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Connectivity in large mobile ad-hoc networks

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A probabilistic model for large mobile ad-hoc networks

- large bounded domain D in $\mathbb{R}^d$ (telecommunication area)
- many particles $X^{(1)}, \dots, X^{(N)}$ (the users) move randomly and independently
- relay principle: messages are transmitted via a sequence of hops from user to user; each hop distance is $\leq R$

Our main question today: Connectivity

- Is there a hop-path from a given user to any other user?
- How many pairs of users have a connection at a given time, over a given time lag? (in terms of percentages)
- What is the amount of time during a given time lag over which a large percentage of the users are connected with each other?
- What is the large-time asymptotics of the probability that a non-zero percentage has a bad service (i.e., low connectivity)?

We will attack such questions for $N \to \infty$ in terms of a law of large numbers.

Connection time

Thermodynamic limit: $D = D_N \uparrow \mathbb{R}^d$ with volume $\asymp N$.

Equivalently: D independent of N, but the transmission radius R is replaced by $R_N = N^{-1/d}$.

Communication zone at time s:

$$D_s^{(N)} = D \cap \bigcup_{i=1}^{N} B\big(X_s^{(i)}, N^{-1/d}\big)$$

Connectivity at time s:

$$x \overset{N}{\underset{s}{\longleftrightarrow}} y \qquad \Longleftrightarrow \qquad x \text{ and } y \text{ lie in the same component of } D_s^{(N)}.$$

Connection time:

$$\tau_T^{(N)} := \int_0^T \mathrm{d}s\, \mathbb{1}\{X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\},$$

We will investigate the asymptotics of $\tau_T^{(N)}$ in the limit $N \to \infty$ in probability w.r.t. $\mathbb{P}(\cdot \mid X^{(1)}, X^{(2)})$ for almost all $(X^{(1)}, X^{(2)})$.

Assumption on the movement scheme

The location of $X_s^{(1)}$ has a continuous density $f_s\colon D \to [0,\infty)$, and $\mathbb{P}(X_s^{(1)} = x \,|\, X_{\widetilde{s}}^{(1)} = y) = 0$ for $s < \widetilde{s}$.

Sufficient: the existence of a jointly continuous density for $(X_s^{(1)}, X_{\widetilde{s}}^{(1)})$.

Notions from continuum percolation

$\overline{\theta}(\lambda)$ percolation probability for a Poisson point process in $\mathbb{R}^2$ with intensity λ.
$\lambda_{\mathrm{c}} = \inf\{\lambda > 0\colon \overline{\theta}(\lambda) > 0\}$ the critical threshold.

Write $\quad x \overset{>}{\underset{s}{\longleftrightarrow}} y \quad :\iff \quad$ there exists a path from x to y within $\{f_s > \lambda_{\mathrm{c}}\}$.

(Analogously with $\geq$ and $\leq$ and $<$ instead of $>$.) For $\diamond \in \{\geq, >\}$, define

$$\tau_T^{(\diamond)}(X^{(1)}, X^{(2)}) = \int_0^T \mathrm{d}s\, \mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}\overline{\theta}^{(\diamond)}\big(f_s(X_s^{(1)})\big)\overline{\theta}^{(\diamond)}\big(f_s(X_s^{(2)})\big),$$

with $\overline{\theta}^{(>)}(\lambda) = \overline{\theta}(\lambda-)$ and $\overline{\theta}^{(\geq)}(\lambda) = \overline{\theta}(\lambda+)$ the left- and right-continuous versions of $\overline{\theta}$.

Main result: asymptotics of the connection time

Theorem: Bounds on the connection time

For almost every paths $X^{(1)}, X^{(2)}$, in probability with respect to $\mathbb{P}_{1,2} := \mathbb{P}(\cdot \mid X^{(1)}, X^{(2)})$,

$$\tau_T^{(>)}(X^{(1)}, X^{(2)}) \leq \liminf_{N\to\infty} \tau_T^{(N)} \leq \limsup_{N\to\infty} \tau_T^{(N)} \leq \tau_T^{(\geq)}(X^{(1)}, X^{(2)}). \tag{1}$$

Comments:

- Global (deterministic) effect: The two walkers are only connected at time s if $\{X_s^{(1)} \overset{\diamond}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$, i.e., if they belong to the same component of the region where the density of users is high enough.
- Local (stochastic) effect: They have a connection only if they locally belong to the 'infinitely large cluster', which has probability $\overline{\theta}(f_s(X_s^{(1)}))\overline{\theta}(f_s(X_s^{(2)}))$.
- Does the limit exist? Under many abstract conditions, all the four expressions in (1) are equal to each other almost surely, but it is cumbersome to formulate and prove reasonably general and explicit ones. (Difficulty here: geometry of the set $\{f_s = \lambda_c\}$)

On the proof: convergence of the expectation (I)

Auxiliary events: $$G^{(i)}_{N,s,\delta} = \Big\{ X^{(i)}_s \overset{N}{\underset{s}{\longleftrightarrow}} \partial\big[X^{(i)}_s + (-\delta/2, \delta/2)^d\big]\Big\}.$$

Then connection of $X^{(1)}_s$ and $X^{(2)}_s$ is roughly equal to $G^{(1)} \cap G^{(2)}$, if they are in the same component of $\{f_s > \lambda_c\}$ or $\{f_s < \lambda_c\}$:

Lemma 1: Approximation with $G^{(i)}$

For $\mathbb{P}$-almost all $X^{(1)}$ and $X^{(2)}$, for almost any $s \in [0,T]$ and on the event $\{f_s(X^{(1)}_s) \neq \lambda_c\} \cap \{f_s(X^{(2)}_s) \neq \lambda_c\} \cap \{X^{(1)}_s \neq X^{(2)}_s\}$,

$$0 = \overline{\lim_{\delta\downarrow 0}}\, \overline{\lim_{N\to\infty}}\, \mathbb{P}_{1,2}\Big[\Big(X^{(1)}_s \overset{N}{\underset{s}{\longleftrightarrow}} X^{(2)}_s\Big) \triangle \Big(G^{(1)}_{N,s,\delta} \cap G^{(2)}_{N,s,\delta} \cap \{X^{(1)}_s \overset{>}{\underset{s}{\longleftrightarrow}} X^{(2)}_s\}\Big)\Big].$$

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- For sufficiently small δ, the δ-boxes around $x = X^{(1)}_s$ and $y = X^{(2)}_s$ are disjoint.
- Use results of [Penrose (1995)]: With probability tending to one, the only way to realise $G^{(1)}$ is that $X^{(1)}_s$ lies in the unique giant cluster of its δ-box. Analogously for $X^{(2)}_s$ and $G^{(2)}$.
- Construct a sequence of mutually overlapping δ-boxes from x to y inside $\{f_s > \lambda_c\}$ and argue similarly for this set.

On the proof: convergence of the expectation (II)

Lemma 2: Convergence of the probability of $G^{(i)}$

Under the same assumptions,

$$\begin{aligned}\overline{\theta}(f_s(X_s^{(1)})-)\overline{\theta}(f_s(X_s^{(2)})-) &\leq \liminf_{\delta\downarrow 0}\liminf_{N\to\infty}\mathbb{P}_{1,2}\Big(G^{(1)}_{N,s,\delta}\cap G^{(2)}_{N,s,\delta}\Big)\\ &\leq \limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}_{1,2}\Big(G^{(1)}_{N,s,\delta}\cap G^{(2)}_{N,s,\delta}\Big)\\ &\leq \overline{\theta}(f_s(X_s^{(1)})+)\overline{\theta}(f_s(X_s^{(2)})+).\end{aligned}$$

- Asymptotic independence of $G^{(1)}$ and $G^{(2)}$ in the limit $N\to\infty$, followed by $\delta\downarrow 0$.

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- Asymptotic independence of $G^{(1)}$ and $G^{(2)}$ in the limit $N\to\infty$, followed by $\delta\downarrow 0$.

Sofar, we have proved that the expectations of $\tau_T^{(N)}$ and

$$\tau_T^{(N,\delta,\diamond)} = \int_0^T \mathrm{d}s\, \mathbb{1}_{G^{(1)}_{N,s,\delta}}\mathbb{1}_{G^{(2)}_{N,s,\delta}}\mathbb{1}\{|X_s^{(1)}-X_s^{(2)}|\geq 3\delta\}\,\mathbb{1}\{X_s^{(1)} \underset{s}{\overset{\diamond}{\longleftrightarrow}} X_s^{(2)}\}$$

are asymptotically equal in the limit $N\to\infty$, followed by $\delta\downarrow 0$
(in terms of an upper bound with $\diamond = \geq$ and a lower bound for $\diamond = >$).

On the proof: Vanishing variance

We use the second-moment method and still have to prove:

Proposition: $\tau_T^{(N,\delta,\diamond)}$ is asymptotically deterministic

Almost surely, under $\mathbb{P}_{1,2} := \mathbb{P}(\cdot \mid X^{(1)}, X^{(2)})$, the variance of $\tau_T^{(N,\delta,\diamond)}$ vanishes in the limit $N \to \infty$, followed by $\delta \downarrow 0$.

- Write out the variance, using integrals $\int_0^T \mathrm{d}s$ and $\int_0^T \mathrm{d}\widetilde{s}$.
- Left to show: independence of $G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}$ and $G_{N,\widetilde{s},\delta}^{(1)} \cap G_{N,\widetilde{s},\delta}^{(2)}$.

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- Write out the variance, using integrals $\int_0^T \mathrm{d}s$ and $\int_0^T \mathrm{d}\widetilde{s}$.
- Left to show: independence of $G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}$ and $G_{N,\widetilde{s},\delta}^{(1)} \cap G_{N,\widetilde{s},\delta}^{(2)}$.
- Let $\mathcal{C}_{x,\delta}^{(s,N)}$ be the biggest component of the $N^{-1/d}$-balls around $X_s^{(1)}, \dots, X_s^{(N)}$ in $X_s^{(1)} + (-\delta, \delta)$. Abbreviate $x = X_s^{(1)}$, $\widetilde{x} = X_{\widetilde{s}}^{(1)}$, $y = X_s^{(2)}$ and $\widetilde{y} = X_{\widetilde{s}}^{(2)}$.
- Idea: the dependence of $\mathcal{C}_{x,\delta}^{(s,N)} \cup \mathcal{C}_{y,\delta}^{(s,N)}$ and $\mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)} \cup \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)}$ comes from only very few of the walkers (only $O(\delta^{2d} N)$).
- Main argument: our assumption that $\mathbb{P}(X_s^{(1)} = x \mid S_{\widetilde{s}} = y) = 0$ implies that the mass of those walkers that are at time $\widetilde{s}$ close to y and at time s close to x is small.

Drawbacks of the model

For mathematical reasons, we made a number of unrealistic assumptions:

- Number of hops for transmitting a given message is unbounded.
- All travels of any user are global.
- There are no fixed additional relays installed.
- No interference nor capacity problems are considered, just the possibility of transferring a single message.

More realistic movement schemes:

Contained in our assumptions: the random waypoint model with arbitrary dependence on time and arbitrary density of the starting site.

Improved version: Pick i.i.d. homes in D and an N-dependent way point measure centred at the home with action radius $\asymp N^{-\alpha}$ and infinite support after scaling.

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More realistic relay systems:

Bounded-hop percolation: Insert base stations on a grid $D \cap S_N \mathbb{Z}^d$ with some $S_N \to 0$, forbid sequences of more than k hops for a message, and look only at connection with some of the base stations. (Some first results on ergodicity of the static model in [HIRSCH (2015)].)

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Include interference:

Investigate how many signals can be successfully received if many signals are floating around. Criterion: Signal-to-interference ratio (SIR) and variants.

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Consider capacity:

Handle each trajectory of a message like a stochastic process and insert upper bounds for the number of messages per relay and time unit.