrm{D}}_\varepsilon)$, e.g. $\mathrm{e}_{11} = O(\varepsilon)$.)

Then $\mathcal{R}^\varepsilon = \mathcal{R}$ is in fact independent of ε.

The scaled RIS is given via $(\boldsymbol{Q}, \mathcal{E}^\varepsilon, \mathcal{R})$.

The scaled energy density $W_\varepsilon(E,p) = W(S_\varepsilon E S_\varepsilon, p)$ satisfies

Lemma (Γ-convergence of densities). In $\mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathbb{R}^{3\times3}_{\mathrm{dev}}$ we have

$$W_\varepsilon \xrightarrow{\Gamma} W_{\mathsf{pl}} : (\mathrm{E}, p) \mapsto \begin{cases} \min_{b\in\mathbb{R}^3} W(\mathrm{E}{+}e_3 \overset{\mathsf{sym}}{\otimes} b, p) & \text{for } \mathrm{E}e_3 = 0, \\ \infty & \text{for } \mathrm{E}e_3 \neq 0. \end{cases}$$

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We have $\mathcal{E}^\varepsilon(t,q) = \mathcal{B}^\varepsilon(q) - \langle \ell(t), q\rangle$ with $\mathcal{B}^\varepsilon(u,p) = \int_\Omega W_\varepsilon(\mathrm{E}(u),p)\,\mathrm{d}y$.

Now set $\mathcal{E}^0(t,q) \overset{\mathsf{def}}{=} \mathcal{B}^0(q) - \langle \ell(t), q\rangle$ with $\mathcal{B}^0(u,p) = \int_\Omega W_{\mathsf{pl}}(\mathrm{E}(u),p)\,\mathrm{d}y$.

Theorem (Mosco convergence, Liero'10)

We have $\mathcal{B}^\varepsilon \xrightarrow{M} \mathcal{B}^0$ in $\boldsymbol{Q} = \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3) \times \mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}})$.

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- Lower bound is a simple Joffe argument using $W_\varepsilon \xrightarrow{\Gamma} W_{\mathsf{pl}}$.
- The existence of recovery sequences follows the purely elastic case, cf. Bourquin/Ciarlet/Geymonat/Raoult'92

Theorem: (Elastoplastic plate model [Liero-M'11])
Under the above assumptions we have $(\boldsymbol{Q}, \mathcal{E}^\varepsilon, \mathcal{R}) \xrightarrow{\text{evol}} (\boldsymbol{Q}, \mathcal{E}^0, \mathcal{R})$.

This is a **plate model** because $\mathcal{B}^0(u,p) < \infty \iff u \in \boldsymbol{U}_{\mathsf{KL}}$.

Kirchhoff–Love displ. $\boldsymbol{U}_{\mathsf{KL}} \overset{\text{def}}{=} \{\, u \in \mathrm{H}^1_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^3) \mid \mathrm{E}(u)e_3 = 0 \text{ a.e. in } \Omega \,\}$

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$$\mathrm{E}(u)e_3 = 0 \quad \Leftrightarrow \quad \partial_3 u_1 + \partial_1 u_3 = \partial_3 u_2 + \partial_2 u_3 = \partial_3 u_3 = 0$$

$$\boldsymbol{U}_{\mathsf{KL}} = \{\, u = (V_1 - y_3\partial_1 V_3, V_2 - y_3 \partial_2 V_3, V_3) \mid V = (V_1, V_2, V_3) \in \boldsymbol{V} \,\},$$

where $\boldsymbol{V} \overset{\text{def}}{=} \mathrm{H}^1_{\gamma_\mathrm{D}}(\omega;\mathbb{R}^2) \times \mathrm{H}^2_{\gamma_\mathrm{D}}(\omega)$

$V = (V_j)$ is defined only on the two-dimensional midsurface ω

However, $p \in \boldsymbol{Z} = \mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{dev}})$ remains on a $3\mathrm{D}$ domain!

Explicit form of plate model for $W(e,p) = \frac{\lambda}{2}(\mathrm{tr}\, e)^2 + \mu|e|^2 + \frac{h}{2}|p|^2$.

Set $\mathbb{E}(V) := \begin{pmatrix} E_{11}(V) & E_{12}(V) \\ E_{12}(V) & E_{22}(V) \end{pmatrix}$, $\mathbb{P} := \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$,

$\Sigma(\mathbb{E}) = \frac{2\lambda\mu}{\lambda+2\mu}\, \mathrm{tr}\, \mathbb{E}\, I_2 + 2\mu\mathbb{E} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$,

$[\mathbb{P}]_0 = \int_{-1}^{1} \mathbb{P}(y_3)\, \mathrm{d}y_3$, $[\mathbb{P}]_1 = \int_{-1}^{1} y_3 \mathbb{P}(y_3)\, \mathrm{d}y_3$, $[\![\, \mathbb{E} \,\|\, b \,]\!] = \begin{pmatrix} \mathbb{E} & \begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix} \\ b_1\; b_2 & b_3 \end{pmatrix} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$

$$
\begin{array}{lll}
(1) & 0 = -\,\mathrm{div}\,\big(\Sigma_0(2\mathbb{E}(V)) - [\mathbb{P}]_0)\big) - G^\ell_{\mathsf{memb}}(t,\cdot) & \text{in } \omega \\
(2) & 0 = \mathrm{div}\,\mathrm{div}\,\big(\Sigma_0(\tfrac{2}{3}\mathrm{D}^2 V_3 + [\mathbb{P}]_1)\big) - g^\ell_{\mathsf{bend}}(t,\cdot) - \mathrm{div}\, G^\ell_{\mathsf{bend}}(t,\cdot) & \text{in } \omega \\
(3) & 0 \in \partial R(\dot p) + \mathrm{dev}\,\big([\![\, \Sigma_0(\mathbb{P} - \mathbb{E}(V) + x_3 \mathrm{D}^2 V_3) \,\|\, 0 \,]\!]\big) + hp & \text{in } \Omega
\end{array}
$$

(1) Membrane equation, in-place displacements (V_1, V_2), 2nd order elliptic, 2D
(2) Bending equation, out-of-plane displacement V_3, 4th order elliptic, 2D
(3) Plastic flow rule, 0th order differential inclusion for plastic tensor p, 3D

Overview

Time discretization was the start of the energetic formulation ([M.&Theil'99, Ortiz&Repetto'99, Miehe et al.'02]):

$0 = t_0 < t_1 < \cdot < t_{N-1} < T_N = T$ partition of time interval

Definition (Time-incremental minimization = backward Euler scheme)

Given $q_0 \in \mathcal{Q}$ find iteratively $q_1, q_1, \ldots, q_N$ via

$$q_k \in \operatorname{Arg\,min}_{\widetilde{q} \in \mathcal{Q}} \mathcal{E}(t_k, \widetilde{q}) + \mathcal{D}(q_{k-1}, \widetilde{q})\big)$$

- independent of timestep $t_k - t_{k-1}$ (rate independence)
- discrete counterparts of (S) & (E):

(S)$_{\text{discr}}$ $\mathcal{E}(t_k, q_k) \leq \mathcal{E}(t_k, \widehat{q}) + \mathcal{D}(q_k, \widehat{q})$ for all $\widehat{q} \in \mathcal{Q}$

(E)$_{\text{discr}}$ $$\int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s \leq \mathcal{E}(t_k, q_k) - \mathcal{E}(t_{k-1}, q_{k-1}) + \mathcal{D}(q_{k-1}, q_k) \leq \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\,\mathrm{d}s$$

Additionally choose discrete subspaces

$\mathcal{Q}_h = \mathcal{F}_h \times \mathcal{Z}_h \subset \mathcal{F} \times \mathcal{Z} = \mathcal{Q}$ with

Energetic density (recovery sequence):
For all q with $\mathcal{E}(t,q) < \infty$ there exist $q_h \in \mathcal{Q}_h$, $h > 0$, with
$q_h \rightharpoonup q$ and $\mathcal{E}(t,q_h) \to \mathcal{E}(t,q)$ for $h \to 0$.

Typical case (e.g., elastoplasticity):

- $\mathcal{Q} = \mathrm{H}^1_0(\Omega;\mathbb{R}^d) \times \mathrm{L}^2(\Omega;\mathbb{R}^m)$
- $\mathcal{Q}_h$ piecewise affine functions on triangulations $\mathcal{T}_h$ of Ω.

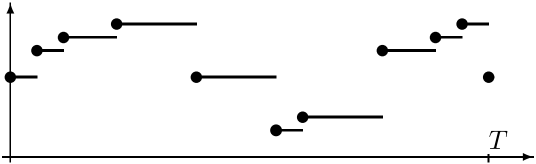
$\Longrightarrow$ Q_h dense in the strong topology and
$\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}$ continuous in strong topology

$\Pi = \{0 = t_0^\Pi < t_1^\Pi < \cdots < t_{N_\Pi}^\Pi = T\}$ partition with fineness $\Phi(\Pi) = \max\{\, t_j - t_{j-1} \mid j = 1, ..., n_\Pi \,\}$.

Space-time discretized problem

$$q_k^{h,\Pi} \in \operatorname*{Arg\,min}_{q \in \mathcal{Q}_h} \big(\mathcal{E}(t_k^\Pi, q) - \mathcal{E}(t_{k-1}^\Pi, q_{k-1}^{h,\Pi}) + \mathcal{D}(q_{k-1}^\Pi, q) \big)$$

Temporally piecewise interpolant $\overline{q}^{h,\Pi} : [0,T] \to \mathcal{Q}_h \subset \mathcal{Q}$ with $\overline{q}^{h,\Pi}(t) = q_k^{h,\Pi}$ for $t \in [t_k, t_{k+1})$ and $\overline{q}^{h,\Pi}(T) = q_{n_\Pi}^{h,\Pi}$.



Theorem (Main convergence result [M-Roubíček'09 M2AN])

- $\mathcal{Q} = \mathcal{F} \times \mathcal{Z} \subset F \times Z$ *reflexive Banach spaces*
- $\mathcal{E}(t,\cdot){:}\mathcal{Q} \to \mathbb{R}_\infty$, $\mathcal{D}{:}\mathcal{Z}{\times}\mathcal{Z} \to [0,\infty]$ *coercive, wlsc*
- $\partial_t\mathcal{E}(\cdot,q) \in \mathrm{C}^1([0,T])$ *and* $|\partial_t\mathcal{E}(t,q)| \le c_1(\mathcal{E}(t,q){+}c_0)$,
- $(\mathcal{E},\mathcal{D},(\mathcal{Q}_h)_{h>0})$ *has* ***mutual recovery sequences***.

For stable $q^0 \in \mathcal{Q}$ *choose* $(q^0_h)_{h>0}$ *with* $Q_h \ni q^0_h \rightharpoonup q_0$ *and* $\mathcal{E}(0,q_h) \to \mathcal{E}(0,q)$, *and define* $\overline{q}^{h,\Pi} : [0,T] \to \mathcal{Q}_h$ *as above.*

Then, there exists a subseq. $(h_j,\Pi_j)_{j\in\mathbb{N}}$ *with* $h_j, \Phi(\Pi_j) \to 0$ *and an energetic solution* $q{:}[0,T] \to \mathcal{Q}$ *with* $q(0) = q^0$ *such that for all* t

- $\overline{z}^{h_j,\Pi_j}(t) \rightharpoonup z(t)$,
- $\mathcal{E}(t,\overline{q}^{h_j,\Pi_j}(t)) \to \mathcal{E}(t,q(t))$,
- $\mathrm{Diss}_\mathcal{D}(\overline{q}^{h_j,\Pi_j},[0,t]) \to \mathrm{Diss}_\mathcal{D}(q,[0,t])$,
- $\partial_t\mathcal{E}(\cdot,\overline{q}^{h_j,\Pi_j}(\cdot)) \to \partial_t\mathcal{E}(\cdot,q(\cdot))$.

- No uniqueness assumption needed
- No assumptions on the smoothness of solutions is made
  ($\rightsquigarrow$ no convergence rates to be expected)

Result in short.

(1) The numerical approximations are relatively compact.

(2) All limits of subsequences of numerical approximations
(as $h, \Phi(\Pi) \to 0$) are true solutions.

(3) There are no spurious or ghost solutions.

(1) $\widehat{=}$ stability of the algorithm
(2) $\widehat{=}$ consistency of the algorithm

- No uniqueness assumption needed
- No assumptions on the smoothness of solutions is made ($\rightsquigarrow$ no convergence rates to be expected)

Definition (Mutual recovery sequences)

Define the stable sets
$\mathcal{S}^h(t) = \{\, q_h \in \mathcal{Q}_h \mid \forall \widehat{q}_h \in \mathcal{Q}_h \colon \mathcal{E}(t, q_h) \leq \mathcal{E}(t, \widehat{q}_h) + \mathcal{D}(q_h, \widehat{q}_h) \,\}$

We say that $(\mathcal{E}, \mathcal{D}, (\mathcal{Q}_h))$ has **mutual recovery sequences** if for all $q, \widetilde{q} \in \mathcal{Q}$ and $(q_h)_{h>0}$ with $q_h \in \mathcal{S}^h(t)$ and $\sup_{h>0} \mathcal{E}(t, q_h) < \infty$ there exists $(\widetilde{q}_h)_{h>0}$ such that $\widetilde{q}_h \rightharpoonup \widetilde{q}$ and

$$\limsup_{h\to 0} \big(\mathcal{E}(t, \widetilde{q}_h) - \mathcal{E}(t, q_h) + \mathcal{D}(q_h, \widehat{q}_h)\big) \leq \mathcal{E}(t, \widetilde{q}) - \mathcal{E}(t, q) + \mathcal{D}(q, \widetilde{q}).$$

Application to linearized elastoplasticity (Moreau, Suquet,...)

$\mathcal{Q} = \mathcal{F} \times \mathcal{Z} = \boldsymbol{F} \times \boldsymbol{Z}$ with $\boldsymbol{F}, \boldsymbol{Z}$ Hilbert spaces
$\mathcal{E}(t,u,z) = \frac{1}{2}\langle\!\langle \mathcal{A}\binom{u}{z}, \binom{u}{z}\rangle\!\rangle - \langle u, \ell(t)\rangle$
$\mathcal{D}(q_{\mathrm{old}}, q_{\mathrm{new}}) = \Psi(z_{\mathrm{new}} - z_{\mathrm{old}})$, where $K := \partial\Psi(0) \subset \boldsymbol{Z}^*$ (convex cone)

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$\mathcal{Q} = \mathcal{F} \times \mathcal{Z} = \boldsymbol{F} \times \boldsymbol{Z}$ with $\boldsymbol{F}, \boldsymbol{Z}$ Hilbert spaces

$\mathcal{E}(t,u,z) = \frac{1}{2}\langle\!\langle \mathcal{A}\binom{u}{z}, \binom{u}{z} \rangle\!\rangle - \langle u, \ell(t)\rangle$

$\mathcal{D}(q_{\text{old}}, q_{\text{new}}) = \Psi(z_{\text{new}} - z_{\text{old}})$, where $K := \partial\Psi(0) \subset \boldsymbol{Z}^*$ (convex cone)

Problem: Ψ not weakly continuous,
in general not even strongly continuous!

Projectors (or interpolants) $P_h : Q \to Q_h$ with $Q_h \ni P_h q \to q$.

(MRS) For $q_h \rightharpoonup q$ and $\widetilde{q}$ there exists $\widetilde{q}_h \in Q_h$ with $\widetilde{q}_h \rightharpoonup \widetilde{q}$ and

$$\mathcal{E}(t,\widetilde{q}_h) - \mathcal{E}(t,q_h) + \Psi(\widetilde{q}_h - q_h) \to \mathcal{E}(t,\widetilde{q}) - \mathcal{E}(t,q) + \Psi(\widetilde{q} - q)$$

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Choose $\widetilde{q}_h = q_h + P_h(\widetilde{q} - q)$,

then $\widetilde{q}_h \rightharpoonup \widetilde{q}$ and $\widetilde{q}_h - q_h \to \widetilde{q} - q$.

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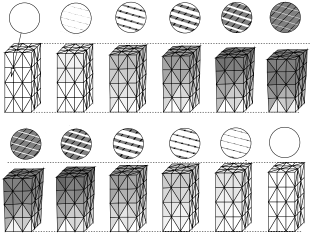
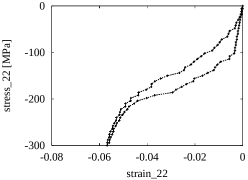
$$\begin{aligned}\mathcal{E}(t,\widetilde{q}_h)-\mathcal{E}(t,q_h) &= \tfrac{1}{2}\langle\!\langle \mathcal{A}\underbrace{(\widetilde{q}_h+q_h)}_{\rightharpoonup \widetilde{q}+q} - 2\ell(t), \underbrace{P_h(\widetilde{q}-q)}_{\to \widetilde{q}-q}\rangle\!\rangle \\ &\to \tfrac{1}{2}\langle\!\langle \mathcal{A}(\widetilde{q}+q) - 2\ell(t), \widetilde{q}-q\rangle\!\rangle \\ &= \mathcal{E}(t,\widetilde{q})-\mathcal{E}(t,q)\end{aligned}$$

Hence, (MRS) is constructed.

Numerical simulations based on this algorithm:

Kružík-M-Roubíček. Modelling of microstructure and its evolution in shape-memory alloy single-crystals, in particular in CuAlNi. *Meccanica* 2005.





hysteresis loop in stress-strain diagram

M-Roubíček-Zeman. Complete damage in elastic and viscoelastic media and its energetics. *Comp. Meth. Appl. Mech. Eng.* 2010.

Bartels-Mielke-Roubíček. Quasistatic small-strain plasticity in the limit of small hardening and its numerical approximation, *SIAM J. Numer. Anal.* 2012.

Conclusion

- Several results on evolutionary Γ-convergence for generalized gradient systems are available $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$
- Depending on the problem certain formulations are more useful:
  - energy-dissipation balance (EDB) $= (\Psi, \Psi^*)$-formulation
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Thank you for your attention

WIAS preprints at http://www.wias-berlin.de/people/mielke/