Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

Eigenvalue order statistics and mass concentration in the parabolic Anderson model

Based on joint works with Marek Biskup (UC Los Angeles) and Renato dos Santos (NYU Shanghai)

Wolfgang König

WIAS Berlin and TU Berlin

Definition

Let Δ be the standard Laplace operator on $\mathbb{Z}^d$, and let $\xi = (\xi(z))_{z\in\mathbb{Z}^d}$ be a random potential, then $\Delta + \xi$ is called a *random Schrödinger operator*.
Explicitly, $(\Delta + \xi)f(z) = \Delta f(z) + \xi(z)f(z)$ for L^2-functions f.

Great mathematical interest in the eigenfunctions stems from the famous

Prediction by P.W. ANDERSON (1958):

Anderson localisation: In a large part of the spectrum of $\Delta + \xi$, all values should be eigenvalues, and the corresponding eigenfunction should be exponentially localised around some (randomly distributed) site.

- Anderson localisation has been confirmed for many random potentials ξ for spectral values close to the boundary of the spectrum, or for $\Delta + \beta\xi$ if $|\beta|$ is large enough.
- Two proof methods (1990s, early 2000s)): *Fractional moment method* and *Multiscale analysis*.

We are interested in the upper edge of the spectrum of $\Delta + \xi$ in a large box B, i.e., in the principal part, including the principal eigenvalue (with zero Dirichlet boundary condition),

$$\lambda_1(B) = \sup\big\{\langle g, (\Delta+\xi)g\rangle \colon g \in \ell^2(\mathbb{Z}^d), \mathrm{supp}(g) \subset B, \|g\|_2 = 1\big\}$$
$$= -\inf\Big\{\|\nabla g\|_2^2 - \sum_z \xi(z) g^2(z) \colon g \in \ell^2(\mathbb{Z}^d), \mathrm{supp}(g) \subset B, \|g\|_2 = 1\Big\}.$$

Introduce all the eigenvalues, $\lambda_1(B) > \lambda_2(B) \geq \lambda_3(B) \geq \dots \geq \lambda_{|B|}(B)$.

Our questions:

- What is the upper-tail behaviour of $\lambda_1(B)$, in particular when coupled with $|B| \to \infty$?
- Is there an extreme-value order statistics for the top eigenvalues in this limit?
- What is the domain of attraction, what are the scaling parameters?
- Does the point process $\sum_{k=1}^{|B|} \delta_{\lambda_k(B)}$ converge, after normalisation?
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Recall: If $M_N = \max\{X_1, \dots, X_N\}$ is the maximum of N i.i.d. random variables, then, if some a_N, b_N exist such that $(M_N - a_N)/b_N$ converges towards a non-degenerate variable in law, then this is either Gumbel, or Fréchet or Weibull. Also $\sum_{k=1}^{N} \delta_{(X_k - a_N)/b_N}$ converges.

Heat equation with random potential; parabolic Anderson model (PAM):

$$\frac{\partial}{\partial t}u(t,z) = \Delta u(t,z) + \xi(z)u(t,z), \qquad \text{for } (t,z) \in (0,\infty)\times\mathbb{Z}^d, \tag{1}$$

$$u(0,z) = \delta_0(z), \qquad \text{for } z \in \mathbb{Z}^d. \tag{2}$$

Interpretations / Motivations:

- Random mass transport through a random field of sinks and sources.
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Eigenvalue expansion

$$u(t,z) \sim u_{B_t}(t,z) = \sum_k \mathrm{e}^{t\lambda_k(B_t)} \varphi_k(0)\varphi_k(z),$$

where $\varphi_1, \varphi_2, \varphi_3 \dots$ are the corresponding orthonormal eigenfunctions in $B_t = t \times [-\frac{1}{2}, \frac{1}{2}]^d$.

- In the limit $t \to \infty$, it is by far not automatic that only $\lambda_1(B_t)$ survives. Rather, the maximum of $\mathrm{e}^{t\lambda_k(B_t)}\, \varphi_k(0)$ over k will be decisive. Therefore, we must know the joint behaviour of *all* top eigenvalues.

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- All the eigenfunctions φ_k decay exponentially fast away from some random site x_k. Then the distance $|x_k - 0|$ determines the value of $\varphi_k(0) \approx \mathrm{e}^{-c|x_k|}$.
- The random sites x_k form (after rescaling) a homogeneous Poisson point process, and the eigenvalues $\lambda_k(B_t)$ behave like i.i.d.

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- Earlier work on the asymptotics of the expectation of $e^{t\lambda(B_t)}$ showed [Gärtner, K., Molchanov 2007] that the typical shape of the eigenfunction approaches a deterministic shape, which is given as the maximizer of $\varphi \mapsto \lambda(\varphi) - I(\varphi)$, where $I(\varphi)$ describes the rate function for the probability that the potential looks like φ on a large scale.

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- All the details of the above heavily depend on the upper-tail behaviour of $\xi(0)$, i.e., on the asymptotics of $\mathrm{Prob}(\xi(0) > r)$ as $r \to \mathrm{esssup}\,\xi(0)$.

- Surveys: [K. 16] and [Astrauskas 16]
- Eigenvalue order statistics and Poisson point process convergence at spectral top (single-site eigenfunctions): [Astrauskas 08, 12, 13]
- Eigenvalue order statistics and Poisson point process convergence at spectral top (non-generate eigenfunctions, double-exp. dist) [Biskup, K. 16], see below
- Poisson point process convergence in Anderson localisation regime: [Molchanov 81], [Minami 96], [Kilip, Nakano 07], [Germinet, Klopp 13, 14]
- Concentration of the PAM in one single site: [K., Lacoin, Mörters, Sidorova 09], [Lacoin, Mörters 12], [Fiodorov, Muirhead 14], [Sidorova, Twarowski 16]
- Concentration of the PAM in one non-degenerate island: [Biskup, K., dos Santos 18], see below, related partial result [Ding, Xu 18] for bounded potential
- first steps for white noise in $d = 2$ by [Allez/Chouk], [Chouk/van Zuijlen (2019)] and [Perkowski/K./van Zuijlen (2019+)].

Open:

Eigenvalue order statistics, Poisson point process convergence at spectral top for bounded potentials, and concentration of the PAM in one island for other potential distributions, e.g. Gaussian fields in $\mathbb{R}^d$ (smooth or white noise).

We are working here for ξ double-exponentially distributed, i.e., for some $\varrho \in (0,\infty)$,

$$\mathrm{Prob}(\xi(0) > r) = \exp\big\{ -\mathrm{e}^{r/\varrho}\big\}, \qquad r \in \mathbb{R}.$$

Theorem 1 [Biskup/K., CMP 16]

There is a number $\chi = \chi_\varrho \in (0, 2d)$ and a sequence $(a_L)_{L\in\mathbb{N}}$ with $a_L = \varrho \log\log |B_L| - \chi + o(1)$ as $L \to \infty$ and, for any $L \in \mathbb{N}$, a sequence $(X_k^{(L)})_k$ in B_L such that, in probability,

$$\lim_{L\to\infty} \sum_{z\colon |z - X_k^{(L)}| \le \log L} \varphi_k(z)^2 = 1, \qquad k \in \mathbb{N},$$

and the law of

$$\sum_{k\in\mathbb{N}} \delta_{\left(\frac{1}{L} X_k^{(L)}, (\lambda_k(B_L) - a_L)\log L\right)}$$

converges weakly to a Poisson process on $B_1 \times \mathbb{R}$ with intensity measure $\mathrm{d}x \otimes \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$.

- Hence, the top eigenvalues in B_L are of order $\log\log L$ and leave gaps of order $1/\log L$ (rather than $1/|B_L|$ as in the bulk of the spectrum).
- The localisation centres are separated by $\asymp L$ and are homogeneously distributed.
- We have an assertion reminding on Anderson localisation at the edge of the spectrum.

Theorem 2 [Biskup, K., dos Santos, PTRF 18]

Put $r_L = L\log L \log\log\log L$ and

$$\Psi_{L,t}(z,\lambda) = \frac{t}{r_L}(\lambda - a_L)\log L - \frac{|z|}{L},$$

and pick k such that $\Psi_{L,t}(X_k^{(L)}, \lambda_k(B_L))$ is maximal. Put $Z_t = X_k^{(L)}$.
Then, with L_t defined by $r_{L_t} = t$, for any $R_t \gg \log t$,

$$\lim_{R\to\infty}\lim_{t\to\infty}\frac{1}{U(t)}\sum_{z\colon |z-Z_t|\le R} u(t,z) = 1 \qquad \text{in probability.}$$

- Hence, the total mass essentially comes from a single $\gg \log t$-island in the centred box with radius $L_t \asymp t/(\log t \log\log\log t)$.
- The Poisson process convergence holds only in the box B_{L_t}, and the contribution from $B^{\mathrm{c}}_{t\log^2 t}$ is easily seen to be negligible. The intermediate region is delicate.
- The two terms in $\Psi_{L,t}$ come from the eigenvalue and the probabilistic cost for the random walk in the Feynman-Kac formula. The choice of r_L comes from an optimisation of $\mathbb{P}_0(|X_s| \asymp L)\mathrm{e}^{(t-s)\lambda_1} \approx \mathrm{e}^{-L\log(L/s)}\mathrm{e}^{(t-s)a_L}$ over $s\in[0,t]$.

A system is said to age if its significant changes come after longer and longer (or shorter and shorter) time lags.
Hence, one can see from the frequency of changes how much time has elapsed.

Ageing properties of the PAM can now be studied in terms of the time lags between jumps of the concentration site.

These ones, in turn, may be described as follows.

Theorem 3: Scaling limit of concentration location [BKdS 18]

As $t \to \infty$, the process $(Z_{\theta t}/L_t)_{\theta \in [1,\infty)}$ converges in distribution to a process $(\overline{Z}(\theta))_{\theta \in [1,\infty)}$, whose marginals $\overline{Z}(\theta)$ have d independent components, which are centered and Laplace-distributed (i.e., with density $z \mapsto \mathrm{e}^{-|z|/\theta}$).
Furthermore, $(Z_t)_{t \in [0,\infty)}$ is aging in the sense that, for any $s > 0$,

$$\lim_{t \to \infty} \mathrm{Prob}\Big(Z_t = Z_{t+\theta t} \text{ for every } \theta \in [0, s] \Big)$$

exists and is a non-trivial function of s.

Put $\varepsilon_R = 2d\Big(1+\frac{A}{2d}\Big)^{1-2R}$.

The top eigenvalues in $B = B_L$ remain the top eigenvalues after discarding potential values significantly less than the eigenvalues.

Fix $A > 0$ and $R \in \mathbb{N}$ and put $U = \bigcup_{z\in B\colon \xi(z)\geq\lambda_1(B)-2A} B_R(z)$. Then

$$\lambda_k(B) \geq \lambda_1(B) - A/2 \qquad \Longrightarrow \qquad |\lambda_k(B) - \lambda_k(U)| \leq \varepsilon_R.$$

- The corresponding ℓ^2-normalized eigenvector $\varphi = \varphi_k$ decays rapidly away from U.
- Proof uses the martingale $(\varphi(Y_n)\prod_{l=0}^{n-1}\frac{2d}{2d+\lambda-\xi(Y_l)})_{n\in\mathbb{N}}$ (with $(Y_n)_n$ an SRW).

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- Furthermore, we use that $\partial_{\xi(z)} \lambda_k(B) = \varphi(z)^2$.
- Introduce $\xi_s = \xi - s\mathbb{1}_{B \setminus U}$ for $s \in [0, \infty]$. Then

$$|\partial_s \lambda_k(\xi_s, B)| = \sum_{z \in B \setminus U} \varphi_{k,\xi_s}(z)^2,$$

which is very small. Integrating over $s \in [0, \infty]$ gives the estimate.

The top eigenvalues are the principal eigenvalues in local regions, and the corresponding eigenfunctions are exponentially localised.

A bit more precisely, with the help of the variational characterisation of the asymptotics of the PAM [GÄRTNER/K./MOLCHANOV 07], one proves the following.

- U consists of connected components of bounded size, which are far away from each other.

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- For any component C, if $\lambda_1(C)$ is close to $a_L \approx \rho \log \log L$, then $\lambda_1(C)$ is bounded away from $\lambda_2(C)$.

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- If λ is an eigenvalue of $\Delta + \xi$ larger than $\lambda_1(B_L) - A/2$ and φ a corresponding ℓ^2-normalised eigenfunction such that the distance of λ to the nearest eigenvalue (spectral gap) is larger than $3\varepsilon_R$, then φ decays exponentially away from one of the components of U.

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- The proof uses that the path $[0, \infty] \ni s \mapsto \lambda_k(\xi_s, B_L)$ (with $\xi_s = \xi - s\mathbb{1}_{B_L \setminus U}$) does not cross other eigenvalues and therefore admits a continuous choice of corresponding eigenfunctions. The one for $s = \infty$ puts all its mass in one component, and the one for $s = 0$ is uniformly close.

The scale a_L satisfies $\mathrm{Prob}(\lambda_1(B_R) > a_L) = 1/|B_L|$.
Hence we may expect finitely many sites in B_L where the local eigenvalue is $\approx a_L$.

$\lambda_1(B_R)$ lies in the max-domain of a Gumbel random variable

As $L \to \infty$, for any $s \in \mathbb{R}$,

$$\mathrm{Prob}(\lambda_1(B_R) > a_L + s/\log L) = \mathrm{e}^{-s}\frac{1}{|B_L|}(1 + o(1)).$$

- The event $\{\lambda_1(B_R) > a\}$ is more or less the same as the event that some shift of the potential $\xi(\cdot)$ is larger than $a + \chi + \psi(\cdot)$ for some well-chosen function ψ.

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- Shifting ξ by an amount of $s/\log L$ yields an additional factor of e^{-s}, using properties of ψ and of the distribution of ξ and some information from the variational characterisation.