ROUTEING PROPERTIES IN A GIBBSIAN MODEL
FOR HIGHLY DENSE MULTIHOP NETWORKS

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Abstract: We investigate a probabilistic model for routeing in a multihop ad-hoc communication network, where each user sends a message to the base station. Messages travel in hops via the other users, used as relays. Their trajectories are chosen at random according to a Gibbs distribution that favours trajectories with low interference, measured in terms of sum of the signal-to-interference ratios for all the hops, and collections of trajectories with little total congestion, measured in terms of the number of pairs of hops arriving at each relay. This model was introduced in our earlier paper [KT17], where we expressed, in the limit of a high density of users, the distribution of the optimal trajectories as the minimizer of a characteristic variational formula.

In the present work, in the special case in which congestion is not penalized, we derive qualitative properties of this minimizer. We encounter and analytically identify the emerging typical scenarios in three extreme regimes. We analyse the typical number of hops and the typical length of a hop, and the deviation of the trajectory from the straight line in two regimes: (1) in the limit of a large communication area and large distances, and (2) in the limit of a strong interference weight. In both regimes, the typical trajectory turns out to quickly approach a straight line, in regime (1) with hops of equal lengths. Interestingly, in regime (1), the typical length of a hop diverges logarithmically as the distance of the transmitter to the base station diverges. We further analyse (3) the local and global repulsive effect of a densely populated subarea on the trajectories. Our findings are illustrated by numerical examples. We also discuss a game-theoretic relation of our Gibbsian model with a joint optimization of message trajectories opposite to a selfish optimization, in case congestion is also penalized.

1. INTRODUCTION

In this work, we continue our research [KT17] on a spatial Gibbsian model for message routeing in a multihop ad-hoc network. In [KT17] we prepared for an analysis of the qualitative properties of the model by deriving simplifying formulas that describe the situation in a densely populated area in the sense of a law of large numbers. In the present work, we carry out this analysis and describe a

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number of characteristic properties of the message trajectories. In particular, we are interested in the interplay between probabilistic properties like *entropy* and energetic properties like *interference* and *congestion* and how this interplay develops geometric properties like number and lengths of the hops or straightness of the trajectories. Our goal is to identify some rules of thumbs in the relationships between all these quantities in asymptotic regimes in which they become particularly pronounced, like large areas and long trajectories, strong influence of interference, or local regions with a particularly high population. While our previous paper used mainly probabilistic methods, the present paper entirely employs analytic tools.

1.1. **The main features of the model.** Let us introduce our telecommunication model. The communication area is a bounded set in $\mathbb{R}^d$, and it has a unique base station at the origin o. Many users are randomly distributed according to some measure. Each user sends out a message to the base station along a random multihop trajectory that uses other users as relays and has at most $k_{\max}$ hops. What we are interested in is a joint distribution of all these message trajectories, conditional on the (random) locations of the users. It is based on a *Gibbsian ansatz*: the trajectories are *a priori* distributed independently and uniformly (i.e., each message chooses first a hop number $k \in \{1, \ldots, k_{\max}\}$ and then a k-hop trajectory, both uniformly at random), and there are two exponential weight terms that penalize *interference* and *congestion*, respectively. More precisely, the first one weights the interference of each of the hops, measured in terms of the well-known *signal-to-interference ratio (SIR)*, while the other term penalizes the entire trajectory collection for high total congestion in the system, measured as constant times the sum of the number of pairs of incoming hops at the relays.

The highest probability is attached to those trajectory families that realize the best compromise between entropy (i.e., probability) and energy (i.e., interference plus congestion). In Section 6.2, we give a thorough discussion of the motivation for studying such a type of model, in particular the randomness of the trajectories.

The interesting feature is that the total penalty is given to the entire trajectory collection in terms of a probability weight, in the spirit of a common welfare, instead of selfish routeing optimization. In Section 7, we give a game-theoretic discussion of the two weight terms in the exponent in the light of traffic theory; more precisely we ask under what circumstances the optimization of the sum of these two terms can be called selfish or non-selfish. In Section 6.2.3, we also make a connection between this optimization and our model from the viewpoint of stochastic algorithms. According to the results of [M18], realizing our Gibbsian system numerically using Monte Carlo Markov chain methods is on average much more effective than finding the optimum. This gives another motivation for our model.

The idea of an optimal trade-off between entropy and energy is most clearly realized in a certain limiting sense in [KT17, Theorem 1.4], which will be the starting point of the present paper and will be summarized in Section 2. There, we carried out the limit of a high density of users, and we derived a kind of law of large numbers for the "typical" trajectory distribution, i.e., the joint trajectory distribution that has the highest probability under the Gibbs measure. This was done using large deviations theory, and the optimal trajectory collection was obtained as the minimizer(s) of a characteristic *variational formula* in [KT17, Theorem 1.2]. Roughly speaking, the variational formula is of the form "minimize the sum of entropy and energy among all admissible trajectory families".

1.2. **Goals.** Our goal in the present paper is to understand the global effects that are induced in the Gibbsian system exclusively by entropy and energy into geometric properties of the trajectory collection. As our model depends on various parameters (size and form of the communication area, density of users, choice of the interference term, strength of interference and congestion weighting, etc.), this can be done rigorously only in certain *limiting* regimes, namely:

(1) large communication area and long distances (and large hop numbers),
(2) strong interference penalization, and
(3) high local density of users on a subset of the communication area.

We are interested in geometric properties such as typical hop lengths, average number of hops, and the typical shape of the trajectory. In regimes (1) and (2), we expect that the typical trajectories approach straight lines, and in (1) there is an additional question about the typical length of a hop and the number of hops. Here, we would like to understand how the quality of service becomes bad in a large telecommunication area and how many and how large hops the messages would like to make if the constraint $k_{\max}$ on the maximum number of hops is dropped.

However, the regime (3) and our questions here are of a different nature. We would like to determine if the presence of a subarea with a particularly high population density has a significant (positive or negative) impact on the effective use of the relaying system: on the one hand, the trajectories have more available relays in such an area, but on the other hand, the interference achieves high values there. This is a trade-off between entropy and energy that we want to understand.

Let us point out that we are going to work on these questions only in the case where only interference is penalized, but not congestion. We decided this because the description of the minimizer(s) of the variational formula in [KT17, Proposition 1.3] is enormously implicit and cumbersome in general, but reduces, if the congestion term is dropped, to relatively simple formulas that are amenable for analytical investigations [KT17, Proposition 1.5]. In particular, only in this setting we know that the minimizer is unique. We believe that the main qualitative properties persist to the case where also congestion is penalized, as this is purely combinatorial and not spatial. In Section 7 we will discuss non-selfishness and other game-theoretic properties of the Gibbsian model in the presence of both terms.

1.3. **Our findings.** In regimes (1)–(2), we will see that the typical trajectory follows a straight line with exponential decay of probabilities of macroscopic deviations from this shape. Moreover, in regime (1) we will also find simple formulas for the asymptotic number of hops and the average length of a hop, which turns out to be the same for each hop of the trajectory. One of our most striking findings is that, in regime (1), the typical hop length diverges as a power of the logarithm of the distance between the transmitter and the base station, and hence the typical number of hops is sub-linear in the distance. This effect seems to come from the facts that the total mass of the intensity measure of the communication area diverges and that, *a priori*, i.e., before switching on the interference weight, every message trajectory of a given length has the same weight, even very unreasonable ones that have long spatial detours, e.g., many loops.

However, in regime (3), we encounter different effects. First we see the following global effect on the total number of relaying hops in the entire system: if the communication area is small (in the sense that all the interferences in the system do not vary much), then the total number of relaying hops vanishes exponentially fast in the diverging parameter of the dense population, regardless of the choice of the densely populated subset, as long as it has positive Lebesgue measure. In some cases, we also detect a local effect on the relaying hops if the densely populated subset is very small: we demonstrate that a certain neighbourhood of that subset is definitely unfavourable for relaying hops for practically all the other users. This is a very clear effect coming from the high interference of the densely populated area, which expels the trajectories away.

Some of our results are easy to guess, and the main value of our work is the explicit characterization of the quantities and the derivation of exponential bounds for deviations. We formulated our results in quite simple settings, by putting the communication area equal to a ball and the user density equal to the Lebesgue measure, but it is clear that they can be extended into various directions with respect to more complex shapes and/or user distributions.

Based on our explicit formulas, we also provide simulations in Section 8. They illustrate that most of the effects that we derived analytically in limiting settings, i.e., for large values of the parameters, already appear in a very pronounced way for quite moderate values of the parameters.

1.4. **Related literature.** The quality of service in highly dense relay-augmented ad-hoc networks has received particular interest in the last years. A multihop network with users distributed according to a Poisson point process with diverging intensity was investigated in [HJKP18]. Using large deviations methods, that paper derives the asymptotic behaviour of rare frustration events such as many users having an unlikely bad quality of service for an unusually long period of time. [HJP18] also describes frustration probabilities in a network, where relays have a bounded capacity, and users become frustrated when their connection to a relay is refused because it is already occupied; see also [HJ17].

One difference between these works and the Gibbsian model of the present paper introduced in [KT17] is that the latter one uses a notion of quality of service for the entire system rather than for single transmissions. In particular, trajectories with bad SIR are *a priori* not excluded. There is a random mechanism for choosing the message trajectories of all users, given the user locations, and our results hold almost surely with respect to the point process of user locations in the high-density limit. For these results, users need not form a Poisson point process, and they can even be located deterministically [KT17, Section 1.7.4]. This is also a difference from [HJKP18, HJP18, HJ17], where user locations are not fixed and their randomness is (at least partially) responsible for unlikely frustration events.

For literature remarks on the notion and use of SIR, in particular for multiple hops, and about the interference penalization term, see Section 6.1 later.

Gibbs sampling was used for various aspects of modelling telecommunication networks, e.g., in [CBK16] for optimal placement of contents in a cellular network, and in [BC12] for power control and associating users to base stations. These Monte Carlo Markov chain methods are used to decrease some kind of cost in the system via a random mechanism, with no easily implementable deterministic methods being available. Our Gibbsian model also has this property if both interference and congestion are penalized. The recent master's thesis of Morgenstern [M18] investigated the use of a Gibbs sampler or a Metropolis algorithm for an experimental realization of our Gibbsian system; see Section 6.2.3 for a summary.

1.5. **Organization of this paper.** In Section 2, we present our Gibbsian model and the results of [KT17] that are relevant for the investigations of the current paper.

Each of the following three sections is devoted to one of our three theoretical investigations, which form the core of this paper, i.e., the analysis of the large-distance limit (1) in Section 3, the limit of strong interference penalization (2) in Section 4 and the limit of high local density of users (3) in Section 5. Each of these sections gives the question, the results, the proofs and a discussion in the respective setting.

Section 6 contains modelling discussions and conclusions. Here we discuss the notion of SIR and the relevance of the interference and congestion penalty terms, provide motivations for the Gibbsian ansatz and sketch some possible extensions of the model.

Section 7 discusses the relevance and properties of our Gibbsian model and the related optimization problem in the light of game-theoretic considerations in traffic theory.

Finally, Section 8 gives numerical plots and studies about qualitative properties of our model.

2. The Gibbsian model and its behaviour in the high-density limit

In this section, we recall the Gibbsian model of [KT17] and its properties in the limit of high density of users. We present the model in Section 2.1, describe its behaviour in the high-density limit in Section 2.2 and comment on the notion of the typical trajectory sent out by a user in Section 2.3. The main objects we will consider in this paper are defined in Sections 2.2 and 2.3, while the nomenclature and interpretation of these objects originate from the preceding Section 2.1.

2.1. **The Gibbsian model.**
We introduce the model that we study in the present paper. This model was introduced in [KT17, Section 1.2.4]; it is a special case of the general model of [KT17].

For any $n \in \mathbb{N}$ and for any measurable subset V of $\mathbb{R}^n$, let $\mathcal{M}(V)$ denote the set of all finite nonnegative Borel measures on V. We write $[k] = \{1, \dots, k\}$ for $k \in \mathbb{N}$.

We are working in $\mathbb{R}^d$ with $d \in \mathbb{N}$ fixed. Let $W \subset \mathbb{R}^d$ be compact, the area of the telecommunication system, containing the origin o of $\mathbb{R}^d$. Let $\mu \in \mathcal{M}(W)$ be an absolutely continuous measure on W with $\mu(W) > 0$. For $\lambda > 0$, we let $X^\lambda = (X_i)_{i \in I^\lambda} = (X_i)_{i=1}^{N(\lambda)}$ be a Poisson point process in W with intensity measure $\lambda\mu$. We refer to the X_i as to the users of the wireless network.

For $i \in I^\lambda$ and $k \in [k_{\max}]$, we call a vector of the form

$$s^i = (s^i_{-1})_{i=-1}^k \in \{k\} \times \{X_i\} \times W^{k-1} \times \{o\} \tag{2.1}$$

with $s^i_{-1} = k$, $s^i_0 = X_i$, $s^i_j \in X^\lambda$ for all $j \in [k-1]$ and $s^i_k = o$ a *message trajectory* from X_i to o with $s^i_{-1} = k$ hops. That is, s^i starts from X_i and ends in o after k hops. For $l \in [k-1]$, the user s^i_l is called the *lth relay* of the trajectory s^i. We write $\mathcal{S}^i_{k_{\max}}(X^\lambda) = \bigcup_{k \in [k_{\max}]}(\{k\} \times \{X_i\} \times W^{k-1} \times \{o\})$. The trajectory of the unique message submitted by X_i to o will be an $\mathcal{S}^i_{k_{\max}}(X^\lambda)$-valued random variable, in particular, its number of hops will also be random. We further define $\mathcal{S}_{k_{\max}}(X^\lambda) = \prod_{i \in I^\lambda} \mathcal{S}^i_{k_{\max}}(X^\lambda)$.

With this definition of message trajectories, we only consider *uplink communication*, i.e., users transmitting messages to the base station. The *downlink*, i.e., the reversed direction, works very similarly. We believe that all the results of [KT17] as well as the ones of the present paper have an analogue for the downlink with an analogous proof, and we refrain from presenting details.

Next, interference is introduced as follows. We choose a *path-loss function*, which describes the propagation of signal strength over distance. This is a monotone decreasing, continuous function $\ell \colon [0, \infty) \to (0, \infty)$. A typical choice is ℓ corresponding to ideal Hertzian propagation, i.e. $\ell(r) = \min\{1, r^{-\alpha}\}$, for some $\alpha > 0$ (see e.g. [GT08, Section II.]). The *signal-to-interference ratio (SIR)* of a transmission from $X_i \in X^\lambda$ to $x \in W$ in the presence of the users in X^λ is defined [HJKP18] as

$$\mathrm{SIR}(X_i, x, X^\lambda) = \frac{\ell(|X_i - x|)}{\frac{1}{\lambda} \sum_{j \in I^\lambda} \ell(|X_j - x|)}. \tag{2.2}$$

The denominator of the r.h.s of (2.2) is the *interference*. Now, given a trajectory collection $s = (s^i)_{i \in I^\lambda} \in \mathcal{S}_{k_{\max}}(X^\lambda)$, we put

$$\mathfrak{S}(s) = \sum_{i \in I^\lambda} \sum_{l=1}^{s^i_{-1}} \mathrm{SIR}(s^i_{l-1}, s^i_l, X^\lambda)^{-1}. \tag{2.3}$$

For a discussion about the notion of SIR and the normalization term $1/\lambda$ in the interference respectively for an interpretation of $\mathfrak{S}(\cdot)$, we refer the reader to Section 6.1.1 respectively 6.1.2.

Next, congestion is defined as follows. For $s \in \mathcal{S}_{k_{\max}}(X^\lambda)$ we put

$$m_i(s) = \sum_{j \in I^\lambda} \sum_{l=1}^{s^j_{-1}-1} \mathbb{1}\{s^j_l = s^i_0\}, \quad i \in I^\lambda, \tag{2.4}$$

as the number of times user $s_0^i = X_i$ is used as a relay in the trajectory collection s, and we define

$$\mathfrak{M}(s) = \sum_{i \in I^\lambda} m_i(s)(m_i(s) - 1). \tag{2.5}$$

Note that $m_i(s)(m_i(s) - 1)$ is the number of ordered pairs of hops arriving at the relay $X_i = s_0^i$, and $m_i(s)(m_i(s) - 1) = 0$ if $m_i(s) \in \{0, 1\}$, i.e., $\mathfrak{M}(\cdot)$ only penalizes multiple hops arriving at the same relay.

These definitions of $\mathfrak{S}(\cdot)$ and $\mathfrak{M}(\cdot)$ yield a snap-shot type model where all messages are transmitted, relayed and received in the same time. In Section 6.1.3, we will discuss the possibility of developing a time-dependent version of this model.

The central object studied in [KT17] is the following Gibbs distribution on the set of collections of trajectories. For any $s = (s^i)_{i \in I^\lambda} \in \mathcal{S}_{k_{\max}}(X^\lambda)$ put

$$\mathrm{P}_{\lambda, X^\lambda}^{\gamma, \beta}(s) := \frac{1}{Z_\lambda^{\gamma, \beta}(X^\lambda)} \Big(\prod_{i \in I^\lambda} \frac{1}{N(\lambda)^{s_{-1}^i - 1}} \Big) \exp \Big\{ -\gamma \mathfrak{S}(s) - \beta \mathfrak{M}(s) \Big\}, \tag{2.6}$$

where $\gamma > 0$ and $\beta \geq 0$ are parameters. This is the Gibbs distribution with a uniform and independent *a priori* measure (see [KT17, Section 1.2.2] for details), subject to an exponential weight with the interference term in (2.3) and the congestion term in (2.5). Here

$$Z_\lambda^{\gamma, \beta}(X^\lambda) = \sum_{r \in \mathcal{S}_{k_{\max}}(X^\lambda)} \Big(\prod_{i \in I^\lambda} \frac{1}{N(\lambda)^{r_{-1}^i - 1}} \Big) \exp \Big\{ -\gamma \mathfrak{S}(r) - \beta \mathfrak{M}(r) \Big\} \tag{2.7}$$

is the normalizing constant, which is referred to as *partition function*. Note that $\mathrm{P}_{\lambda, X^\lambda}^{\gamma, \beta}(\cdot)$ is random and defined conditional on X^λ, and it is a probability measure on $\mathcal{S}_{k_{\max}}(X^\lambda)$.

2.2. **The limiting behaviour of the telecommunication system.** We study the above wireless communication system in the *high-density limit* $\lambda \to \infty$. As we already mentioned in Section 1.2, for the main part of the present paper, we will focus on the case $\beta = 0$ where congestion is not penalized. The case $\beta > 0$ will be considered again in the context of the relation of our model to Monte Carlo Markov chains in Section 6.2.3 and the game-theoretic investigations of Section 7.

Now we summarize the results of [KT17] in the case $\beta = 0$ that are relevant for the present paper. For $k \in \mathbb{N}$, elements of the product space $W^k = W^{\{0,1,\ldots,k-1\}}$ are denoted as $(x_0, \ldots, x_{k-1})$. For $l = 0, \ldots, k-1$, the l-th marginal of a measure $\nu_k \in \mathcal{M}(W^k)$ is denoted by $\pi_l \nu_k \in \mathcal{M}(W)$, i.e., $\pi_l \nu_k(A) = \nu_k(W^{\{0,\ldots,l-1\}} \times A \times W^{\{l+1,\ldots,k-1\}})$ for any Borel set A of W.

We assume that the *empirical measure* of X^λ normalized by $1/\lambda$, i.e., the measure

$$L_\lambda = \frac{1}{\lambda} \sum_{i \in I^\lambda} \delta_{X_i}, \tag{2.8}$$

converges to μ almost surely in the high-density limit $\lambda \to \infty$. This condition is satisfied e.g. if $\lambda \mapsto X^\lambda$ is increasing; for further details see [KT17, Section 1.7.4].

For fixed $k \in [k_{\max}]$ and for a trajectory collection $s = (s^i)_{i \in I^\lambda} \in \mathcal{S}_{k_{\max}}(X^\lambda)$, we define the empirical measure of all the k-hop trajectories of s as

$$R_{\lambda, k}(s) = \frac{1}{\lambda} \sum_{i \in I^\lambda} \delta_{(s_0^i, \ldots, s_{k-1}^i)} \mathbb{1}\{s_{-1}^i = k\} \in \mathcal{M}(W^k). \tag{2.9}$$

Since each user sends out exactly one message, we have

$$\sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda, k}(s) = L_\lambda. \tag{2.10}$$

This assumption can be relaxed, see Section 6.3 for a discussion about this. Note that (2.3) can be expressed in terms of $(R_{\lambda,k}(s))_{k\in[k_{\max}]}$ as follows

$$\mathfrak{S}(s) = \sum_{k=1}^{k_{\max}} \int_W R_{\lambda,k}(s)(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) \sum_{l=1}^{k} \frac{\int_W \ell(|y-x_l|)L_\lambda(\mathrm{d}y)}{\ell(|x_{l-1}-x_l|)}, \quad x_k = o. \tag{2.11}$$

Since $L_\lambda \Rightarrow \mu$ as $\lambda \to \infty$ and (2.10) holds for any $\lambda > 0$, subsequential limits of $(R_{\lambda,k}(s))_{k\in[k_{\max}]}$ in the coordinatewise weak topology are easily seen to be of the form $\Sigma = (\nu_k)_{k\in[k_{\max}]}$ with $\nu_k \in \mathcal{M}(W^k)$, satisfying

$$\sum_{k=1}^{k_{\max}} \pi_0 \nu_k = \mu, \tag{2.12}$$

cf. [KT17, Section 3.4]. For such Σ, we define the following continuous analogue of (2.11)

$$\mathrm{S}(\Sigma) = \sum_{k=1}^{k_{\max}} \int_W \nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) \sum_{l=1}^{k} g(x_{l-1},x_l), \quad x_k = o,$$

with

$$g(x,y) = \frac{\int_W \mu(\mathrm{d}z)\ell(|z-y|)}{\ell(|x-y|)}. \tag{2.13}$$

Moreover, we need the following *entropy term* that describes counting complexity [KT17, Sections 2.2, 3.1]:

$$\mathrm{J}(\Sigma) = \sum_{k=1}^{k_{\max}} \int_{W^k} \mathrm{d}\nu_k \log \frac{\mathrm{d}\nu_k}{\mathrm{d}\mu^{\otimes k}} + \log \mu(W) \sum_{k=1}^{k_{\max}} (k-1)\nu_k(W) \in [0,\infty], \tag{2.14}$$

with the convention that $0\log 0 = 0\log(0/0) = 0$ and $\mathrm{J}(\Sigma) = \infty$ whenever $\nu_k \not\ll \mu^{\otimes k}$ for some k.

The key result [KT17, Proposition 1.5, parts (3), (4)] about the limiting behaviour of the telecommunication system that we will use this paper is the following.

Proposition 2.1 (Law of large numbers for the empirical measures). *Let $\gamma > 0$ and $k_{\max} \in \mathbb{N}$. Let S be a random variable with distribution $\mathrm{P}_{\lambda,X^\lambda}^{\gamma,0}$. Then, almost surely with respect to X^λ, as $\lambda \to \infty$, the distribution of $\Sigma_\lambda(S) = (R_{\lambda,k}(S))_{k\in[k_{\max}]}$ converges coordinatewise weakly to the unique minimizer of the variational formula*

$$\inf_{\Sigma=(\nu_k)_{k=1}^{k_{\max}}:\ \sum_{k=1}^{k_{\max}} \pi_0\nu_k=\mu} \Big(\mathrm{J}(\Sigma) + \gamma\mathrm{S}(\Sigma)\Big). \tag{2.15}$$

For $k_{\max} > 1$, the minimizer is given as $\Sigma = (\nu_k)_{k=1}^{k_{\max}}$, where

$$\nu_k(\mathrm{d}x_0,\dots,\mathrm{d}x_{k-1}) = \mu(\mathrm{d}x_0)A(x_0) \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma\sum_{l=1}^{k} g(x_{l-1},x_l)}, \quad x_k = o,\ k \in [k_{\max}], \tag{2.16}$$

and the normalizing function A is defined as

$$\frac{1}{A(x_0)} = \sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \frac{\mu(\mathrm{d}x_l)}{\mu(W)} \mathrm{e}^{-\gamma\sum_{l=1}^{k} g(x_{l-1},x_l)}, \qquad x_0 \in W, \tag{2.17}$$

so that (2.12) holds.

For $k_{\max} = 1$, the unique minimizer of (2.15) is $\Sigma = (\nu_1)$ with $\nu_1 = \mu$, as this is the unique Σ satisfying (2.12).

The variational formula (2.15) has indeed the form of "entropy plus energy", as anticipated before. In the minimizer (2.16), the starting points of the k-hop message trajectories, $k \in [k_{\max}]$, are chosen according to the measure $\mu(\mathrm{d}x_0)A(x_0)$ and the lth relays according to the measure $\mu(\mathrm{d}x_l)/\mu(W)$ for

all $l \in [k-1]$, exponentially weighted by the limiting interference penalization term $\gamma \sum_{l=1}^{k} g(x_{l-1}, x_l)$. For further details about the limiting behaviour of the system, see [KT17, Sections 1.6, 1.7].

2.3. **Interpretation of the limiting trajectory distribution.** The purpose of the present paper is to make further qualitative assertions about the "typical" trajectory from a given transmission site $x_0 \in W$ to the origin, after having taken the limit $\lambda \to \infty$. A definition of the "typical" trajectory as a random variable is not immediate, due to the nature of this setting. In the present paper, we will focus on the probability measure on $\bigcup_{k \in [k_{\max}]}(\{k\} \times W^{k-1})$ given by its Radon–Nikodym derivative

$$T_{x_0}(k, x_1, \ldots, x_{k-1}) = \frac{\nu_k(\mathrm{d}x_0, \mathrm{d}x_1, \ldots, \mathrm{d}x_{k-1})}{\left(\sum_{k=1}^{k_{\max}} \pi_0 \nu_k(\mathrm{d}x_0)\right)\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_{k-1})} = \frac{\nu_k(\mathrm{d}x_0, \mathrm{d}x_1, \ldots, \mathrm{d}x_{k-1})}{\mu(\mathrm{d}x_0)\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_{k-1})}, \quad (2.18)$$

with respect to $\sum_{k \in [k_{\max}]}(\delta_k \otimes \mu^{\otimes(k-1)})$. This function is the main object of our study in the present paper. We normalized T_{x_0} in such a way that $\sum_{k \in [k_{\max}]} \int_{W^{k-1}} T_{x_0}(k, x_1, \ldots, x_{k-1})\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_{k-1}) = 1$. According to Proposition 2.1,

$$T_{x_0}(k, x_1, \ldots, x_{k-1}) = A(x_0)\mu(W)^{-(k-1)} \prod_{l=1}^{k-1} \mathrm{e}^{-\gamma \sum_{l=1}^{k} g(x_{l-1}, x_l)}, \quad (2.19)$$

where we recall (2.13). We will use the convention that the 0th coordinate of T_{x_0} is the one corresponding to k and the lth is the one corresponding to x_l, for $l \in [k-1]$. This way, the marginal $\pi_0 T_{x_0}$ is a measure on $[k_{\max}]$.

We note that also the measure $M = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k$ carries interesting information about the system. Indeed, in [KT17, Section 1.3] it was explained that, at a position $x \in W$, the typical number of incoming hops of a user at x is Poisson distributed with parameter $M(\mathrm{d}x)/\mu(\mathrm{d}x)$, and the total mass $M(W)$ is the amount of relaying hops in the entire system, with the convention that it is zero if every message hops directly into o without any relaying hop. Part of our analysis will also be devoted explicitly to M, see Section 5.

3. LARGE COMMUNICATION AREAS WITH LARGE TRANSMITTER–RECEIVER DISTANCES

This section is devoted to the analysis of the highly dense telecommunication system described in Section 2.2 in regime (1), i.e., in the limit of a large communication area coupled with a large distance of the user from the base station. In Section 3.1, we present our main results and in Section 3.2 we prove them. Section 3.3 includes discussions related to this regime.

3.1. **The typical number, length and direction of hops in a large-distance limit.** In this section, the main object of interest is the shape of the optimal trajectory from a certain site to the origin, in particular the typical length of any of the hops, the number of hops and the spatial progress of the trajectory, in particular whether or not it runs along the straight line or how strongly it deviates from it. We will answer these questions for the special choice that W is a closed ball around the origin, μ is the Lebesgue measure on W, and the path-loss function ℓ corresponds to ideal Hertzian propagation so that $b = \int_{\mathbb{R}^d} \ell(|x|)\mathrm{d}x < \infty$, that is, $\ell(r) = \min\{1, r^{-\alpha}\}$ for some $\alpha > d$.

Furthermore, in order to obtain a transparent picture and to derive a neat result, we will have to assume that the starting site of our trajectory is far away from the origin. In such a setting, it is plausible to expect that, as the radius of the ball tends to infinity, a proportion of users that tends to one takes the same order of magnitude of number of hops. This also gives information about the typical length and direction of each hop in large but still compact communication areas.

We will see that this setting exhibits the interesting property that the typical number of hops diverges to infinity as the distance of the user x_0 from o tends to infinity, however, in a sublinear way, more precisely, like the distance divided by a power of its logarithm. Second, using the asymptotics of

the value of this largest summand in (2.17), one can conclude about the typical length of the hops and about how much they deviate from the straight line between the transmitter and the receiver o. In our specific setting, we will be able to obtain precise and explicit asymptotics for all these quantities.

We denote the radius of the communication area $W = \overline{B_r(o)}$ by r, and we recall that $k_{\max}$ is the maximal hop number. We consider the limit of large r and large $k_{\max}$. We consider one user placed at $x_0 \in W$ with a distance from the origin $|x_0| = r_0$ being large, such that $r > r_0$, but $r \asymp r_0$. (We write "$\asymp$" if the quotient of the two sides stays bounded and bounded away from zero.) Then one can say that for large r, x_0 is a "typical" location of a user in $\overline{B_r(o)}$, chosen uniformly at random.

In our first result, Theorem 3.1, we examine the "typical" number of hops of a trajectory from x_0 to o as a random variable under the marginal distribution $\pi_0 T_{x_0}$ on $\mathbb{N}$. According to (2.19), in the present setting, this is given by

$$\pi_0 T_{x_0}(k) = A(x_0) a_k(x_0) \qquad \text{where } a_k(x_0) = \int_{(B_r(o))^{k-1}} \prod_{l=1}^{k-1} \left(\omega_d r^{-d} e^{-\gamma g(x_{l-1}, x_l)} \, dx_l \right), \qquad x_k = o,$$
(3.1)

where ω_d is the volume of the unit ball in $\mathbb{R}^d$, and we recall that $g(x_{l-1}, x_l) = \frac{\int_W dy \, \ell(|y-x_l|)}{\ell(|x_{l-1}-x_l|)}$. It is *a priori* not clear what the relation between r and $k_{\max}$ in the limiting setting should be in order to obtain interesting assertions. Nevertheless, while A depends on $k_{\max}$ via the identity $1/A(x_0) = \sum_{k=1}^{k_{\max}} a_k(x_0)$, we observe that the terms $a_k(x_0)$ can also be defined for $k > k_{\max}$ analogously to (3.1) for $k \leq k_{\max}$. Thus, it will be our first task to find the asymptotics of $a_k(x_0)$ without any reference to $k_{\max}$. We encounter a large deviation principle on a quite surprising scale.

Theorem 3.1 (Large deviations for the hop number). *Fix $t \in (0, \infty)$. Then, in the limit $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, for any choice of $r_0 \mapsto k(r_0) \in \mathbb{N}$,*

$$\frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \begin{cases} = -(dt + b\gamma t^{1-\alpha}) + o(1) & \text{if } \frac{k(r_0)}{r_0 \log^{-1/\alpha} r_0} \to t, & (3.2) \\ \leq -b\gamma t^{1-\alpha} + o(1) & \text{if } \frac{k(r_0)}{r_0 \log^{-1/\alpha} r_0} \leq t + o(1) & (3.3) \\ \leq -dt + o(1) & \text{if } \frac{k(r_0)}{r_0 \log^{-1/\alpha} r_0} \geq t + o(1). & (3.4) \end{cases}$$

where we recall that $b = \int_{\mathbb{R}^d} dy \, \ell(|y|)$.

The upper bound in (3.3) follows from the convexity of $1/\ell(|\cdot|)$ and a comparison between the functionals $(x, y) \mapsto g(x, y)$ and $(x, y) \mapsto b/\ell(|x - y|)$.

Note that Theorem 3.1 identifies the growth of $\log a_{k(r_0)}(x_0)$ on the scale $r_0 \log^{1-1/\alpha} r_0$ for $k(r_0)$ on the scale $r_0 \log^{-1/\alpha} r_0$; indeed, the second and third line rule out small and large values of $k(r_0)$ on that scale, and the first line identifies the precise dependence on the prefactor. In more technical terms, $a_k(x_0)$ satisfies, with $k(r_0) \asymp r_0 \log^{-1/\alpha}(r_0)$, a large deviation principle on the scale $r_0 \log^{1-1/\alpha} r_0$ with rate function $t \mapsto dt + b\gamma t^{1-\alpha}$. It is easily seen that this rate function has a unique minimizer

$$t^* = \arg\min_{t \in (0,\infty)} \left(dt + b\gamma t^{1-\alpha} \right) = \left(\frac{b\gamma(\alpha - 1)}{d} \right)^{1/\alpha}$$
(3.5)

with minimum value

$$dt^* + b\gamma(t^*)^{1-\alpha} = \frac{(b\gamma)^{1/\alpha}}{(\alpha - 1)d} \left[d + \left((\alpha - 1)d \right)^{1/\alpha} \right].$$

As a consequence, we have the following law of large numbers.

Corollary 3.2. *In the limit $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, any maximizer $k^*(r_0)$ of $\mathbb{N} \ni k \mapsto a_k(x_0)$ satisfies*

$$k^*(r_0) \sim t^* \frac{r_0}{\log^{1/\alpha} r_0}.$$
(3.6)

Further, if $k_{\max} \geq k^*(r_0)$ for at least one such maximizer for all sufficiently large $r_0 > 0$, then we have

$$\frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \frac{1}{A(x_0)} \to -(dt^* + b\gamma(t^*)^{1-\alpha}). \tag{3.7}$$

If $k_{\max}$ is smaller than all the minimizers, then the asymptotics of $A(x_0)$ depend on those of $a_{k_{\max}}(x_0)$ rather than on $a_{k^*(r_0)}(r_0)$, and (3.7) has to be adapted accordingly. We note that (3.7) requires only a lower bound on $k_{\max}$, and in Corollary 3.2, $k_{\max}$ could be equal to $+\infty$ for each r_0; see Section 3.3.3 for a discussion about allowing arbitrary many hops in our model. (3.7) says that the asymptotic logarithmic behaviour of $1/A(x_0)$ on scale $r_0 \log^{1-1/\alpha} r_0$ coincides with the one of the single maximal summand $a_{k^*(r_0)}(r_0)$. Formulated in terms of the marginal distribution $\pi_0 T_{x_0}$ of T_{x_0} on the length k of the path from x_0 to o, since the behaviour of the Lebesgue measure restricted to $\overline{B_r(o)}$ is subexponential in r_0 in the large-distance limit that we are considering, we have that

$$\pi_0 T_{x_0} \Big([t^* - \varepsilon, t^* + \varepsilon]^{\mathrm{c}} \frac{r_0}{\log^{1/\alpha}(r_0)} \Big)$$

tends to zero exponentially fast on the scale $r_0 \log^{1-1/\alpha} r_0$. In Section 3.3.1 we give an explanation of how these scales arise.

In the proof of the lower bound of (3.6), considering a uniform hop length distribution was sufficient, i.e., $t^* r_0 / \log^{1/\alpha} r_0$ hops along the same straight line directed from x_0 to o with length $r_0/k(r_0) \sim \frac{1}{t^*} \log^{1/\alpha} r_0$ each. We now show, again in terms of a large deviations estimate on the scale $r_0 \log^{1-1/\alpha} r_0$, that macroscopic deviations from this optimal hop length on the scale $\log^{1/\alpha} r_0$ have extremely small probability. For a finite set A, we write $\#A$ for the cardinality of A.

Proposition 3.3. *For $\varepsilon, \delta > 0$ and $k \in \mathbb{N}$, let*

$$D_{\varepsilon,\delta}(k, x_0) = \Big\{ (x_1, \ldots, x_{k-1}) \in B_r(o)^{k-1} \colon \exists I \subseteq [k-1] \colon \#I \geq \delta k,$$

$$\frac{1}{\#I} \sum_{l \in I} \frac{|x_{l-1} - x_l| - \big||x_{l-1}| - |x_l|\big|}{\log^{1/\alpha} r_0} > \varepsilon \Big\}, \qquad x_k = o. \tag{3.8}$$

Then, in the limit $r_0 \to \infty$ with $r > r_0 = |x_0| \asymp r$, for $k(r_0) \sim t^* r_0 / \log^{1/\alpha} r_0$,

$$\limsup \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log T_{x_0}\big(k(r_0), D_{\varepsilon,\delta}(k(r_0), x_0)\big) < 0. \tag{3.9}$$

In words, the probability that there are $\asymp k(r_0)$ hops $x_l - x_{l-1}$, $l \in I$ for some index set I, in the trajectory of relays such that their average hop length $\frac{1}{\#I} \sum_{i=1}^{\#I} |x_{l_i} - x_{l_i-1}|$ deviates from the optimal hop length $\frac{1}{t^*} \log^{1/\alpha} r_0 \approx r_0/k^*(r_0)$ on that scale, decays exponentially fast to zero on the scale $r_0 \log^{1-1/\alpha} r_0$. (In the denominator of the summands in (3.8), we have removed the factor $\frac{1}{t^*}$ in order to simplify notation.)

We presented the results of this section for the path-loss functions ℓ of the form $\ell(r) = \min\{1, r^{-\alpha}\}$, $\alpha > d$, which makes the notation in the proofs less heavy. However, these assertions require only two properties of ℓ: the integrability of $\ell(|\cdot|)$ over $\mathbb{R}^d$ and the convexity of $1/\ell(|\cdot|)$, see Section 3.3.2.

The proofs of Theorem 3.1, Corollary 3.2 and Proposition 3.3 are carried out in Sections 3.2.1, 3.2.2 and 3.2.3, respectively. A discussion about these results and their proofs can be found in Section 3.3.1.

Certainly, our results of this section hold for much more general communication areas W, not only for balls. Essential for our approach is only that a – in every space dimension diverging – neighbourhood of the straight line between x_0 and o is contained in W in the limit considered. The parameter d appearing in the rate function goes back to our assumption that the volume of W grows like the d-th power of r; however, other powers than d in $[1, d]$ are also possible by putting other geometric assumptions on W.

3.2. **Proof of the results of Section 3.1.** All the three results of Section 3.1 tell about the limit $r_0 \to \infty$ with $r > r_0 \asymp r$, where $x_0 \in W = \overline{B_r(o)}$ has Euclidean norm $|x_0| = r_0$. Throughout this section, we will use the notation $\lim_{r,r_0}$ for this limit and refer to it as "our limit".

3.2.1. *Proof of Theorem 3.1.* Our strategy for proving the three assertions (3.2), (3.3) and (3.4) is the following. First we verify the lower bound in (3.2). Then we prove (3.3) and afterwards (3.4), and we combine these two proofs in order to conclude the upper bound in (3.2).

Proof of (3.2), *lower bound.* Let us first consider $k(r_0)$ satisfying just $k(r_0) = o(r_0)$. We obtain a lower bound for $a_{k(r_0)}(x_0)$ defined in (3.1) by restricting the x_l-integral to the ball with radius one around $\frac{k(r_0)-l}{k(r_0)}x_0$ for $l = 1, \ldots, k(r_0) - 1$. Then, eventually, $1 \leq |x_{l-1} - x_l| \leq |x_0|/k(r_0) + 2$ for $l = 1, \ldots, k(r_0)$. Note that $g(x_{l-1}, x_l) \leq b/\ell(|x_{l-1} - x_l|) = b|x_{l-1} - x_l|^\alpha$, where we recall that $b = \int_{\mathbb{R}^d} \mathrm{d}y\, \ell(|y|)$. Hence, for any $\varepsilon \in (0,1)$, eventually,

$$g(x_{l-1}, x_l) \leq b|x_{l-1} - x_l|^\alpha \leq (1 + \varepsilon)br_0^\alpha/k(r_0)^\alpha, \qquad l = 1, \ldots, k(r_0),$$

where in the first step we used that $\frac{r_0}{k(r_0)} = \frac{|x_0|}{k(r_0)}$ tends to infinity in our limit. This gives

$$a_{k(r_0)}(x_0) \geq (\omega_d r^d)^{-k(r_0)+1} \mathrm{e}^{-\gamma bk(r_0)(1+\varepsilon)(r_0/k(r_0))^\alpha} \geq \mathrm{e}^{-(d+\varepsilon)k(r_0)\log r_0 - \gamma bk(r_0)(1+\varepsilon)(r_0/k(r_0))^\alpha},$$

where the second inequality holds eventually, since $r_0 \asymp r$. Now an elementary optimization on $k(r_0)$ shows that $k(r_0) \asymp r_0 \log^{-1/\alpha} r_0$ is the relevant scale. Then, in the particular case that $k(r_0) \sim tr_0 \log^{-1/\alpha} r_0$ for some $t \in (0, \infty)$, carrying out the limit and making $\varepsilon \downarrow 0$ afterwards, we have

$$\liminf_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \geq -\left(dt + \gamma bt^{1-\alpha}\right),$$

which is the lower bound in (3.2). $\triangle$

Proof of (3.3). This proof uses that $1/\ell$ is convex and that the numerator $\int_W \ell(|y - x_l|)\, \mathrm{d}y$ can be well approximated by b for sufficiently many l. These arguments lead to the following lemma.

Lemma 3.4. *Let $\varepsilon > 0$. If $k(r_0) \leq \frac{1}{2}r_0$ for all r_0 sufficiently large, then eventually in our limit,*

$$\sum_{l=1}^{k(r_0)} g(x_{l-1}, x_l) \geq (1-\varepsilon)^\alpha (b - \varepsilon)k(r_0)^{1-\alpha}r_0^\alpha \tag{3.10}$$

holds simultaneously for all $x_0, \ldots, x_{k(r_0)-1} \in B_r(o)$ with $|x_0| = r_0$ and $x_{k(r_0)} = o$.

Proof. Let us now define an auxiliary function $s \colon (0, \infty) \to (0, \infty)$ such that $r - s(r) \to \infty$ and $0 < r - s(r) = o(r)$ in our limit. Fix $\varepsilon \in (0, \frac{1}{4})$. The idea is to pick r sufficiently large so that

$$\fint_{\overline{B_r(o)}} \ell(|y - x|)\, \mathrm{d}y \geq b - \varepsilon, \qquad \forall x \in \overline{B_{s(r)}(o)}. \tag{3.11}$$

Let us assume that we are given a trajectory $(x_0, x_1, \ldots, x_{k(r_0)-1}, x_{k(r_0)})$ with $k(r_0) \leq \frac{1}{2}r_0$, $|x_0| = r_0$ and $x_{k(r_0)} = o$. Let us define the index of the last hop outside $B_{s(r)}(o)$:

$$K(r_0, r) = \max\{l \in \{0, 1, \ldots, k(r_0) - 1\} \colon |x_l| \geq s(r)\}, \tag{3.12}$$

which we want to understand as $K(r_0, r) = 0$ if there is no such hop. Let $r_0 > 0$ be sufficiently large so that $s(r) > (1 - \varepsilon)r$ and (3.11) holds. Then we have

$$\sum_{l=1}^{k(r_0)} g(x_{l-1}, x_l) \geq \sum_{l=K(r_0,r)+1}^{k(r_0)} g(x_{l-1}, x_l) \geq \sum_{l=K(r_0,r)+1}^{k(r_0)} \frac{b - \varepsilon}{\ell(|x_{l-1} - x_l|)} \tag{3.13}$$

$$\geq \frac{(b - \varepsilon)(k(r_0) - K(r_0, r))}{\ell\left(\frac{1}{k(r_0) - K(r_0,r)} \sum_{l=K(r_0,r)+1}^{k(r_0)} |x_{l-1} - x_l|\right)} \tag{3.14}$$

$$\geq \frac{(b - \varepsilon)(k(r_0) - K(r_0, r))}{\ell\left(\frac{1}{k(r_0) - K(r_0,r)} (1 - \varepsilon)r_0\right)} \tag{3.15}$$

$$\geq (1 - \varepsilon)^\alpha (b - \varepsilon)(k(r_0) - K(r_0, r))^{1-\alpha} r_0^\alpha \geq (1 - \varepsilon)^\alpha (b - \varepsilon)(k(r_0))^{1-\alpha} r_0^\alpha. \tag{3.16}$$

In (3.13) we used the fact that $x_{K(r_0,r)}, \ldots, x_{k(r_0)-1}, x_{k(r_0)}$ lie in $B_{s(r)}(o)$ and therefore (3.11) can be applied for the numerator of each $g(x_{l-1}, x_l)$ with $l > K(r_0, r)$. Next, (3.14) is an application of Jensen's inequality for $1/\ell(|\cdot|)$, and (3.15) uses the following fact. Either $K(r_0, r) = 0$, in which case

$$\sum_{l=K(r_0,r)+1}^{k(r_0)} |x_{l-1} - x_l| \geq \sum_{l=K(r_0,r)+1}^{k(r_0)} (|x_{l-1}| - |x_l|) \geq |x_0| = r_0 \geq 2k(r_0) \geq k(r_0) - K(r_0, r), \tag{3.17}$$

or $K(r_0, r) > 0$, and thus

$$\sum_{l=K(r_0,r)+1}^{k(r_0)} |x_{l-1} - x_l| \geq \sum_{l=K(r_0,r)+1}^{k(r_0)} (|x_{l-1}| - |x_l|) \geq s(r)$$
$$\geq (1 - \varepsilon)r > (1 - \varepsilon)r_0 \geq k(r_0) \geq k(r_0) - K(r_0, r). \tag{3.18}$$

In both cases, the argument in $\ell(\cdot)$ is ≥ 1, and we can write the term in terms of the α-norm and the first step in (3.16) also follows. Hence, we have derived (3.10). $\qquad\square$

By Lemma 3.4, for any $\varepsilon > 0$, we have eventually in our limit, under the assumptions of the lemma

$$\int_{\overline{B_r(o)}^{k(r_0)-1}} \left(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(\overline{B_r(o)})}\right) \mathrm{e}^{-\gamma \sum_{l=1}^{k(r_0)} g(x_{l-1}, x_l)}$$

$$\leq \int_{\overline{B_r(o)}^{k(r_0)-1}} \left(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(\overline{B_r(o)})}\right) \mathrm{e}^{-\gamma(1-\varepsilon)^\alpha(b-\varepsilon)k(r_0)^{1-\alpha}r_0^\alpha} = \mathrm{e}^{-\gamma(1-\varepsilon)^\alpha(b-\varepsilon)k(r_0)^{1-\alpha}r_0^\alpha}.$$

Now, let $t > 0$ and $r_0 \mapsto k(r_0)$ such that $k(r_0) \leq (t + o(1))r_0/\log^{1/\alpha} r_0$ (in particular $k(r_0) \leq \frac{1}{2}r_0$ eventually).

$$\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \leq \limsup_{r,r_0} \frac{-(b-\varepsilon)(1-\varepsilon)^\alpha \gamma k(r_0)^{1-\alpha} r_0^\alpha}{r_0 \log^{1-1/\alpha} r_0}$$

$$= \limsup_{r,r_0} -(b-\varepsilon)(1-\varepsilon)^\alpha \gamma \left(\frac{k(r_0)\log^{1/\alpha}(r_0)}{r_0}\right)^{1-\alpha} \leq -(b-\varepsilon)(1-\varepsilon)^\alpha \gamma t^{1-\alpha} \tag{3.19}$$

for all $\varepsilon > 0$, and thus

$$\limsup_{r,r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) \leq -b\gamma t^{1-\alpha},$$

which is (3.3). $\qquad\triangle$

Proof of (3.4). Note that for any $x \in \overline{B_r(o)}$, we have

$$\int_{B_r(o)} \ell(|y - re_1|) \, dy \leq \int_{B_r(o)} \ell(|y - x|) \, dy,$$

where $e_1 = (1, 0, \ldots, 0)$ is the first unit vector of $\mathbb{R}^d$.

Let us introduce the quantity $b_0 = \lim_{r \to \infty} \int_{B_r(o)} \ell(|y - re_1|) \, dy = \sup_{r \in (0, \infty)} \int_{B_r(o)} \ell(|y - re_1|) \, dy \in (0, b)$. Now, for any $k \colon (0, \infty) \to \mathbb{N}$, in our limit,

$$
\begin{aligned}
\mathrm{Leb}(B_r(o))^{-(k(r_0)-1)} a_{k(r_0)}(x_0) &= \int_{(B_r(o))^{k(r_0)-1}} \Big(\prod_{l=1}^{k(r_0)-1} dx_l \Big) e^{-\gamma \sum_{l=1}^{k(r_0)} \frac{\int_{B_r(o)} \ell(|y-x_l|) \, dy}{\ell(|x_{l-1}-x_l|)}} \\
&\leq \int_{(\mathbb{R}^d)^{k(r_0)-1}} \prod_{l=1}^{k(r_0)-1} \Big(dx_l \, e^{-\gamma \frac{b_0 - o(1)}{\ell(|x_{l-1}-x_l|)}} \Big) \\
&\leq \Big(\int_{\mathbb{R}^d} e^{-\gamma \frac{b_0 - o(1)}{\ell(|y|)}} \, dy \Big)^{k(r_0)-1} = O(1)^{k(r_0)} = \exp\big(o(k(r_0) \log r_0)\big),
\end{aligned}
$$
$$(3.20)$$

where the first step in the last line follows from an elementary substitution and a reversion of the order of integration. Now, recall that in our limit $r \asymp r_0$. If $t > 0$ and $k(r_0) \geq (t + o(1)) r_0 / \log^{1/\alpha} r_0$, we have that

$$\mathrm{Leb}(B_r(o))^{-(k(r_0)-1)} = \exp(-(dt + o(1)) k(r_0) \log r_0) \leq \exp\big(-(dt + o(1)) r_0 \log^{1-1/\alpha} r_0 \big).$$

This implies (3.4). $\triangle$

Proof of (3.2), *upper bound.* We combine our arguments from the proofs of the upper bounds in (3.3) and (3.4) in order to obtain the upper bound in (3.2). Indeed, for $t > 0$ and $k(r_0) \sim t r_0 / \log^{1/\alpha} r_0$ and $\varepsilon > 0$, let us write $g(x_{l-1}, x_l) = \varepsilon g(x_{l-1}, x_l) + (1 - \varepsilon) g(x_{l-1}, x_l)$, estimate the first term like in (3.20) and the second term with the help of (3.10). This gives eventually

$$
\begin{aligned}
a_{k(r_0)}(x_0) &\leq \int_{W^{k(r_0)-1}} \Big(\prod_{l=1}^{k(r_0)-1} \frac{dx_l}{\mathrm{Leb}(B_r(o))} \Big) e^{-\varepsilon \gamma \sum_{l=1}^{k(r_0)} \frac{b_0 - o(1)}{\ell(|x_{l-1}-x_l|)}} e^{-(1-\varepsilon)(1-\varepsilon)^{\alpha}(b-\varepsilon)\gamma t^{1-\alpha} r_0 \log^{1-1/\alpha} r_0} \\
&\leq \exp\Big(-(dt - \varepsilon) r_0 \log^{1-1/\alpha} r_0 - (1 - \varepsilon)^{\alpha+1} \gamma (b - \varepsilon) t^{1-\alpha} r_0 \log^{1-1/\alpha} r_0 \Big).
\end{aligned}
$$
$$(3.21)$$

Carrying out our limit and letting $\varepsilon \downarrow 0$ implies the upper bound in (3.2). This finishes the proof of Theorem 3.1. $\square$

3.2.2. *Proof of Corollary 3.2.* The identity (3.6) follows immediately from Theorem 3.1. As for (3.7), let $k^*(r_0)$ be the smallest maximizer of $k \mapsto a_k(x_0)$, and let $r_0 \mapsto k_{\max}(r_0)$ satisfy the assumption of the corollary, i.e., $k_{\max}(r_0) \geq k^*(r_0)$. The lower bound easily follows from (3.2) by estimating $1/A(x_0)$ from below by the single summand $a_{k^*(r_0)}(x_0)$ and using (3.6). As for an upper bound, we first write

$$
\limsup_{r, r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \frac{1}{A(x_0)} \leq \limsup_{r, r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \Big(\sum_{k=1}^{\lfloor \frac{1}{2} r_0 \rfloor} a_k(x_0) + \sum_{k=\lfloor \frac{1}{2} r_0 \rfloor + 1}^{\infty} a_k(x_0) \Big)
$$

$$
= \max \Big\{ \limsup_{r, r_0} \frac{1}{\frac{1}{2} r_0 \log^{1-1/\alpha} r_0} \log \Big(\sum_{k=1}^{\lfloor \frac{1}{2} r_0 \rfloor} a_k(x_0) \Big), \limsup_{r, r_0} \frac{1}{r_0 \log^{1-1/\alpha} r_0} \log \Big(\sum_{k=\lfloor \frac{1}{2} r_0 \rfloor + 1}^{\infty} a_k(x_0) \Big) \Big\}.
$$

Then the proof of (3.4) implies that there exists a constant $D > 0$ such that we have

$$\sum_{k=\lfloor\frac{1}{2}r_0\rfloor+1}^{\infty} a_k(x_0) \leq \sum_{k=\lfloor\frac{1}{2}r_0\rfloor+1}^{\infty} (Dr_0^d)^{-k} = \frac{(Dr_0^d)^{-\lfloor\frac{1}{2}r_0\rfloor+1}}{1-\frac{1}{Dr_0^d}} \leq \exp(-(\tfrac{1}{2}-o(1))r_0\log r_0),$$

therefore

$$\limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log\Big(\sum_{k=\lfloor\frac{1}{2}r_0\rfloor+1}^{\infty} a_k(x_0)\Big) = -\infty.$$

Moreover, the assumption in Corollary 3.2 that $k_{\max}(r_0) \geq k^*(r_0)$ for all sufficiently large $r_0 > 0$ together with (3.2) yields

$$\limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log\Big(\sum_{k=1}^{\lfloor\frac{1}{2}r_0\rfloor} a_k(x_0)\Big)$$

$$\leq \limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log(\lfloor r_0/2\rfloor) + \limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log a_{k^*}(x_0) = -(dt^* + \gamma bt^{*1-\alpha}),$$

where we recall that $t^* = (b\gamma(\alpha-1)/d)^{1/\alpha}$ is the unique minimizer of $t \mapsto dt+t^{1-\alpha}$ on $(0,\infty)$, cf. (3.5). We conclude the upper bound in (3.7). $\qquad\square$

3.2.3. *Proof of Proposition 3.3.* Let $\varepsilon, \delta > 0$ be fixed. First, let us note that by the definition of T_{x_0} and the fact that the behaviour of the Lebesgue measure restricted to $\overline{B_r(o)}$ is subexponential in our limit, (3.9) is equivalent to

$$\limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log\int_{D_{\varepsilon,\delta}(k(r_0),x_0)} \Big(\prod_{l=1}^{k(r_0)-1} \frac{\mathrm{d}x_l}{\mathrm{Leb}(B_r(o))}\Big) \mathrm{e}^{-\gamma\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)}$$

$$< \limsup_{r,r_0} \frac{1}{r_0\log^{1-1/\alpha} r_0} \log a_{k(r_0)}(x_0) = -\big(dt^* + b\gamma t^{*1-\alpha}\big), \tag{3.22}$$

with $k(r_0) \sim t^* r_0 \log^{-1/\alpha} r_0$ and $x_{k(r_0)} = o$, where in the last step we used (3.2). For this, it suffices to show that there exists $\varepsilon_1 > 0$ such that for any choice of $x_0 \mapsto (x_1,\ldots,x_{k(r_0)-1}) = (x_1(x_0),\ldots,x_{k(r_0)-1}(x_0)) \in D_{\varepsilon,\delta}(k(r_0),x_0))$ writing $I = I(x_0,x_1,\ldots,x_{k(r_0)-1})$ as in (3.8), we have

$$\liminf_{r,r_0} \frac{\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)}{k(r_0)\log r_0} = \liminf_{r,r_0} \frac{\sum_{l=1}^{k(r_0)} g(x_{l-1},x_l)}{t^* r_0\log^{1-1/\alpha} r_0} \geq bt^{*-\alpha} + \varepsilon_1. \tag{3.23}$$

Indeed, then one can argue analogously to (3.21) to conclude the first inequality in (3.22).

Now we prove (3.23). We will first replace the functional $(x,y) \mapsto g(x,y)$ by $(x,y) \mapsto \frac{b}{\ell(|x-y|)}$ everywhere, and then argue for $g(x,y)$, estimating the numerator of $g(\cdot,\cdot)$ similarly to Lemma 3.4.

We have, first using Jensen's inequality for the convex function $|\cdot|^\alpha$, then the definition of $D_{\varepsilon,\delta}(k(r_0),x_0)$ together with the fact that $\alpha > 1$,

$$\frac{1}{\#I}\sum_{l\in I} |x_l - x_{l-1}|^\alpha \geq \Big(\frac{1}{\#I}\sum_{l\in I} |x_l - x_{l-1}|\Big)^\alpha \geq \Big(\frac{1}{\#I}\sum_{l\in I} \big||x_l| - |x_{l-1}|\big|\Big) + \varepsilon\log^{1/\alpha} r_0\Big)^\alpha$$

$$\geq \Big(\frac{1}{\#I}\sum_{l\in I} \big||x_l| - |x_{l-1}|\big|\Big)^\alpha + (\varepsilon\log^{1/\alpha} r_0)^\alpha. \tag{3.24}$$

Similarly, by Jensen's inequality and the triangle inequality,

$$\frac{\sum_{l\in[k(r_0)]\setminus I} |x_l - x_{l-1}|^\alpha}{k(r_0)-\#I} \geq \Big(\frac{1}{k(r_0)-\#I}\sum_{l\in[k(r_0)]\setminus I} |x_l-x_{l-1}|\Big)^\alpha \geq \sum_{l\in[k(r_0)]\setminus I} \Big(\frac{1}{k(r_0)-\#I}\big||x_l|-|x_{l-1}|\big|\Big)^\alpha.$$

Hence, more applications of Jensen's inequality yield

$$\frac{1}{k(r_0)} \sum_{l \in [k(r_0)]} |x_l - x_{l-1}|^\alpha$$

$$= \frac{\#I}{k(r_0)} \frac{1}{\#I} \sum_{l \in I} |x_l - x_{l-1}|^\alpha + \frac{k(r_0) - \#I}{k(r_0)} \frac{1}{k(r_0) - \#I} \sum_{l \in [k(r_0)] \setminus I} |x_l - x_{l-1}|^\alpha$$

$$\geq \frac{\#I}{k(r_0)} \Big(\frac{1}{\#I} \sum_{l \in I} \big||x_l| - |x_{l-1}|\big| \Big)^\alpha + \frac{k(r_0) - \#I}{k(r_0)} \Big(\frac{\sum_{l \in [k(r_0)] \setminus I} \big||x_l| - |x_{l-1}|\big|}{k(r_0) - \#I} \Big)^\alpha + \frac{\#I(\varepsilon \log^{1/\alpha} r_0)^\alpha}{k(r_0)}$$

$$\geq \Big(\frac{1}{k(r_0)} \sum_{l \in [k(r_0)]} \big||x_{l-1}| - |x_l|\big| \Big)^\alpha + \delta \varepsilon^\alpha \log r_0 \geq \Big(\frac{r_0}{k(r_0)} \Big)^\alpha + \delta \varepsilon^\alpha \log r_0$$

$$= (t^{*-\alpha} + \delta \varepsilon^\alpha) \log r_0, \tag{3.25}$$

where in the penultimate step we used that $\#I \geq \delta k(r_0)$.

Now, we turn to $\ell(|\cdot|)$ instead of $|\cdot|^{-\alpha}$. Hence, we have to distinguish $|\cdot| \leq 1$ and $|\cdot| > 1$. Let us define $I' = I'(x_0, (x_1, \dots, x_{k(r_0)-1})) \subseteq [k(r_0)]$ as the set of $l \in [k(r_0)]$ such that $|x_l - x_{l-1}| \leq 1$. Without loss of generality, I' is not empty. Then, after passing to a subsequence, if needed, we have that $\#I' \sim \delta' k(r_0)$ for some $\delta' \in [0, 1]$. Thus,

$$\Big| \frac{1}{\#I'} \sum_{l \in I'} \frac{|x_{l-1} - x_l| - \big||x_{l-1}| - |x_l|\big|}{\log^{1/\alpha} r_0} \Big| = O(1/\log^{1/\alpha} r_0) = o(1). \tag{3.26}$$

Let us assume for a moment that $I \cap I' = \varnothing$ and $\delta' < 1$. Splitting into I' and $[k(r_0)] \setminus I'$, we obtain

$$\frac{1}{k(r_0)} \sum_{l \in [k(r_0)]} \frac{1}{\ell(|x_l - x_{l-1}|)} \geq \frac{1}{k(r_0)} \Big(O(\#I') + \sum_{l \in [k(r_0)] \setminus I'} |x_l - x_{l-1}|^\alpha \Big)$$

$$\geq \delta' - o(1) + \frac{1 - \delta' - o(1)}{k(r_0) - \#I'} \sum_{l \in [k(r_0)] \setminus I'} |x_l - x_{l-1}|^\alpha. \tag{3.27}$$

We want to apply to the last term a lower bound analogous to (3.25), i.e., for the sum over $[k(r_0)] \setminus I'$ instead of $[k(r_0)]$. For this, we need that the sum of the $\big||x_{l-1}| - |x_l|\big|$ satisfies a lower bound against $\approx r_0$. Using that $I \cap I' = \varnothing$, we indeed see this as follows:

$$\sum_{l \in [k(r_0)] \setminus I'} \big||x_{l-1}| - |x_l|\big| \geq -(\delta' + o(1))k(r_0) + \sum_{l \in [k(r_0)]} \big||x_{l-1}| - |x_l|\big| \geq r_0(1 - o(1)).$$

Now, making a computation analogous to (3.25) for the right-hand side of (3.27), we obtain in our limit

$$\frac{1}{k(r_0)} \sum_{l \in [k(r_0)]} \frac{1}{\ell(|x_l - x_{l-1}|)} \geq \delta' - o(1) + (1 - \delta' - o(1)) \Big[\Big(\frac{r_0}{\#([k(r_0)] \setminus I')} \Big)^\alpha + \frac{\delta}{1 - \delta'} \varepsilon^\alpha \log r_0 \Big]$$

$$\geq \Big((1 - \delta')^{1-\alpha} t^{*-\alpha} + \delta \varepsilon^\alpha - o(1) \Big) \log r_0 \geq (t^{*-\alpha} + \delta \varepsilon^\alpha - o(1)) \log r_0. \tag{3.28}$$

The case $I \cap I' \neq \varnothing$ can be handled analogously as long as $\delta' < 1$. Indeed, in this case, (3.26) implies that $\liminf_{r, r_0} \#(I \setminus I')/k(r_0)$ and $\liminf_{r, r_0} \frac{1}{k(r_0)} \sum_{l \in I \setminus I'} (|x_{l-1} - x_l| - \big||x_{l-1} - x_l|\big|)$ are positive. Thus, a lower estimate on $\frac{1}{k(r_0)} \sum_{l \in [k(r_0)]} (|x_{l-1} - x_l| - \big||x_{l-1} - x_l|\big|)$ can still be obtained analogously to (3.27). Further, we observe that this lower bound tends to infinity as $\delta' \uparrow 1$.

Hence, we have in any case that (3.28) holds with $\delta \varepsilon^\alpha$ replaced by some positive number. From this, (3.23) follows for $(x, y) \mapsto g(x, y)$ replaced by $(x, y) \mapsto \frac{b}{\ell(|x-y|)}$ for some $\varepsilon_1 > 0$.

In order to conclude (3.23), we now proceed similarly to the proof of (3.3), that is, we use uniform convergence of the interferences to b within $B_r(o)$ away from the boundary. Let us recall the auxiliary function s and the index $K(r_0, r)$ at (3.12). We essentially show that either a non-negligible part of the deviations from the straight line induced by the definition of $D_{\varepsilon,\delta}(k(r_0), x_0)$ takes place after the $K(r_0, r)$th hop, or the first $K(r_0, r)$ hops have a very high interference penalization value, and in both cases, (3.23) holds.

For each x_0 with $|x_0| = r_0$, let us choose $(x_1(x_0), \ldots, x_{k(r_0)-1}(x_0)) \in D_{\varepsilon,\delta}(k(r_0), x_0)$. We use the notation $\tau(r_0) = \tau(x_0, x_1(x_0), \ldots, x_{k(r_0)-1}(x_0))$ for $\tau(r_0) = \frac{K(r_0,r)}{k(r_0)}$. Let us further write $I(r_0) = I(x_0, x_1(x_0), \ldots, x_{k(r_0)-1}(x_0))$ for a choice of a set I according to (3.8). According to (3.28), without loss of generality we can assume that $I' = I'(x_0, x_1(x_0), \ldots, x_{k(r_0)-1}(x_0)) = \varnothing$ for all x_0 considered.

In our limit, $\int_{B_r(o)} \ell(|z - y|)\mathrm{d}z = b - o(1)$ uniformly in $y \in \overline{B_{s(r)}(o)}$. Thus, in case $\tau(r_0) = 0$, (3.28) implies that (3.23) holds with some ε_1. Hence, in order to conclude (3.23), we can assume that $\tau(r_0) \neq 0$ eventually in our limit. Further, by our assumptions on the function s, for any $\varepsilon_2 > 0$, eventually $s(r) > (1 - \varepsilon_2)r_0$. Now, on the one hand, since $x_l(x_0) \in \overline{B_{s(r)}(o)}$ for all $l > K(r_0, r)$, similarly to (3.28), the convexity of $1/\ell(|\cdot|)$ implies the following

$$\frac{1}{k(r_0)} \sum_{l \in [k(r_0)]} g(x_{l-1}, x_l) \geq \frac{1}{k(r_0)} \sum_{l=K(r_0,r)+1}^{k(r_0)} g(x_{l-1}, x_l) \geq \frac{1}{k(r_0)} \sum_{l=K(r_0,r)+1}^{k(r_0)} \frac{b - o(1)}{\ell(|x_{l-1} - x_l|)} \tag{3.29}$$
$$\geq \kappa(\varepsilon_2)(1 - \tau(r_0))^{1-\alpha}(b - o(1))t^{*-\alpha} \log r_0$$

for some function $\kappa \colon [0, 1] \to \mathbb{R}$ with $\lim_{\varrho \downarrow 0} \kappa(\varrho) = 1$. Now, taking first our limit and then $\varepsilon_2 \downarrow 0$, we see that if $\liminf_{r,r_0} \tau(r_0) > 0$, then the proof of our goal (3.23) is finished. Now assume that $\liminf_{r,r_0} \tau(r_0) = 0$. After passing to a subsequence, we can assume that $\lim_{r,r_0} \tau(r_0) = 0$.

Let us first consider the case that $\liminf_{r,r_0} \frac{1}{r_0} \sum_{l=K(r_0,r)+1}^{k(r_0)}(|x_{l-1}| - |x_l|) \geq 1$ (observe that the total sum over all $l \in [k(r_0)]$ is telescoping and hence equal to r_0) and

$$\liminf_{r,r_0} \frac{1}{\#I(r_0)} \sum_{l \in I(r_0)\colon l > K(r_0,r)} \frac{|x_{l-1} - x_l| - \big||x_{l-1}| - |x_l|\big|}{\log^{1/\alpha} r_0} > 0. \tag{3.30}$$

Then one can employ an estimate analogous to (3.25) in order to conclude (3.23). Next, we investigate the case that $\limsup_{r,r_0} \frac{1}{r_0} \sum_{l=K(r_0,r)+1}^{k(r_0)}(|x_{l-1}| - |x_l|) < 1$. Then we have

$$\liminf_{r,r_0} \frac{1}{r_0} \sum_{l=1}^{K(r_0,r)} \big||x_{l-1}| - |x_l|\big| \geq \liminf_{r,r_0} \frac{1}{r_0} \sum_{l=1}^{K(r_0,r)} (|x_{l-1}| - |x_l|) > \varepsilon_3$$

for some $\varepsilon_3 > 0$. Thus, using that $\int_{B_r(o)} \ell(|z - y|)\mathrm{d}z \geq b_0 - o(1)$ uniformly for $y \in \overline{B_r(o)}$ in our limit (where b_0 was defined before (3.20)), a convexity argument similar to (3.25) yields

$$\liminf_{r,r_0} \frac{1}{k(r_0) \log r_0} \sum_{l \in [k(r_0)]} g(x_{l-1}, x_l) \geq \liminf_{r,r_0} \frac{1}{t^{*\alpha} k(r_0)^{1-\alpha} r_0^\alpha} \sum_{l=1}^{K(r_0,r)} g(x_{l-1}, x_l)$$
$$\geq \liminf_{r,r_0} \varepsilon_3^\alpha t^{*-\alpha}(b_0 - o(1))\Big(\frac{K(r_0,r)}{k(r_0)}\Big)^{1-\alpha} = \infty. \tag{3.31}$$

Hence, in order to finish the proof of (3.23), it remains to consider the case that $\liminf_{r,r_0} \frac{1}{r_0} \sum_{l=K(r_0,r)+1}^{k(r_0)}(|x_{l-1}| - |x_l|) \geq 1$ but (3.30) fails. After passing to a subsequence, we can

assume that $\lim_{r,r_0} w(r_0) \geq \varepsilon$, where we put

$$w(r_0) = \frac{1}{\#I(r_0)} \sum_{l \in I(r_0) \cap [K(r_0,r)]} \frac{|x_{l-1} - x_l| - \big||x_{l-1}| - |x_l|\big|}{\log^{1/\alpha} r_0}.$$

Using also that $\#I(r_0) \geq \delta k(r_0) \sim \delta t^* r_0 \log^{1/\alpha} r_0$, we have

$$\varepsilon - o(1) \leq \frac{1}{\#I(r_0)} \sum_{l \in I(r_0) \cap [K(r_0,r)]} \frac{|x_{l-1} - x_l|}{\log^{1/\alpha} r_0} \leq \Big(\frac{1}{\delta t^*} + o(1)\Big) \sum_{l \in I(r_0) \cap [K(r_0,r)]} \frac{|x_{l-1} - x_l|}{r_0}.$$

Thus, a convexity argument similar to (3.25) implies

$$\liminf_{r,r_0} \frac{1}{k(r_0) \log r_0} \sum_{l \in [k(r_0)]} g(x_{l-1}, x_l) \geq \liminf_{r,r_0} \frac{1}{k(r_0) \log r_0} \sum_{l \in I(r_0) \cap [K(r_0,r)]} g(x_{l-1}, x_l)$$

$$\geq \liminf_{r,r_0} \frac{1}{k(r_0) \log r_0} (b_0 - o(1)) \Big(\frac{r_0 \delta t^* \varepsilon}{\#(I(r_0) \cap [K(r_0,r)])}\Big)^\alpha \#(I(r_0) \cap [K(r_0,r)])$$

$$\geq \liminf_{r,r_0} \tau(r_0)^{1-\alpha} b_0 (\delta \varepsilon)^\alpha = \infty.$$

This shows that (3.23) holds with a suitable choice of $\varepsilon_1 > 0$. $\qquad\square$

3.3. **Discussion about the results of Section 3.** This section discusses the relevance and extensions of the results of Section 3.1. In Section 3.3.1 we interpret our large-distance limit, in Section 3.3.2 we explain how the choice of the path-loss function influences our results, and in Section 3.3.3 we comment on allowing an arbitrary number of hops.

3.3.1. *The large-distance limit.* In Section 3.1, we consider the typical trajectory in a large homogeneous multihop communication system with one base station in the area W, after the high-density limit has been taken. According to the basic rules in this system, virtually every hop in the area W is homogeneously admitted (even those that do not bring the message any closer to the base station or even further away), but an exponential interference weight is given to the joint configuration of all the trajectories. It may appear somewhat irrelevant to consider a limit of large area, large distances and many hops, since with an increasing number of hops the technical difficulties and annoying side-effects become larger, but our work is meant to reveal the basic effects emerging in such a setting, in particular the effect of the interference penalization, and our result in terms of a large deviation principle gives also bounds on deviations from the extreme regime.

Since the interference term in particular gives small weights to large hops, it may be expected that the typical trajectory turns out to follow a straight line with all the hops being of the same length, but it may also come as a surprise that the typical hop length diverges like a power of the logarithm of the distance. The reason for this is the fact that *a priori* all the hops (within the area) are admitted and that, in the distribution T_{x_0} of the typical trajectory, as W grows in size, a very small weight term $1/\mathrm{Leb}(W) = 1/\mu(W)$ for each hop appears. This favours a small number of hops. The best compromise between this effect and the interference effect turns out to be on a logarithmic scale.

One could think of a model in which the search for the next hop is done only in a neighbourhood of the current location, which would presumably lead to the removal of the small weight term $1/\mathrm{Leb}(W)$ per hop and finally to a number of hops that is linear in the distance from the origin, but this would make the decay of the path-loss function ℓ irrelevant and describe a fundamentally different organization of message routeing in the telecommunication system. Such an organization is found e.g. in the continuum percolation setting of [YCG11], where the optimal number of hops turns out to be asymptotically linear in the distance from the user to the origin in a large-distance limit. Further, [YCG11, Theorem 2.1] claims that the probability of having trajectories of a significantly unusual length decays exponentially fast, which is analogous to our Proposition 3.3.

3.3.2. *The role of the choice of the path-loss function in the large-distance limit.* We derived our large-distance statements for the path-loss function $\ell(r) = \min\{1, r^{-\alpha}\}$ for $\alpha > d$, since this ℓ describes the propagation of signal strength realistically, see e.g. [BB09, GT08, HJKP18]. However, following the proofs of the results of Section 3.1 presented in Section 3.2, we conclude that analogous results hold whenever the path-loss function ℓ has the following two properties: $\int_{\mathbb{R}^d} \ell(|x|)\mathrm{d}x < \infty$ and $1/\ell$ is convex. If ℓ satisfies these assumptions, then in our large-distance limit, in the optimal strategy (cf. Section 3.2.1), the user takes $\asymp k(r_0)$ hops, where $r_0 \mapsto k(r_0)$ satisfies

$$\log(r_0) \sim \ell\Big(\frac{r_0}{k(r_0)}\Big). \tag{3.32}$$

This shows that the optimal scale depends only on the tail behaviour of ℓ. Thus, for example, the results of Section 3.1 also hold for the path-loss function $\ell(r) = (K + r)^{-\alpha}$, $K > 0$, $\alpha > d$. In general, (3.32) shows that under the two above assumptions on ℓ, the optimal scale diverges to ∞ and is sublinear. The faster ℓ decays, the slower $r_0/k(r_0)$ grows. E.g., if $\ell(r) = \mathrm{e}^{-\alpha r}$ for some $\alpha > 0$, then the correct scale is $k(r_0) \asymp r_0/\log\log r_0$.

3.3.3. *Allowing an unbounded number of hops.* Note that the interference term is linear in the number of hops, hence this number is upper bounded by some geometric random variable and thus almost surely finite, even without the upper bound $k_{\max}$. For a similar reason, the measures ν_k in (2.16) are also well-defined and are the unique minimizers of the variational formula (2.15) for $k_{\max} = \infty$. Also, it is clear that the proof of Corollary 3.2 also works if we choose $k_{\max} = \infty$ for each r_0. However, the proof techniques of [KT17, Proposition 2.1] do not generalize to the case $k_{\max} = \infty$ or $k_{\max}$ being a function of λ and tending to infinity as $\lambda \to \infty$. Thus, as long as an analogue of this proposition has not been proven for $k_{\max} = \infty$, these results have no verified connection with a Gibbsian model. Proving such an analogue may be a mathematically interesting task, nevertheless, from a modelling point of view, we find the necessity of taking an arbitrary number of hops in a fixed compact communication area questionable.

4. Strong penalization for the interference

This section is devoted to regime (2), i.e., the limit of strong penalization of interference. Our main result corresponding to this, Proposition 4.1, is stated in Section 4.1 and proven in Section 4.2.

4.1. **Strong interference penalization makes message trajectories straight.** Proposition 3.3 shows that in the large-distance limit, with μ being the Lebesgue measure in a large ball W, the typical message trajectory from the transmitter x_0 to $x_k = o$ under T_{x_0} does not deviate much from the straight line with high probability. In this proposition, $|x_0|$, $k = k(|x_0|)$ and the radius of W are assumed to tend to infinity in a certain coupled way. From an application point of view, it is also desirable to see a similar effect for a fixed compact communication area W, a fixed starting site x_0 and a fixed upper bound $k_{\max} \in \mathbb{N}$ on the hop number. One way to find such an effect is to consider the limit of a large interference penalization parameter γ. It is easily seen from (2.19) that this limiting behaviour should be entirely described by the minimizer of $W^{k-1} \ni (x_1, \ldots, x_{k-1}) \mapsto \sum_{l=1}^{k} g(x_{l-1}, x_l)$. In this section, for $k \in [k_{\max}]$, we write ν_k^γ for the measure ν_k introduced in (2.16) and $T_{x_0}^\gamma$ for the measure T_{x_0} defined in (2.18) corresponding to the parameter γ. Our next result gives criteria under which this minimizer follows a straight line and we have exponential estimates for deviations of trajectories from that.

Let us consider the case where W is a closed ball $\overline{B_r(o)}$ with $r > 0$, and the path-loss function ℓ is strictly monotone decreasing (and satisfies the original condition that it is continuous and positive on $[0, \infty)$). A typical choice [BB09, Section 22.1.2] is $\ell(r) = (1 + r)^{-\alpha}$. Further, let us assume that the intensity measure is rotationally invariant, i.e., $\mu \circ O^{-1} = \mu$ for any orthogonal $d \times d$ matrix O. Under these conditions, we conclude that any minimizer of $W^{k-1} \ni (x_1, \ldots, x_{k-1}) \mapsto \sum_{l=1}^{k} g(x_{l-1}, x_l)$ is of

the form $x_l = c_l x_0$ for $l = 1, \ldots, k-1$ with positive constants $1 > c_1 > \ldots > c_{k-1} > 0$. Moreover, the total probability mass carried by trajectories deviating from the straight line through the transmitter and o in Euclidean distance at least by some fixed positive quantity decays exponentially fast as $\gamma \to \infty$.

More precisely, writing $[[x, y]] = \{\alpha x + (1-\alpha)y | \alpha \in \mathbb{R}\}$ for the line through $x, y \in \mathbb{R}^d$, we state the following.

Proposition 4.1. *Let* $r > 0, W = \overline{B_r(o)}, k_{\max} \geq 2, \ell$ *and* μ *be fixed. Let us assume that* ℓ *is strictly monotone decreasing and* μ *is rotationally invariant.*

(A) For $x_0 \in W$, *let us write*

$$\mathfrak{m}_{k_{\max}}(x_0) = \min_{k \in [k_{\max}]} \min_{x_1,\ldots,x_{k-1} \in W} \sum_{l=1}^{k} g(x_{l-1}, x_l), \qquad x_k = o.$$

Then, for any minimizer $k \in [k_{\max}]$ *and* $x_1, \ldots, x_{k-1}$, *there exist* $1 > c_1 > \ldots > c_{k-1} > 0$ *such that* $x_l = c_l x_0$ *for all* $l \in [k-1]$.

(B) For $k \in [k_{\max}]$ *and* $\varepsilon > 0$, *let us define*

$$D_k^\varepsilon(x_0) = \{(x_1, \ldots, x_{k-1}) \in W^{k-1} \mid \exists l \in \{1, \ldots, k-1\} \colon \mathrm{dist}(x_l, [[x_0, o]]) > \varepsilon\}. \tag{4.1}$$

Then, we have

$$\sup_{x_0 \in W} \sup_{k \in [k_{\max}]} \limsup_{\gamma \to \infty} \frac{1}{\gamma} \log \sup_{(x_1,\ldots,x_{k-1}) \in D_k^\varepsilon(x_0)} T_{x_0}^\gamma(k, x_1, \ldots, x_{k-1}) < 0. \tag{4.2}$$

The proof of the first part of this proposition is based on simple geometric arguments, while the proof of the second part additionally uses the Laplace method. Note that in the first part, a minimizer always exists because W is compact and g is continuous. The proof is carried out in Section 4.2.

We expect that Proposition 4.1 (A) is not true in general if ℓ is not strictly monotone decreasing. Indeed, in this case, modifying the position of a relay in a path that is optimal with respect to interference penalization may not change the penalization at all. This indicates that if $\mathfrak{m}_{k_{\max}}(x_0)$ is attained for some x_0 by a path along $[x_0, o]$, it may also be attained by a non-straight path.

4.2. **Proof of Proposition 4.1.** Throughout the proof, given any number of hops $k \in [k_{\max}]$, we will always assume that $x_k = o$.

We start with proving part (A). Let us fix $x_0 \in \overline{B_r(o)}$. The fact that $(x, y) \mapsto g(x, y)$ is bounded away from 0 implies that for $x_0 = o$, $\mathfrak{m}_{k_{\max}}(x_0)$ is uniquely attained at the 1-hop trajectory from x_0 to $x_1 = o$. Thus, we can assume that $x_0 \neq o$.

Let now $k \in [k_{\max}]$ and $(x_1, \ldots, x_{k-1}) \in \overline{B_r(o)}^{k-1}$. Let us assume that $\sum_{l=1}^{k} g(x_{l-1}, x_l) = \mathfrak{m}_{k_{\max}}(x_0)$. We show that there are $1 > c_1 > \ldots > c_{k-1} > 0$ such that $x_j = c_j x_0$ for all $j \in [k-1]$, proceeding in the following steps.

(i) Let $\mathcal{H}$ denote the closed half-space of $\mathbb{R}^d$ that contains x_0 and whose boundary is orthogonal to the vector from x_0 to o and contains o. Then $(x_1, \ldots, x_{k-1}) \in \mathcal{H}^{k-1}$.

(ii) $(x_1, \ldots, x_{k-1}) \in (\mathcal{H} \cap [x_0, o])^{k-1}$, where we write $[x, y] = \{\alpha x + (1-\alpha)y \colon \alpha \in [0, 1]\}$ for the closed segment between $x, y \in \mathbb{R}^d$.

(iii) $|x_0| > |x_1| > \ldots > |x_{k-1}| > 0$.

We prove these claims respectively as follows.

(i) Assume that the assertion does not hold, then let us define another trajectory $(x_1', \ldots, x_{k-1}') \in \mathcal{H}^{k-1}$ via $x_l' = x_l$ if $x_l \in \mathcal{H}$ and x_l' being the image of x_l under reflection across the boundary

hyperplane of $\mathcal{H}$ otherwise, for all $l \in [k-1]$. The rotation invariance of μ and W, combined with $|x_l| = |x_l'|$, implies that

$$\int_W \mu(\mathrm{d}y)\ell(|x_l - y|) = \int_W \mu(\mathrm{d}y)\ell(|x_l' - y|), \qquad \forall l \in [k_{\max}]. \tag{4.3}$$

But, since $|x_{l-1} - x_l| \geq |x_{l-1}' - x_l'|$ and ℓ is strictly decreasing,

$$\ell(|x_{l-1} - x_l|) \leq \ell(|x_{l-1}' - x_l'|), \tag{4.4}$$

where equality holds if and only if x_{l-1}, x_l are both in $\mathcal{H}$ or both in $\mathbb{R}^d \setminus \mathcal{H}$. We conclude that $\sum_{l=1}^k g(x_{l-1}, x_l) > \sum_{l=1}^k g(x_{l-1}', x_l')$, which contradicts $(x_1, \ldots, x_{k-1})$ being the minimizer in (4.1).

(ii) The case $d = 1$ is trivial. Let us consider the case $d \geq 2$. Assume $(x_1, \ldots, x_{k-1}) \in \mathcal{H}^{k-1}$. Let us define another trajectory $(x_1', \ldots, x_{k-1}') \in (\mathcal{H} \cap [x_0, o])^{k-1}$ such that for all $l \in [k-1]$, x_l' satisfies $|x_l'| = |x_l|$ and $x_l' \in [x_0, o]$. That is, $x_l' = x_0|x_l|/|x_0|$. Then, the radial symmetry of μ implies that (4.3) holds. Furthermore, the fact that ℓ is strictly decreasing but $|x_{l-1} - x_l| \geq |x_{l-1}' - x_l'|$ implies that also (4.4) is true in this case, where equality holds if and only if $x_l = x_l'$ for all $l \in [k-1]$, i.e., if $x_l \in [x_0, o]$ for all $l \in [k-1]$.

(iii) Let $(x_1, \ldots, x_{k-1}) \in [x_0, o]^{k-1}$. In the following argument, we cancel in this trajectory all hops that increase the distance from o. This results in a smaller sum of the SIR terms. Indeed, let us define $i_0 = 0$ and $i_j = \inf\{l \in [k]: |x_l| < |x_{i_{j-1}}|\}$, $j = 1, \ldots, k$. Let m be the largest index j such that $i_j < \infty$, then it is clear that $1 \leq m \leq k$ since $x_0 \neq o$. Now, let us define an m-hop trajectory with relay sequence $(y_1, \ldots, y_{m-1}) = (x_{i_1}, \ldots, x_{i_{m-1}})$, writing $y_0 = x_0$ and $y_m = o$. Let us further define $\varepsilon' = \min_{x,y \in \overline{B_r(o)}} g(x, y) > 0$. Then, since for any $j \in [m-1]$ we have that $|x_{i_j - 1} - x_{i_j}| \geq |x_{i_{j-1}} - x_{i_j}|$, we conclude that

$$\sum_{j=1}^m g(y_{j-1}, y_j) = \sum_{j=1}^m g(x_{i_{j-1}}, x_{i_j}) \leq \sum_{j=1}^m g(x_{i_j - 1}, x_{i_j}) \leq \sum_{l=1}^k g(x_{l-1}, x_l) - (k-m)\varepsilon'.$$

Thus, $(x_1, \ldots, x_{k-1})$ can only minimize (4.1) if $k = m$, that is, if $|x_0| > |x_1| > \ldots > |x_{k-1}| > 0$.

This finishes the proof of part (A) of Proposition 4.1.

As for part (B), we note that the case $d = 1$ is trivial since $D_k^\varepsilon(x_0) = \varnothing$ for all $x_0 \in \overline{B_r(o)}$. Throughout the rest of the proof, let $d \geq 2$. First, we fix $x_0 \in \overline{B_r(o)}$ and $k \in [k_{\max}]$, and we verify that

$$\limsup_{\gamma \to \infty} \frac{1}{\gamma} \log \sup_{(x_1, \ldots, x_{k-1}) \in D_k^\varepsilon(x_0)} T_{x_0}^\gamma(k, x_1, \ldots, x_{k-1}) < -\kappa \tag{4.5}$$

for some $\kappa > 0$ that neither depends on x_0 nor on k. This will imply (4.2).

Again, it is easy to see that if $x_0 = o$, then (4.5) holds for some $\kappa > 0$, let us therefore assume that $x_0 \neq o$. We first verify that there exists $\delta = \delta(\varepsilon) > 0$, independent of x_0 and k, such that

$$\mathfrak{m}_{k_{\max}}^\varepsilon(x_0) = \inf_{(x_1, \ldots, x_{k-1}) \in D_k^\varepsilon(x_0)} \sum_{l=1}^k g(x_{l-1}, x_l) \geq \mathfrak{m}_{k_{\max}} + \delta(\varepsilon). \tag{4.6}$$

In the construction of $(x_1, \ldots, x_{k-1}) \mapsto (x_1', \ldots, x_{k-1}')$ in the proof of (i) above, the fact that $\mathrm{dist}(x_l, [[x_0, o]]) = \mathrm{dist}(x_l', [[x_0, o]])$ for all $l \in [k-1]$ and $k \in [k_{\max}]$ implies that if $(x_1, \ldots, x_{k-1}) \in D_k^\varepsilon(x_0)$, then $(x_1', \ldots, x_{k-1}') \in D_k^\varepsilon(x_0) \cap \mathcal{H}^{k-1}$. It follows that the infimum in (4.6) can be realized along sequences of trajectories that have all their relays $x_1, \ldots, x_{k-1}$ in $\mathcal{H}$.

Let now $(x_1, \ldots, x_{k-1}) \in D_k^\varepsilon(x_0) \cap \mathcal{H}^{k-1}$, and consider the construction of $(x_1, \ldots, x_{k-1}) \mapsto (x_1', \ldots, x_{k-1}')$ in the proof of (ii) above. We observe the following. Since $x_0 \in [x_0, o]$ and

$(x_1, \ldots, x_{k-1}) \in D_k^\varepsilon(x_0)$, there exists $l_1 \in [k]$ such that

$$\text{dist}(x_{l_1}, [[x_0, o]]) > \text{dist}(x_{l_1-1}, [[x_0, o]]) + \frac{\varepsilon}{k} \geq \text{dist}(x_{l_1-1}, [[x_0, o]]) + \frac{\varepsilon}{k_{\max}},$$

where each $[[x_0, o]]$ can also be replaced by $[x_0, o]$. One easily sees that this bound holds uniformly in $x_0 \in W$ and $k \in [k_{\max}]$.

Now, the Pythagoras theorem together with the fact that ℓ is strictly monotone decreasing yields that in this case there exists $\delta'(\varepsilon) > 0$ such that $\ell(|x_{l_1-1} - x_{l_1}|) < \ell(|x'_{l_1-1} - x'_{l_1}|) - \delta'(\varepsilon)$. Note that $\delta'(\varepsilon)$ depends only on ℓ, r and ε but not on k or l_1. On the other hand, by the rotational symmetry of μ, the identity (4.3) holds for all $l \in [k]$ for this choice of the relays x_l and x'_l. Therefore, we conclude that there exists a constant $\delta = \delta(\varepsilon) > 0$ such that for all $n \in \mathbb{N}$ we have

$$\sum_{l=1}^k g(x_{l-1}, x_l) > \sum_{l=1}^k g(x'_{l-1}, x'_l) + \delta(\varepsilon) \geq \mathfrak{m}_{k_{\max}} + \delta(\varepsilon).$$

This implies (4.6), and the construction shows that $\delta(\varepsilon) > 0$ can be chosen independently of x_0 and k.

We now finish the proof of part (B). Let us use the notation $A^\gamma(x_0) = A(x_0)$ for the normalization term in (2.17) corresponding to γ and recall the notation $T_{x_0}^\gamma = T_{x_0}$ from Proposition 4.1. It is clear from the Laplace method [DZ98, Section 4.3] that we have

$$A^\gamma(x_0) = e^{\gamma \mathfrak{m}_{k_{\max}}(x_0) + o(\gamma)} \quad \text{as } \gamma \to \infty.$$

For any $(x_1, \ldots, x_{k-1}) \in D_k^\varepsilon$, using (2.16) and (4.6), we can estimate

$$T_{x_0}^\gamma(k, x_1, \ldots, x_{k-1}) = \frac{\nu_k^\gamma(\mathrm{d}x_0, \ldots, \mathrm{d}x_{k-1})}{\mu(\mathrm{d}x_0)\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_{k-1})} \leq e^{\gamma \mathfrak{m}_{k_{\max}}(x_0) - \gamma \mathfrak{m}_{\max}^\varepsilon(x_0) + o(\gamma)} \leq e^{o(\gamma) - \gamma \delta(\varepsilon)}.$$

We conclude (4.5) (with $\kappa > 0$ being independent of $x_0 \in W$ and $k \in [k_{\max}]$). Thus, part (B) of Proposition 4.1 follows.

5. High local density of users

This section describes the behaviour of the system in regime (3), i.e., in the limit of a high local density of users in a subset of the communication area. We explain both global and local aspects of this limit, respectively in Section 5.1 and Section 5.2.

We consider the following question about the behaviour of our model given by (2.16), assuming always that $k_{\max} \geq 2$.

> Does the density of trajectories increase unboundedly in a densely populated subarea,
> or do the messages avoid such an area for the sake of having lower interference?

In order to give substance to this question, we replace our user density measure μ by

$$\mu^a = \mu + a\text{Leb}|_\Delta \in \mathcal{M}(W), \qquad a \in (0, \infty), \tag{5.1}$$

where $\text{Leb}|_\Delta$ is the Lebesgue measure concentrated on a compact set $\Delta \subseteq W$, seen as a measure on W, where we assume that $\text{Leb}(\Delta) > 0$. We think of Δ as of a set of very high concentration of users and will consider the behaviour of the optimal path trajectory in the limit $a \to \infty$. We will from now on label all objects that depend on μ^a instead of μ with the index a. We will study the measure

$$M^a = \sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \pi_l \nu_k^a, \tag{5.2}$$

where ν_k^a is defined according to (2.16). It can be interpreted as the measure of all the incoming hops at a given location (see also Section 2.3). Note that the total mass $M^a(W)$ is zero if all messages go

directly to the base station without any relaying hop; hence it is a measure for the total amount of hops. Explicitly, we have

$$M^a(\mathrm{d}x) = \mu^a(\mathrm{d}x) \int_W \mu^a(\mathrm{d}x_0) \frac{\sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \int_{W^{k-2}} \prod_{l' \in [k-1]\setminus\{l\}} \mu^a(\mathrm{d}x_{l'})\, \mathrm{e}^{-\gamma \sum_{l'=1}^k g^a(x_{l'-1},x_{l'})}\Big|_{x_l=x}}{\sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \mu^a(\mathrm{d}x_l) \mathrm{e}^{-\gamma \sum_{l=1}^{k-1} g^a(x_{l-1},x_l)}}.$$
(5.3)

Now we are interested in the behaviour of the measure M^a as $a \to \infty$. Since $(x,y) \mapsto \ell(|x-y|)$ is bounded away from 0 on $W \times W$, we first note that the large-a behaviour of the interference term is given by

$$\lim_{a \to \infty} \frac{1}{a} g^a(x,y) = \frac{\int_\Delta \mathrm{d}z\, \ell(|y-z|)}{\ell(|x-y|)} =: g_\Delta(x,y), \quad x,y \in W. \tag{5.4}$$

The limiting function g_Δ measures the interference only in relation with the interference coming from Δ. This ratio will turn out to be relevant and the effective interference term in the limit $a \to \infty$.

5.1. Global effects.
Our first result is that, when the path-loss function $(x,y) \mapsto \ell(|x-y|)$ does not vary much on $W \times W$, the presence of the highly dense area Δ has a strongly repellent effect everywhere in the system and suppresses all the relaying hops; indeed, the total mass of the measure M^a tends to zero exponentially fast as $a \to \infty$, under our general assumptions on the path-loss function ℓ.

Proposition 5.1 (Criterion for exponential decay of the amount of relays). *We have*

$$\sup_{x \in W} \limsup_{a \to \infty} \frac{1}{a} \log \frac{\mathrm{d}M^a}{\mathrm{d}\mathrm{Leb}}(x) < 0 \tag{5.5}$$

if and only if

$$\min_{x_0 \in W} \left[\min_{x_1 \in W} \big(g_\Delta(x_0, x_1) + g_\Delta(x_1, o) \big) - g_\Delta(x_0, o) \right] > 0, \tag{5.6}$$

Remark 5.2.   (i) The inequality (5.5) implies an exponential decay of the total mass of M^a, i.e.,

$$\limsup_{a \to \infty} \frac{1}{a} \log M^a(W) < 0.$$

(ii) Since μ^a is clearly subexponential in $a \to \infty$, (5.5) is equivalent to a uniform exponential decay of the Radon–Nikodym derivative of M^a with respect to μ^a instead of $\mathrm{Leb}|_W$.

(iii) The condition in (5.6) says that the effective interference penalty for a two-hop trajectory is uniformly worse than the one of a direct hop to the origin. This condition involves only one- and two-hop trajectories and is valid even when $k_{\max}$ is much larger than 2.

(iv) Multiplying by two of the three denominators in (5.6) and using that the map $W \times W \ni (x,y) \mapsto \ell(|x-y|)$ is bounded and bounded away from zero, we easily see that (5.6) holds if and only if

$$\min_{x_0, x_1 \in W} \left[\ell(|x_1|) \int_\Delta \ell(|z-x_1|)\, \mathrm{d}z + \ell(|x_0-x_1|) \int_\Delta \ell(|z|)\, \mathrm{d}z - \frac{\ell(|x_1|)\ell(|x_0-x_1|)}{\ell(|x_0|)} \int_\Delta \ell(|z|)\mathrm{d}z \right] > 0. \tag{5.7}$$

(v) A sufficient condition for (5.6) to hold is as follows. Let $p \in (0,1]$ be such that $p\ell_{\max} = \ell_{\min}$, where $\ell_{\max} = \max_{x,y \in W} \ell(|x-y|)$ and $\ell_{\min} = \min_{x,y \in W} \ell(|x-y|)$ are the maximal and the minimal path-loss values in the system, respectively. Then, a lower bound for the left-hand side of (5.7) is $\ell_{\max}^2 \mathrm{Leb}(\Delta)(2p^2 - \frac{1}{p})$. This is positive as long as p is larger than $2^{-1/3} \approx 0.794$.

Similarly, an upper bound on the left-hand side of (5.7) in terms of p is $\ell_{\max}^2 \mathrm{Leb}(\Delta)(2-p^3)$, but this is larger than zero for all $p \in (0,1]$, hence such a general estimate cannot be used for disproving (5.7) in any case.

(vi) In our numerical results in Examples 8.1 and 8.2 with $W = \Delta$, the condition (5.6) does not hold.

Proof of Proposition 5.1. Consider the quantity on the left-hand side of (5.5). Taking the limit $a \to \infty$, we obtain for fixed $x, x_0 \in W$ for the numerator of (5.3)

$$\lim_{a \to \infty} \frac{1}{a} \log \Big[\sum_{k=1}^{k_{\max}} \sum_{l=1}^{k-1} \int_{W^{k-2}} \prod_{l' \in [k-1] \setminus \{l\}} \mu^a(\mathrm{d}x_{l'}) \exp \Big(-\gamma \sum_{l'=1}^{k} g^a(x_{l'-1}, x_{l'}) \Big|_{x_l = x} \Big) \Big]$$

$$= -\gamma \min_{k \in [k_{\max}] \setminus \{1\}} \min_{l \in [k-1]} \min_{x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{k-1} \in W} \sum_{l'=1}^{k} g_\Delta(x_{l'-1}, x_{l'}) \Big|_{x_l = x}.$$

(5.8)

On the other hand, for the denominator of (5.3) for x_0 fixed, we have

$$\lim_{a \to \infty} \frac{1}{a} \log \Big[\sum_{k=1}^{k_{\max}} \int_{W^{k-1}} \prod_{l=1}^{k-1} \mu^a(\mathrm{d}x_l) \exp \Big(-\gamma \sum_{l=1}^{k-1} g^a(x_{l-1}, x_l) \Big) \Big]$$

$$= -\gamma \min_{k \in [k_{\max}]} \min_{x_1, \ldots, x_{k-1} \in W} \sum_{l=1}^{k} g_\Delta(x_{l-1}, x_l).$$

(5.9)

These two assertions follow from the Laplace method [DZ98, Section 4.3] in a standard way, since the a-dependence of the integrating measure μ^a is clearly subexponential. Hence, we obtain that

$$\lim_{a \to \infty} \frac{1}{a} \log M^a(\mathrm{d}x) = -\gamma \min_{x_0 \in W} \Big[\min_{k \in [k_{\max}] \setminus \{1\}} \min_{l \in [k-1]} \min_{x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{k-1} \in W} \sum_{l'=1}^{k} g_\Delta(x_{l'-1}, x_{l'}) \Big|_{x_l = x}$$

$$- \min_{k \in [k_{\max}]} \min_{x_1, \ldots, x_{k-1} \in W} \sum_{l=1}^{k} g_\Delta(x_{l-1}, x_l) \Big].$$

(5.10)

Note that after taking supremum over $x \in W$ on the right-hand side of (5.10), we obtain a negative number if and only if

$$\min_{x_0 \in W} \Big[\min_{k \in [k_{\max}] \setminus \{1\}} \min_{x_1, \ldots, x_{k-1} \in W} \sum_{l=1}^{k} g_\Delta(x_{l-1}, x_l) - g_\Delta(x_0, o) \Big] > 0.$$

(5.11)

Now, assume that the condition (5.6) does not hold. Then we may pick $x_0', x_1' \in W$ with $(g_\Delta(x_0', x_1') + g_\Delta(x_1', o) - g_\Delta(x_0', o)) \leq 0$. But this implies that (5.11) is false, as is shown by taking $k = 2$, $x_0 = x_0'$ and $x = x_1'$. We conclude that (5.5) does not hold.

Conversely, let us assume that (5.5) is not satisfied and let us conclude that (5.6) also does not hold. Using (5.5) and (5.10), we can choose $x_0 \in W$, $k \in [k_{\max}] \setminus \{1\}$ and $x_1, \ldots, x_{k-1} \in W$ such that

$$\sum_{l=1}^{k} g_\Delta(x_{l-1}, x_l) \leq g_\Delta(x_0, o), \qquad x_k = o.$$

(5.12)

Let k be minimal for x_0 with this property. We show that there exists $x_0', x_1' \in W$ such that $g_\Delta(x_0', x_1') + g_\Delta(x_1', o) \leq g_\Delta(x_0', o)$, therefore (5.6) does not hold. Indeed, if this is not the case for $x_0' = x_{k-2}$ and $x_1' = x_{k-1}$, then we have

$$\sum_{l=1}^{k-2} g_\Delta(x_{l-1}, x_l) + g_\Delta(x_{k-2}, o) \leq \sum_{l=1}^{k} g_\Delta(x_{l-1}, x_l) \leq g_\Delta(x_0, o) < \sum_{l=1}^{k-2} g_\Delta(x_{l-1}, x_l) + g_\Delta(x_{k-2}, o),$$

where in the last step we used the minimality of k for x_0. This is in contradiction with (5.6) and thus the proof is concluded. $\square$

5.2. **Local effects.** The condition (5.6) can be applied to any $\Delta \subseteq W$ with $\mathrm{Leb}(\Delta) > 0$, in particular also to $\Delta = W$. In this sense, Proposition 5.1 is non-spatial. We now discuss among what conditions the spatial effect that the quality of service (interference penalization with interference coming only from Δ) is significantly worse for messages relaying through a neighbourhood of Δ than through an area sufficiently far away from Δ occurs in our model. For simplicity, we consider only the case $k_{\max} = 2$, a very small set Δ and a special choice of the path-loss function. We will give arguments that suggest that, for any large a, it is strictly suboptimal to relay through a neighbourhood of Δ as opposed to circumventing Δ sufficiently far.

Analogously to (5.8)–(5.9), the large-a limit for the mass of all relaying hops from x_0 into a set $A \subset W$ (assumed being equal to the closure of its interior) and further to o is given by

$$- \lim_{a \to \infty} \frac{1}{a} \log T_{x_0}^a(2, A) = \gamma \Big[\Xi_{x_0}(A) - \min \big\{ g_\Delta(x_0, o), \Xi_{x_0}(W) \big\} \Big], \tag{5.13}$$

where

$$\Xi_{x_0}(A) = \min_{x_1 \in A} [g_\Delta(x_0, x_1) + g_\Delta(x_1, o)].$$

We want to discuss under what conditions $\Xi_{x_0}(A)$ is smaller for sets A that are bounded away from Δ than for A being a neighbourhood of Δ. For simplicity, let us do that for $W = \mathbb{R}^d$ and very small sets $\Delta = B_r(y_0)$ with $r \ll 1$ only, i.e., we approximate

$$g_\Delta(x, y) \approx |\Delta| \frac{\ell(|y - y_0|)}{\ell(|y - x|)}, \qquad x, y \in \mathbb{R}^d. \tag{5.14}$$

Hence, we will put $\Delta = \{y_0\}$ and discuss the function

$$f_{x_0, y_0}(\varepsilon) = \min_{x_1 \in W : |x_1 - y_0| = \varepsilon} \left[\frac{\ell(|x_1 - y_0|)}{\ell(|x_0 - x_1|)} + \frac{\ell(|y_0|)}{\ell(|x_1|)} \right], \qquad \varepsilon \geq 0.$$

This is an approximation of $\Xi_{x_0}(\partial B_\varepsilon(y_0))$. We will see that, under quite general conditions, $f_{x_0, y_0}(\varepsilon) < f_{x_0, y_0}(0)$ for all $\varepsilon \in [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$. This means that, for all sufficiently large a, the probability weight for trajectories $x_0 \to B_{\varepsilon_0 - \delta}(y_0) \to o$ is exponentially smaller than the one for trajectories $x_0 \to B_{\varepsilon_0}(y_0)^{\mathrm{c}} \to o$ for any $\varepsilon_0 > \delta > 0$.

To do this, use the triangle inequality and the monotonicity of ℓ to see that

$$f_{x_0, y_0}(\varepsilon) \leq \widetilde{f}_{x_0, y_0}(\varepsilon) := \frac{\ell(\varepsilon)}{\ell(|x_0 - y_0| + \varepsilon)} + \frac{\ell(|y_0|)}{\ell(|y_0| + \varepsilon)}.$$

Note that $\widetilde{f}_{x_0, y_0}(0) = f_{x_0, y_0}(0)$ and that

$$\widetilde{f}'_{x_0, y_0}(0) = \frac{\ell'(0)}{\ell(|x_0 - y_0|)} - \frac{\ell(0)\ell'(|x_0 - y_0|)}{\ell(|x_0 - y_0|)^2} - \frac{\ell'(|y_0|)}{\ell(|y_0|)}.$$

Note that for the choice $\ell(r) = (1 + r)^{-\alpha}$ for some $\alpha > 0$, this is negative as soon as $|x_0 - y_0|(1 + |x_0 - y_0|)^{\alpha - 1} > (1 + |y_0|)^{-1}$, i.e., as soon as y_0 is sufficiently far away from o, given the distance of the transmission site x_0 from y_0. This proves the announced conclusion that a two-hop transmission from x_0 to the origin is strictly not optimal if the relaying hop uses a neighbourhood of y_0; here we used no information about the spatial relation of the three sites x_0, y_0 and o, but the fact that $\ell'(0) < 0$. However, for the path-loss function $\ell(r) = \min\{1, r^{-\alpha}\}$, this argument does not work, since $\widetilde{f}'_{x_0, y_0}(0) > 0$ (because $\ell'(0) = 0$).

6. MODELLING DISCUSSIONS AND CONCLUSIONS

In this section, we explain our motivation for some aspects of the model and for the questions that we address. In Section 6.1 we interpret the SIR-related quantities of the model, in Section 6.2 we argue about the relevance of our Gibbsian ansatz, and in Section 6.3 we explain possible extensions of the model via allowing users to send no or multiple messages.

6.1. **The notion of SIR and the interference term.** This section includes the following modelling discussions. In Section 6.1.1, we explain the conventional definition of SIR and its relation to our notion of SIR and discuss about the continuity of the path-loss function at 0. In Section 6.1.2, we comment on the relevance of our choice of the interference penalization term $\mathfrak{S}(s)$ in (2.3). Finally, in Section 6.1.3, we sketch more realistic versions of our model involving time.

6.1.1. *The notion of SIR and its adaptation to the high-density setting.* Note that the conventional definition of interference of a transmission from X_i to x is $\sum_{j \in I^\lambda \setminus \{i\}} \ell(|X_j - x|)$, in contrast to our definition in (2.2), where we added a factor of $\frac{1}{\lambda}$, following [HJKP18, Section 1]. According to this convention, we should say "total received power" instead of "interference", cf. [KB14, Section II.]. As we are interested in the limit $\lambda \to \infty$, where it makes no difference whether or not we add $\frac{1}{\lambda}\ell(|X_i - x|)$ to the denominator, we stick to our notions "SIR" and "interference". For the same reason, our model does not include noise. However, note also our additional factor of $1/\lambda$, which we think is appropriate, at least mathematically, to our setting, in which we consider the high-density limit $\lambda \to \infty$. We actually scale the "usual" SIR by the density parameter. Indeed, in order to cope with an enormous number of messages in a system with one base station and a fixed bandwidth, one can either distribute the messages over a longer time stretch or decompose the messages into many smaller ones. The factor of $1/\lambda$ is a crude approximation of a combination of these two strategies.

The assumption that the path-loss function ℓ is continuous at 0 comes from [GT08, HJKP18] and differs from the works [GK00, KB14], which make mathematical use of the perfect scaling of the path-loss function $\ell(r) = r^{-\alpha}$, which is for this reason one of the standard choices. However, for small r, this is an unrealistic choice, cf. [GK00, Section I.A], [GT08, Section I.].

6.1.2. *The interference term.* The interference term $\mathfrak{S}(s)$ in (2.3) quantifies the quality of the transmission of the messages in case they use the trajectories s^i from X_i to o. The choice of the *reciprocals* of the SIRs comes from the fact that the *bandwidth* used for a transmission is defined [SPW07] as

$$\frac{\varrho}{\log_2(1 + \text{SIR}(\cdot))}, \tag{6.1}$$

where ϱ is the data transmission rate, and SIR is defined as in (2.2) without the factor of $1/\lambda$ in the denominator of (2.2). This quantity is of order $1/\lambda$ for λ large, under the assumption that $L_\lambda \Rightarrow \mu$. In the high-density setting $\lambda \to \infty$ that we study, (6.1) can be approached well by (a constant times) the reciprocals of the SIR, since $\log(1 + x) \sim x$ as $x \to 0$. [SPW07, Section 3] suggests that in case of multihop communication, the used bandwidth equals the sum of the used bandwidth values corresponding to the individual hops, which explains our choice of the sum over l in (2.3). We note that the idea of using a sum of reciprocals of SIR values as a cost function to be minimized appeared also in [BC12].

6.1.3. *Time-dependent versions of the Gibbsian model.* We note that the notion of interference can be made more realistic according to [GK00, Section I.A] via introducing time dependence in our model. E.g., one introduces $k_{\max}$ discrete time slots indexed by $[k_{\max}]$, and for $l \in [k_{\max}]$, the lth hop of any message trajectory is assumed to happen at time l. Then, the interference of a transmission at time l is obtained from the starting points of all hops that happen at the same time. The SIR is defined analogously to (2.2) but with this notion of interference, which depends on the entire message trajectories rather than only on the users. Time-dependent versions of our model can be set up in various ways; for example, one could allow for messages standing still or for a longer time horizon and users transmitting multiple messages over time. The new notion of SIR comes with significant changes in the behaviour of the system in the high-density limit, and we decided to defer such investigations to a later work.

6.2. **Why a Gibbsian ansatz?** Let us give some motivation for our choice of the probability of the form $\exp\{-(\gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s))\}$ (modulo a counting factor) for a trajectory collection s in (2.6).

6.2.1. *Penalization of interference.* In a mathematical description of a telecommunication system, one typically requires that the SIR be larger than a given threshold $\tau > 0$, in order that the signal can be successfully transmitted. However, our model is designed to investigate the global throughput, where we do not want to consider any single message, but the total quality of transmission in the entire system. This quantity $\mathfrak{S}(s)$ is the sum of all the reciprocal values of the SIRs of all the (hops of the) messages, which we explained in Section 6.1.2. It is exponentially weighted with a negative factor, which "softly" keeps all the SIRs at high values on average.

One can also modify our Gibbs distribution in such a way that trajectory families exhibiting hops with $\mathrm{SIR}(s_{l-1}^i, s_l^i, X^\lambda)$ less than or equal to τ have probability zero, simply by changing the interference penalization value (2.3) to ∞ for such families, similarly to [BC12, Section III.A]. For τ large enough, almost surely, the modified model is well-posed for all $\lambda > 0$ sufficiently large. This means a change from the penalization function $x \mapsto \gamma/x$ (applied to $\mathrm{SIR}(s_{l-1}^i, s_l^i, X^\lambda)$) into the function $x \mapsto \infty \times \mathbb{1}_{[0,\tau]}(x)$. We expect that an analogue of Proposition 2.1 in [KT17, Section 5] is valid, but additional topological problems have to be addressed.

One of our motivations is to explore the physical effect of the penalization of the joint probability of the random paths, which are *a priori* randomly picked with equal probability: Does the (soft) requirement of a good transmission quality force the trajectories already to choose geometrically the shortest route? What hop lengths do they choose? We would like to understand the interplay between entropy and interference-energy and the result coming out of this by optimizing their relation.

6.2.2. *Penalization of congestion.* The congestion term $\mathfrak{M}(s)$ in (2.5) counts the ordered pairs of incoming hops arriving at the relays in the system. This is certainly an important characteristics of the quality of service, as too high an accumulation of many messages at relays results in a delay. Hence, it is natural to suppress the occurrence of such events, in order to increase the value of the model for realistic modeling.

An important property of this term is that it introduces dependence between the trajectories of different messages, unlike the interference term. Indeed, while $\mathfrak{S}(s)$ can be decomposed into a sum of terms depending on the respective trajectories, each summand in $\mathfrak{M}(s)$ involves many different trajectories.

6.2.3. *Relation to an optimization problem via Monte Carlo Markov chains.* In the light of the above, it is certainly interesting to minimize the cost function $s \mapsto \gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s)$ for fixed $\beta, \gamma \in (0, \infty)$. Computationally, this is in general a hard problem for high densities λ because the cardinality of $\mathcal{S}_{k_{\max}}(X^\lambda)$ increases super-exponentially in $N(\lambda)$, and $N(\lambda)$ is of linear order in λ. Thus, computing all values of $s \mapsto \gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s)$ and then extracting the maximum is only feasible for small λ.

Now, our Gibbsian trajectory distribution opens the possibility to optimize this cost function via the well-known approach of *simulated annealing*. Furthermore, for λ large, it is substantially less complex to realize the Gibbs distribution using Monte Carlo Markov chains than to directly minimize the cost function.

Indeed, our Gibbs distribution favours trajectory collections with small values of the cost function. Now, let us investigate the computational complexity of the numerical realization of the Gibbs distribution $\mathrm{P}_{\lambda,X^\lambda}^{\gamma,\beta}$, using Monte Carlo Markov chains (see e.g. [H02]). The recent master's thesis of Morgenstern [M18] investigates this question. Given the intensity λ and the realization of the point process of users X^λ, the author finds irreducible and aperiodic Markov chains on the state space

$\mathcal{S}_{k_{\max}}(X^\lambda)$, both of Gibbs sampler and Metropolis types, having the Gibbs distribution as their stationary distribution. These chains therefore converge towards $\mathrm{P}^{\gamma,\beta}_{\lambda,X^\lambda}$ as the number of Markovian steps tends to infinity.

The following results have been verified in [M18]:

- The chains can be constructed in such a way that computing their transition matrices takes only a polynomial number of operations in $N(\lambda)$.
- For both types of chains, in the limit $N(\lambda) \asymp \lambda \to \infty$, the mixing time is at most exponential in λ. This, together with the previous observation, provides an at most exponential upper bound on the number of operations needed in order to simulate the Gibbs distribution up to a given error $\varepsilon > 0$ in total variation distance. This is certainly much more efficient than evaluating all the trajectory collections.
- In a variant of the Gibbsian model where any user (relay) can receive at most a fixed number $m_{\max} \in \mathbb{N}$ of incoming hops, the mixing time is even polynomial in λ. This is a realistic modelling since, in most applications in multihop networks, each relay has a bounded capacity for receiving incoming hops (see e.g. [HJP18, HJ17]). Note that this variant is always well-defined if $m_{\max} \geq k_{\max} - 1$. The results of [KT17, Section 1] also show that for large $m_{\max}$, this variant behaves similarly to the original model in the high-density limit.
- These Monte Carlo Markov chains can also be used in order to find the optimum of the cost function $s \mapsto \gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s)$ for a fixed λ and a fixed realization of X^λ, using simulated annealing. Here, one lets the transition probability of the t-th step of the chain depend on t via replacing (γ, β) by (γ_t, β_t) such that $\gamma_t, \beta_t \to \infty$ sufficiently slowly as $t \to \infty$. [M18, Theorem 7.1] shows that if one chooses $\beta_t = \frac{\beta}{\gamma}\gamma_t \leq c_0 \log t$ for a suitably chosen $c_0 = c_0(\lambda) \asymp \lambda/N(\lambda)^2$, then the Markov chain converges to the uniform distribution on the set of minimizers of the cost function.

6.3. **Extensions: sending no or multiple messages.** One easily sees from the proofs in [KT17, Sections 2–5] that Proposition 2.1 can be extended to the situation where users send no message or multiple messages. This models the standard situation in which large messages are cut into many smaller ones, who independently find their ways through the system.

For this, we have to enlarge the trajectory probability space: to each user $X_i \in X^\lambda$, we attach the number $P_i \in \mathbb{N}_0$ of transmitted messages, and for each $j \in \{1, \ldots, P_i\}$, there is an independent trajectory $X_i \to o$. The empirical trajectory measure $R_{\lambda,k}(\cdot)$ must be augmented by these trajectories. The main additional assumption then is that $\sum_{k=1}^{k_{\max}} \pi_0 R_{\lambda,k}(\cdot)$ converges to some measure $\mu_0 \in \mathcal{M}(W)$ with $0 \neq \mu_0 \ll \mu$.

The interference term also has to be changed. The number P_i can be interpreted as a *signal power* of the user X_i. Thus, according to [BB09, Sections 2.3.1, 5.1], the SIR of his transmission of a message to $x \in W$ should be defined as follows

$$\mathrm{SIR}((X_i, P_i),\ x,\ ((X_j, P_j))_{j \in I^\lambda}) = \frac{\ell(|X_i - x|)P_i}{\frac{1}{\lambda}\sum_{j \in I^\lambda} \ell(|X_j - x|)P_j}.$$

One could also incorporate (possibly random) sizes of the messages, which would require an additional enlargement of the trajectory space.

7. Game-theoretic interpretation of the optimization problem

In Section 6.2.3 we explained how our model that we introduced in Section 2.1 can be employed for obtaining a stochastic simulation algorithm for finding minimizer(s) s of $\gamma\mathfrak{S}(s) + \beta\mathfrak{M}(s)$, i.e., including the congestion term. In this section, we give a more thorough discussion of this optimization problem

from a game-theoretical point of view. In particular, we explain in which sense the optimum in our model is selfish or non-selfish and give a number of explicit examples for illustration. Note that in the term $\mathfrak{S}(s)$ there is no interaction between the trajectories (only with the users), but in the term $\mathfrak{M}(s)$. We therefore keep both $\beta > 0$ and $\gamma > 0$ fixed.

Let $X^\lambda = \{X_1, \ldots, X_n\}$ be fixed, where $n \in \mathbb{N}$. For the rest of this section, we simplify the notation as follows. We write $\mathcal{S} = \mathcal{S}_{k_{\max}}(X^\lambda)$ and for $i \in [n]$, $\mathcal{S}^i = \mathcal{S}^i_{k_{\max}}(X^\lambda)$. Let now $s = (s^i)^n_{i=1} \in \mathcal{S}$ be a collection of message trajectories. We recall that for $i \in [n]$, s^i_{-1} is the number of hops taken by the ith trajectory s^i sent out from X_i to o. Then, in terms of interference and congestion, the *individual cost* $C_i(s)$ of s^i with respect to the entire collection s is the individual interference penalization of s^i, together with the congestion penalization at all the relays that s^i uses:

$$C_i(s) = \gamma \sum_{l=1}^{s^i_{-1}} \mathrm{SIR}^{-1}(s^i_{l-1}, s^i_l, X^\lambda) + \beta \sum_{l=1}^{s^i_{-1}-1} \sum_{j=1}^{n} (m_j(s) - 1) \mathbb{1}\{s^i_l = X_j\}. \tag{7.1}$$

The *total cost* of the trajectory collection s is defined as

$$C(s) = \sum_{i=1}^{n} C_i(s) = \gamma \mathfrak{S}(s) + \beta \mathfrak{M}(s) = \gamma \sum_{i=1}^{n} \sum_{l=1}^{s^i_{-1}} \mathrm{SIR}^{-1}(s^i_{l-1}, s^i_l, X^\lambda) + \beta \sum_{j=1}^{n} m_j(s)(m_j(s) - 1).$$

We say that s is *system-optimal* if $C(s) \leq C(s')$ for all $s' \in \mathcal{S}$.

For a collection $s = (s^i)^n_{i=1}$ of trajectories we write $s = s_i(s^i, s^{-i})$, where $s^{-i} = (s^j)_{j \in [n], j \neq i}$. Now, given i and $s^{-i} = (s^j)_{j \neq i}$ with $s^j \in \mathcal{S}^j$ for all $j \neq i$, a *best response* of the ith user for s^{-i} is $u^i \in \mathcal{S}^i$ such that $C_i(s_i(u^i, s^{-i})) \leq C_i(s_i(s^i, s^{-i})) = C_i(s)$ for all $s^i \in \mathcal{S}^i$. We say [NRTV07, Section 1.3.3] that $s = (s^i)^n_{i=1}$ is a *pure Nash equilibrium* if s^i is a best response for $s^{-i} = (s^j)_{j \neq i}$ for all $i \in [n]$.

Claim 7.1. *For $\beta, \gamma, \lambda > 0$, given X^λ (with $n > 0$), a pure Nash equilibrium always exists.*

Proof. The claim follows from the well-known result [NRTV07, Theorem 18.12] that unweighted atomic congestion games always have a pure Nash equilibrium. Indeed, the cost functions C_i, $i \in [n]$, and C are the individual respectively total costs in an unweighted atomic congestion game (atomic instance) [NRTV07, Section 18], which is defined as follows. For each $i \in [n]$, the set of all possible paths $s^i \in \mathcal{S}^i$ of length at most $k_{\max}$ from X_i to o via users in $X_j \in X^\lambda$ without visiting the same X_j twice can be seen as the set of the strategies of the ith user (player) X_i. Indeed, for the sake of optimization of individual and total costs, we can neglect trajectories with loops since removing any loop from the trajectory of the ith user strictly decreases C_i and does not increase C_j for $j \neq i$, neither C. Since almost surely the points of X^λ are pairwise distinct, the sets of possible strategies of different users are disjoint, in other words, the game is unweighted. Further, each user has a finite number of strategies.

The cost function in this game is defined as follows. Each hop from X_i to X_j has a constant cost equal to $\gamma \mathrm{SIR}^{-1}(X_i, X_j, X^\lambda)$, and each used relay X_j has a linear cost equal to $\beta(m_j(s) - 1)$, depending on the trajectory collection s. This way, by (7.1), the cost of the strategy of X_i corresponding to $s \in \mathcal{S}$ equals $C_i(s)$. Thus, the claim follows. $\square$

Now, if there exists a system-optimal $s \in \mathcal{S}$ such that $C(s) < C(s')$ for all Nash equilibria s', then we call s a *non-selfish optimum*, since there exists $i \in [n]$ such that s^i is not the best response of the ith user for the remaining coordinates of the trajectory collection. Example 7.2 shows a two-dimensional example that has a non-selfish optimum, and Remark 7.4 tells more about the relation of the individual and the total costs.

Example 7.2. Let $d = 2$, $\lambda = 1$ and $k_{\max} = 2$, and let $X^\lambda = X^1 = \{X_1, X_2, X_3\}$, ℓ and $\gamma > 0$ be chosen in the following way. X_1, X_2, X_3 and o, X_2, X_3 are vertices of two equilateral triangles with X_1 being in the interior of the latter triangle, so that $|X_1 - X_2| = |X_1 - X_3|$ and $|X_2| = |X_3|$, so
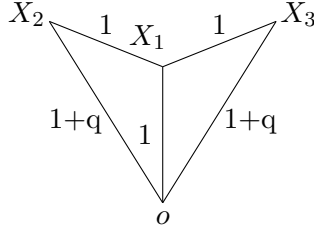
FIGURE 1. Interference weights per hop in Example 7.2. In the relevant cases, the congestion at X_1 is $\beta y(y-1)$, where y is the number of elements of $\{X_2, X_3\}$ relaying through X_1.

Number of hops of s^2	Number of hops of s^3	$C_2(s)$	$C_3(s)$	$C(s)$
1	1	$2+q$	$2+q$	$5+2q$
2	1	2	$2+q$	$5+q$
2	2	$2+\beta$	$2+\beta$	$5+2\beta$

TABLE 1. Individual and total costs in standard representatives of the relevant cases in Example 7.2.

that $\gamma\mathrm{SIR}^{-1}(X_1, o, X^1) = \gamma\mathrm{SIR}^{-1}(X_i, X_1, X^1) = 1$ and $\gamma\mathrm{SIR}^{-1}(X_i, o, X^1) = 1+q$ for all $i \in \{2,3\}$ for some $q > 0$ (see Figure 1).

The boundedness of $\ell(|\cdot - \cdot|)$ away from 0 on $W \times W$ implies that for any $\beta > 0$ and $i \in \{2,3\}$, any $s^i \in \mathcal{S}^i$ that uses some X_j with $j \in \{2,3\}$ as a relay is suboptimal both with respect to total and individual costs. Indeed, leaving out this relay and moving on to the next hop of the same trajectory instead decreases $C_i(s)$ without increasing any $C_m(s)$, $m \neq i$. Using analogous arguments, one easily concludes that in any system-optimal trajectory and also in any Nash equilibrium, X_1 submits directly to o, and the two users X_2, X_3 use either the direct link to o or the two-hop path via X_1 to o. Table 1 shows the individual costs and the total cost in some standard representatives of these cases.

The positive parameters q and β can be chosen such that the following holds. Given that X_2 uses its two-hop path $X_2 \to X_1 \to o$, the best response of X_3 is to also use its two-hop path $X_3 \to X_1 \to o$ and vice versa, so that both users using their two-hop paths forms the unique Nash equilibrium, but the system optima are the trajectory collections in which only one of them relays via X_1 and the other one submits directly to o. According to Table 1, this holds if $q > 0$ and $\beta \in (q/2, q)$. Thus, in such cases, the optimum is non-selfish.

Similar effects occur in all dimensions $d \geq 2$, with $d+1$ users $X_1, X_2, \ldots, X_{d+1}$ situated so that $|X_j - X_1| = |X_i - X_1|$ and $|X_1| < |X_i| = |X_j|$ for all $i, j \geq 2$. In such cases, one can choose the parameters in such a way that for all $j \geq 2$, knowing that X_1 transmits directly to o and each X_i, $j \neq i \geq 2$ relays through X_1, the best response of X_j is to use also the relayed link via X_1, but with respect to total costs it would be better if X_j transmitted directly to o. Note that if this holds, it may still happen that neither of these two joint strategies is system-optimal. ◇

Remark 7.3. In the setting of our Gibbsian model, Nash equilibria are not necessarily unique. Consider Example 7.2 in the boundary case $\beta = q$. Then one easily checks that the system exhibits three different Nash equilibria, namely the three ones that appear in Table 1. Also for $\beta > q$, there are two Nash equilibria, namely the ones where exactly one of s^2, s^3 transmits directly to o and the other one via X_1, by the symmetry between X_2 and X_3.

Remark 7.4. Now we show that in general, if plugging in an additional relay to a trajectory decreases the total cost, then it also decreases the individual cost of the transmitter of that trajectory. Thus,
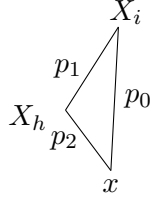
FIGURE 2. A situation opposite to Example 7.2 is not possible.

in particular, a situation opposite to Example 7.2 is not possible, that is, for any choice of ℓ, β, γ, it cannot happen that $C(s)$ is a non-selfish system optimum while $C_2(s)$ or $C_3(s)$ is a Nash equilibrium.

Indeed, consider Figure 2 with $\lambda > 0$, $X_i, X_h \in X^\lambda$ and $x \in X^\lambda \cup \{o\}$, where the direct hop from X_i to x has interference penalization $p_0 > 0$, while the two-hop path via X_h has interference penalization $p_1 + p_2$ with $p_1, p_2 > 0$. Now, if $s^{-i} = (s^j)_{j \neq i}$ is given and the number of incoming hops at X_h coming from all trajectories but the one of X_i equals $m \geq 0$, then the direct link from X_i to x has individual cost $p_0 + K$ and the $X_i \to X_h \to o$ relayed link has individual cost $m + p_1 + p_2 + K$, for some $K \geq 0$. On the other hand, the total cost of the collection with the $X_i \to x$ direct link is $2m + p_1 + p_2 + K'$, and the one with the $X_i \to X_h \to x$ relayed link is $p_0 + K'$, for some $K' \geq 0$. Thus, if plugging in the relay X_h increases the individual cost C_i, then it also increases the total cost C. This implies the claim.

8. NUMERICAL STUDIES

In this section, we give numerical illustrations of various properties of the minimizer $(\nu_k)_{k \in [k_{\max}]}$ of (2.16), which describes the limiting empirical trajectory measure according to Proposition 2.1. We consider $k_{\max} = 2$, $d = 1, 2$, ℓ satisfying $\ell(r) \sim r^{-4}$ as $r \to \infty$, W being a ball sufficiently large such that both direct communication and two-hop communication are non-negligible, and μ being the Lebesgue measure on W. We do not consider congestion, i.e., we put $\beta = 0$. We will look at the areas where one-hop and two-hop communication dominate, respectively, and the approximation of a straight line of the latter trajectories for $d = 2$. We will see that the effects that we proved in Section 4 in the limit $\gamma \to \infty$ are already very pronounced for $\gamma = 1$.

First, let us choose $\ell(r) = \min\{1, r^{-4}\}$. Let $W = \overline{B_\varrho(o)}$ be a ball around the origin o. We will pick ϱ sufficiently large so that the effect of the path-loss function ℓ is strong enough in the sense that we can study areas in W from which a direct hop to o is preferred and areas from which a two-hop trajectory is preferred. We are interested in seeing how sharp the transition between these two areas is. By rotational invariance, we expect that the first area is a centred ball and the second the complement of a ball in W. Hence, we do not lose much when going to $d = 1$. Using the arguments of the first paragraph of Section 4.1, for large γ, we expect the transition close to the point where the interference term gives the transition from optimality of one-hop trajectories to two-hop trajectories, i.e., at the radius $|x_0|$, where the number

$$g(x_0, o) - \min_{x_1 \in W} \big(g(x_0, x_1) + g(x_1, o) \big) \tag{8.1}$$

switches the sign. Let r_0^* denote that point. Our main question is whether already for moderate values of γ, we see a pronounced transition in the measures $\nu_1(\mathrm{d}x_0)$ and $\pi_0 \nu_2(\mathrm{d}x_0)$ of the form that $\nu_1(\mathrm{d}x_0) \approx \mu(\mathrm{d}x_0)$ for all x_0 with $|x_0| < r_0^*$ and $\pi_0 \nu_2(\mathrm{d}x_0) \approx \mu(\mathrm{d}x_0)$ for all x_0 with $|x_0| > r_0^*$, with a fast change around r_0^*. Observe that both ν_1 and $\pi_0 \nu_2$ have densities that are positive throughout the interior of the support of μ.
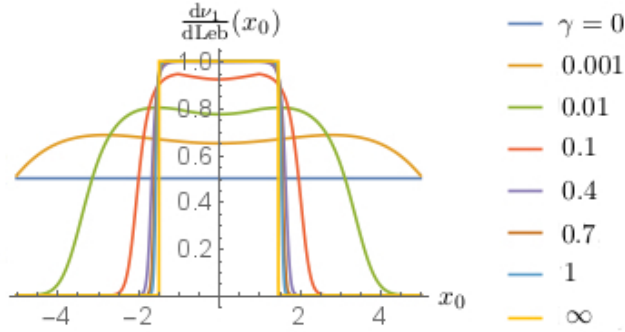
FIGURE 3. The graphs of $x_0 \mapsto \frac{d\nu_1}{d\mathrm{Leb}}(x_0)$ as in Example 8.1 for $\gamma = 0, 0.001, 0.01, 0.1, 0.4, 0.7, 1, \infty$.

In the following one-dimensional numerical example, the answer is yes, already for $\gamma = 1$. The plots presented here were created using Wolfram *Mathematica*.

Example 8.1. Let $k_{\max} = 2$, $d = 1$, $W = [-5,5] = \overline{B_5(o)} \subset \mathbb{R}$, $\mu = \mathrm{Leb}|_W$ and $\ell(r) = \min\{1, r^{-4}\}$. According to Proposition 2.1, the minimizing measures $\Sigma = (\nu_1, \nu_2)$ are given as follows. With

$$\frac{1}{A(x_0)} = \exp\left(-\gamma\frac{\int_{-5}^{5}\ell(|y|)\mathrm{d}y}{\ell(|x_0|)}\right) + \frac{1}{10}\int_{-5}^{5}\mathrm{d}x_1\exp\left(-\gamma\left(\frac{\int_{-5}^{5}\ell(|y-x_1|)\mathrm{d}y}{\ell(|x_0-x_1|)} + \frac{\int_{-5}^{5}\ell(|y|)\mathrm{d}y}{\ell(|x_1|)}\right)\right), \quad (8.2)$$

we have

$$\nu_1(\mathrm{d}x_0) = \mathrm{d}x_0 A(x_0)\exp\left(-\gamma\frac{\int_{-5}^{5}\ell(|y|)\mathrm{d}y}{\ell(|x_0|)}\right) \quad (8.3)$$

and

$$\nu_2(\mathrm{d}x_0, \mathrm{d}x_1) = \frac{1}{10}\mathrm{d}x_0\mathrm{d}x_1 A(x_0)\exp\left(-\gamma\left(\frac{\int_{-5}^{5}\ell(|y-x_1|)\mathrm{d}y}{\ell(|x_0-x_1|)} + \frac{\int_{-5}^{5}\ell(|y|)\mathrm{d}y}{\ell(|x_1|)}\right)\right). \quad (8.4)$$

All integrals are numerically tractable for $\gamma \in [0,1]$. As seen in Figure 3, already for $\gamma = 1$, the density of ν_1 is very close to the step function with a jump at the point r_0^* where (8.1) switches its sign. Also the density of two-hops paths, $(x_0, x_1) \mapsto \frac{d\nu_2}{d\mathrm{Leb}^{\otimes 2}}(x_0, x_1)$, is extremely small for $|x_0 - x_1|$ large, already for $\gamma = 1$, so that we prefer to plot it on a logarithmic scale, see Figure 4.

It is not easy to read from Figure 4 what value(s) maximize(s) $\frac{d\nu_2}{d\mathrm{Leb}^{\otimes 2}}(x_0, x_1)$ over $x_1 \in W$ for given $x_0 \in W$. Such a maximizer can be interpreted as an *optimal relay* of the transmitter x_0. Numerical simulations of the optimal relays turn out to be very noisy. Nevertheless, we see that this maximization problem has an interesting property. Since the normalized intensity measure $\mu/\mu(W)$ is the uniform distribution on W, for any $\gamma > 0$ and $x_0 \in W$ fixed, the set of optimal relays of x_0 equals the set of minimizers of

$$x_1 \mapsto g(x_0, x_1) + g(x_1, o) = \frac{\int_W \ell(|z - x_1|)\mathrm{d}z}{\ell(|x_0 - x_1|)} + \frac{\int_W \ell(|z|)\mathrm{d}z}{\ell(|x_1|)}. \quad (8.5)$$

The interpretation of this phenomenon is that the optimal value with respect to interference penalization corresponds to the largest value of $x_1 \mapsto \frac{d\nu_2}{d\mathrm{Leb}^{\otimes 2}}(x_0, x_1)$ for any $x_0 \in W$. Now, the second term in (8.5) is optimal if $|x_1| \leq 1$. Further, since $x_1 \mapsto \int_W \ell(|z - x_1|)\mathrm{d}z$ is strictly monotone decreasing in $|x_1|$, one easily sees that if $|x_0| \leq 1$, then $x_1 = \mathrm{sgn}\, x_0$ optimizes the first term over $\{|x_1| \leq 1\}$. This implies that if $|x_0| < 1$, then $\frac{d\nu_2}{d\mathrm{Leb}^{\otimes 2}}(x_0, x_1)$ is strictly larger for $x_1 = \mathrm{sgn}\, x_0$ than for any x_1 with $|x_1| < 1$. That is, all optimal relays of x_0 are further away from o than x_0 itself, and the first hop of an optimal message trajectory increases the distance from o! Nevertheless, we recall from Figure 3 that for $|x_0| \leq 1$, one-hop communication dominates two-hop communication already for $\gamma = 1$, i.e., the density of ν_2 at an optimal two-hop path is still very low.
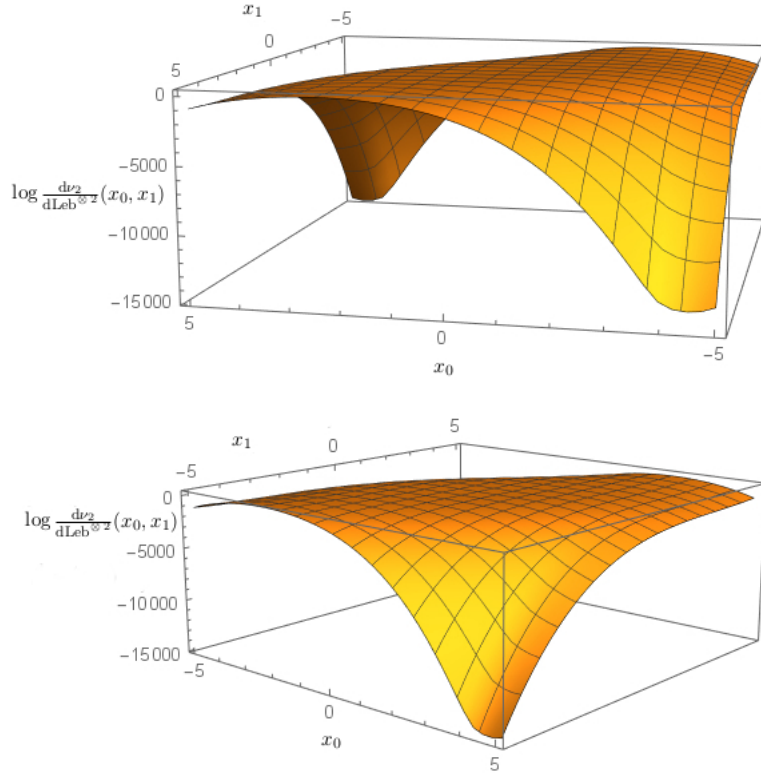
FIGURE 4. The graphs of $(x_0, x_1) \mapsto \log \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0, x_1)$ as in Example 8.1 for $\gamma = 1$ from two different views.

Since $x_1 \mapsto \int_W \ell(|z - x_1|)\mathrm{d}z$ varies much slower than $x_1 \mapsto \ell(|x_1|)$ for $|x_1| > 1$, we expect according to (8.5) that in general for $|x_0| \leq 2$, all optimal relays of x_0 must be situated very close to 1. In contrast, for $|x_0|$ significantly larger than 2, the optimal relays of x_0 are all about $x_0/2$. Indeed, in this regime, the optimization of $x_1 \mapsto \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0, x_1)$ can be restricted to the set of x_1 such that both $|x_0 - x_1| = |x_0| - |x_1| > 1$ and $|x_1| > 1$ hold. On this set, the following three properties are satisfied: $x_1 \mapsto \int_W \ell(|z - x_1|)\mathrm{d}z$ does not vary much (at $x_1 = o$, its value is around 2.6613, while for $|x_1| = 4$, it is about 2.3328), ℓ is taken at values $|x_1|$ and $|x_0 - x_1|$ where it is strictly monotone decreasing, and $1/\ell$ is convex. In the context of Example 8.2, where these properties hold everywhere (i.e., ℓ has no constant part), we will argue why they imply that the optimal relay of x_0 is close to $x_0/2$ for all $x_0 \in W$.

Our numerical results suggest that (5.6) does not hold in this example for $\Delta = W$, since otherwise, by Proposition 5.1, $\pi_0\nu_2$ would be close to the zero measure on the entire communication area W for large γ. The same is true for Example 8.2. $\diamond$

In the next, two-dimensional, example, we analyse the concentration of the measure ν_2 (2.16) on the straight line between transmitter and receiver in the setting of Section 4, i.e., in case of a rotationally invariant intensity μ and a strictly monotone decreasing path-loss function ℓ.

Example 8.2. We choose $d = 2$, $k_{\max} = 2$, $W = \overline{B_7(o)} \subset \mathbb{R}^2$, $\mu = \mathrm{Leb}|_W$ and $\ell(r) = (1 + r)^{-4}$. We note that ℓ is strictly monotone decreasing. Now, the one-hop trajectory measure ν_1 is a measure on $W \subset \mathbb{R}^2$ and the two-hop one ν_2 is a measure on $W^2 \subset \mathbb{R}^4$; they are defined as in (2.16), analogously to the concrete case (8.2)–(8.4), with a suitable adaptation to the new parameters.

Now, Proposition 4.1 (A) implies that for any $x_0 \in W$, all maximizers of the function $x_1 \mapsto \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0, x_1)$ are situated along the straight line segment $[x_0, o]$. The higher the value of γ is, the
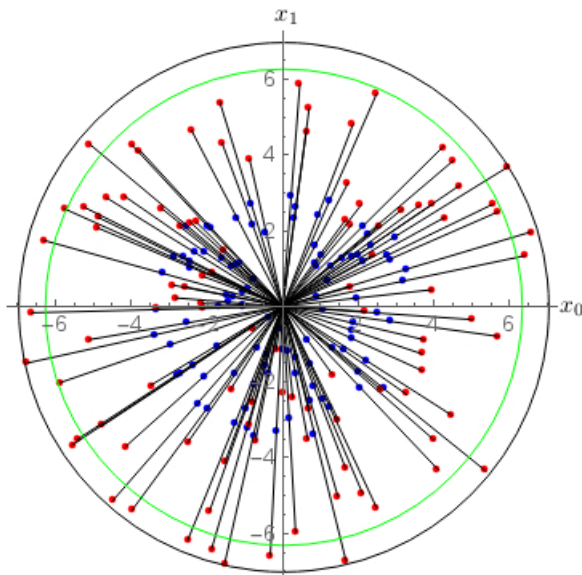
FIGURE 5. 100 uniformly distributed users x_0 in $W = \overline{B_7(o)}$ (in red), their approximate optimal relays $\approx x_0/2$ (blue) and the circle of radius ≈ 6.299 separating the regimes where one-hop and two-hop communication dominates, respectively (green). The outer black circle is the boundary of W.

stronger the density $\frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0, \cdot)$ concentrates around the set of maximizers, according to part B of the same proposition. As in Example 8.1, since $\mu/\mu(W)$ is the uniform distribution on W, the maximum of $x_1 \mapsto \frac{\mathrm{d}\nu_2}{\mathrm{dLeb}^{\otimes 2}}(x_0, x_1)$ is taken at the minimizer(s) of $x_1 \mapsto g(x_0, x_1) + g(x_1, o)$ for any $\gamma > 0$.

Since these minimizers lie on $[x_0, o]$, we now numerically search for an approximate minimizer of the form $x_1 = \alpha x_0$ with $\alpha \in [0, 1]$. It is highly expectable that the optimal α will be very close to $1/2$ for any $x_0 \in W$. Indeed, since $1/\ell$ is convex, for $c > 0$, $x_1 \mapsto c/\ell(|x_0 - \alpha x_0|) + c/\ell(|\alpha x_0 - o|)$ attains its minimum at $\alpha = 1/2$. On the other hand, for any $x_0 \in W$, the numerator of $g(x_0, \cdot)$, i.e., $x_1 \mapsto \int_W \ell(|z - x_1|)\mu(\mathrm{d}z)$, is close to a constant on $\frac{1}{2}W = \{x_0/2 \colon x_0 \in W\}$. Its value at o is approximately 1.0022 and its minimal value in $\frac{1}{2}W$ at $|x_1| = 7/2$ is around 0.9798. Therefore, α much smaller than $1/2$ cannot be the optimal choice. Outside $\frac{1}{2}W$, the interference is not any more close to constant: its minimal value at $|x_1| = 7$ is about 0.4770, slightly less than the half of the value at o. Nevertheless, for α much larger than $1/2$ (close to 1), the additional penalty for a longer second hop is much larger than the gain due to the larger path-loss value and better interference of the first hop. Thus, we expect that such an α also cannot be optimal.

We simulate 100 points x_0 according to the uniform distribution $\mu/\mu(W)$ in W and we compute the minimizer of $\alpha \mapsto g(x_0, \alpha x_0) + g(\alpha x_0, x_0)$ for each of these points. We observe that in all these cases, the optimal α is very close to 0.5: it lies strictly between 0.49 and 0.51. We note that this holds both in the regime $\{|x_0| < r_0^*\}$ where one-hop and in the regime $\{|x_0| > r_0^*\}$ where two-hop communication dominates for large γ. In this example, the value of r_0^* lies between 6.2989 and 6.299, so that 18 points out of 100 in the sample belong to the regime $\{|x_0| > r_0^*\}$. These results are visualized in Figure 5.

References

[BB09]   F. Baccelli and B. Blaszczyszyn, *Stochastic Geometry and Wireless Networks: Volume II: Applications.* Now Publishers Inc. (2009).

[BC12]   F. Baccelli and C. S. Chen, Self-Optimization in Mobile Cellular Networks: Power Control and User Association, *IEEE International Conference on Communications*, Cape Town (2010).

[DZ98]   A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, 2nd edition, Springer, Berlin (1998).

[CBK16]   A. Chattopadhyay, B. Blaszczyszyn and H. P. Keeler, Gibbsian On-Line Distributed Content Caching Strategy for Cellular Networks, *arXiv:1610.02318* (2016).

[GK00]   P. Gupta and P. R. Kumar, The Capacity of Wireless Networks, *IEEE Trans. Inform. Theory* **46:2**, (2000).

[GT08]   A. J. Ganesh and G. L. Torrisi, Large Deviations of the Interference in a Wireless Communication Model, *IEEE Trans. Inform. Theory* **54:8**, (2008).

[H02]   O. Häggström, *Finite Markov Chains and Algorithmic Applications,* Cambridge University Press (2002).

[HJ17]   C. Hirsch and B. Jahnel, Large deviations for the capacity in dynamic spatial relay networks, *arXiv:1712.03763* (2017).

[HJKP18] C. Hirsch, B. Jahnel, P. Keeler and R. Patterson, Large deviations in relay-augmented wireless networks, *Queueing Systems* **88**, 349–387 (2018). *arXiv:1510.04146*.

[HJP18]   C. Hirsch, B. Jahnel and R. Patterson, Space-time large deviations in capacity-constrained relay networks, *Latin American Journal of Probability and Statistics* **15**, 587–615 (2018), *arXiv:1609.06856*.

[KB14]   H. P. Keeler and B. Blaszczyszyn, SINR in wireless networks and the two-parameter Poisson-Dirichlet process, *IEEE Wireless Communications Letters, IEEE*, **3:5**, 525–528 (2014).

[KT17]   W. König and A. Tóbiás, A Gibbsian model for message routing in highly dense multi-hop networks, *arXiv:1704.03499v2* (2017).

[M18]   S. Morgenstern, *Markov Chain Monte Carlo for Message Routing*, master's thesis, TU Berlin (2018).

[NRTV07] N. Nisan, T. Roughgarden, É. Tardos and V. Vazirani (editors), *Algorithmic Game Theory*, Cambridge University Press (2007).

[SPW07]   H. Song, M. Peng and W. Wang, Node Selection in Relay-based Cellular Networks, *IEEE 2007 International Symposium on Microwave, Antenna, Propagation, and EMC Technologies For Wireless Communications*.

[YCG11]   C.-L. Yao, G. Chen, and T.-D. Guo, Large deviations for the graph distance in supercritical continuum percolation, *J. Appl. Probab.*, 48(1):154–172 (2011).