A TWO CITIES THEOREM FOR THE PARABOLIC ANDERSON MODEL

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Abstract: The parabolic Anderson problem is the Cauchy problem for the heat equation $\partial_t u(t,z) = \Delta u(t,z) + \xi(z)u(t,z)$ on $(0,\infty) \times \mathbb{Z}^d$ with random potential $(\xi(z) \colon z \in \mathbb{Z}^d)$. We consider independent and identically distributed potentials, such that the distribution function of $\xi(z)$ converges polynomially at infinity. If u is initially localised in the origin, i.e., if $u(0,z) = \mathbb{1}_0(z)$, we show that, as time goes to infinity, the solution is completely localised in *two* points almost surely and in *one* point with high probability. We also identify the asymptotic behaviour of the concentration sites in terms of a weak limit theorem.

1. Introduction and main results

1.1 The parabolic Anderson model and intermittency.

We consider the heat equation with random potential on the integer lattice $\mathbb{Z}^d$ and study the Cauchy problem with localised initial datum,

$$\begin{aligned} \partial_t u(t,z) &= \Delta u(t,z) + \xi(z)u(t,z), && (t,z) \in (0,\infty) \times \mathbb{Z}^d, \\ u(0,z) &= \mathbb{1}_0(z), && z \in \mathbb{Z}^d, \end{aligned} \tag{1.1}$$

where

$$(\Delta f)(z) = \sum_{y \sim z} [f(y) - f(z)], \qquad z \in \mathbb{Z}^d, f \colon \mathbb{Z}^d \to \mathbb{R}$$

is the discrete Laplacian, and the potential $(\xi(z) \colon z \in \mathbb{Z}^d)$ is a collection of independent identically distributed random variables.

The problem (1.1) and its variants are often called the *parabolic Anderson problem.* It originated in the work of the physicist P. W. Anderson on entrapment of electrons in crystals with impurities, see [An58]. The parabolic version of the problem appears in the context of chemical kinetics and population dynamics, and also provides a simplified qualitative approach to problems in magnetism and turbulence. The references [GM90], [Mo94] and [CM94] provide applications, background and heuristics around the parabolic Anderson model. Interesting recent mathematical progress can be found, for example, in [GH06], [HKM06], and [BMR07], and [GK05] is a recent survey article.

One main reason for the great interest in the parabolic Anderson problem lies in the fact that it exhibits an *intermittency effect*: It is believed that, at late times, the overwhelming contribution to the total mass of the solution u of the problem (1.1) comes from a small

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number of spatially separated regions of small diameter, which are often called the *relevant islands*. As the upper tails of the potential distribution gets heavier, this effect is believed to get stronger, the number of relevant islands and their size are believed to become smaller. Providing rigorous evidence for intermittency is a major challenge for mathematicians, which has lead to substantial research efforts in the past 15 years.

An approach, which has been proposed in the physics literature, see [Z+87] or [GK05], suggests to study large time asymptotics of the moments of the total mass

$$U(t) = \sum_{z\in\mathbb{Z}^d} u(t,z), \qquad t > 0\,. \tag{1.2}$$

Denoting expectation with respect to ξ by $\langle\,\cdot\,\rangle$, if all exponential moments $\langle\exp(\lambda\xi(z))\rangle$ for $\lambda > 0$ exist, then so do all moments $\langle U(t)^p\rangle$ for $t > 0$, $p > 0$. Intermittency becomes manifest in a faster growth rate of higher moments. More precisely, the model is called intermittent if

$$\limsup_{t\to\infty} \frac{\langle U(t)^p\rangle^{1/p}}{\langle U(t)^q\rangle^{1/q}} = 0, \qquad \text{for } 0 < p < q. \tag{1.3}$$

Whenever ξ is nondegenerate random, the parabolic Anderson model is intermittent in this sense, see [GM90, Theorem 3.2]. Further properties of the relevant islands, like their asymptotic size and shape of potential and solution, are reflected (on a heuristical level) in the asymptotic expansion of $\log\langle U(t)^p\rangle$ for large t. Recently, in [HKM06], it was argued that the distributions with finite exponential moments can be divided into exactly four different universality classes, with each class having a qualitatively different long-time behaviour of the solution.

It is, however, a much harder mathematical challenge to prove intermittency in the original geometric sense, and to identify asymptotically the number, size and location of the relevant islands. This programme was initiated by Sznitman for the closely related continuous model of a Brownian motion with Poissonian obstacles, and the very substantial body of research he and his collaborators created is surveyed in his monograph [Sz98]. For the problem (1.1) and two universality classes of potentials, the double-exponential distribution and distributions with tails heavier than double-exponential (but still with all exponential moments finite), the recent paper [GKM07] makes substantial progress towards completing the geometric picture: Almost surely, the contribution coming from the complement of a random number of relevant islands is negligible compared to the mass coming from these islands, asymptotically as $t\to\infty$. In the double-exponential case, the radius of the islands stays bounded, in the heavier case the islands are single sites, and in Sznitman's case the radius tends to infinity on the scale $t^{1/(d+2)}$.

Questions about the number of relevant islands remained open in all these cases, and constitute the main concern of the present paper. Both in [GKM07] and [Sz98] it is shown that an upper bound on the number of relevant islands is $t^{o(1)}$, but this is certainly not always best possible. In particular, the questions whether a *bounded number* of islands already carry the bulk of the mass, or when *just one* island is sufficient, are unanswered. These questions are difficult, since there are many local regions that are good candidates for being a relevant island, and the known criterion that identifies relevant islands does not seem to be optimal.

In the present paper, we study the parabolic Anderson model with potential distributions that do not have any finite exponential moment. For such distributions one expects the intermittency effect to be even more pronounced than in the cases discussed above, with a very small number of relevant islands, which are just single sites. Note that in this case intermittency cannot be studied in terms of the moments $\langle U(t)^p \rangle$, which are not finite.

The main result of this paper is that, in the case of potentials with polynomial tails, *almost surely* at all large times there are *at most two* relevant islands, each of which consists of a single site. In other words, the proportion of the total mass $U(t)$ is asymptotically concentrated in just two time-dependent lattice points. Note that, by the intermediate value theorem, the total mass cannot be concentrated in just one site, if this site is changing in time on the lattice. Hence this is the strongest form of localisation that can hold almost surely. However, we also show that, *with high probability*, the total mass $U(t)$ is concentrated in a *single* lattice point.

The *intuitive picture* is that, at a typical large time, the mass, which is thought of as a population, inhabits one site, interpreted as a city. At some rare times, however, word spreads that a better site has been found, and the entire population moves to the new site, so that at the transition times part of the population still lives in the old city, while part has already moved to the new one. This picture inspired the term *'two cities theorem'* for our main result, which was suggested to us by S.A. Molchanov. The present paper is, to the best of our knowledge, the first where such a behaviour is found in a model of mathematical physics.

Concentration of the mass in a single site with high probability has been observed so far only for quite simple mean field models, see [FM90, FG92]. The present paper is the first instance where it has been found in the parabolic Anderson model or, indeed, any comparable lattice-based model. We also study the asymptotic locations of the points where the mass concentrates in terms of a weak limit theorem with an explicit limiting density. Precise statements are formulated in the next section.

1.2 The parabolic Anderson model with Pareto-distributed potential.

We assume that the potentials $\xi(z)$ at all sites z are independent and *Pareto-distributed* with parameter $\alpha > d$, i.e., the distribution function is

$$F(x) = \mathrm{Prob}(\xi(z) < x) = 1 - x^{-\alpha}, \qquad x \geq 1. \tag{1.4}$$

In particular, we have $\xi(z) \geq 1$ for all $z \in \mathbb{Z}^d$, almost surely. Note from [GM90, Theorem 2.1] that the restriction to parameters $\alpha > d$ is necessary and sufficient for (1.1) to possess a unique nonnegative solution $u\colon (0,\infty) \times \mathbb{Z}^d \to [0,\infty)$. Recall that

$$U(t) = \sum_{z \in \mathbb{Z}^d} u(t,z)$$

is the total mass of the solution at time $t > 0$. Our main result shows the almost sure localisation of the solution $u(t,\,\cdot\,)$ in two lattice points $Z_t^{(1)}$ and $Z_t^{(2)}$, as $t \to \infty$.

Theorem 1.1 (Two cities theorem)**.** *Suppose* $u\colon (0,\infty)\times\mathbb{Z}^d \to [0,\infty)$ *is the solution to the parabolic Anderson problem* (1.1) *with i.i.d. Pareto-distributed potential with parameter* $\alpha > d$. *Then there exist processes* $(Z_t^{(1)}\colon t>0)$ *and* $(Z_t^{(2)}\colon t>0)$ *with values in* $\mathbb{Z}^d$, *such that* $Z_t^{(1)} \neq Z_t^{(2)}$ *for all* $t>0$, *and*

$$\lim_{t\to\infty} \frac{u(t,Z_t^{(1)})+u(t,Z_t^{(2)})}{U(t)} = 1 \qquad \textit{almost surely.}$$

Remark 1. At least two sites are needed to carry the total mass in an *almost sure* limit theorem. Indeed, assume that there is a single process $(Z_t\colon t>0)$ such that $u(t,Z_t) > 2U(t)/3$ for all large t. As $u(\,\cdot\,,z)$ is continuous for any $z\in\mathbb{Z}^d$, this leads to a contradiction at jump times of the process $(Z_t\colon t>0)$. From the growth of $U(t)$ one can see that this process is not eventually constant, and thus has jumps at arbitrarily large times. ◇

Our second result concerns convergence in probability. We show that the solution $u(t,\,\cdot\,)$ is localised in *just one* lattice point with high probability.

Theorem 1.2 (One point localisation in probability)**.** *The process* $(Z_t^{(1)}\colon t>0)$ *in Theorem 1.1 can be chosen such that*

$$\lim_{t\to\infty} \frac{u(t,Z_t^{(1)})}{U(t)} = 1 \qquad \textit{in probability.}$$

Remark 2. The proof of this result given in this paper uses strong results provided for the proof of Theorem 1.1. However, it can be proved with less sophisticated tools, and a self-contained proof can be found in our unpublished preprint [KMS06]. ◇

Remark 3. We conjecture that the one-point localization phenomenon holds for a wider class of heavy-tailed potentials, including the stretched exponential case. We also believe that it does *not* hold for *all* potentials in the 'single-peak' class of [HKM06]. ◇

Remark 4. The asymptotic behaviour of $\log U(t)$ for the Anderson model with heavy-tailed potential is analysed in detail in [HMS08]. In the case of a Pareto-distributed potential it turns out that already the leading term in the asymptotic expansion of $\log U(t)$ is random. This is in sharp contrast to potentials with exponential moments, where the leading two terms in the expansion are always deterministic. More precisely, introducing

$$q = \frac{d}{\alpha-d} \qquad \text{and} \qquad \theta = \frac{2^d B(\alpha-d,d)}{q^d(d-1)!}, \tag{1.5}$$

where $B(\cdot,\cdot)$ denotes the Beta function, in [HMS08, Th. 1.2] it is shown that

$$\frac{(\log t)^q}{t^{q+1}} \log U(t) \Longrightarrow Y, \qquad \text{where} \quad \mathbb{P}(Y\le y) = \exp\{-\theta y^{d-\alpha}\}, \tag{1.6}$$

and $\Rightarrow$ denotes weak convergence. Note that the upper tails of Y have the same asymptotic order as the Pareto distribution with parameter $\alpha-d$, i.e., $\mathbb{P}(Y>y) \asymp y^{d-\alpha}$ as $y\to\infty$. The proof of [HMS08, Th. 1.2] also shows that there is a process $(Z_t\colon t>0)$ such that

$$\frac{(\log t)^q}{t^{q+1}} \log u(t,Z_t) \Longrightarrow Y, \qquad \text{where} \quad \mathbb{P}(Y\le y) = \exp\{-\theta y^{d-\alpha}\}. \tag{1.7}$$

Note, however, that a combination of (1.6) with (1.7) does not yield the concentration property in Theorem 1.2 since the asymptotics are only logarithmic. Much more precise techniques are necessary for this purpose. ◇

In Section 1.3 we see how the process $(Z_t^{(1)} : t > 0)$ in Theorem 1.2 can be defined as the maximiser in a random variational problem associated with the parabolic Anderson problem. Our third result is a limit theorem for this process. Recall the definition of q and θ from (1.5), and denote by $|\cdot|$ the ℓ^1-norm on $\mathbb{R}^d$.

Theorem 1.3 (Limit theorem for the concentration site). *The process $(Z_t^{(1)} : t > 0)$ in Theorem 1.2 can be chosen such that, as $t \to \infty$,*

$$Z_t^{(1)} \Big(\frac{\log t}{t}\Big)^{q+1} \Longrightarrow X^{(1)},$$

where $X^{(1)}$ is an $\mathbb{R}^d$-valued random variable with density

$$p^{(1)}(x_1) = \alpha \int_0^\infty \frac{\exp\{-\theta y^{d-\alpha}\}\,\mathrm{d}y}{(y+q|x_1|)^{\alpha+1}}.$$

Remark 5. The proof of this result uses the point process technique developed in [HMS08]. A more elementary proof can be found in our unpublished preprint [KMS06]. ◇

Remark 6. If we choose the processes $(Z_t^{(1)} : t > 0)$ and $(Z_t^{(2)} : t > 0)$ such that, with probability tending to one, $u(t, Z_t^{(1)})$ and $u(t, Z_t^{(2)})$ are the largest and second largest value of $u(t,z)$, we show that , as $t \to \infty$,

$$(Z_t^{(1)}, Z_t^{(2)})\Big(\frac{\log t}{t}\Big)^{q+1} \Longrightarrow (X^{(1)}, X^{(2)}),$$

where $(X^{(1)}, X^{(2)})$ is a pair of $\mathbb{R}^d$-valued random variables with joint density

$$p(x_1, x_2) = \int_0^\infty \frac{\alpha \exp\{-\theta y^{d-\alpha}\}\,\mathrm{d}y}{(y+q|x_1|)^{\alpha}(y+q|x_2|)^{\alpha+1}}.$$

By projecting this result on the first component we obtain the convergence in distribution statement of Theorem 1.3, where the density of $X^{(1)}$ is given by

$$p^{(1)}(x_1) = \int_0^\infty \Big(\int_{\mathbb{R}^d} \frac{\mathrm{d}x_2}{(y+q|x_2|)^{\alpha+1}} \Big) \frac{\alpha \exp\{-\theta y^{d-\alpha}\}}{(y+q|x_1|)^{\alpha}}\,\mathrm{d}y\,.$$

The inner integral equals $y^{d-\alpha-1}\, 2^d q^{-d} B(\alpha+1-d, d)/(d-1)!$. Recalling (1.5) and using the functional equation $B(x+1,y)\,(x+y) = B(x,y)\,x$ for $x, y > 0$, yields

$$p^{(1)}(x_1) = (\alpha-d)\theta \int_0^\infty y^{d-\alpha-1} \frac{\exp\{-\theta y^{d-\alpha}\}}{(y+q|x_1|)^{\alpha}}\,\mathrm{d}y = \alpha \int_0^\infty \frac{\exp\{-\theta y^{d-\alpha}\}\,\mathrm{d}y}{(y+q|x_1|)^{\alpha+1}},$$

using integration by parts in the last step. Moreover, from the proof of Theorem 1.3 one can easily infer the joint convergence

$$\Big(\frac{\log t}{t}\Big)^{q+1} \Big(Z_t^{(1)}, \frac{\log u(t, Z_t^{(1)})}{\log t}\Big) \Longrightarrow (X, Y)\,,$$

where the joint density of (X, Y) is

$$(x, y) \mapsto \alpha \frac{\exp\{-\theta y^{d-\alpha}\}}{(y+q|x|)^{\alpha+1}}.$$

◇

1.3 Overview: The strategy behind the proofs.

Throughout the paper we will say that a statement occurs *eventually for all t* when there exist a time t_0 such that the statement is fulfilled for all $t > t_0$. Note that when a statement is said to hold true almost surely eventually for all t, the corresponding t_0 can be random.

As shown in [GM90, Theorem 2.1], under the assumption $\alpha > d$, the unique nonnegative solution $u\colon (0,\infty)\times\mathbb{Z}^d \to [0,\infty)$ of (1.1) has a *Feynman-Kac representation*

$$u(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t = z\}\Big], \qquad t>0,\ z\in\mathbb{Z}^d, \tag{1.8}$$

where $(X_s\colon s\geq 0)$ under $\mathbb{P}_0$ (with expectation $\mathbb{E}_0$) is a continuous-time simple random walk on the lattice $\mathbb{Z}^d$ with generator Δ starting at the origin. Hence, the total mass of the solution is given by

$$U(t) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\Big].$$

Heuristically, for a fixed time $t>0$, the paths $(X_s\colon 0\leq s\leq t)$ that have the greatest impact on the average $U(t)$ spend most of their time at a site z which,

- has a large potential value $\xi(z)$,
- and can be reached quickly, i.e., is sufficiently close to the origin.

For $\rho\in(0,1)$, the strategy $A_t^{z,\rho}$ of wandering to a site z during the time interval $[0,\rho t)$ and staying at z during the time $[\rho t, t]$ has, for $|z|\gg t$, approximately the probability

$$\mathbb{P}_0\,(A_t^{z,\rho}) \approx \exp\Big\{-|z|\log\frac{|z|}{\mathrm{e}\rho t} + \eta(z)\Big\},$$

where $\eta(z) = \log N(z)$ and $N(z)$ denotes the number of paths of length $|z|$ starting at zero and ending at z (see Proposition 4.2 for details). Then the integral in the Feynman–Kac formula is bounded from below by $t(1-\rho)\xi(z)$ for the paths of the random walk following the strategy $A_t^{z,\rho}$. Hence, we obtain by optimising over z and $\rho\in(0,1)$,

$$\frac1t\log U(t) \gtrapprox \sup_{z\in\mathbb{Z}^d}\sup_{\rho\in(0,1)}\Big[(1-\rho)\xi(z) - \frac{|z|}{t}\log\frac{|z|}{\mathrm{e}\rho t} + \frac{\eta(z)}{t}\Big] = \max_{z\in\mathbb{Z}^d}\Phi_t(z),$$

where

$$\Phi_t(z) = \Big[\xi(z) - \frac{|z|}{t}\log\xi(z) + \frac{\eta(z)}{t}\Big]\mathbb{1}\{t\xi(z)\geq|z|\}. \tag{1.9}$$

The restriction $t\xi(z)\geq|z|$ arises as otherwise the globally optimal value $\rho = |z|/(t\xi(z))$ would exceed one. This bound, stated as Proposition 4.2, is a minor improvement of the lower bound obtained in [HMS08]. In addition, we show that $\max\Phi_t$ also gives an asymptotic upper bound for $\frac1t\log U(t)$, which is much harder and constitutes a significant improvement of the bound obtained in [HMS08], see Proposition 4.4. Altogether

$$\frac1t\log U(t)\approx\max_{z\in\mathbb{Z}^d}\Phi_t(z),$$

and it is plausible that the optimal sites at time t are the sites, where the two largest values of the random functional Φ_t are attained. This is indeed the definition of the processes $(Z_t^{(1)}\colon t\geq 0)$ and $(Z_t^{(2)}\colon t\geq0)$, which is underlying our three main theorems.

The remainder of the paper is organised as follows:

In Section 2 we provide several technical results for later use. In particular, we study the behaviour of $\eta(z)$ and of the upper order statistics of the potential ξ, and we derive spectral estimates similar to those obtained in [GKM07].

In Section 3 we study the asymptotic properties of the sites $Z_t^{(1)}$, $Z_t^{(2)}$, and $Z_t^{(3)}$, where Φ_t attains its three largest values, as well as the properties of $\Phi_t(Z_t^{(i)})$, for $i = 1, 2, 3$. Here we prove Proposition 3.4, which states that, almost surely, the gap $\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)})$ is eventually large enough. This is the main reason for $u(t, z)$ being concentrated at just two sites $Z_t^{(1)}$ and $Z_t^{(2)}$. Observe that a similar statement about the gap $\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)})$ is not true as, by continuity, there are arbitrarily large times t such that $\Phi_t(Z_t^{(1)}) = \Phi_t(Z_t^{(2)})$, which is the main technical reason for the absence of one point almost sure localisation.

In Section 4 we study the total mass of the solution and its relation to Φ_t. We split $U(t)$ into five parts according to five groups of paths, and show that only one of them makes an essential contribution, namely the one corresponding to paths which visit either $Z_t^{(1)}$ or $Z_t^{(2)}$ and whose length is not too large. Then we prove Propositions 4.2 and Proposition 4.4, which are the very precise upper and lower approximations of $\frac{1}{t}\log U(t)$ by $\Phi_t(Z_t^{(1)})$ needed for Theorem 1.1.

In Section 5 we prove Theorem 1.1. We split the probability space into three disjoint events:

- the gap $\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)})$ is small and the sites $Z_t^{(1)}$ and $Z_t^{(2)}$ are close;
- the gap $\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)})$ is small but the sites $Z_t^{(1)}$ and $Z_t^{(2)}$ are far away;
- the gap $\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)})$ is large.

Correspondingly, we prove Propositions 5.1, 5.2, and 5.3, which justify Theorem 1.1 for each event. In each case, we decompose $u(t, z)$ in two components (differently for different events) and show that one of them localises around $Z_t^{(1)}$ and $Z_t^{(2)}$, and the other one is negligible.

Finally, in Section 6 we prove Theorems 1.2 and 1.3. We use the point processes technique developed in [HMS08], which readily gives Theorem 1.3. Theorem 1.2 is obtained using a combination of the point processes approach and Theorem 1.1.

2. Notation and preliminary results

For $z \in \mathbb{Z}^d$, we define $N(n, z)$ to be the number of paths of length n in $\mathbb{Z}^d$ starting at the origin and passing through z. Recall that $N(z) = N(|z|, z)$, where here and throughout the paper $|\cdot|$ denotes the ℓ^1-norm. For $n \geq |z|$, we define

$$\eta(n, z) = \log N(n, z) \qquad \text{and} \qquad \eta(z) = \log N(z).$$

It is easy to see that $0 \leq \eta(z) \leq |z| \log d$. We define two important scaling functions

$$r_t = \left(\frac{t}{\log t}\right)^{q+1} \qquad \text{and} \qquad a_t = \left(\frac{t}{\log t}\right)^{q}, \tag{2.1}$$

where r_t will turn out to be the appropriate scaling for $Z_t^{(i)}$ and a_t for $\Phi_t(Z_t^{(i)})$, $i = 1, 2, 3$.

For each $r > 0$, denote $\xi_r^{(1)} = \max_{|z| \leq r} \xi(z)$ and

$$\xi_r^{(i)} = \max\left\{\xi(z)\colon |z| \leq r, \xi(z) \neq \xi_r^{(j)}\ \forall j < i\right\}$$

for $2 \leq i \leq \ell_r$, where ℓ_r is the number of points in the ball $\{|z| \leq r\}$. Hence,

$$\xi_r^{(1)} > \xi_r^{(2)} > \xi_r^{(3)} > \cdots > \xi_r^{(\ell_r)}$$

are precisely the potential values in this ball.

Fix $0 < \rho < \sigma < \frac{1}{2}$ so that $\sigma < 1 - \frac{\rho}{d}$, and $\nu > 0$. We define four auxiliary scaling functions

$$f_t = (\log t)^{-\frac{1}{d}-\nu}, \quad g_t = (\log t)^{\frac{1}{\alpha-d}+\nu}, \quad k_t = \lfloor \lfloor r_t g_t \rfloor^{\rho} \rfloor, \quad m_t = \lfloor \lfloor r_t g_t \rfloor^{\sigma} \rfloor,$$

and two sets

$$F_t = \big\{ z \in \mathbb{Z}^d \colon |z| \le r_t g_t, \exists i < k_t \text{ such that } \xi(z) = \xi^{(i)}_{r_t g_t} \big\},$$
$$G_t = \big\{ z \in \mathbb{Z}^d \colon |z| \le r_t g_t, \exists i < m_t \text{ such that } \xi(z) = \xi^{(i)}_{r_t g_t} \big\},$$

which will be used throughout this paper. In words, F_t, respectively G_t, is the set of those sites in the ball $\{|z| \le r_t g_t\}$ in which the $k_t - 1$, respectively $m_t - 1$, largest potential sites are attained. Hence $F_t \subset G_t$, and F_t, respectively G_t, has precisely $k_t - 1$, respectively $m_t - 1$, elements.

2.1 Two technical lemmas.

We start by proving an estimate on $\eta(n, z)$, which we will use later in order to prove that if z is a point where the potential is high, then a path passing through z only contributes to the Feynman–Kac formula if its length is close to $|z|$.

Lemma 2.1. *There is a constant K such that for all $n \ge |z|$,*

$$\eta(n, z) - \eta(z) \le (n - |z|) \log \frac{2den}{n - |z|} + K.$$

Proof. We fix $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$ and without loss of generality assume that $z_i \ge 0$. Denote by $\mathcal{P}_{n,z}$ the set of paths of length n starting at the origin and passing through z. Each $y \in \mathcal{P}_{n,z}$ can be described by the vector $(y_1, \dots, y_n)$ of its increments, where $|y_i| = 1$ for all i. Since the path y passes through z, there is a subsequence $(y_{i_1}, \dots, y_{i_{|z|}})$ corresponding to a path from $\mathcal{P}_{|z|,z}$. Thus, every path from $\mathcal{P}_{n,z}$ can be obtained from a path in $\mathcal{P}_{|z|,z}$ by adding $n - |z|$ elements to its coding sequence. As there are only $2d$ possible elements and $\binom{n}{n-|z|}$ possibilities where the elements can be added, we obtain an upper bound

$$N(n, z) \le N(z)(2d)^{n-|z|} \binom{n}{n - |z|} \le N(z) \frac{(2dn)^{n-|z|}}{(n - |z|)!} \le N(z) \mathrm{e}^K \Big(\frac{2den}{n - |z|} \Big)^{n-|z|},$$

with K such that $m! \ge \mathrm{e}^{-K}(m/\mathrm{e})^m$ for all m. Taking the logarithm completes the proof. $\square$

In the next lemma we derive some properties of the upper order statistics of the potential ξ, which will be used later to prove that $\Phi_t(Z_t^{(1)})$ is an approximate upper bound for $\frac{1}{t} \log U(t)$.

Lemma 2.2. *There exists $c > 0$ such that, with probability one, eventually for all t*

- *(i)* $t^c < \xi^{(k_t)}_{r_t g_t} < t^{q-c}$ *and* $\xi^{(m_t)}_{r_t g_t} / \xi^{(k_t)}_{r_t g_t} < t^{-c}$*;*
- *(ii)* $F_t \cap \{|z| \le t^{(q+1)\sigma + c}\} = \varnothing$*;*
- *(iii)* G_t *is totally disconnected, that is, if* $x, y \in G_t$ *then* $|x - y| \ne 1$*.*

Proof. (*i*) Note that $\widehat{\xi}(z) = \alpha \log \xi(z)$ defines a field of independent exponentially distributed random variables. It has been proved in [HMS08, (4.7)] that, for each $\kappa \in (0, 1)$,

$$\lim_{n\to\infty} \frac{\log \xi_n^{(\lfloor n^\kappa \rfloor)}}{\log n} = \frac{d - \kappa}{\alpha} \qquad \text{almost surely.}$$

Substituting $n = r_t g_t$ and $\kappa = \rho$, respectively $\kappa = \sigma$, we obtain

$$\lim_{t\to\infty} \frac{\log \xi^{(k_t)}_{r_t g_t}}{\log t} = \frac{(d-\rho)(q+1)}{\alpha} \quad \text{and} \quad \lim_{t\to\infty} \frac{\log \xi^{(m_t)}_{r_t g_t}}{\log t} = \frac{(d-\sigma)(q+1)}{\alpha}. \tag{2.2}$$

The result follows, since $\frac{(d-\rho)(q+1)}{\alpha} \in (0, q)$ for $\rho \in (0,1)$ and $\frac{(d-\rho)(q+1)}{\alpha} > \frac{(d-\sigma)(q+1)}{\alpha}$ for $\rho < \sigma$.

(ii) Because $\sigma < 1 - \frac{\rho}{d}$, we can pick c and ε small enough such that $\sigma + \frac{cd}{q\alpha} + \frac{2\varepsilon}{q} < 1 - \frac{\rho}{d}$. Then by [HMS08, Lemma 3.5] we obtain

$$\max_{|z|\le t^{(q+1)\sigma+c}} \xi(z) \le t^{\frac{d}{\alpha}[(q+1)\sigma+c]+\varepsilon} = t^{q\sigma+\frac{cd}{\alpha}+\varepsilon} < t^{\frac{(d-\rho)q}{d}-\varepsilon}$$

eventually, which, together with the first part of (2.2) implies the statement.

(iii) For each $n \in \mathbb{N}$, denote $h_n = \lfloor n^\sigma \rfloor$ and

$$\widehat{G}_n = \big\{ z \in \mathbb{Z}^d \colon |z| \le n, \exists i < h_n \text{ such that } \xi(z) = \xi^{(i)}_n \big\}.$$

Since $G_t = \widehat{G}_{\lfloor r_t g_t \rfloor}$, it suffices to show that $\widehat{G}_n$ is totally disconnected eventually.

First, consider the case $d \ge 2$. The set $\widehat{G}_n$ consists of h_n different points belonging to the ball $B_n = \{|z| \le n\}$. Denote them by $a_0, \dots, a_{h_n-1}$, where a_i is such that $\xi(a_i) = \xi^{(i)}_n$. For $i \ne j$, the pair (a_i, a_j) is uniformly distributed over all pairs of distinct points in B_n. Hence the probability of a_i and a_j being neighbours, written $a_i \sim a_j$, can be estimated by

$$\mathrm{Prob}\big(a_i \sim a_j\big) \le \max_{|z|\le n} \mathrm{Prob}\big(a_i \sim z \,|\, a_j = z\big) \le \frac{2d}{\ell_n - 1},$$

where ℓ_n is the number of points in B_n. Summing over all pairs, we get

$$\mathrm{Prob}\big(\widehat{G}_n \text{ is not totally disconnected}\big) \le \sum_{0\le i<j<h_n} \mathrm{Prob}\,(a_i \sim a_j) \le \frac{2dh_n^2}{\ell_n - 1} \le Cn^{2\sigma - d}, \tag{2.3}$$

for some $C > 0$. As $\sigma < 1/2$ and $d \ge 2$, this sequence is summable. By the Borel–Cantelli lemma $\widehat{G}_n$ is eventually totally disconnected.

The situation is more delicate if $d = 1$. Pick $\sigma' \in (\sigma, 1/2)$ and denote $h'_n = \lfloor n^{\sigma'} \rfloor$ and

$$\widehat{G}'_n = \big\{ z \in \mathbb{Z}^d \colon |z| \le n, \exists i < h'_n \text{ such that } \xi(z) = \xi^{(i)}_n \big\}.$$

Further, let $p_n = 2^{\lfloor \log_2 n \rfloor}$ such that $p_n \le n < 2p_n$.
It is easy to see that $\widehat{G}'_{p_n}$ is totally disconnected eventually. Indeed, (2.3) remains true with $\hat{G}_n$ and h_n replaced by $\hat{G}'_n$ and h'_n, respectively and one just needs to notice that $\sum_{n=1}^\infty 2^{n(2\sigma'-d)} < \infty$ for $d = 1$ and $\sigma' < 1/2$.

The final step is to prove that $\widehat{G}_n \subset \widehat{G}'_{2p_n}$. Let $\varkappa_n$ be the cardinality of $\widehat{G}'_{2p_n} \cap B_n$ and observe that for this purpose it suffices to show that $\varkappa_n \ge h_n$. Indeed, on this set the $\varkappa_n$ largest values of ξ over B_n are achieved. We actually prove a stronger statement, showing that there are at least h_{2p_n} points from $\widehat{G}'_{2p_n}$ in the ball B_{p_n}. From now on we drop the subscript n. We write

$$\widehat{G}'_{2p} = \{a'_0, \dots, a'_{h'_{2p}-1}\},$$

where a'_i is such that $\xi(a'_i) = \xi^{(i)}_{2p}$. Let $X = (X_i \colon 0 \le i < h'_{2p})$ with $X_i = \mathbb{1}\{|a'_i| \le p\}$ and

$$|X| = \sum_{i=0}^{h'_{2p}-1} X_i.$$

Since $h'_{2p} = o(p)$ and $|B_p| = 2p+1$, $|B_{2p}| = 4p+1$, we obtain, using that the points in $\widehat{G}'_{2p}$ are uniformly distributed over B_{2p} without repetitions, that for large p

$$\mathrm{Prob}\,(X_j = 1 \,|\, X_i = x_i\ \forall i < j) < 3/4 \quad \text{and} \quad \mathrm{Prob}\,(X_j = 0 \,|\, X_i = x_i\ \forall i < j) < 3/4$$

for all $j < h'_{2p}$ and all $(x_0, \dots, x_{j-1}) \in \{0,1\}^j$. Hence, for all $x \in \{0,1\}^{h'_{2p}}$,

$$\mathrm{Prob}(X = x) \le (3/4)^{h'_{2p}}.$$

This yields

$$\begin{aligned}\mathrm{Prob}\,(|X| < h_{2p}) &= \sum_{i=0}^{h_{2p}-1} \sum_{|x|=i} \mathrm{Prob}\,(X = x) \le \sum_{i=0}^{h_{2p}-1} \binom{h'_{2p}}{i} (3/4)^{h'_{2p}} \le h_{2p} (h'_{2p})^{h_{2p}-1} (3/4)^{h'_{2p}} \\ &\le \exp\left\{-h'_{2p} \log(4/3) + h_{2p} \log h'_{2p} + \log h_{2p}\right\} = \mathrm{e}^{-c(2p)^{\sigma'}},\end{aligned}$$

for some $c > 0$. Since this sequence is summable, we have $|X| \ge h_{2p}$ eventually. $\square$

2.2 Spectral estimates.

In this section we exploit ideas developed in [GKM07]. Let $A \subset \mathbb{Z}^d$ be a bounded set and denote by $Z_A \in A$ the point, where the potential ξ takes its maximal value over A. Denote by

$$\mathfrak{g}_A = \xi(Z_A) - \max_{z \in A \setminus \{Z_A\}} \xi(z)$$

the gap between the largest value and the rest of the potential on A. Denote by A^* the connected component of A containing Z_A. Let γ_A and v_A be the principal eigenvalue and eigenfunction of $\Delta + \xi$ with zero boundary conditions in A^* extended by zero to the whole set A. We assume that v_A is normalised to $v_A(Z_A) = 1$. Recall that under $\mathbb{P}_z$ and $\mathbb{E}_z$ the process $(X_t \colon t \in [0,\infty))$ is a simple random walk with generator Δ started from $z \in \mathbb{Z}^d$. The entrance time to a set A is denoted $\tau_A = \inf\{t \ge 0 \colon X_t \in A\}$, and we write τ_z instead of $\tau_{\{z\}}$. Then, as in [GKM07, (4.4)], the eigenfunction v_A admits the probabilistic representation

$$v_A(z) = \mathbb{E}_z\Big[\exp\Big\{\int_0^{\tau_{Z_A}} [\xi(X_s) - \gamma_A]\,\mathrm{d}s\Big\} \mathbb{1}\{\tau_{Z_A} < \tau_{A^c}\}\Big], \qquad z \in A. \tag{2.4}$$

It turns out that v_A is concentrated around the maximal point Z_A of the potential.

Lemma 2.3. *There is a decreasing function $\varphi \colon (2d, \infty) \to \mathbb{R}_+$ such that $\lim_{x\to\infty} \varphi(x) = 0$ and, for any bounded set $A \subset \mathbb{Z}^d$ satisfying $\mathfrak{g}_A > 2d$,*

$$||v_A||_2^2 \sum_{z \in A \setminus \{Z_A\}} v_A(z) \le \varphi(\mathfrak{g}_A).$$

Proof. It suffices to consider $z \in A^*$. By the Rayleigh–Ritz formula we have

$$\begin{aligned}\gamma_A &= \sup\Big\{\langle (\Delta + \xi) f, f\rangle \colon f \in \ell^2(\mathbb{Z}^d),\ \mathrm{supp}(f) \subset A^*,\ \|f\|_2 = 1\Big\} \\ &\ge \sup\{\langle (\Delta + \xi)\delta_z, \delta_z\rangle \colon z \in A^*\} = \sup\{\xi(z) - 2d \colon z \in A^*\} \\ &= \xi(Z_A) - 2d.\end{aligned}$$

Since the paths of the random walk (X_s) in (2.4) do not leave A and avoid the point Z_A where the maximum of the potential is achieved, we can estimate the integrand using the gap $\mathfrak{g}_A$. Hence, we obtain

$$v_A(z) \le \mathbb{E}_z\Big[\exp\big\{\tau_{Z_A}(\xi(Z_A) - \mathfrak{g}_A - \gamma_A)\big\}\Big] \le \mathbb{E}_z\Big[\exp\big\{-\tau_{Z_A}(\mathfrak{g}_A - 2d)\big\}\Big].$$

Under $\mathbb{P}_z$ the random variable τ_{Z_A} is stochastically bounded from below by a sum of $|z-Z_A|$ independent exponentially distributed random variables with parameter $2d$. If τ denotes such a random time, we therefore have

$$v_A(z) \le \Big(\mathbb{E}\big[\mathrm{e}^{-\tau(\mathfrak{g}_A-2d)}\big]\Big)^{|z-Z_A|} = \Big(\frac{2d}{\mathfrak{g}_A}\Big)^{|z-Z_A|}.$$

The statement of the lemma follows easily with

$$\varphi(x) = \Big(\sum_{z\in\mathbb{Z}^d}(2d/x)^{2|z|}\Big)\Big(\sum_{z\in\mathbb{Z}^d\setminus\{0\}}(2d/x)^{|z|}\Big),$$

which obviously satisfies the required conditions. □

Let now $B\subset\mathbb{Z}^d$ be a bounded set containing the origin and $\Omega\subset B$. Denote

$$\mathfrak{g}_{\Omega,B} = \min_{z\in\Omega}\xi(z) - \max_{z\in B\setminus\Omega}\xi(z).$$

and denote, for any $(t,z)\in(0,\infty)\times\mathbb{Z}^d$,

$$u_{\Omega,B}(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\,\mathbb{1}\{X_t=z\}\,\mathbb{1}\{\tau_\Omega\le t,\tau_{B^{\mathrm{c}}}>t\}\Big].$$

Lemma 2.4. *Assume that $\mathfrak{g}_{\Omega,B}>2d$. Then, for all $z\in\mathbb{Z}^d$ and $t>0$,*

(a) $u_{\Omega,B}(t,z) \le \sum_{y\in\Omega} u_{\Omega,B}(t,y)\,||v_{(B\setminus\Omega)\cup\{y\}}||_2^2\, v_{(B\setminus\Omega)\cup\{y\}}(z)$,

(b) $\dfrac{\sum_{z\in B\setminus\Omega} u_{\Omega,B}(t,z)}{\sum_{z\in B} u_{\Omega,B}(t,z)} \le \varphi(\mathfrak{g}_{\Omega,B})$.

Proof. (*a*) This is a slight generalisation of [GKM07, Th. 4.1] with a ball replaced by an arbitrary bounded set B; we repeat the proof here for the sake of completeness. For each $y\in\Omega$, by time reversal and using the Markov property at time s, we obtain a lower bound for $u_{\Omega,B}(t,y)$ by requiring that the random walk (now started at y) is at y at time u and has not entered $\Omega\setminus\{y\}$ before. We have

$$\begin{aligned} u_{\Omega,B}(t,y) &= \mathbb{E}_y\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t=0\}\mathbb{1}\{\tau_{\Omega\setminus\{y\}}\le t,\tau_{B^{\mathrm{c}}}>t\}\Big]\\ &\ge \mathbb{E}_y\Big[\exp\Big\{\int_0^u \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_u=y\}\mathbb{1}\{\tau_{\Omega\setminus\{y\}}> u,\tau_{B^{\mathrm{c}}}>u\}\Big]\\ &\quad\times \mathbb{E}_y\Big[\exp\Big\{\int_0^{t-u} \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_{t-u}=0\}\mathbb{1}\{\tau_{B^{\mathrm{c}}}>t-u\}\Big]. \end{aligned} \tag{2.5}$$

Using an eigenvalue expansion for the parabolic problem in $(B\setminus\Omega)\cup\{y\}$ represented by the first factor on the right hand side of the formula above, we obtain the bound

$$\mathbb{E}_y\Big[\exp\Big\{\int_0^u \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_u=y\}\mathbb{1}\{\tau_{\Omega\setminus\{y\}}> u,\tau_{B^{\mathrm{c}}}>u\}\Big] \ge \mathrm{e}^{u\gamma_{(B\setminus\Omega)\cup\{y\}}}\frac{v_{(B\setminus\Omega)\cup\{y\}}(y)^2}{||v_{(B\setminus\Omega)\cup\{y\}}||_2^2},$$

where we have used that $Z_{(B\setminus\Omega)\cup\{y\}}=y$ since $\mathfrak{g}_{\Omega,B}>0$. Substituting the above estimate into (2.5) and taking into account that $v_{(B\setminus\Omega)\cup\{y\}}(y)=1$, we obtain

$$\mathbb{E}_y\Big[\exp\Big\{\int_0^{t-u} \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_{t-u}=0\}\mathbb{1}\{\tau_{B^{\mathrm{c}}}>t-u\}\Big] \le \mathrm{e}^{-u\gamma_{(B\setminus\Omega)\cup\{y\}}}||v_{(B\setminus\Omega)\cup\{y\}}||_2^2 u_{\Omega,B}(t,y).$$

The claimed estimate is obvious for $z\notin B$. For $z\in\Omega$, it follows from $v_{(B\setminus\Omega)\cup\{z\}}(z)=1$, which is implied by $\mathfrak{g}_{\Omega,B}>0$ and hence $Z_{(B\setminus\Omega)\cup\{z\}}=z$. Let us now assume that $z\in B\setminus\Omega$.

Using time reversal, the strong Markov property at time τ_Ω, and the previous lower bound with $u = \tau_y$ we obtain

$$\begin{aligned}
u_{\Omega,B}(t,z) &= \sum_{y\in\Omega} \mathbb{E}_z\Big[\exp\Big\{\int_0^{\tau_y} \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{\tau_y=\tau_\Omega\le t, \tau_{B^c}>\tau_y\} \\
&\qquad\qquad \mathbb{E}_y\Big[\exp\Big\{\int_0^{t-u}\xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_{t-u}=0\}\mathbb{1}\{\tau_{B^c}>t-u\}\Big]_{u=\tau_y}\Big] \\
&\le \sum_{y\in\Omega} u_{\Omega,B}(t,y)||v_{(B\backslash\Omega)\cup\{y\}}||_2^2\,\mathbb{E}_z\Big[\exp\Big\{\int_0^{\tau_y}\big[\xi(X_s)-\gamma_{(B\backslash\Omega)\cup\{y\}}\big]\,\mathrm{d}s\Big\}\mathbb{1}\{\tau_y<\tau_{B^c}\} \\
&= \sum_{y\in\Omega} u_{\Omega,B}(t,y)||v_{(B\backslash\Omega)\cup\{y\}}||_2^2\, v_{(B\backslash\Omega)\cup\{y\}}(z).
\end{aligned}$$

(b) It suffices to apply Lemma 2.3 to $A=(B\backslash\Omega)\cup\{y\}$, note that $\mathfrak{g}_A\ge\mathfrak{g}_{\Omega,B}$, and use monotonicity of φ. Using (a), we obtain

$$\begin{aligned}
\sum_{z\in B\backslash\Omega} u_{\Omega,B}(t,z) &\le \sum_{y\in\Omega} u_{\Omega,B}(t,y) \sum_{z\in B\backslash\Omega} ||v_{(B\backslash\Omega)\cup\{y\}}||_2^2\, v_{(B\backslash\Omega)\cup\{y\}}(z) \\
&\le \sum_{y\in\Omega} u_{\Omega,B}(t,y)\,\varphi(\mathfrak{g}_{(B\backslash\Omega)\cup\{y\}}) \le \varphi(\mathfrak{g}_{\Omega,B})\sum_{y\in B} u_{\Omega,B}(t,y),
\end{aligned}$$

which completes the proof. □

3. Properties of the maximisers $Z_t^{(i)}$ and values $\Phi_t(Z_t^{(i)})$

In this section we introduce the three maximisers $Z_t^{(1)}$, $Z_t^{(2)}$ and $Z_t^{(3)}$ and analyse some of their crucial properties. In Section 3.1 we concentrate on the long term behaviour of the maximisers themselves, and in Section 3.2 we prove that the maximal value $\Phi_t(Z_t^{(1)})$ is well separated from $\Phi_t(Z_t^{(3)})$.

3.1 The maximisers $Z_t^{(1)}$, $Z_t^{(2)}$ and $Z_t^{(3)}$.

Recall that $Z_t^{(1)}$, $Z_t^{(2)}$ and $Z_t^{(3)}$ denote the first three maximisers of the random functional Φ_t defined in (1.9). More precisely, we define $Z_t^{(i)}$ to be such that

$$\Phi_t(Z_t^{(1)}) = \max_{z\in\mathbb{Z}^d}\Phi_t(z), \qquad \Phi_t(Z_t^{(2)}) = \max_{z\in\mathbb{Z}^d\backslash\{Z_t^{(1)}\}}\Phi_t(z), \qquad \Phi_t(Z_t^{(3)}) = \max_{z\in\mathbb{Z}^d\backslash\{Z_t^{(1)},Z_t^{(2)}\}}\Phi_t(z). \tag{3.1}$$

Lemma 3.1. *With probability one, $Z_t^{(1)}$, $Z_t^{(2)}$ and $Z_t^{(3)}$ are well-defined for any $t>0$.*

Proof. Fix $t>0$. Let $\varepsilon\in(0,1-\frac{d}{\alpha})$. By [HMS08, Lemma 3.5] there exists a random radius $\rho(t)>0$ such that, almost surely,

$$\xi(z)\le\xi_{|z|}^{(1)}\le|z|^{\frac{d}{\alpha}+\varepsilon}\le\frac{|z|}{t}, \qquad \text{for all } |z|>\rho(t). \tag{3.2}$$

Consider $|z|>\max\{\rho(t),edt\}$. If $t\xi(z)<|z|$ then $\Phi_t(z)=0$. Otherwise, using $\eta(z)\le|z|\log d$ and estimating $\xi(z)$ in two different ways, we obtain

$$\Phi_t(z)\le\xi_{|z|}^{(1)}-\frac{|z|}{t}\log\frac{|z|}{dt}\le\frac{|z|}{t}\Big[1-\log\frac{|z|}{dt}\Big]<0.$$

Thus, Φ_t takes only finitely many positive values and therefore the maxima in (3.1) exist. □

Remark 7. The maximisers $Z_t^{(1)}$, $Z_t^{(2)}$ and $Z_t^{(3)}$ are in general not uniquely defined. However, almost surely, if t_0 is sufficiently large, they are uniquely defined, for all $t \in (t_0, \infty) \setminus T$, where T is a countable random set, and for $t \in T$ it can only happen that $Z_t^{(1)} = Z_t^{(2)} \neq Z_t^{(3)}$ or $Z_t^{(1)} \neq Z_t^{(2)} = Z_t^{(3)}$. Thus, the non-uniqueness only occurs at the time when the maximal (or the second maximal value) relocates from one point to the other. It can be seen from the further proofs (see Lemma 3.2) that T consists of isolated points. ⋄

To prove Proposition 3.4 below, we need to analyse the functions $t \mapsto \Phi_t(Z_t^{(i)})$, $i = 1, 2, 3$, locally. It turns out that they have some regularity and that, using rather precise asymptotics for $|Z_t^{(i)}|$ and $\Phi_t(Z_t^{(i)})$, we can have good control on their increments.

Lemma 3.2. *Let $\varepsilon > 0$. For $i = 1, 2, 3$, almost surely eventually for all t,*

(*i*) $\Phi_t(Z_t^{(i)}) > a_t\,(\log t)^{-\varepsilon}$ *and* $\xi(Z_t^{(i)}) > a_t (\log t)^{-\varepsilon}$;

(*ii*) $t\xi(Z_t^{(i)}) > |Z_t^{(i)}|$;

(*iii*) $r_t (\log t)^{-\frac{1}{d}-\varepsilon} < |Z_t^{(i)}| < r_t\,(\log t)^{\frac{1}{\alpha-d}+\varepsilon}$;

(*iv*) $\Phi_u(Z_u^{(i)}) - \Phi_t(Z_t^{(i)}) \le \frac{u-t}{t}\,a_u\,(\log u)^{\frac{1}{\alpha-d}+\varepsilon}$ *for all* $u > t$;

(*v*) $u \mapsto \Phi_u(Z_u^{(i)})$ *is increasing on* (t, ∞).

Proof. As an auxiliary step, let us show that, for any $c > 0$ and any $i \in \mathbb{N}$,

$$\xi_r^{(i-1)} > r^{\frac{d}{\alpha}} (\log r)^{-c} \qquad \text{eventually.} \tag{3.3}$$

Obviously, the distribution of $\xi_r^{(i-1)}$ is given by

$$\text{Prob}\left(\xi_r^{(i-1)} \le x\right) = \sum_{k=0}^{i-1} \binom{\ell_r}{k} x^{-\alpha k} (1 - x^{-\alpha})^{\ell_r - k},$$

where $\ell_r \sim \kappa_d r^d$ is the number of points in the ball $\{|z| \le r\}$, and κ_d is a positive constant. Using that $\binom{\ell_r}{k} \le \ell_r^k \sim \kappa_d^k r^{dk}$, we get

$$\begin{aligned}
\text{Prob}\big(\xi_r^{(i-1)} \le r^{\frac{d}{\alpha}} (\log r)^{-c}\big) &\le (1 + o(1)) \sum_{k=0}^{i-1} \kappa_d^k (\log r)^{c\alpha k} (1 - r^{-d} (\log r)^{c\alpha})^{\ell_r - k} \\
&\le (1 + o(1))\, i \kappa_d^{i-1} (\log r)^{c\alpha(i-1)} (1 - r^{-d} (\log r)^{c\alpha})^{\ell_r - i + 1} \\
&= \exp\left\{-\kappa_d (\log r)^{c\alpha} (1 + o(1))\right\},
\end{aligned}$$

which is summable along the subsequence $r_n = 2^n$. Hence, by the Borel–Cantelli lemma the inequality (3.3) holds eventually along $(r_n)_{n\in\mathbb{N}}$. As $\xi_r^{(i-1)}$ is increasing, we obtain eventually

$$\xi_r^{(i-1)} \ge \xi_{2^{\lfloor \log_2 r \rfloor}}^{(i-1)} \ge \big(2^{\lfloor \log_2 r \rfloor}\big)^{\frac{d}{\alpha}} (\log 2^{\lfloor \log_2 r \rfloor})^{-c} \ge 2^{-\frac{d}{\alpha}} r^{\frac{d}{\alpha}} (\log r - \log 2)^{-c} > r^{\frac{d}{\alpha}} (\log r)^{-2c},$$

which is equivalent to (3.3).

Now we prove parts $(i) - (v)$ of the lemma. We assume throughout the proof that t is sufficiently large to use all statements which hold eventually.

(i) Let z_1, z_2, z_3 be the points where the three largest values of ξ in $\{|z| \le r_t (\log t)^{-\varepsilon}\}$ are achieved. Take $c < \varepsilon(\alpha - d)/(2\alpha)$ and observe that (3.3) implies for each i eventually

$$\xi(z_i) > r_t^{\frac{d}{\alpha}} (\log t)^{-\frac{\varepsilon d}{\alpha}} (\log r_t - \varepsilon \log\log t)^{-c} > a_t (\log t)^{-\frac{\varepsilon d}{\alpha} - 2c}.$$

By [HMS08, Lemma 3.5] we also have

$$\log \xi(z_i) \le \log \xi^{(1)}_{r_t(\log t)^{-\varepsilon}} < \log r_t \le (q+1)\log t.$$

We obtain, observing that $t\xi(z_i) > ta_t(\log t)^{-\frac{\varepsilon d}{\alpha}-2c} > r_t(\log t)^{-\varepsilon} \ge |z_i|$, that

$$\Phi_t(z_i) \ge \xi(z_i) - \frac{|z_i|}{t}\log \xi(z_i) > a_t(\log t)^{-\frac{\varepsilon d}{\alpha}-2c} - \frac{r_t}{t}(q+1)(\log t)^{1-\varepsilon} > a_t(\log t)^{-\varepsilon}$$

as $\frac{\varepsilon d}{\alpha} + 2c < \varepsilon$ and $(r_t/t)\log t = a_t$. Since the inequality is fulfilled for the three points z_1, z_2 and z_3, it is also fulfilled for the maximisers $Z_t^{(1)}$, $Z_t^{(2)}$ and $Z_t^{(3)}$, completing the proof of the first inequality in (i). As $\Phi_t(Z_t^{(i)}) \ne 0$ we must have $\xi(Z_t^{(i)}) \ge |Z_t^{(i)}|/t$, and hence

$$\xi(Z_t^{(i)}) = \Phi_t(Z_t^{(i)}) + \frac{|Z_t^{(i)}|}{t}\log\xi(Z_t^{(i)}) - \frac{\eta(Z_t^{(i)})}{t} \ge \Phi_t(Z_t^{(i)}) + \frac{|Z_t^{(i)}|}{t}\log\frac{|Z_t^{(i)}|}{dt} > \Phi_t(Z_t^{(i)}) - d/\mathrm{e}.$$

The second inequality in (i) follows now from the lower bound for $\Phi_t(Z_t^{(i)})$.

(ii) This is an obvious consequence of (i) as $\Phi_t(Z_t^{(i)}) \ne 0$.

(iii) To prove the upper bound, let us pick $c \in (0, \frac{\varepsilon(\alpha-d)}{2\alpha})$. Then for each z such that $|z| \ge r_t(\log t)^{\frac{1}{\alpha-d}+\varepsilon}$ we obtain by [HMS08, Lemma 3.5], eventually,

$$\frac{\xi(z)}{|z|} \le |z|^{\frac{d}{\alpha}-1}(\log|z|)^{\frac{1}{\alpha}+c} \le o(1/t).$$

Hence (ii) implies that $z \ne Z_t^{(i)}$, which implies the upper bound on $|Z_t^{(i)}|$.

To prove the lower bound, suppose that $|Z_t^{(i)}| \le r_t(\log t)^{-\frac{1}{d}-\varepsilon}$. By [HMS08, Lemma 3.5],

$$\xi(Z_t^{(i)}) \le |Z_t^{(i)}|^{\frac{d}{\alpha}}(\log|Z_t^{(i)}|)^{\frac{1}{\alpha}+c} \le a_t(\log t)^{-\frac{d\varepsilon}{\alpha}+2c},$$

which contradicts (i) if we pick $c \in (0, \frac{\varepsilon d}{2\alpha})$.

(iv) Let t be large enough so that the previous eventual estimates hold for all $u \ge t$. Then, for each $s \in [t, u]$, according to (iii), we have that $\Phi_s(Z_s^{(i)})$ is the ith largest value of Φ_s over a collection of finitely many points. Hence $s \mapsto \Phi_s(Z_s^{(i)})$ is a continuous piecewise smooth function. On the smooth pieces, using again [HMS08, Lemma 3.5] and (iii) with $\varepsilon/2$, we can estimate its derivative by

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi_s(Z_s^{(i)}) = \frac{|Z_s^{(i)}|}{s^2}\log\xi(Z_s^{(i)}) - \frac{\eta(Z_s^{(i)})}{s^2} \le \frac{|Z_s^{(i)}|}{s^2}\log|Z_s^{(i)}|^{\frac{d}{\alpha}+c} < \frac{a_s}{s}(\log s)^{\frac{1}{\alpha-d}+\varepsilon}.$$

Finally, we obtain

$$\Phi_u(Z_u^{(i)}) - \Phi_t(Z_t^{(i)}) = \int_t^u \frac{\mathrm{d}}{\mathrm{d}s}\Phi_s(Z_s^{(i)})\,\mathrm{d}s \le \frac{u-t}{t}a_u(\log u)^{\frac{1}{\alpha-d}+\varepsilon},$$

which completes the proof.

(v) Using $\eta(z) \le |z|\log d$ in the second, and (i) in the last step, we see that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi_s(Z_s^{(i)}) = \frac{|Z_s^{(i)}|}{s^2}\log\xi(Z_s^{(i)}) - \frac{\eta(Z_s^{(i)})}{s^2} \ge \frac{|Z_s^{(i)}|}{s^2}\log\frac{\xi(Z_s^{(i)})}{d} > 0,$$

eventually for all t. ∎

3.2 Lower bound for $\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)})$.

In this section we prove that $\Phi_t(Z_t^{(1)})$ and $\Phi_t(Z_t^{(3)})$ are well separated from each other. The crucial estimate for this is provided in Lemma 3.3.

First, it is important to make the density of the random variable $\Phi_t(z)$ explicit. Observe that, on the set $\{t\xi(z) \geq z\}$, the event $\{\Phi_t(z) < x\}$ has the form $\{\chi_a(\xi(z)) \leq x - \eta(z)/t\}$, where we abbreviated $a = |z|/t$ and introduced the map $\chi_a(x) = x - a\log x$. Note that χ_a is an increasing bijection from $[a,\infty)$ to $[a - a\log a, \infty)$, hence on $\{t\xi(z) \geq z\}$ we can describe $\{\Phi_t(z) < x\}$ using the inverse function $\psi_a \colon [a - a\log a, \infty) \to [a,\infty)$ of χ_a. In order to also include the complement of $\{t\xi(z) \geq z\}$, we extend ψ_a to a function $\mathbb{R} \to [a,\infty)$ by putting $\psi_a(x) = a$ for $x < a - a\log a$. Then we have, for each t, z, and $x > 0$,

$$\big\{\Phi_t(z) \leq x\big\} = \big\{\xi(z) \leq \psi_{\frac{|z|}{t}}\big(x - \eta(z)/t\big)\big\}. \tag{3.4}$$

Lemma 3.3. *Fix $\beta > 1 + \frac{1}{\alpha-d}$ and let $\lambda_t = (\log t)^{-\beta}$. Then there exists a constant $c > 0$ such that*

$$\mathrm{Prob}\,\big(\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) \leq 2a_t\lambda_t\big) \leq c\lambda_t^2, \qquad \text{for } t > 0.$$

Proof. This proof, though tedious, is fairly standard and is carried out in *four steps.* In the first step, we show that there exists a constant $C_1 > 0$ such that, for all sufficiently large t, and all $s \geq (\log t)^{-1/2}$,

$$\begin{aligned}\mathrm{Prob}\big(&\Phi_t(Z_t^{(1)}) \in \mathrm{d}(a_t s),\, \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) \leq 2a_t\lambda_t\big)\\ &\leq C_1\, a_t^3\, \lambda_t^2\, \mathrm{Prob}\big(\Phi_t(Z_t^{(1)}) \leq a_t s\big)\Big[\sum_{z\in\mathbb{Z}^d}\big(a_t s + \tfrac{|z|}{t}\log\tfrac{|z|}{dt}\big)^{-\alpha-1}\Big]^3 \mathrm{d}s\,.\end{aligned} \tag{3.5}$$

In the second step we evaluate the infinite sum and show that there exists $C_2 > 0$ such that

$$\sum_{z\in\mathbb{Z}^d}\big(a_t s + \tfrac{|z|}{t}\log\tfrac{|z|}{dt}\big)^{-\alpha-1} \leq C_2\, a_t^{-1}\, s^{d-\alpha-1}\,. \tag{3.6}$$

To bound the right hand side of (3.5) further, we show in the third step that there exists a constant $C_3 > 0$ such that, for all $(\log t)^{-1/2} \leq s \leq 1$,

$$\mathrm{Prob}\big(\Phi_t(Z_t^{(1)}) \leq a_t s\big) \leq \exp\big\{-C_3\, s^{d-\alpha}\big\}\,. \tag{3.7}$$

In the fourth step we combine these three equations and integrate over s to get the result.

For the *first step* we use independence to obtain

$$\begin{aligned}\mathrm{Prob}\big(&\Phi_t(Z_t^{(1)}) \in \mathrm{d}(a_t s),\, \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) \leq 2a_t\lambda_t\big)\\ &\leq \sum_{\substack{z_1,z_2,z_3\in\mathbb{Z}^d\\ \text{distinct}}} \mathrm{Prob}\big(\Phi_t(z_1) \in \mathrm{d}(a_t s);\, \Phi_t(z_i) \in a_t[s - 2\lambda_t, s] \text{ for } i = 2,3;\\ &\qquad\qquad\qquad \Phi_t(z) \leq a_t s \text{ for } z \notin \{z_1,z_2,z_3\}\big)\\ &\leq \Big(\sum_{z\in\mathbb{Z}^d}\frac{\mathrm{Prob}(\Phi_t(z) \in \mathrm{d}(a_t s))}{\mathrm{Prob}(\Phi_t(z) \leq a_t s)}\Big)\Big(\sum_{z\in\mathbb{Z}^d}\frac{\mathrm{Prob}(\Phi_t(z) \in a_t[s - 2\lambda_t, s])}{\mathrm{Prob}(\Phi_t(z) \leq a_t s)}\Big)^2\\ &\qquad\qquad \times \prod_{z\in\mathbb{Z}^d}\mathrm{Prob}\big(\Phi_t(z) \leq a_t s\big)\,.\end{aligned} \tag{3.8}$$

All the denominators in (3.8) converge to one, uniformly in z and $s \geq (\log t)^{-1/2} - 2\lambda_t$. Indeed, by (3.4), we get

$$\begin{aligned}\mathrm{Prob}\big(\Phi_t(z) \leq a_t s\big) &= \mathrm{Prob}\big(\xi(z) \leq \psi_{\frac{|z|}{t}}(a_t s - \tfrac{\eta(z)}{t})\big) \\ &\geq \mathrm{Prob}\big(\xi(z) \leq a_t s - \tfrac{|z|}{t}\log d + \tfrac{|z|}{t}\log\tfrac{|z|}{t}\big) \geq \mathrm{Prob}\big(\xi(z) \leq a_t s - \tfrac{d}{\mathrm{e}}\big) \geq 1 + o(1)\,,\end{aligned}$$

using that $\eta(z) \leq |z|\log d$, $x\log(x/d) \geq -d/\mathrm{e}$, and $\psi_a(x) \geq x + a\log a$ (with $a = |z|/t$), where the latter is obvious for $x \leq a - a\log a$ and follows from $\psi_a(x) = x + a\log\psi_a(x) \geq x + a\log a$ otherwise.

Further, we use (3.4) to observe that, by a coordinate transformation, the density of $\Phi_t(z)$ at x is given as

$$\psi'_{\frac{|z|}{t}}\big(x - \tfrac{\eta(z)}{t}\big)\,\alpha\,\big(\psi_{\frac{|z|}{t}}(x - \tfrac{\eta(z)}{t})\big)^{-\alpha-1} \qquad \text{if} \quad x - \tfrac{\eta(z)}{t} > \tfrac{|z|}{t} - \tfrac{|z|}{t}\log\tfrac{|z|}{t}.$$

If t is large enough, the latter condition is satisfied for $x = a_t s$, all z and $s \geq (\log t)^{-1/2} - 2\lambda_t$, and moreover, using again $\psi_a(x) \geq x + a\log a$, we have

$$\psi_{\frac{|z|}{t}}\big(a_t s - \tfrac{\eta(z)}{t}\big) \geq a_t s - \tfrac{\eta(z)}{t} + \tfrac{|z|}{t}\log\tfrac{|z|}{t} \geq a_t s + \tfrac{|z|}{t}\log\tfrac{|z|}{dt}\,.$$

Hence, if t is big enough to satisfy $a_t[(\log t)^{-1/2} - 2\lambda_t] > t^{q/2}$ we get

$$\tfrac{t}{|z|}\,\psi_{\frac{|z|}{t}}(a_t s - \tfrac{\eta(z)}{t}) \geq \tfrac{1}{|z|}\,t^{1+q/2} + \log\tfrac{|z|}{dt} \geq \min_{r>0}\big\{t^{q/2}r - \log(rd)\big\} = \log\tfrac{\mathrm{e}t^{q/2}}{d}.$$

Differentiating the equality $\psi_a(x) - a\log\psi_a(x) = x$ with respect to x, for $x > a$, we obtain $\psi'_a(x) = (1 - a/\psi_a(x))^{-1}$. This implies that, as $t \uparrow \infty$,

$$\psi'_{\frac{|z|}{t}}(a_t s - \tfrac{\eta(z)}{t}) = \Big(1 - \tfrac{|z|}{t}/\psi_{\frac{|z|}{t}}\big(a_t s - \tfrac{\eta(z)}{t}\big)\Big)^{-1} \longrightarrow 1 \quad \text{uniformly in } z \text{ and } s.$$

Hence

$$\mathrm{Prob}\big(\Phi_t(z) \in \mathrm{d}(a_t s)\big) \leq (\alpha + o(1))\,a_t\,\big(a_t s + \tfrac{|z|}{t}\log\tfrac{|z|}{dt}\big)^{-\alpha-1}\,\mathrm{d}s\,. \tag{3.9}$$

Integrating (3.9) over the interval $[s - 2\lambda_t, s]$ yields

$$\mathrm{Prob}\big(\Phi_t(z) \in a_t[s - 2\lambda_t, s]\big) \leq (\alpha + o(1))\,2a_t\lambda_t\,\big(a_t(s - 2\lambda_t) + \tfrac{|z|}{t}\log\tfrac{|z|}{dt}\big)^{-\alpha-1}.$$

Using that $x\log(x/d) \geq -d/\mathrm{e}$ we obtain

$$\frac{a_t(s - 2\lambda_t) + \frac{|z|}{t}\log\frac{|z|}{dt}}{a_t s + \frac{|z|}{t}\log\frac{|z|}{dt}} = 1 - \frac{2a_t\lambda_t}{a_t s + \frac{|z|}{t}\log\frac{|z|}{dt}} \geq 1 - \frac{2a_t\lambda_t}{a_t s - d/\mathrm{e}} \geq 1 + o(1),$$

hence, uniformly in $z \in \mathbb{Z}^d$ and $s \geq (\log t)^{-1/2} - 2\lambda_t$,

$$\mathrm{Prob}\big(\Phi_t(z) \in a_t[s - 2\lambda_t, s]\big) \leq (\alpha + o(1))\,2a_t\lambda_t\,\big(a_t s + \tfrac{|z|}{t}\log\tfrac{|z|}{dt}\big)^{-\alpha-1}. \tag{3.10}$$

Inserting (3.9) and (3.10) in (3.8), and estimating all denominators uniformly by a constant factor yields (3.5).

In the *second step* we estimate the infinite sum in (3.6). Recalling that $r_t^d = a_t^\alpha$ and that the number of points in the ball $\{|z| \leq r\}$ is equal to $\kappa_d r^d(1 + o(1))$ we obtain

$$\begin{aligned}\sum_{|z|\leq r_t/\log t}\big(a_t s + \tfrac{|z|}{t}\log\tfrac{|z|}{dt}\big)^{-\alpha-1} &\leq \sum_{|z|\leq r_t/\log t}\big(a_t s - \tfrac{d}{\mathrm{e}}\big)^{-\alpha-1} \\ &\leq (\kappa_d + o(1))\,\frac{r_t^d}{(a_t s)^{\alpha+1}[\log t]^d} = o\big(a_t^{-1}s^{d-\alpha-1}\big).\end{aligned} \tag{3.11}$$

We have $\log \frac{|z|}{dt} \geq (1+o(1))\, q\, [\log t]$ uniformly over all $z \in \mathbb{Z}^d$ with $|z| \geq r_t/\log t$. Therefore,

$$\sum_{|z|\geq r_t/\log t} \big(a_t s + \tfrac{|z|}{t}\log\tfrac{|z|}{dt}\big)^{-\alpha-1} \leq (1+o(1))\,(a_t s)^{-\alpha-1} \sum_{|z|\geq r_t/\log t} \big(1+q\,\tfrac{|z|}{r_t s}\,\log t\big)^{-\alpha-1}$$

$$= (1+o(1))\,\frac{(r_t s)^d}{(a_t s)^{\alpha+1}} \int_{\mathbb{R}^d} \big(1+q\,|x|\big)^{-\alpha-1}\,\mathrm{d}x \leq C_2 a_t^{-1} s^{d-\alpha-1}.$$

Combining this with (3.11) yields (3.6).

In the *third step* we show (3.7) by a direct calculation. First, let us show that for $\varepsilon > 0$

$$\psi_{\frac{|z|}{t}}(a_t s) \leq a_t s + \frac{|z|}{t}\,(q+\varepsilon)\log t \tag{3.12}$$

for all large t, $\frac{r_t}{\log t} \leq |z| \leq r_t \log t$ and $(\log t)^{-\frac{1}{2}} \leq s \leq 1$. By definition,

$$\psi_{\frac{|z|}{t}}(a_t s) = a_t s + \frac{|z|}{t}\log\psi_{\frac{|z|}{t}}(a_t s),$$

hence it suffices to prove that $\psi_{\frac{|z|}{t}}(a_t s) \leq t^{q+\varepsilon}$.
Assume this is false for some large t, z and s. Then using the monotonicity of $x \mapsto x - a\log x$ for $x \geq a$, we obtain

$$t^{q+\frac{\varepsilon}{2}} \geq a_t \geq a_t s = \psi_{\frac{|z|}{t}}(a_t s) - \frac{|z|}{t}\log\psi_{\frac{|z|}{t}}(a_t s) \geq t^{q+\varepsilon} - \frac{|z|}{t}\,(q+\varepsilon)\log t \geq t^{q+\varepsilon} - t^{q+\frac{\varepsilon}{2}},$$

which is a contradiction. Now we can compute

$$\mathrm{Prob}\big(\Phi_t(Z_t^{(1)}) \leq a_t s\big) = \prod_{z\in\mathbb{Z}^d} \mathrm{Prob}\big(\Phi_t(z) \leq a_t s\big) \leq \prod_{\frac{r_t}{\log t}\leq|z|\leq r_t\log t} \mathrm{Prob}\Big(\xi(z) \leq \psi_{\frac{|z|}{t}}(a_t s - \frac{\eta(z)}{t})\Big)$$

$$\leq \prod_{\frac{r_t}{\log t}\leq|z|\leq r_t\log t} \mathrm{Prob}\Big(\xi(z) \leq a_t s + \frac{|z|}{t}\,(q+\varepsilon)\log t\Big)$$

using (3.4), $\psi_{\frac{|z|}{t}}(a_t s - \frac{\eta(z)}{t}) \leq \psi_{\frac{|z|}{t}}(a_t s)$ and (3.12). Inserting the explicit form of the distribution function we get

$$\mathrm{Prob}\big(\Phi_t(Z_t^{(1)}) \leq a_t s\big) \leq \exp\big\{-(1+o(1)) \sum_{\frac{r_t}{\log t}\leq|z|\leq r_t\log t} \big(a_t s + \tfrac{|z|}{t}(q+\varepsilon)\log t\big)^{-\alpha}\big\}$$

$$\leq \exp\Big\{-(1+o(1))\, s^{d-\alpha} \int_{\mathbb{R}^d} \frac{\mathrm{d}u}{(1+(q+\varepsilon)|u|)^{\alpha}}\Big\}$$

using a Riemann sum approximation as in the second step. This proves (3.7).

Coming to the *fourth step*, we now use (3.5), (3.6) and (3.7) to get

$$\begin{aligned}
&\mathrm{Prob}\big(\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) \leq 2a_t\lambda_t\big)\\
&\quad\leq \mathrm{Prob}\big(\Phi_t(Z_t^{(1)}) \leq a_t(\log t)^{-1/2}\big)\\
&\qquad + \int_{(\log t)^{-1/2}}^{\infty} \mathrm{Prob}\big(\Phi_t(Z_t^{(1)}) \in \mathrm{d}(a_t s),\, \Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) \leq 2a_t\lambda_t\big)\\
&\quad\leq \exp\big\{-C_3\,(\log t)^{\frac{\alpha-d}{2}}\big\}\\
&\qquad + C_1 C_2^3\,\lambda_t^2\,\Big[\int_{(\log t)^{-1/2}}^{1} \frac{\exp\{-C_3\, s^{d-\alpha}\}\,\mathrm{d}s}{s^{3(\alpha-d+1)}} + \int_1^{\infty} \frac{\mathrm{d}s}{s^{3(\alpha-d+1)}}\Big].
\end{aligned}$$

The first term is $o(\lambda_t^2)$ by choice of λ_t, and the expression in the square bracket is bounded by an absolute constant. This completes the proof. $\square$

Now we turn the estimate of Lemma 3.3 into an almost sure bound.

Proposition 3.4. *Almost surely, eventually for all t,*

$$\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) \geq a_t\lambda_t.$$

Proof. Let $\varepsilon \in (0, 2\beta-1)$ be such that $\beta > 1 + \frac{1}{\alpha-d} + 2\varepsilon$ and let $t_n = \mathrm{e}^{n^\gamma}$, where $\gamma \in (\frac{1}{2\beta}, \frac{1}{2\beta-\varepsilon})$. Note that $\gamma < 1$. Since $\lambda_{t_n}^2 = n^{-2\gamma\beta}$ is summable, Lemma 3.3 and the Borel–Cantelli lemma imply that

$$\Phi_{t_n}(Z_{t_n}^{(1)}) - \Phi_{t_n}(Z_{t_n}^{(3)}) \geq 2a_{t_n}\lambda_{t_n} \qquad \text{eventually for all } n.$$

For each $t \in [t_n, t_{n+1})$ we obtain by Lemma 3.2(*iv,v*)

$$\begin{aligned}\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) &\geq \Phi_{t_n}(Z_{t_n}^{(1)}) - \Phi_{t_{n+1}}(Z_{t_{n+1}}^{(3)}) \\ &= \big[\Phi_{t_n}(Z_{t_n}^{(1)}) - \Phi_{t_n}(Z_{t_n}^{(3)})\big] - \big[\Phi_{t_{n+1}}(Z_{t_{n+1}}^{(3)}) - \Phi_{t_n}(Z_{t_n}^{(3)})\big] \\ &\geq 2a_{t_n}\lambda_{t_n} - \frac{t_{n+1}-t_n}{t_n}\, a_{t_{n+1}}(\log t_{n+1})^{\frac{1}{\alpha-d}+\varepsilon}. \end{aligned} \tag{3.13}$$

Notice that eventually

$$\frac{t_{n+1}-t_n}{t_n} = \mathrm{e}^{(n+1)^\gamma - n^\gamma} - 1 = \gamma n^{\gamma-1}(1+o(1)) = \gamma(\log t_n)^{\frac{\gamma-1}{\gamma}}(1+o(1)) \leq (\log t_n)^{-2\beta+1+\varepsilon}.$$

Denote by $n(t)$ the integer such that $t \in [t_{n(t)}, t_{n(t)+1})$. Since $t_{n+1}/t_n \to 1$ we have $t_{n(t)} \sim t$ and $t_{n(t)+1} \sim t$. Substituting this and the last estimate into (3.13), we obtain

$$\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(3)}) \geq 2a_t\lambda_t(1+o(1)) - a_t(\log t)^{\frac{1}{\alpha-d}-2\beta+1+2\varepsilon}(1+o(1)) \geq a_t\lambda_t$$

eventually since $\lambda_t = (\log t)^{-\beta}$ and $(\log t)^{\frac{1}{\alpha-d}-\beta+1+2\varepsilon} = o(1)$, which makes the second term negligible compared to the first one. $\square$

4. Total mass of the solution

In this section we show that the total mass $U(t)$ of the solution can be well approximated by $\exp\{t\Phi_t(Z_t^{(1)})\}$. The main tool is the Feynman–Kac formula in (1.8) and a technical lemma provided in Section 4.1. In Section 4.2 we prove the lower bound for $\frac{1}{t}\log U(t)$. In Section 4.3 we split the set of all paths into five path classes four of which turn out to give negligible contribution to the Feynman–Kac formula for $U(t)$. In Section 4.4 we show that the remaining class yields a useful upper bound for $\frac{1}{t}\log U(t)$.

4.1 A technical lemma.

We bound contributions to the Feynman–Kac formula for $U(t)$ coming from path classes that are defined according to their number of steps and the maximum along their path. Denote by J_t the number of jumps of the random walk $(X_s : s \geq 0)$ up to time t. Recall the notation from the beginning of Section 2 and let $H = (H_t)_{t>0}$ be some family of sets $H_t \subset F_t$, and let $h = (h_t)_{t>0}$ be some family of functions $h_t : \mathbb{Z}^d \to \mathbb{N}_0$. Denote by

$$U_{H,h}(t) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\big\{\exists z \in F_t\backslash H_t\colon \max_{s\in[0,t]} \xi(X_s) = \xi(z), h_t(z) \leq J_t \leq r_t g_t\big\}\Big]$$

the contribution to the total mass that comes from paths which attain their maximal potential value in some $z \in F_t \setminus H_t$ with step number in $\{h_t(z), \dots, \lfloor r_t g_t \rfloor\}$.

Lemma 4.1. *There is $\delta > 0$ such that, almost surely, for $t \to \infty$,*

(a) $\displaystyle \frac{1}{t}\log U_{H,h}(t) \le \max_{z\in F_t\setminus H_t}\Big\{\Phi_t(z) + \frac{1}{t}\max_{n\ge h_t(z)}\Big[\eta(n,z) - \eta(z) - \tfrac{n-|z|}{2}\log\xi(z)\Big]\Big\} + O(t^{q-\delta}),$

(b) $\displaystyle \frac{1}{t}\log U_{H,h}(t) \le \max_{z\in F_t\setminus H_t}\Phi_t(z) + O(t^{q-\delta}).$

Proof. We write $U_{H,h}(t)$ as

$$U_{H,h}(t) = \sum_{z\in F_t\setminus H_t} U_{H,h}(t,z), \tag{4.1}$$

where we define, for any $z\in\mathbb{Z}^d$,

$$U_{H,h}(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\Big\{\max_{s\in[0,t]}\xi(X_s) = \xi(z), h_t(z)\le J_t\le r_t g_t\Big\}\Big].$$

Denote by

$$\mathcal{P}_{n,z} = \big\{y = (y_0,y_1,\dots,y_n)\in(\mathbb{Z}^d)^{n+1}\colon y_0 = 0,\ |y_i - y_{i-1}| = 1,\ \max_{0\le i\le n}\xi(y_i) = \xi(z)\big\}$$

the set of all discrete time paths in $\mathbb{Z}^d$ of length n starting at the origin, going through z, such that the maximum of the potential over the path is attained at z. Let $(\tau_i)_{i\in\mathbb{N}_0}$ be a sequence of independent exponentially distributed random variables with parameter $2d$. Denote by E the expectation with respect to (τ_i). Averaging over all random paths following the same path y (with individual timings) we obtain

$$U_{H,h}(t,z) = \sum_{n=h_t(z)}^{\lfloor r_t g_t\rfloor}\sum_{y\in\mathcal{P}_{n,z}} U_{H,h}(t,z,y), \tag{4.2}$$

where

$$U_{H,h}(t,z,y) = (2d)^{-n}\mathsf{E}\left[\exp\Big\{\sum_{i=0}^{n-1}\tau_i\xi(y_i) + \Big(t - \sum_{i=0}^{n-1}\tau_i\Big)\xi(y_n)\Big\}\mathbb{1}\Big\{\sum_{i=0}^{n-1}\tau_i \le t < \sum_{i=0}^{n}\tau_i\Big\}\right].$$

Note that, as y can have self-intersections, some of the values of ξ over y may coincide. We would like to avoid the situation when the maximum of ξ over y is taken at more than one point. Therefore, for each path y, we slightly change the potential over y. Namely, we denote by $i(y) = \min\{i\colon \xi(y_i) = \xi(z)\}$ the index of the first point where the maximum of the potential over the path is attained. Then we define the modified version of the potential $\xi^y\colon \{0,\dots,n\}\to\mathbb{R}$ by

$$\xi_i^y = \begin{cases}\xi(y_i), & \text{if } i\ne i(y),\\ \xi(y_i)+1, & \text{if } i = i(y).\end{cases}$$

Repeating the computations (4.16) and (4.17) from [HMS08] we obtain

$$U_{H,h}(t,z,y) \le \mathrm{e}^{t\xi^y_{i(y)} - 2dt}\prod_{i\ne i(y)}\frac{1}{\xi^y_{i(y)} - \xi^y_i} \le \mathrm{e}^{t\xi(z)}\prod_{i=1}^{n}\frac{1}{1+\xi(z)-\xi(y_i)}. \tag{4.3}$$

Let us now find a lower bound for the number of sites on the path where the potential is small compared to $\xi(z)$ or, more precisely, we estimate the number of indices $1\le i\le n$ such that $\xi(y_i)\in G_t^{\mathrm{c}}$. *First*, we erase loops that the path y may have made before reaching z for the first time and extract from $(y_0,\dots,y_{i(y)})$ a self-avoiding path $(y_{i_0},\dots,y_{i_{l(y)}})$ starting at the origin, ending at z and having length $l(y)\ge|z|$, where we take $i_0 = 0$ and

$$i_{j+1} = \min\{i\colon y_l \ne y_{i_j}\ \forall l\in[i,i(y)]\}.$$

Since this path visits $l(y)$ different points, at least $l(y)-m_t$ of them belong to $G_t^{\rm c}$. By Lemma 2.2 (ii) we have $|z|>t^{(q+1)\sigma+c}>m_t$ and hence $l(y)-m_t$ is eventually positive. *Second*, for each $0\le j\le l(y)-1$, consider the path $(y_{i_j+1},\dots,y_{i_{j+1}-1})$, which was removed during erasing the j-th loop. Obviously, it contains an even number $i_{j+1}-i_j-1$ of steps, as $y_{i_j}=y_{i_{j+1}-1}$ and y_{i_j} and $y_{i_{j+1}-1}$ are neighbours. Notice that, as G_t is totally disconnected by Lemma 2.2 (iii), at least half of the steps, $(i_{j+1}-i_j-1)/2$, belong to $G_t^{\rm c}$. *Third*, consider the remaining piece $(y_{i(y)+1},\dots,y_n)$. Again, since G_t is totally disconnected, there will be at least $(n-i(y))/2$ points belonging to $G_t^{\rm c}$. Summing up these three observations, we obtain that y makes at least

$$l(y)-m_t+\sum_{j=0}^{l(y)-1}\frac{i_{j+1}-i_j-1}{2}+\frac{n-i(y)}{2}=l(y)-m_t+\frac{n-l(y)}{2}\ge|z|-m_t+\frac{n-|z|}{2}$$

steps that belong to $G_t^{\rm c}$.

Now we can continue estimating $U_{H,h}(t,z,y)$. Recall that the potential is at most $\xi^{(m_t)}_{r_tg_t}$ on the set $G_t^{\rm c}$. If we drop the terms corresponding to the points from G_t in (4.3), we obtain

$$U_{H,h}(t,z,y)\le {\rm e}^{t\xi(z)}\big[\xi(z)-\xi^{(m_t)}_{r_tg_t}\big]^{-\left(|z|-m_t+\frac{n-|z|}{2}\right)}.$$

Substituting this into (4.2) and using $|\mathcal{P}_{n,z}|\le N(n,z)$, we get

$$\begin{aligned}\frac1t\log U_{H,h}(t,z)&\le\frac1t\log\sum_{n=h_t(z)}^{r_tg_t}\sum_{y\in\mathcal{P}_{n,z}}{\rm e}^{t\xi(z)}\big[\xi(z)-\xi^{(m_t)}_{r_tg_t}\big]^{-\left(|z|-m_t+\frac{n-|z|}{2}\right)}\\&\le\frac1t\log\max_{h_t(z)\vee|z|\le n\le r_tg_t}\Big\{N(n,z)\,{\rm e}^{t\xi(z)}\big[\xi(z)-\xi^{(m_t)}_{r_tg_t}\big]^{-\left(|z|-m_t+\frac{n-|z|}{2}\right)}\Big\}+o(1)\\&=\max_{h_t(z)\vee|z|\le n\le r_tg_t}\Big\{\xi(z)+\frac{\eta(n,z)}{t}-\frac1t\Big[|z|-m_t+\frac{n-|z|}{2}\Big]\log\big(\xi(z)-\xi^{(m_t)}_{r_tg_t}\big)\Big\}+o(1).\end{aligned}$$

In order to simplify the expression under the maximum, we decompose

$$\begin{aligned}&\Big[|z|-m_t+\frac{n-|z|}{2}\Big]\log\big(\xi(z)-\xi^{(m_t)}_{r_tg_t}\big)\\&\qquad=\Big[|z|+\frac{n-|z|}{2}\Big]\log\xi(z)+\Big[|z|-m_t+\frac{n-|z|}{2}\Big]\log\Big(1-\frac{\xi^{(m_t)}_{r_tg_t}}{\xi(z)}\Big)-m_t\log\xi(z)\end{aligned}$$

and show that the last two terms are negligible. Indeed, for the second term, we use Lemma 2.2 (i) in the second step to obtain, for each $\delta<c$,

$$\Big|\Big[|z|-m_t+\frac{n-|z|}{2}\Big]\log\Big(1-\frac{\xi^{(m_t)}_{r_tg_t}}{\xi(z)}\Big)\Big|\le n\Big|\log\Big(1-\frac{\xi^{(m_t)}_{r_tg_t}}{\xi^{(k_t)}_{r_tg_t}}\Big)\Big|\le r_tg_tt^{-c}=O(t^{q+1-\delta})$$

uniformly for all $n\ge|z|$. For the third term, we use [HMS08, Lemma 3.5] and obtain $\log\xi(z)\le O(\log t)$ uniformly for all $|z|\le r_tg_t$. For $\delta<(q+1)(1-\sigma)$ this implies that

$$m_t\log\xi(z)\le O((r_tg_t)^\sigma\log t)=O(t^{q+1-\delta}).$$

Hence, there is a small positive δ such that

$$\begin{aligned}\frac{1}{t}\log U_{H,h}(t,z) &\le \max_{h_t(z)\vee|z|\le n\le r_tg_t}\Big\{\xi(z)+\frac{\eta(n,z)}{t}-\frac{1}{t}\Big[|z|+\tfrac{n-|z|}{2}\Big]\log\xi(z)\Big\}+O(t^{q-\delta})\\ &=\Big[\xi(z)+\frac{\eta(z)}{t}-\frac{|z|}{t}\log\xi(z)\Big] \qquad (4.4)\\ &\qquad+\frac{1}{t}\max_{h_t(z)\vee|z|\le n\le r_tg_t}\Big\{\eta(n,z)-\eta(z)-\tfrac{n-|z|}{2}\log\xi(z)\Big\}+O(t^{q-\delta}).\end{aligned}$$

To prove (a), observe that for each $z\in F_t$ we have $\xi(z)>\mathrm{e}d$. Hence either $t\xi(z)\ge|z|$ or

$$\xi(z)+\frac{\eta(z)}{t}-\frac{|z|}{t}\log\xi(z)\le\xi(z)-\frac{|z|}{t}\log\frac{\xi(z)}{d}\le\xi(z)\Big[1-\log\frac{\xi(z)}{d}\Big]<0.$$

In any case we obtain, using (4.1) and (4.4),

$$\begin{aligned}\frac{1}{t}\log U_{H,h}(t) &= \max_{z\in F_t\setminus H_t}\Big[\frac{1}{t}\log U_{H,h}(t,z)\Big]+o(1)\\ &\le \max_{z\in F_t\setminus H_t}\Big[\Phi_t(z)+\frac{1}{t}\max_{h_t(z)\le n}\Big\{\eta(n,z)-\eta(z)-\tfrac{n-|z|}{2}\log\xi(z)\Big\}\Big]+O(t^{q-\delta}).\end{aligned}$$

To prove (b), we show that the second term on the right hand side of (4.4) is negligible. Let $z\in F_t$. By Lemma 2.2 (i) we have $\xi(z)>t^c$ eventually. Further, for $n\ge|z|$, we use Lemma 2.1 and the substitution $r=n/|z|-1$ to get

$$\begin{aligned}\max_{h_t(z)\vee|z|\le n\le r_tg_t}\Big\{\eta(n,z)-\eta(z)-\tfrac{n-|z|}{2}\log\xi(z)\Big\} &\le \max_{n\ge|z|}\Big[(n-|z|)\log\tfrac{2den}{(n-|z|)\sqrt{\xi(z)}}\Big]+K\\ &\le |z|\max_{r\ge0}\Big[r\log\frac{2d\mathrm{e}(r+1)}{rt^{c/2}}\Big]+K. \qquad (4.5)\end{aligned}$$

If t is large enough, the expression in the square brackets is negative for $r\ge1$, hence the maximum is attained at some $r<1$. Using this to estimate the numerator and optimising the estimate, we obtain

$$\max_{r\ge0}\Big[r\log\frac{2d\mathrm{e}(r+1)}{rt^{c/2}}\Big]\le\max_{r\ge0}\Big[r\log\frac{4d\mathrm{e}}{rt^{c/2}}\Big]=4dt^{-c/2}. \qquad (4.6)$$

Finally, since $|z|\le r_tg_t$, we obtain, combining (4.5) and (4.6) and, if necessary, decreasing δ so that it satisfies $\delta<c/2$,

$$\max_{h_t(z)\vee|z|\le n\le r_tg_t}\Big\{\eta(n,z)-\eta(z)-\tfrac{n-|z|}{2}\log\xi(z)\Big\}\le r_tg_t\,4dt^{-c/2}+K=O(t^{q+1-\delta}).$$

Using this on the right hand side of (a) completes the proof. □

4.2 A lower bound for the growth of the mass.

We now derive a *lower* bound for $U(t)$, which is a slight improvement on the bound given in [HMS08, Lemma 2.2]. This argument does not rely on Lemma 4.1.

Proposition 4.2. *Almost surely, eventually for all t*

$$\frac{1}{t}\log U(t)\ge\Phi_t(Z_t^{(1)})-2d+o(1).$$

Proof. The proof follows the same lines as in [HMS08, Lemma 2.2], so that we will shorten some computations if they are the same as there. Let $\rho\in(0,1]$ and $z\in\mathbb{Z}^d$ with $|z|\ge2$. Denote by

$$A_t^{z,\rho}=\{J_{\rho t}=|z|,X_s=z\,\forall s\in[\rho t,t]\}$$

the event that the random walk X reaches the point z before time ρt, making the minimal possible number of jumps, and stays at z for the rest of the time. Denote by $P_\lambda(\cdot)$ the Poisson distribution with parameter λ. Then, using Stirling's formula, we obtain

$$\mathbb{P}_0(A_t^{z,\rho}) = \frac{N(z)P_{2d\rho t}(|z|)P_{2d(1-\rho)t}(0)}{(2d)^{|z|}} = \exp\left\{\eta(z) - |z|\log\frac{|z|}{\mathrm{e}\rho t} - 2dt + O(\log|z|)\right\},$$

where the last error term is bounded by the multiple of $\log|z|$ with an absolute constant. As $\xi(z) \geq 0$ almost surely for all z, we obtain, for all ρ and z as above,

$$\begin{aligned} U(t) = \mathbb{E}_0\left[\exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\}\right] &\geq \mathrm{e}^{t(1-\rho)\xi(z)}\mathbb{P}_0(A_t^{z,\rho}) \\ &\geq \exp\left\{t(1-\rho)\xi(z) + \eta(z) - |z|\log\frac{|z|}{\mathrm{e}\rho t} - 2dt + O(\log|z|)\right\}. \end{aligned}$$

Since $\log|z| = o(t)$ for $|z| \leq t^\beta$ for any fixed positive β, this implies

$$\frac{1}{t}\log U(t) \geq \max_{0<\rho\leq 1}\max_{1\leq|z|\leq t^\beta}\left[(1-\rho)\xi(z) + \frac{\eta(z)}{t} - \frac{|z|}{t}\log\frac{|z|}{\mathrm{e}\rho t}\right] - 2d + o(1). \tag{4.7}$$

Let $\widehat{\eta} \in (\frac{d}{\alpha}, 1)$ and $\beta = (1-\widehat{\eta})^{-1}(1+\varepsilon)$, $\varepsilon > 0$. By [HMS08, Lemma 3.5] there is r_0 such that $\xi_r^{(1)} \leq r^{\widehat{\eta}}$ for all $r > r_0$. We thus have, using the bound $\eta(z) \leq |z|\log d$ and a similar computation as in [HMS08, Lemma 2.2],

$$\begin{aligned} \max_{|z|>\max\{r_0,t^\beta\}}\left[(1-\rho)\xi(z) + \frac{\eta(z)}{t} - \frac{|z|}{t}\log\frac{|z|}{\mathrm{e}\rho t}\right] &\leq \max_{|z|>\max\{r_0,t^\beta\}}\left[(1-\rho)\xi_{|z|}^{(1)} - \frac{|z|}{t}\log\frac{|z|}{d\mathrm{e}\rho t}\right] \\ &\leq \max_{|z|>\max\{r_0,t^\beta\}}\left[|z|^{\widehat{\eta}}\left(1-\rho-t^\varepsilon\log\frac{t^{\beta-1}}{d\mathrm{e}\rho}\right)\right] < 0, \end{aligned} \tag{4.8}$$

eventually for all t. Recall that $\frac{1}{t}\log U(t) \geq 0$ and take t large enough so that $t^\beta > r_0$. Then (4.8) implies that the maximum in (4.7) can be taken over all z instead of $|z| \leq t^\beta$. It is easy to see that this maximum is attained at $\rho = \frac{|z|}{t\xi(z)}$ unless this value exceeds one. Substituting this ρ into (4.7) we obtain

$$\begin{aligned} \frac{1}{t}\log U(t) &\geq \max_{z\in\mathbb{Z}^d}\left[\xi(z) + \frac{\eta(z)}{t} - \frac{|z|}{t}\log\xi(z)\right]\mathbb{1}\{t\xi(z) > |z|\} - 2d + o(1) \\ &= \Phi_t(Z_t^{(1)}) - 2d + o(1), \end{aligned}$$

which completes the proof. □

4.3 Negligible parts of the total mass.
In this section we show that the main contribution to the Feynman–Kac formula for $U(t)$ comes from those paths that pass through $Z_t^{(1)}$ or $Z_t^{(2)}$ and do not make significantly more than $|Z_t^{(1)}| \wedge |Z_t^{(2)}|$ steps. For this purpose, we define five path classes and show that the latter four of them each give a negligible contribution to the total mass $U(t)$.

In the sequel, we assume that δ is taken small enough so that Lemma 4.1 holds and $\delta < q$. We decompose the set of all paths $[0,t] \to \mathbb{Z}^d$ into the following five classes

$$A_i = \begin{cases} \Big\{J_t \le r_t g_t, \exists z \in \{Z_t^{(1)}, Z_t^{(2)}\}\colon \max\limits_{s\in[0,t]} \xi(X_s) = \xi(z), J_t < |z|(1+t^{-\delta/2})\Big\}, & i=1,\\ \Big\{J_t \le r_t g_t, \exists z \in \{Z_t^{(1)}, Z_t^{(2)}\}\colon \max\limits_{s\in[0,t]} \xi(X_s) = \xi(z), J_t \ge |z|(1+t^{-\delta/2})\Big\}, & i=2,\\ \Big\{J_t \le r_t g_t, \exists z \in F_t\backslash\{Z_t^{(1)}, Z_t^{(2)}\}\colon \max\limits_{s\in[0,t]} \xi(X_s) = \xi(z)\Big\}, & i=3,\\ \Big\{J_t \le r_t g_t, \max\limits_{s\in[0,t]} \xi(X_s) \le \xi_{r_t g_t}^{(k_t)}\Big\}, & i=4,\\ \Big\{J_t > r_t g_t\Big\}, & i=5, \end{cases}$$

and split the total mass into five components $U(t) = \sum_{i=1}^5 U_i(t)$, where

$$U_i(t) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}_{A_i}\Big], \qquad 1 \le i \le 5.$$

Lemma 4.3. *Almost surely,* $\lim_{t\to\infty} U_i(t)/U(t) = 0$ *for* $2 \le i \le 5$.

Proof. Case $i=2$*:* Denote $h_t(z) = |z|(1+t^{-\delta/2})$ and $H_t = F_t\backslash\{Z_t^{(1)}, Z_t^{(2)}\}$. By Lemma 4.1 (a),

$$\begin{aligned} \frac{1}{t}\log U_2(t) &\le \frac{1}{t}\log U_{H,h}(t)\\ &\le \max_{z\in\{Z_t^{(1)},Z_t^{(2)}\}} \Big\{\Phi_t(z) + \frac{1}{t}\max_{n\ge|z|(1+t^{-\delta})}\Big[\eta(n,z) - \eta(z) - \tfrac{n-|z|}{2}\log\xi(z)\Big]\Big\} + O(t^{q-\delta}). \end{aligned} \tag{4.9}$$

For each $z \in \{Z_t^{(1)}, Z_t^{(2)}\}$ we have by Lemma 3.2(i), for any $c>0$, that $\xi(z) > (2d\mathrm{e})^2 t^{q-c}$ eventually. Together with Lemma 2.1 this implies

$$\begin{aligned} \max_{n\ge|z|(1+t^{-\delta/2})}\Big[\eta(n,z) - \eta(z) - \tfrac{n-|z|}{2}\log\xi(z)\Big] &\le \max_{n\ge|z|(1+t^{-\delta/2})}\Big[(n-|z|)\log\tfrac{2den\xi(z)^{-\frac12}}{n-|z|}\Big] + K\\ \le \max_{n\ge|z|(1+t^{-\delta/2})}\Big[(n-|z|)\log\tfrac{nt^{-\frac{q-c}{2}}}{n-|z|}\Big] + K &= |z|\max_{r\ge t^{-\delta/2}}\Big[r\log\tfrac{(r+1)t^{-\frac{q-c}{2}}}{r}\Big] + K. \end{aligned}$$

It is easy to check that, eventually for all t, the function under the maximum is decreasing on $(t^{-\frac{q-c}{2}}, \infty)$ if $c<q$. Since $\delta < q$ we can choose c so small that $\delta < q-c$. The maximum is then attained at $r = t^{-\delta/2}$ and, as $|z| \ge t^{q+1-\frac{\delta}{4}}$ by Lemma 3.2 (iii), we obtain

$$\max_{n\ge|z|(1+t^{-\delta/2})}\Big[\eta(n,z) - \eta(z) + \tfrac{n-|z|}{2}\log\xi(z)\Big] \le -|z|\,t^{-\delta/2}\log\big(t^{\frac{q-c-\delta}{2}}\big) + K \le -t^{q+1-\frac{3\delta}{4}}$$

eventually for all t. Combining this with (4.9) and using Proposition 4.2 we finally get

$$\frac{1}{t}\log\frac{U_2(t)}{U(t)} \le \max_{i=1,2}\big\{\Phi_t(Z_t^{(i)}) - t^{q-\frac{3\delta}{4}}\big\} - \Phi_t(Z_t^{(1)}) + O(t^{q-\delta}) = -t^{q-\frac{3\delta}{4}} + O(t^{q-\delta}) \to -\infty.$$

Case $i=3$*:* Pick $h_t(z) = 0$ and $H_t = \{Z_t^{(1)}, Z_t^{(2)}\}$. By Lemma 4.1 (b) we obtain

$$\frac{1}{t}\log U_3(t) \le \frac{1}{t}\log U_{H,h}(t) \le \max_{z\in\mathbb{Z}^d\backslash\{Z_t^{(1)},Z_t^{(2)}\}} \Phi_t(z) + O(t^{q-\delta}) = \Phi_t(Z_t^{(3)}) + O(t^{q-\delta}).$$

It remains to apply Propositions 4.2 and 3.4 to get eventually

$$\frac{1}{t}\log\frac{U_3(t)}{U(t)} \le \Phi_t(Z_t^{(3)}) - \Phi_t(Z_t^{(1)}) + O(t^{q-\delta}) \le -a_t\lambda_t + O(t^{q-\delta}) \to -\infty.$$

Case $i = 4$: We estimate the integral in the Feynman–Kac formula by $t\xi^{(k_t)}_{r_t g_t}$. Lemma 2.2 (i) implies that there is $c > 0$ such that eventually

$$\frac{1}{t}\log U_4(t) \le \xi^{(k_t)}_{r_t g_t} \le t^{q-c}.$$

On the other hand, it follows from [HMS08, Th. 1.1] that, for each $\widehat{c} > 0$, we have $\frac{1}{t}\log U(t) \ge t^{q-\widehat{c}}$ eventually. Since $\widehat{c}$ can be taken smaller than c, the statement is proved.

Case $i = 5$: We decompose the Feynman–Kac formula according to the number J_t of jumps. Observe that the integral there can be estimated by $t\xi^{(1)}_{J_t}$ and use that J_t has Poisson distribution with parameter $2dt$. Thus, we obtain

$$U_5(t) = \sum_{n>r_t g_t} \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{J_t = n\}\Big] \le \sum_{n>r_t g_t} \exp\Big\{t\xi^{(1)}_n - 2dt + \log\tfrac{(2dt)^n}{n!}\Big\}.$$

Pick $0 < \varepsilon < \nu/(q+1)$ and assume that t is large enough. It follows from [HMS08, Lemma 3.5] that $\xi^{(1)}_n < n^{\frac{d}{\alpha}}(\log n)^{\frac{1}{\alpha}+\varepsilon}$ for all $n > r_t g_t$. Further, it follows from Stirling's formula that $n! > (n/\mathrm{e})^n$ for all $n > r_t g_t$. Then, for all $n > r_t g_t$, we obtain, using monotonicity in n,

$$t\xi^{(1)}_n - 2dt + \log\tfrac{(2dt)^n}{n!} < tn^{\frac{d}{\alpha}}(\log n)^{\frac{1}{\alpha}+\varepsilon} + n\log\tfrac{2d\mathrm{e}t}{n} \le -n^{\frac{d}{\alpha}}.$$

Combining the last two displays we obtain that $U_5(t) = o(1)$. □

4.4 An upper bound for the growth of the mass

Lemmas 4.1 and 4.3 make it possible to prove an upper bound for $\frac{1}{t}\log U(t)$, which is asymptotically equal to the lower bound of Proposition 4.2.

Proposition 4.4. *Fix $\delta > 0$ as in Lemma 4.1. Then, almost surely, eventually for all t,*

$$\frac{1}{t}\log U(t) \le \Phi_t(Z^{(1)}_t) + O(t^{q-\delta}).$$

Proof. Consider $H_t = \varnothing$ and $h_t = 0$. Then $U_{H,h}(t) = U_1(t) + U_2(t) + U_3(t)$. Since for the remaining two functions we have $U_4(t) + U_5(t) \le U(t)o(1)$ by Lemma 4.3, we obtain by Lemma 4.1 (b) that $\frac{1}{t}\log U(t) \le \frac{1}{t}\log U_{H,h}(t)(1 + o(1)) \le \Phi_t(Z^{(1)}_t) + O(t^{q-\delta})$. □

5. Almost sure localisation in two points

In this section, we prove Theorem 1.1. In Section 5.1 we introduce a decomposition into three events, formulate our main steps and provide some technical preparation. The remaining Sections 5.2–5.4 give the proofs of the localisation on the three respective events.

5.1 Decomposition into three events.

In the proof of Theorem 1.1, we distinguish between three disjoint events constituting a partition of the full probability space:

- $\Phi_t(Z^{(1)}_t) - \Phi_t(Z^{(2)}_t)$ is small and the sites $Z^{(1)}_t$ and $Z^{(2)}_t$ are close to each other;
- $\Phi_t(Z^{(1)}_t) - \Phi_t(Z^{(2)}_t)$ is small but the sites $Z^{(1)}_t$ and $Z^{(2)}_t$ are far away from each other;
- $\Phi_t(Z^{(1)}_t) - \Phi_t(Z^{(2)}_t)$ is large.

and prove the two point localisation on each event by different arguments. To be precise, for $i = 1, 2$, denote by

$$\Gamma^{(i)}_t = \big\{z \in \mathbb{Z}^d \colon |z - Z^{(i)}_t| + \min\{|z|, |Z^{(i)}_t|\} < |Z^{(i)}_t|(1 + t^{-\delta/2})\big\}$$

the set containing all sites z such that there is a path of length less than $|Z_t^{(i)}|(1+t^{-\delta/2})$ starting from the origin passing through both z and $Z_t^{(i)}$. Further, denote

$$\Gamma_t = \big\{z\in\mathbb{Z}^d\colon |z-Z_t^{(1)}|+\min\{|z|,|Z_t^{(1)}|\}<|Z_t^{(1)}|(1+6t^{-\delta/2})\big\}.$$

In Sections 5.2–5.4 we prove the following three propositions.

Proposition 5.1. *Almost surely,*

$$\lim_{t\to\infty}\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d\backslash\{Z_t^{(1)},Z_t^{(2)}\}}u(t,z)\Big]\mathbb{1}\{\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})<a_t\lambda_t/2, Z_t^{(2)}\in\Gamma_t^{(1)}\}=0.$$

Proposition 5.2. *Almost surely,*

$$\lim_{t\to\infty}\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d\backslash\{Z_t^{(1)},Z_t^{(2)}\}}u(t,z)\Big]\mathbb{1}\{\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})<a_t\lambda_t/2, Z_t^{(2)}\notin\Gamma_t^{(1)}\}=0.$$

Proposition 5.3. *Almost surely,*

$$\lim_{t\to\infty}\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d\backslash\{Z_t^{(1)}\}}u(t,z)\Big]\mathbb{1}\{\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})\geq a_t\lambda_t/2\}=0.$$

Obviously, Theorem 1.1 follows immediately from the three propositions. For each of them, the idea of the proof is to decompose u into a sum of two functions u_1 and u_2 (which is different in different cases) such that u_2 is negligible and localisation of u_1 can be shown with the help of our spectral bounds derived in Section 2.2. If the gap between $\Phi_t(Z_t^{(1)})$ and $\Phi_t(Z_t^{(2)})$ is small (Cases 1 and 2) then both points $Z_t^{(1)}$ and $Z_t^{(2)}$ contribute to the total mass. However, the strategy of the proof is different, since in the second case the points $Z_t^{(1)}$ and $Z_t^{(2)}$ do not interact as they are far away from each other, whereas in the first case they do. If the gap between $\Phi_t(Z_t^{(1)})$ and $\Phi_t(Z_t^{(2)})$ is large (Case 3) only the site $Z_t^{(1)}$ contributes to the total mass. In the remaining part of this section, we prove a lemma, which is used in the proof of each of the three propositions.

Lemma 5.4. *There is $c\in(0,q)$ such that, almost surely eventually for all t,*

(*i*) $\xi(z)<\xi(Z_t^{(1)})-t^{q-c}$ *for all* $z\in\Gamma_t\backslash\{Z_t^{(1)},Z_t^{(2)}\}$,

(*ii*) $\xi(z)<\xi(Z_t^{(1)})-t^{q-c}$ *for all* $z\in\Gamma_t\backslash\{Z_t^{(1)}\}$ *if* $\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})\geq a_t\lambda_t/2$,

(*iii*) $\xi(z)<\xi(Z_t^{(2)})-t^{q-c}$ *for all* $z\in\Gamma_t^{(2)}\backslash\{Z_t^{(1)},Z_t^{(2)}\}$ *if* $\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})<a_t\lambda_t/2$,

(*iv*) $\Gamma_t^{(1)}\subset\Gamma_t$. *If* $Z_t^{(2)}\in\Gamma_t^{(1)}$ *then* $\Gamma_t^{(2)}\subset\Gamma_t$.

Proof. We prove (*i*)–(*iii*) simultaneously, first making the following observations:

(1) By Proposition 3.4 we have $\Phi_t(Z_t^{(1)})-\Phi_t(z)>a_t\lambda_t/2$ for all $z\notin\{Z_t^{(1)},Z_t^{(2)}\}$.

(2) $\Phi_t(Z_t^{(1)})-\Phi_t(z)\geq\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})\geq a_t\lambda_t/2$ for all $z\neq Z_t^{(1)}$ by assumption.

(3) Using Proposition 3.4 and our assumption we obtain $\Phi_t(Z_t^{(2)})-\Phi_t(Z_t^{(3)})>a_t\lambda_t/2$. Hence $\Phi_t(Z_t^{(2)})-\Phi_t(z)\geq\Phi_t(Z_t^{(2)})-\Phi_t(Z_t^{(3)})>a_t\lambda_t/2$ for all $z\notin\{Z_t^{(1)},Z_t^{(2)}\}$.

Thus, to show (*i*)–(*iii*), it suffices to prove that there exists $c\in(0,q)$ such that eventually

$$\xi(Z_t^{(i)})-\xi(z)>t^{q-c}$$

for each $i\in\{1,2\}$ and each z satisfying

$$\Phi_t(Z_t^{(i)})-\Phi_t(z)\geq a_t\lambda_t/2\quad\text{ and }\quad|z-Z_t^{(i)}|+\min\{|z|,|Z_t^{(i)}|\}<|Z_t^{(i)}|(1+6t^{-\delta/2}).\tag{5.1}$$

Assume that the statement is false. Then given $c < q$ there is an arbitrarily large t and $z \in \mathbb{Z}^d$ satisfying (5.1) with $\xi(Z_t^{(i)}) - \xi(z) \le t^{q-c}$. Then

$$\begin{aligned}\Phi_t(Z_t^{(i)}) - \Phi_t(z) = \big[\xi(Z_t^{(i)}) - \xi(z)\big] + \frac{|Z_t^{(i)}|}{t}\log\frac{\xi(z)}{\xi(Z_t^{(i)})} \\ + \frac{|z| - |Z_t^{(i)}|}{t}\log\xi(z) + \frac{\eta(Z_t^{(1)}) - \eta(z)}{t}.\end{aligned} \tag{5.2}$$

We can bound the second summand by zero if $\xi(z) \le \xi(Z_t^{(i)})$. For $\xi(z) > \xi(Z_t^{(i)})$, we use the inequality $\log x \le x - 1$ for $x > 0$ to obtain, by Lemma 3.2 (ii), eventually

$$\frac{|Z_t^{(i)}|}{t}\log\frac{\xi(z)}{\xi(Z_t^{(i)})} \le \frac{|Z_t^{(i)}|(\xi(z) - \xi(Z_t^{(i)}))}{t\xi(Z_t^{(i)})} < \xi(z) - \xi(Z_t^{(i)}).$$

In both cases we obtain the estimate for the first two terms

$$\big[\xi(Z_t^{(i)}) - \xi(z)\big] + \frac{|Z_t^{(i)}|}{t}\log\frac{\xi(z)}{\xi(Z_t^{(i)})} \le \max\{\xi(Z_t^{(i)}) - \xi(z), 0\} \le t^{q-c} = o(a_t\lambda_t).$$

We prove that the remaining two terms in (5.2) are of order $o(a_t\lambda_t)$ as well. First, assume that $|z| < |Z_t^{(i)}|$. Then (5.1) implies

$$|Z_t^{(i)}| \le |z - Z_t^{(i)}| + |z| < |Z_t^{(i)}|(1 + 6t^{-\delta/2}). \tag{5.3}$$

Notice that $\eta(Z_t^{(i)}) \le \eta(|z - Z_t^{(i)}| + |Z_t^{(i)}|, z)$ as to any path of length $|Z_t^{(i)}|$ passing through $Z_t^{(i)}$ we can add a path of length $|z - Z_t^{(i)}|$ in such a way that it passes through z. Using Lemma 2.1, Lemma 3.2 (iii) and (5.3) we obtain

$$\begin{aligned}\eta(Z_t^{(1)}) - \eta(z) &\le \eta(|z - Z_t^{(i)}| + |Z_t^{(i)}|, z) - \eta(z)\\ &\le \big(|z - Z_t^{(i)}| + |Z_t^{(i)}| - |z|\big)\log\tfrac{2de(|z-Z_t^{(i)}|+|Z_t^{(i)}|)}{|z-Z_t^{(i)}|+|Z_t^{(i)}|-|z|} + K\\ &\le \big(6t^{-\delta/2}|Z_t^{(i)}| + 2(|Z_t^{(i)}| - |z|)\big)\log\tfrac{2de((2+t^{-\delta/2})|Z_t^{(i)}|-|z|)}{|Z_t^{(i)}|-|z|} + K\\ &\le 2(|Z_t^{(i)}| - |z|)\log\tfrac{5de|Z_t^{(i)}|}{|Z_t^{(i)}|-|z|} + O(t^{q+1+\varepsilon-\delta/4}).\end{aligned}$$

Substituting this as well as the estimate for the first two terms into (5.2) we obtain

$$\Phi_t(Z_t^{(i)}) - \Phi_t(z) \le \tfrac{2(|Z_t^{(i)}|-|z|)}{t}\log\tfrac{5de|Z_t^{(i)}|}{(|Z_t^{(i)}|-|z|)\sqrt{\xi(z)}} + o(a_t\lambda_t).$$

By Lemma 3.2 (i) we have $\xi(Z_t^{(i)}) > t^{q-c/4}$ as t is large enough. By assumption we then have $\xi(z) \ge \xi(Z_t^{(i)}) - t^{q-c} > t^{q-c/2}$. Hence the expression under the logarithm is positive only if $|Z_t^{(i)}| - |z| < 5de\,|Z_t^{(i)}|\,t^{-q/2+c/4}$, which is smaller than $t^{q/2+1+c/2}$ by Lemma 3.2 (iii). Since $c < q$ we obtain $\Phi_t(Z_t^{(i)}) - \Phi_t(z) \le o(a_t\lambda_t)$.

Finally, consider $|z| \ge |Z_t^{(i)}|$. For the third term in (5.2) we notice that (5.1) implies that $|z| \le |Z_t^{(i)}|(1 + 6t^{-\delta/2})$. Then we use Lemma 3.2 (iii) and [HMS08, Lemma 3.5], which gives

$$\tfrac{|z|-|Z_t^{(i)}|}{t}\log\xi(z) \le 6t^{-\delta/2-1}|Z_t^{(i)}|\,\log\big(|Z_t^{(i)}|^{\frac{d}{\alpha}+\delta}(1 + 6t^{-\delta/2})^{\frac{d}{\alpha}+\delta}\big) = o(a_t\lambda_t).$$

For the last term in (5.2) we obtain from (5.1) that $\eta(Z_t^{(i)}) \le \eta(|Z_t^{(i)}|(1 + 6t^{-\delta/2}), z)$. Hence, by Lemma 2.1,

$$\begin{aligned}\eta(Z_t^{(1)}) - \eta(z) &\le \eta(|Z_t^{(i)}|(1 + 6t^{-\delta/2}), z) - \eta(z)\\ &\le \big(|Z_t^{(i)}|(1 + 6t^{-\delta/2}) - |z|\big)\log\tfrac{2de|Z_t^{(i)}|(1+6t^{-\delta/2})}{|Z_t^{(i)}|(1+6t^{-\delta/2})-|z|} + K.\end{aligned}$$

Notice that, for $a>0$, $x\mapsto x\log\frac{a}{x}$ is an increasing function on $(0,a/\mathrm{e})$. Since $|z|\ge|Z_t^{(i)}|$ we have $|Z_t^{(i)}|(1+6t^{-\delta/2})-|z|\le 6t^{-\delta/2}|Z_t^{(i)}|$, which is smaller than $2d|Z_t^{(i)}|(1+6t^{-\delta/2})$. Hence we obtain

$$\eta(Z_t^{(1)})-\eta(z)\le 6t^{-\delta/2}|Z_t^{(i)}|\log\tfrac{de(1+6t^{-\delta/2})}{3t^{-\delta/2}}\le O(t^{q+1-\delta/4}).$$

This proves that the last term in (5.2) is also bounded by $o(a_t\lambda_t)$. It remains to notice that we have proved $\Phi_t(Z_t^{(i)})-\Phi_t(z){\le}o(a_t\lambda_t)$, which contradicts to our assumption that $\Phi_t(Z_t^{(i)})-\Phi_t(z)\ge a_t\lambda_t/2$. This proves (i)–(iii).

(iv) The first statement is trivial. To prove the second one, we pick $z\in\Gamma_t^{(2)}$. For any such z there exists a path of length less than $|Z_t^{(2)}|(1+t^{\delta/2})$ starting at the origin and going through z and $Z_t^{(2)}$. If $|z|\le|Z_t^{(2)}|$ we can choose the path in such a way that it ends at $Z_t^{(2)}$. If $|z|>|Z_t^{(2)}|$ then $|z-Z_t^{(2)}|<|Z_t^{(2)}|\,t^{\delta/2}$ and so there is a path of length less than $|Z_t^{(2)}|(1+2t^{\delta/2})$ starting at the origin, going through z and ending in $Z_t^{(2)}$. In either of the cases, there is then a path of length less than $|Z_t^{(2)}|(1+2t^{\delta/2})+|Z_t^{(1)}-Z_t^{(2)}|$ starting in the origin, passing through z, $Z_t^{(2)}$ and ending at $Z_t^{(1)}$.

Observe that since $Z_t^{(2)}\in\Gamma_t^{(1)}$, there is a path of length less than $|Z_t^{(1)}|(1+t^{-\delta/2})$ going through $Z_t^{(1)}$ and $Z_t^{(2)}$, which in particular implies $|Z_t^{(2)}|<|Z_t^{(1)}|(1+t^{-\delta/2})$. If $|Z_t^{(2)}|<|Z_t^{(1)}|$, then $Z_t^{(2)}\in\Gamma_t^{(1)}$ implies $|Z_t^{(2)}|+|Z_t^{(1)}-Z_t^{(2)}|<|Z_t^{(1)}|(1+t^{-\delta/2})$ and so

$$|Z_t^{(2)}|(1+2t^{\delta/2})+|Z_t^{(1)}-Z_t^{(2)}|\le|Z_t^{(1)}|(1+3t^{-\delta/2})<|Z_t^{(1)}|(1+5t^{-\delta/2}).$$

If $|Z_t^{(2)}|\ge|Z_t^{(1)}|$, then $Z_t^{(2)}\in\Gamma_t^{(1)}$ implies $|Z_t^{(1)}-Z_t^{(2)}|<|Z_t^{(1)}|t^{-\delta/2}$ and so

$$\begin{aligned}|Z_t^{(2)}|(1+2t^{\delta/2})+|Z_t^{(1)}-Z_t^{(2)}|&\le|Z_t^{(1)}|(1+t^{-\delta/2})(1+2t^{-\delta/2})+|Z_t^{(1)}|t^{-\delta/2}\\&<|Z_t^{(1)}|(1+5t^{-\delta/2}).\end{aligned}$$

In each case we obtain that $z\in\Gamma_t$, which completes the proof. ∎

5.2 First event: $\Phi_t(Z_t^{(1)})$ is close to $\Phi_t(Z_t^{(2)})$, and $Z_t^{(1)}$ is close to $Z_t^{(2)}$.
In this section we prove Proposition 5.1. Let us decompose u into $u=u_1+u_2$ with

$$\begin{aligned}u_1(t,z)&=\mathbb{E}_0\Big[\exp\Big\{\int_0^t\xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t=z\}\mathbb{1}\{\tau_{\{Z_t^{(1)},Z_t^{(2)}\}}\le t,\tau_{\Gamma_t^{\mathrm{c}}}>t\}\Big],\\u_2(t,z)&=\mathbb{E}_0\Big[\exp\Big\{\int_0^t\xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t=z\}\mathbb{1}\{\tau_{\{Z_t^{(1)},Z_t^{(2)}\}}>t\text{ or }\tau_{\Gamma_t^{\mathrm{c}}}\le t\}\Big].\end{aligned}$$

We show that u_1 is localised in $Z_t^{(1)}$ and $Z_t^{(2)}$ (see Lemma 5.5) and that the contribution of u_2 is negligible (see Lemma 5.6).

Lemma 5.5. *Almost surely,*

$$\lim_{t\to\infty}\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d\setminus\{Z_t^{(1)},Z_t^{(2)}\}}u_1(t,z)\Big]\mathbb{1}\{\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})<a_t\lambda_t/2,Z_t^{(2)}\in\Gamma_t^{(1)}\}=0.$$

Proof. We further split u_1 into three contributions $u_1=u_{1,1}+u_{1,2}+u_{1,3}$ with

$$u_{1,j}(t,z)=\mathbb{E}_0\Big[\exp\Big\{\int_0^t\xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t=z\}\mathbb{1}_{C_{1j}}\Big]$$

with

$$C_{1j} = \begin{cases} \big\{\tau_{Z_t^{(1)}} \le t, \tau_{\Gamma_t^{\rm c}} > t\big\}, & j=1, \\ \big\{\tau_{Z_t^{(1)}} > t, \tau_{Z_t^{(2)}} \le t, \tau_{[\Gamma_t^{(2)}]^{\rm c}} > t\big\}, & j=2, \\ \big\{\tau_{Z_t^{(1)}} > t, \tau_{Z_t^{(2)}} \le t, \tau_{[\Gamma_t^{(2)}]^{\rm c}} \le t, \tau_{\Gamma_t^{\rm c}} > t\big\}, & j=3. \end{cases}$$

Observe that the sets C_{11}, C_{12}, C_{13} are disjoint on the event $\{Z_t^{(2)} \in \Gamma_t^{(1)}\}$ since $\Gamma_t^{(2)} \subset \Gamma_t$ by Lemma 5.4 (iv). Furthermore, on this event, their union is equal to the event

$$\big\{\tau_{\{Z_t^{(1)}, Z_t^{(2)}\}} \le t,\ \tau_{\Gamma_t^{\rm c}} > t\big\}$$

appearing in the definition of $u_1(t,z)$. Hence, we indeed have $u_1 = u_{1,1} + u_{1,2} + u_{1,3}$.

We now fix a sufficiently large t and argue on the event $\{\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) < a_t\lambda_t/2, Z_t^{(2)} \in \Gamma_t^{(1)}\}$. We also fix some $c \in (0,q)$ and use this to distinguish between two cases.

(1) First, we assume $\xi(Z_t^{(2)}) \le \xi(Z_t^{(1)}) - t^{q-c}$. We show that $u_{1,1}$ and $u_{1,2}$ are localised around $Z_t^{(1)}$ and $Z_t^{(2)}$, respectively, and that the contribution of $u_{1,3}$ is negligible.

Let us fix t large enough and pick $B = \Gamma_t$, $\Omega = \{Z_t^{(1)}\}$ to study $u_{1,1}$ and $B = \Gamma_t^{(2)} \backslash \{Z_t^{(1)}\}$, $\Omega = \{Z_t^{(2)}\}$ to study $u_{1,2}$. For the first choice we have

$$\mathfrak{g}_{\Omega,B} = \xi(Z_t^{(1)}) - \max_{\Gamma_t \backslash \{Z_t^{(1)}\}} \xi(z) = \min\big\{\xi(Z_t^{(1)}) - \max_{\Gamma_t \backslash \{Z_t^{(1)}, Z_t^{(2)}\}} \xi(z),\ \xi(Z_t^{(1)}) - \xi(Z_t^{(2)})\big\} \ge t^{q-c}$$

by our assumption and Lemma 5.4 (i). For the second choice we get

$$\mathfrak{g}_{\Omega,B} = \xi(Z_t^{(2)}) - \max_{z \in \Gamma_t^{(2)} \backslash \{Z_t^{(1)}, Z_t^{(2)}\}} \xi(z) \ge t^{q-c}$$

by Lemma 5.4 (iii), using that $\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) < a_t\lambda_t/2$. Now we apply Lemma 2.4 and use the monotonicity of φ to obtain

$$\frac{\sum_{z \in \Gamma_t \backslash \{Z_t^{(1)}\}} u_{1,1}(t,z)}{\sum_{z \in \Gamma_t} u_{1,1}(t,z)} \le \varphi(t^{q-c}) \qquad \text{and} \qquad \frac{\sum_{z \in \Gamma_t^{(2)} \backslash \{Z_t^{(1)}, Z_t^{(2)}\}} u_{1,2}(t,z)}{\sum_{z \in \Gamma_t^{(2)} \backslash \{Z_t^{(1)}\}} u_{1,2}(t,z)} \le \varphi(t^{q-c}). \quad (5.4)$$

Obviously, the estimate remains true if we increase the denominators and sum over all z the larger function $u(t,z)$, which will produce $U(t)$. For the numerators, notice that $u_{1,1}(t,z) = 0$ for all $z \notin \Gamma_t$ as the paths from C_{11} do not leave Γ_t, and $u_{1,2}(t,z) = 0$ for all $z \notin \Gamma_t^{(2)} \backslash \{Z_t^{(1)}\}$ as the paths from C_{12} do not leave this set. Hence (5.4) implies

$$U(t)^{-1} \sum_{z \in \mathbb{Z}^d \backslash \{Z_t^{(1)}\}} u_{1,1}(t,z) \le \varphi(t^{q-c}) = o(1) \quad \text{and} \quad U(t)^{-1} \sum_{z \in \mathbb{Z}^d \backslash \{Z_t^{(2)}\}} u_{1,2}(t,z) \le \varphi(t^{q-c}) = o(1),$$

which proves the localisation of $u_{1,1}$ and $u_{1,2}$.
To prove that $u_{1,3}$ is negligible, observe that the contributing paths do not visit $Z_t^{(1)}$ and are longer than $|Z_t^{(2)}|(1+t^{-\delta/2})$ (the latter is true as they pass through $Z_t^{(2)}$ and leave $\Gamma_t^{(2)}$). Thus, they do not belong to the set A_1 (defined at the beginning of Section 4.3) and so, using Lemma 4.3, we obtain

$$\sum_{z \in \mathbb{Z}^d} u_{1,3}(t,z) \le U_2(t) + U_3(t) + U_4(t) + U_5(t) = U(t)\, o(1).$$

(2) Now we consider the complementary case $\xi(Z_t^{(2)}) > \xi(Z_t^{(1)}) - t^{q-c}$. Let us pick $B = \Gamma_t$ and $\Omega = \{Z_t^{(1)}, Z_t^{(2)}\}$. We have

$$\begin{aligned}\mathfrak{g}_{\Omega,B} &= \min\{\xi(Z_t^{(1)}), \xi(Z_t^{(2)})\} - \max_{z\in\Gamma_t\setminus\{Z_t^{(1)},Z_t^{(2)}\}} \xi(z)\\ &> \xi(Z_t^{(1)}) - t^{q-c} - \max_{z\in\Gamma_t\setminus\{Z_t^{(1)},Z_t^{(2)}\}} \xi(z) > t^{q-c/2} - t^{q-c},\end{aligned}$$

where we used our assumption on the difference between $\xi(Z_t^{(1)})$ and $\xi(Z_t^{(2)})$ and Lemma 5.4 (i) with the constant $c/2$. By Lemma 2.4 we now obtain, as $t \to \infty$,

$$\frac{\sum_{z\in\Gamma_t\setminus\{Z_t^{(1)},Z_t^{(2)}\}} u_1(t,z)}{\sum_{z\in\Gamma_t} u_1(t,z)} \le \varphi(t^{q-c/2} - t^{q-c}) = o(1)$$

Again, the denominator will only increase if we replace it by $U(t)$. For the numerator, we observe that $u_1(t,z) = 0$ for all $z \notin \Gamma_t$ as the paths corresponding to u_1 do not leave Γ_t. This completes the proof. □

Lemma 5.6. *Almost surely,*

$$\lim_{t\to\infty} \Big[U(t)^{-1} \sum_{z\in\mathbb{Z}^d} u_2(t,z)\Big] \mathbb{1}\{Z_t^{(2)} \in \Gamma_t^{(1)}\} = 0.$$

Proof. We further split u_2 into the three contributions $u_2 = u_{2,1} + u_{2,2} + u_{2,3}$, where

$$u_{2,j}(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t = z\}\mathbb{1}_{C_{2j}}\Big]$$

with

$$C_{2j} = \big\{\tau_{\{Z_t^{(1)},Z_t^{(2)}\}} > t \text{ or } \tau_{\Gamma_t^{\mathrm{c}}} \le t\big\} \cap \begin{cases} (A_1 \cup A_2 \cup A_3) \cap \{\tau_{\Gamma_t^{\mathrm{c}}} \le t\}, & j = 1,\\ (A_1 \cup A_2 \cup A_3) \cap \{\tau_{\Gamma_t^{\mathrm{c}}} > t\}, & j = 2,\\ (A_4 \cup A_5), & j = 3,\end{cases}$$

where we recall the events $A_1, \dots, A_5$ defined at the beginning of Section 4.3. Since $A_1, \dots, A_5$ are pairwise disjoint and $(\cup_{i=1}^5 A_i)^{\mathrm{c}} = \varnothing$, the sets C_{21}, C_{22} and C_{23} are pairwise disjoint as well, and their union is equal to the set

$$\big\{\tau_{\{Z_t^{(1)},Z_t^{(2)}\}} > t \text{ or } \tau_{\Gamma_t^{\mathrm{c}}} \le t\big\}$$

appearing in the definition of $u_2(t,z)$. Hence, we indeed have $u_2 = u_{2,1} + u_{2,2} + u_{2,3}$.

We argue on the event $\{Z_t^{(2)} \in \Gamma_t^{(1)}\}$, but only for $u_{2,1}(t,z)$ this condition will be essential. Each path contributing to $u_{2,1}$ leaves Γ_t and so passes through some point $z \notin \Gamma_t^{(1)} \cup \Gamma_t^{(2)}$ according to Lemma 5.4 (iv). If the path also passes through $Z_t^{(i)}$ for $i = 1$ or $i = 2$ then its length must not be less than $|Z_t^{(i)}|(1 + t^{-\delta/2})$. Hence, by Lemma 4.3,

$$\sum_{z\in\mathbb{Z}^d} u_{2,1}(t,z) \le U_2(t) + U_3(t) = U(t)o(1).$$

To bound $u_{2,2}$ we observe that as $\tau_{\Gamma_t^{\mathrm{c}}} > t$, the alternative $\tau_{\{Z_t^{(1)},Z_t^{(2)}\}} > t$ must be satisfied. Hence we can use Lemma 4.3 to get

$$\sum_{z\in\mathbb{Z}^d} u_{2,2}(t,z) \le U_3(t) = U(t)o(1).$$

Finally, to bound $u_{2,3}$ we simply use Lemma 4.3 and obtain

$$\sum_{z\in\mathbb{Z}^d} u_{2,3}(t,z) \le U_4(t) + U_5(t) = U(t)o(1),$$

which completes the proof. □

5.3 Second event: $\Phi_t(Z_t^{(1)})$ is close to $\Phi_t(Z_t^{(2)})$, but $Z_t^{(1)}$ is far from $Z_t^{(2)}$.

In this section we prove Proposition 5.2. Again, we decompose $u = u_1 + u_2$ such that u_1 is localised in $Z_t^{(1)}$ and $Z_t^{(2)}$, and that u_2 is negligible. In order to show that we further decompose u_1 and u_2 as

$$u_1(t,z) = \sum_{j=1}^{2} u_{1,j}(t,z) \quad \text{and} \quad u_2(t,z) = \sum_{j=1}^{4} u_{2,j}(t,z),$$

where the functions $u_{i,j}$ are defined by

$$u_{i,j}(t,z) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X_s)\,\mathrm{d}s\Big\}\mathbb{1}\{X_t = z\}\mathbb{1}_{C_{ij}}\Big]$$

with

$$C_{1j} = \begin{cases} \big\{\tau_{Z_t^{(1)}} \le t, \tau_{[\Gamma_t^{(1)}]^c} > t\big\}, & j=1, \\ \big\{\tau_{Z_t^{(1)}} > t, \tau_{Z_t^{(2)}} \le t, \tau_{[\Gamma_t^{(2)}]^c} > t\big\}, & j=2, \end{cases}$$

and

$$C_{2j} = \begin{cases} (A_1\cup A_2\cup A_3)\cap \big\{\tau_{Z_t^{(1)}} \le t, \tau_{[\Gamma_t^{(1)}]^c} \le t\big\}, & j=1, \\ (A_1\cup A_2\cup A_3)\cap \big\{\tau_{Z_t^{(1)}} > t, \tau_{Z_t^{(2)}} > t\big\}, & j=2, \\ (A_1\cup A_2\cup A_3)\cap \big\{\tau_{Z_t^{(1)}} > t, \tau_{Z_t^{(2)}} \le t, \tau_{[\Gamma_t^{(2)}]^c} \le t\big\}, & j=3, \\ (A_4\cup A_5)\cap (C_{11}\cup C_{12})^{\mathrm{c}}, & j=4, \end{cases}$$

where we again recall the definition of the disjoint sets $A_1,\dots,A_5$ from Section 4.3. It is easy to see that the six sets C_{11}, C_{12}, C_{21}, C_{22}, C_{23} and C_{24} are pairwise disjoint and exhaustive.

Lemma 5.7. *Almost surely,*

$$\lim_{t\to\infty}\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d\setminus\{Z_t^{(1)},Z_t^{(2)}\}} u_1(t,z)\Big]\mathbb{1}\{\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) < a_t\lambda_t/2, Z_t^{(2)} \notin \Gamma_t^{(1)}\} = 0.$$

Proof. We argue on the event $\{\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) < a_t\lambda_t/2, Z_t^{(2)} \notin \Gamma_t^{(1)}\}$. We now fix t large enough and pick $B = \Gamma_t^{(1)}$, $\Omega = \{Z_t^{(1)}\}$ to study $u_{1,1}$ and $B = \Gamma_t^{(2)}\setminus\{Z_t^{(1)}\}$, $\Omega = \{Z_t^{(2)}\}$ to study $u_{1,2}$. Since $Z_t^{(2)} \notin \Gamma_t^{(1)}$ we have for the first choice

$$\mathfrak{g}_{\Omega,B} = \xi(Z_t^{(1)}) - \max_{z\in\Gamma_t^{(1)}\setminus\{Z_t^{(1)}\}} \xi(z) \ge t^{q-c},$$

using parts (i) and (iv) of Lemma 5.4. For the second choice, we also obtain

$$\mathfrak{g}_{\Omega,B} = \xi(Z_t^{(2)}) - \max_{z\in\Gamma_t^{(2)}\setminus\{Z_t^{(1)},Z_t^{(2)}\}} \xi(z) \ge t^{q-c},$$

by Lemma 5.4 (iii) since the condition $\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) < a_t\lambda_t/2$ is satisfied. By Lemma 2.4 and using monotonicity of φ we now obtain

$$\frac{\sum_{z\in\Gamma_t^{(1)}\setminus\{Z_t^{(1)}\}} u_{1,1}(t,z)}{\sum_{z\in\Gamma_t^{(1)}} u_{1,1}(t,z)} \le \varphi(t^{q-c}) \quad \text{and} \quad \frac{\sum_{z\in\Gamma_t^{(2)}\setminus\{Z_t^{(1)},Z_t^{(2)}\}} u_{1,2}(t,z)}{\sum_{z\in\Gamma_t^{(2)}\setminus\{Z_t^{(1)}\}} u_{1,2}(t,z)} \le \varphi(t^{q-c}).$$

Increasing the denominators to $U(t)$ and taking into account the fact that $u_{1,1}(t,z)=0$ for all $z\notin\Gamma_t^{(1)}$ and $u_{1,2}(t,z)=0$ for all $z\notin\Gamma_t^{(2)}\backslash\{Z_t^{(1)}\}$ completes the proof. $\square$

Lemma 5.8. *Almost surely,*

$$\lim_{t\to\infty}\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d}u_2(t,z)\Big]=0.$$

Proof. Observe that

$$A_1\cap\Big[\big\{\tau_{Z_t^{(1)}}\le t,\tau_{[\Gamma_t^{(1)}]^{\rm c}}\le t\big\}\cup\big\{\tau_{Z_t^{(1)}}>t,\tau_{Z_t^{(2)}}>t\big\}\cup\big\{\tau_{Z_t^{(1)}}>t,\tau_{Z_t^{(2)}}\le t,\tau_{[\Gamma_t^{(2)}]^{\rm c}}\le t\big\}\Big]=\varnothing$$

and therefore the union with A_1 can be skipped in the definition of C_{21}, C_{22} and C_{23}. By Lemma 4.3 we obtain, almost surely,

$$\sum_{z\in\mathbb{Z}^d}u_{2,j}(t,z)\le U_2(t)+U_3(t)=U(t)\,o(1)\qquad\text{for } j=1,2,3$$

Note that, obviously, $\sum_{z\in\mathbb{Z}^d}u_{2,4}(t,z)\le U_4(t)+U_5(t)=U(t)\,o(1)$ almost surely. $\square$

5.4 Third event: The difference between $\Phi_t(Z_t^{(1)})$ and $\Phi_t(Z_t^{(2)})$ is large.

In this section we prove Proposition 5.3. Here we decompose $u=u_1+u_2$ and further $u_2=u_{2,1}+u_{2,2}+u_{2,3}$ where

$$u_1(t,z)=\mathbb{E}_0\Big[\exp\Big\{\int_0^t\xi(X_s)\,{\rm d}s\Big\}\mathbb{1}\{X_t=z\}\mathbb{1}\{\tau_{Z_t^{(1)}}\le t,\tau_{[\Gamma_t^{(1)}]^{\rm c}}>t\}\Big]$$
$$u_{2,j}(t,z)=\mathbb{E}_0\Big[\exp\Big\{\int_0^t\xi(X_s)\,{\rm d}s\Big\}\mathbb{1}\{X_t=z\}\mathbb{1}_{C_{2j}}\Big]$$

with

$$C_{2j}=\begin{cases}(A_1\cup A_2\cup A_3)\cap\big\{\tau_{Z_t^{(1)}}>t\big\}, & j=1,\\ (A_1\cup A_2\cup A_3)\cap\big\{\tau_{Z_t^{(1)}}\le t,\tau_{[\Gamma_t^{(1)}]^{\rm c}}\le t\big\}, & j=2,\\ (A_4\cup A_5)\cap C_1^{\rm c}, & j=3.\end{cases}$$

Again, it is easy seen that u is equal to the sum of the functions u_1 and $u_{2,1}$, $u_{2,2}$ and $u_{2,3}$.

Lemma 5.9. *Almost surely,*

$$\lim_{t\to\infty}\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d\backslash\{Z_t^{(1)}\}}u_1(t,z)\Big]\mathbb{1}\{\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})\ge a_t\lambda_t/2\}=0.$$

Proof. We fix t large enough and argue on the event $\{\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})\ge a_t\lambda_t/2\}$. Pick $B=\Gamma_t^{(1)}$, $\Omega=\{Z_t^{(1)}\}$. We have

$$\mathfrak{g}_{\Omega,B}=\xi(Z_t^{(1)})-\max_{\Gamma_t^{(1)}\backslash\{Z_t^{(1)}\}}\xi(z)\ge t^{q-c}$$

by Lemma 5.4 (ii) since the condition $\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})\ge a_t\lambda_t/2$ is satisfied. Using Lemma 2.4 we obtain

$$\frac{\sum_{z\in\Gamma_t^{(1)}\backslash\{Z_t^{(1)}\}}u_1(t,z)}{\sum_{z\in\Gamma_t^{(1)}}u_1(t,z)}\le\varphi(t^{q-c})=o(1).$$

Increasing the denominators to $U(t)$ and taking into account the fact that $u_1(t,z)=0$ for all $z\notin\Gamma_t^{(1)}$ completes the proof. $\square$

Lemma 5.10. *Almost surely,*

$$\lim_{t\to\infty}\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d}u_2(t,z)\Big]\mathbb{1}\{\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})\geq a_t\lambda_t/2\}=0.$$

Proof. We argue on the event $\{\Phi_t(Z_t^{(1)})-\Phi_t(Z_t^{(2)})\geq a_t\lambda_t/2\}$. Denote $h_t(z)=0$ and $H_t=\{Z_t^{(1)}\}$. By Proposition 4.2 and by Lemma 4.1 (b) we have

$$\begin{aligned}\frac{1}{t}\log\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d}u_{2,1}(t,z)\Big]&\leq\frac{1}{t}\log\Big[U(t)^{-1}\sum_{z\in\mathbb{Z}^d}u_{H,h}(t,z)\Big]\\&\leq\Phi_t(Z_t^{(2)})-\Phi_t(Z_t^{(1)})+O(t^{q-\delta})\leq -a_t\lambda_t/2+O(t^{q-\delta})\to-\infty.\end{aligned}$$

Further, since $A_1\cap\{\tau_{Z_t^{(1)}}\leq t,\tau_{[\Gamma_t^{(1)}]^c}\leq t\}=\varnothing$ the union with A_1 can be skipped in the definition of C_{22}. Then by Lemma 4.3 we obtain, almost surely,

$$\sum_{z\in\mathbb{Z}^d}u_{2,2}(t,z)\leq U_2(t)+U_3(t)=U(t)o(1)$$

Obviously, we also have $\sum_{z\in\mathbb{Z}^d}u_{2,3}(t,z)\leq U_4(t)+U_5(t)=U(t)o(1)$ almost surely. □

6. One-point localisation in law and concentration sites

In this section we prove Theorems 1.2 and 1.3, the convergence assertions for $u(t,Z_t^{(1)})/U(t)$ in probability and for $(Z_t^{(1)},Z_t^{(2)})/r_t$ in distribution. This easily follows from our earlier almost-sure results, using a point process convergence approach. Background on point processes and similar arguments can be found in [HMS08].

Consider the Radon measure $\mu(\,\mathrm{d}y)=\frac{\alpha\,\mathrm{d}y}{y^{\alpha+1}}$ on $(0,\infty]$ and, for any $r>0$, the point process on $\mathbb{R}^d\times(0,\infty]$ given by

$$\zeta_r=\sum_{z\in\mathbb{Z}^d}\varepsilon_{(z/r,X_{r,z})},\qquad\text{where } X_{r,z}=\frac{\xi(z)}{r^{d/\alpha}},\tag{6.1}$$

where we write ε_x for the Dirac measure in x. Furthermore, for any t, consider the point process on $\mathbb{R}^d\times(0,\infty]$ given by

$$\Pi_t=\sum_{z\in\mathbb{Z}^d\colon\,\Phi_t(z)>0}\varepsilon_{(z/r_t,\Phi_t(z)/a_t)}.$$

Finally, define a locally compact Borel set

$$H=\big\{(x,y)\in\dot{\mathbb{R}}^d\times(0,\infty]\colon y\geq q|x|/2\big\},$$

where $\dot{\mathbb{R}}^d$ is the one-point compactification of $\mathbb{R}^d$.

Lemma 6.1. *For each t, Π_t is a point process on*

$$\widehat{H}=\dot{\mathbb{R}}^{d+1}\setminus\Big(\big(\mathbb{R}^d\times(-\infty,0)\big)\cup\{(0,0)\}\Big).$$

As $t\to\infty$, Π_t converges in law to a Poisson process Π on $\widehat{H}$ with intensity measure

$$\nu(\,\mathrm{d}x,\,\mathrm{d}y)=\,\mathrm{d}x\otimes\frac{\alpha}{(y+q|x|)^{\alpha+1}}\mathbb{1}_{\{y>0\}}\,\mathrm{d}y.$$

Proof. Our first goal is to write Π_t as a suitable transformation of ζ_{r_t} on $\widehat{H}$. Introduce $H' = \dot{\mathbb{R}}^{d+1} \setminus \{0\}$ and a transformation $T_t \colon H \to H'$ given by

$$T_t(x,y) = \begin{cases} \big(x, y - q|x| - \delta(t,x,y)\big) & \text{if } x \neq \infty \text{ and } y \neq \infty, \\ \infty & \text{otherwise,} \end{cases}$$

Here δ is an error function satisfying $\delta(t,x,y) \to 0$ as $t \to \infty$ uniformly in $(x,y) \in K_n^{\mathrm{c}}$, where

$$K_n = \{(x,y) \in H \colon |y| \geq n\}.$$

Recalling that $\frac{r_t}{ta_t} = \frac{1}{\log t}$, we see that

$$\begin{aligned} \frac{\Phi_t(z)}{a_t} &= \Big[\frac{\xi(z)}{a_t} - \frac{|z|}{ta_t}\log a_t - \frac{|z|}{ta_t}\,\log\frac{\xi(z)}{a_t} + \frac{\eta(z)}{ta_t}\Big]\mathbb{1}\Big\{\frac{\xi(z)}{a_t} \geq [\log t]^{-1}\frac{|z|}{r_t}\Big\} \\ &= \Big[\frac{\xi(z)}{a_t} - (q+o(1))\,\Big|\frac{z}{r_t}\Big| - \frac{1}{\log t}\Big|\frac{z}{r_t}\Big|\log\frac{\xi(z)}{a_t} + \frac{\eta(z)}{ta_t}\Big]\mathbb{1}\Big\{\frac{\xi(z)}{a_t} \geq [\log t]^{-1}\frac{|z|}{r_t}\Big\}. \end{aligned}$$

The same fact also implies that $\frac{\eta(z)}{ta_t} \leq |\frac{z}{r_t}|\frac{\log d}{\log t}$ for all $z \in \mathbb{Z}^d$ and $t > 0$. Hence, we have

$$\Pi_t = \big(\zeta_{r_t}|_H \circ T_t^{-1}\big)\big|_{\widehat{H}} \qquad \text{eventually for all } t. \tag{6.2}$$

To show the convergence, we define the transformation $T \colon H \to H'$ by $T(x,y) = (x, y - q|x|)$ if $x \neq \infty$ and $y \neq \infty$ and $T(x,y) = \infty$ otherwise. By [HMS08, Lemma 3.7] $\zeta_r|_H$ is a point process in H converging, as $r \to \infty$, in law to a Poisson point process $\zeta|_H$ with intensity measure $\mathrm{Leb}_d \otimes \mu|_H$, where Leb_d denotes the Lebesgue measure on $\mathbb{R}^d$. Using (6.2), it now suffices to show that

$$\zeta_{r_t}|_H \circ T_t^{-1} \Longrightarrow \zeta|_H \circ T^{-1},$$

as the Poisson process on the right has the required intensity by a straightforward change of coordinates. This convergence follows from [HMS08, Lemma 2.5], provided that the conditions (*i*)-(*iii*) stated there are satisfied, which we now check.

(*i*) T is obviously continuous.

(*ii*) For each compact set $K' \subset H'$ there is an open neighbourhood V' of zero such that $K' \subset H' \setminus V'$. Since $T(x,y) \to (0,0)$ as $(x,y) \to (0,0)$ and since $T_t \to T$ uniformly on K_n^{c}, there exists an open neighbourhood $V \subset H$ of zero such that $T(V) \subset V'$ and $T_t(V) \subset V'$ for all t large enough. Hence, for $K = H \setminus V$, we obtain $T^{-1}(K') \subset T^{-1}(H' \setminus V') \subset K$ and similarly $T_t^{-1}(K') \subset K$ for all t.

(*iii*) Recall that $\delta(t,x,y) \to 0$ uniformly on K_n^{c}, and observe that

$$(\mathrm{Leb}_d \otimes \mu)(K_n) = \int_{\mathbb{R}^d} \mathrm{d}x \int_{n \vee (q|x|/2)}^{\infty} \frac{\alpha\,\mathrm{d}y}{y^{\alpha+1}} = (2/q)^{\alpha} \int_{\mathbb{R}^d} \frac{\mathrm{d}x}{((2n/q) \vee |x|)^{\alpha}} \to 0$$

as $n \to \infty$ as $|x|^{-\alpha}$ is integrable away from zero for $\alpha > d$. □

Lemma 6.2. *We have*

$$\Big(\frac{Z_t^{(1)}}{r_t}, \frac{Z_t^{(2)}}{r_t}, \frac{\Phi_t(Z_t^{(1)})}{a_t}, \frac{\Phi_t(Z_t^{(2)})}{a_t}\Big) \Rightarrow \big(X^{(1)}, X^{(2)}, Y^{(1)}, Y^{(2)}\big),$$

where the limit random variable has the density

$$p(x_1, x_2, y_1, y_2) = \frac{\alpha^2 \exp\{-\theta y_2^{d-\alpha}\}}{(y_1 + q|x_1|)^{\alpha+1}(y_2 + q|x_2|)^{\alpha+1}}\,\mathbb{1}\{y_1 \geq y_2\}.$$

Proof. It has been computed in the proof of [HMS08, Prop. 3.8] that $\nu(\mathbb{R}^d \times (y,\infty)) = \theta y^{d-\alpha}$ for $y > 0$. For any relative compact set $A \subset \widehat{H} \times \widehat{H}$ such that $\mathrm{Leb}_{2d+2}(\partial A) = 0$, we obtain by Lemma 6.1,

$$\begin{aligned}
&\mathrm{Prob}\Big(\big(\tfrac{Z_t^{(1)}}{r_t}, \tfrac{Z_t^{(2)}}{r_t}, \tfrac{\Phi_t(Z_t^{(1)})}{a_t}, \tfrac{\Phi_t(Z_t^{(2)})}{a_t}\big) \in A\Big) \\
&\quad = \int_A \mathrm{Prob}\big(\Pi_t(\,\mathrm{d}x_1 \times \,\mathrm{d}y_1) = \Pi_t(\,\mathrm{d}x_2 \times \,\mathrm{d}y_2) = 1, \\
&\qquad\qquad\qquad \Pi_t(\mathbb{R}^d \times (y_1,\infty)) = \Pi_t(\mathbb{R}^d \times (y_2, y_1)) = 0\big) \\
&\quad \to \int_A \mathrm{Prob}\big(\Pi(\,\mathrm{d}x_1 \times \,\mathrm{d}y_1) = 1\big)\,\mathrm{Prob}\big(\Pi(\,\mathrm{d}x_2 \times \,\mathrm{d}y_2) = 1\big) \\
&\qquad\qquad\qquad \mathrm{Prob}\big(\Pi(\mathbb{R}^d \times (y_1,\infty)) = 0\big)\,\mathrm{Prob}\big(\Pi(\mathbb{R}^d \times (y_2, y_1)) = 0\big) \\
&\quad = \int_A \nu(\mathbb{R}^d \times (y_2,\infty))\,\nu(\,\mathrm{d}x_1,\,\mathrm{d}y_1)\nu(\,\mathrm{d}x_2,\,\mathrm{d}y_2) = \int_A p(x_1,x_2,y_1,y_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2.
\end{aligned}$$

It remains to notice that

$$\int\limits_{\mathbb{R}^d \times \mathbb{R}^d \times \{(y_1 > y_2 > 0)\}} p(x_1,x_2,y_1,y_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2 = \mathrm{Prob}\big(\Pi(\mathbb{R}^d \times (0,\infty)) \geq 2\big) = 1$$

since $\Pi(\mathbb{R}^d \times (0,\infty)) = \infty$ with probability one. □

Proof of Theorem 1.2. We use the same decomposition $u(t,z) = u_1(t,z) + u_2(t,z)$ as we used to prove Proposition 5.3. By Lemmas 5.9 and 5.10 it suffices to show that

$$\lim_{t\to\infty} \mathrm{Prob}\,\big(\Phi_t(Z_t^{(1)}) - \Phi_t(Z_t^{(2)}) \geq a_t\lambda_t/2\big) = 1. \tag{6.3}$$

Since, by Lemma 6.2, $(\Phi_t(Z_t^{(1)})/a_t, \Phi_t(Z_t^{(2)})/a_t)$ converges weakly to a random variable $(Y^{(1)}, Y^{(2)})$ with density, we obtain (6.3) because $\lambda_t \to 0$. □

Proof of Theorem 1.3. The result follows from Lemma 6.2 by integrating the density function $p(x_1,x_2,y_1,y_2)$ over all possible values of y_1 and y_2. We obtain

$$p(x_1,x_2) = \int\limits_{\{y_1 > y_2 > 0\}} \frac{\alpha^2 \exp\{-\theta y_2^{d-\alpha}\}\,\mathrm{d}y_1\,\mathrm{d}y_2}{(y_1 + q|x_1|)^{\alpha+1}(y_2 + q|x_2|)^{\alpha+1}} = \int_0^\infty \frac{\alpha \exp\{-\theta y^{d-\alpha}\}\,\mathrm{d}y}{(y + q|x_1|)^{\alpha}(y + q|x_2|)^{\alpha+1}}.$$

This completes the proof of Theorem 1.3. □

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