# Chapter 3
# Introduction to Sobolev Spaces

*Remark 3.1. Contents.* Sobolev spaces are the basis of the theory of weak or variational forms of partial differential equations. A very popular approach for discretizing partial differential equations, the finite element method, is based on variational forms. In this chapter, a short introduction into Sobolev spaces will be given. Recommended literature are the books Adams (1975); Adams & Fournier (2003), and Evans (2010). □

## 3.1 Elementary Inequalities

**Lemma 3.2. Inequality for strictly monotonically increasing function.** *Let $f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ be a continuous and strictly monotonically increasing function with $f(0) = 0$ and $f(x) \to \infty$ for $x \to \infty$. Then, for all $a, b \in \mathbb{R}_+ \cup \{0\}$, it is*

$$ab \le \int_0^a f(x)\ dx + \int_0^b f^{-1}(y)\ dy,$$

*where $f^{-1}(y)$ is the inverse of $f(x)$.*

*Proof.* Since $f(x)$ is strictly monotonically increasing, the inverse function exists.

The proof is based on a geometric argument, see Figure 3.1.

Consider the interval $(0, a)$ on the $x$-axis and the interval $(0, b)$ on the $y$-axis. Then, the area of the corresponding rectangle is given by $ab$, $\int_0^a f(x)\ dx$ is the area below the curve, and $\int_0^b f^{-1}(y)\ dy$ is the area between the positive $y$-axis and the curve. From Figure 3.1, the inequality follows immediately. The equal sign holds only iff $f(a) = b$. ■

*Remark 3.3. Young's[1] inequality.* Young's inequality

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \quad \forall\, a, b \in \mathbb{R}_+ \cup \{0\}, \varepsilon \in \mathbb{R}_+, \tag{3.1}$$

[1] William Henry Young (1863 – 1942)

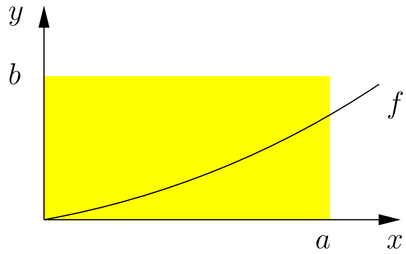


**Fig. 3.1** Sketch to the proof of Lemma 3.2.

follows from Lemma 3.2 with $f(x) = \varepsilon x$, $f^{-1}(y) = \varepsilon^{-1} y$. It is also possible to derive this inequality from the binomial theorem. For proving the generalized Young inequality

$$ab \le \frac{\varepsilon^p}{p} a^p + \frac{1}{q\varepsilon^q} b^q, \quad \forall\, a, b \in \mathbb{R}_+ \cup \{0\}, \varepsilon \in \mathbb{R}_+, \tag{3.2}$$

with $p^{-1}+q^{-1} = 1, p, q \in (1, \infty)$, one chooses $f(x) = x^{p-1}$, $f^{-1}(y) = y^{1/(p-1)}$ and applies Lemma 3.2 with intervals where the upper bounds are given by $\varepsilon a$ and $\varepsilon^{-1} b$. □

*Remark 3.4. Cauchy–Schwarz inequality.*

- The Cauchy[2]–Schwarz[3] inequality (for vectors, for sums)

$$\left|(\underline{x}, \underline{y})\right| \le \|\underline{x}\|_2 \left\|\underline{y}\right\|_2 \ \forall\, \underline{x}, \underline{y} \in \mathbb{R}^n, \tag{3.3}$$

  where $(\cdot, \cdot)$ is the Euclidean product and $\|\cdot\|_2$ the Euclidean norm, is well known.
- One can prove this inequality with the help of Young's inequality. First, it is clear that the Cauchy–Schwarz inequality is correct if one of the vectors is the zero vector. Now, let $\underline{x}, \underline{y}$ with $\|\underline{x}\|_2 = \left\|\underline{y}\right\|_2 = 1$. One obtains with the triangle inequality and Young's inequality (3.1)

$$\left|(\underline{x}, \underline{y})\right| = \left|\sum_{i=1}^n x_i y_i\right| \le \sum_{i=1}^n |x_i|\,|y_i| \le \frac{1}{2}\sum_{i=1}^n |x_i|^2 + \frac{1}{2}\sum_{i=1}^n |y_i|^2 = 1.$$

  Hence, the Cauchy–Schwarz inequality is correct for $\underline{x}, \underline{y}$. Last, one considers arbitrary vectors $\underline{\tilde{x}} \ne \underline{0}, \underline{\tilde{y}} \ne \underline{0}$. Now, one can utilize the homogeneity of the Cauchy–Schwarz inequality. From the validity of the Cauchy–Schwarz inequality for $\underline{x}$ and $\underline{y}$, one obtains by a scaling argument

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[2] Augustin Louis Cauchy (1789 – 1857)

[3] Hermann Amandus Schwarz (1843 – 1921)

$$\left|(\underbrace{\|\underline{\tilde{x}}\|_2^{-1}\,\underline{\tilde{x}}}_{\underline{x}},\underbrace{\|\underline{\tilde{y}}\|_2^{-1}\,\underline{\tilde{y}}}_{\underline{y}})\right| \le 1$$

Both vectors $\underline{x}, \underline{y}$ have the Euclidean norm 1, hence

$$\frac{1}{\|\underline{\tilde{x}}\|_2 \|\underline{\tilde{y}}\|_2}\left|(\underline{\tilde{x}},\underline{\tilde{y}})\right| \le 1 \quad \Longleftrightarrow \quad \left|(\underline{\tilde{x}},\underline{\tilde{y}})\right| \le \|\underline{\tilde{x}}\|_2 \|\underline{\tilde{y}}\|_2 .$$

- The generalized Cauchy–Schwarz inequality or Hölder's[4] inequality

$$\left|(\underline{x},\underline{y})\right| \le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \left(\sum_{i=1}^n |y_i|^q\right)^{1/q}$$

with $p^{-1} + q^{-1} = 1, p, q \in (1, \infty)$, can be proved in the same way with the help of the generalized Young inequality.

□

**Definition 3.5.** *Lebesgue spaces.* The space of functions that are Lebesgue[5] integrable on $\Omega$ to the power of $p \in [1, \infty)$ is denoted by

$$L^p(\Omega) = \left\{ f \ : \ \int_\Omega |f(\boldsymbol{x})|^p \, d\boldsymbol{x} < \infty \right\},$$

which is equipped with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{1/p}.$$

For $p = \infty$, this space is given by

$$L^\infty(\Omega) = \{f \ : \ |f(\boldsymbol{x})| < \infty \text{ almost everywhere in } \Omega\}$$

with the norm

$$\|f\|_{L^\infty(\Omega)} = \text{ess sup}_{\boldsymbol{x}\in\Omega}|f(\boldsymbol{x})|.$$

□

**Lemma 3.6. Hölder's inequality.** *Let $p^{-1} + q^{-1} = 1, p, q \in [1, \infty]$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then it is $uv \in L^1(\Omega)$ and it holds that*

$$\|uv\|_{L^1(\Omega)} \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} . \tag{3.4}$$

*If $p = q = 2$, then this inequality is also known as Cauchy–Schwarz inequality*

[4] Otto Hölder (1859 – 1937)

[5] Henri Lebesgue (1875 – 1941)

$$\|uv\|_{L^1(\Omega)} \le \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} . \tag{3.5}$$

*Proof.* *i)* $p, q \in (1, \infty)$. First, one has to show that $|uv(\boldsymbol{x})|$ can be estimated from above by an integrable function. Setting in the generalized Young inequality (3.2) $\varepsilon = 1$, $a = |u(\boldsymbol{x})|$, and $b = |v(\boldsymbol{x})|$ gives

$$|u(\boldsymbol{x})v(\boldsymbol{x})| \le \frac{1}{p} |u(\boldsymbol{x})|^p + \frac{1}{q} |v(\boldsymbol{x})|^q . \tag{3.6}$$

Since the right-hand side of this inequality is integrable, by assumption, it follows that $uv \in L^1(\Omega)$. Integrating (3.6), Hölder's inequality is proved for the case $\|u\|_{L^p(\Omega)} = \|v\|_{L^q(\Omega)} = 1$

$$\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})| \ d\boldsymbol{x} \le \frac{1}{p} \int_\Omega |u(\boldsymbol{x})|^p \ d\boldsymbol{x} + \frac{1}{q} \int_\Omega |v(\boldsymbol{x})|^q \ d\boldsymbol{x} = 1.$$

The general inequality follows, for the case that both functions do not vanish almost everywhere, with the same homogeneity argument as used for proving the Cauchy–Schwarz inequality of sums. In the case that one of the functions vanishes almost everywhere, (3.4) is trivially satisfied.

*ii)* $p = 1, q = \infty$. It is

$$\int_\Omega |u(\boldsymbol{x})v(\boldsymbol{x})| \ d\boldsymbol{x} \le \int_\Omega |u(\boldsymbol{x})| \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |v(\boldsymbol{x})| \ d\boldsymbol{x} = \|u\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega)} .$$

■

## 3.2 Weak Derivative and Distributions

*Remark 3.7. Contents.* This section introduces a generalization of the derivative which is needed for the definition of weak or variational problems. For an introduction to the topic of this section, e.g., see Haroske & Triebel (2008)

Let $\Omega \subset \mathbb{R}^d$ be a domain with boundary $\Gamma = \partial\Omega$, $d \in \mathbb{N}$, $\Omega \neq \emptyset$. A domain is always an open set. □

**Definition 3.8. The space $C_0^\infty(\Omega)$.** The space of infinitely often differentiable real functions with compact (closed and bounded) support in $\Omega$ is denoted by $C_0^\infty(\Omega)$

$$C_0^\infty(\Omega) = \{v \ : \ v \in C^\infty(\Omega), \ \operatorname{supp}(v) \subset \Omega\},$$

where

$$\operatorname{supp}(v) = \overline{\{\boldsymbol{x} \in \Omega \ : \ v(\boldsymbol{x}) \neq 0\}}.$$

In particular, functions from $C_0^\infty(\Omega)$ vanish in a neighborhood of the boundary. □

**Definition 3.9. Convergence in $C_0^\infty(\Omega)$.** The sequence $\{\phi_n(\boldsymbol{x})\}_{n=1}^\infty$, $\phi_n \in C_0^\infty(\Omega)$ for all $n$, is said to convergence to the zero functions if and only if

a) $\exists K \subset \Omega, K$ compact with $\operatorname{supp}(\phi_n) \subset K$ for all $n$,

b) $D^{\boldsymbol{\alpha}}\phi_n(\boldsymbol{x}) \to 0$ for $n \to \infty$ on $K$ for all multi-indices $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_d$.

It is

$$\lim_{n\to\infty} \phi_n = \phi \quad \Longleftrightarrow \quad \lim_{n\to\infty} (\phi_n - \phi) = 0.$$

□

**Definition 3.10. Weak derivative.** Let $f, F \in L^1_{\mathrm{loc}}(\Omega)$. A function $u$ belongs to $L^1_{\mathrm{loc}}(\Omega)$ if for each compact subset $\Omega' \subset \Omega$, it holds

$$\int_{\Omega'} |u(\boldsymbol{x})| \ d\boldsymbol{x} < \infty.$$

If for all functions $g \in C_0^\infty(\Omega)$, it holds that

$$\int_\Omega F(\boldsymbol{x})g(\boldsymbol{x}) \ d\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega f(\boldsymbol{x}) D^{\boldsymbol{\alpha}} g(\boldsymbol{x}) \ d\boldsymbol{x},$$

then $F(\boldsymbol{x})$ is called weak derivative of $f(\boldsymbol{x})$ with respect to the multi-index $\boldsymbol{\alpha}$. □

*Remark 3.11. On the weak derivative.*

- The notion ‘weak’ is used in mathematics if something holds for all appropriate test elements (test functions).
- One uses the same notations for the derivative as in the classical case : $F(\boldsymbol{x}) = D^{\boldsymbol{\alpha}} f(\boldsymbol{x})$.
- If $f(\boldsymbol{x})$ is classically differentiable on $\Omega$, then the classical derivative is also the weak derivative.
- The assumptions on $f(\boldsymbol{x})$ and $F(\boldsymbol{x})$ are such that the integrals in the definition of the weak derivative are well defined. In particular, since the test functions vanish in a neighborhood of the boundary, the behavior of $f(\boldsymbol{x})$ and $F(\boldsymbol{x})$ if $\boldsymbol{x}$ approaches the boundary is not of importance.
- The main aspect of the weak derivative is due to the fact that the (Lebesgue) integral is not influenced from the values of the functions on a set of (Lebesgue) measure zero. Hence, the weak derivative is defined only up to a set of measure zero. It follows that $f(\boldsymbol{x})$ might be not classically differentiable on a set of measure zero, e.g., in a point, but it can still be weakly differentiable.
- The weak derivative is uniquely determined, in the sense described above.

□

*Example 3.12. Weak derivative.* The weak derivative of the function $f(x) = |x|$ is

$$f'(x) = \begin{cases} -1 & x < 0, \\ 0 & x = 0, \\ 1 & x > 0. \end{cases}$$

In $x = 0$, one can use also any other real number. The proof of this statement follows directly from the definition and it is left as an exercise. □

**Definition 3.13. Distribution.** A continuous linear functional defined on $C_0^\infty(\Omega)$ is called distribution. The set of all distributions is denoted by $(C_0^\infty(\Omega))'$.

Let $u \in C_0^\infty(\Omega)$ and $\psi \in (C_0^\infty(\Omega))'$, then the following notations are used for the application of the distribution to the function

$$\psi(u) = \langle \psi, u \rangle \in \mathbb{R}.$$

□

*Remark 3.14. On distributions.* Distributions are a generalization of functions. They assign each function from $C_0^\infty(\Omega)$ a real number. □

*Example 3.15. Regular distribution.* Let $u \in L^1_{\mathrm{loc}}(\Omega)$. Then, a distribution is defined by

$$\int_\Omega u(\boldsymbol{x})\phi(\boldsymbol{x})\ d\boldsymbol{x} = \langle \psi, \phi \rangle \quad \forall\ \phi \in C_0^\infty(\Omega).$$

This distribution will be identified with $u \in L^1_{\mathrm{loc}}(\Omega)$.

Distributions with such an integral representation are called regular, otherwise they are called singular. □

*Example 3.16. Dirac distribution.* Let $\boldsymbol{\xi} \in \Omega$ be fixed, then

$$\langle \delta_{\boldsymbol{\xi}}, \phi \rangle = \phi(\boldsymbol{\xi}) \quad \forall\ \phi \in C_0^\infty(\Omega)$$

defines a singular distribution, the so-called Dirac[6] distribution or $\delta$-distribution. It is denoted by $\delta_{\boldsymbol{\xi}} = \delta(\boldsymbol{x} - \boldsymbol{\xi})$. □

**Definition 3.17.** *Derivatives of distributions.* Let $\phi \in (C_0^\infty(\Omega))'$ be a distribution. The distribution $\psi \in (C_0^\infty(\Omega))'$ is called derivative in the sense of distributions or distributional derivative of $\phi$ if

$$\langle \psi, u \rangle = (-1)^{|\boldsymbol{\alpha}|} \langle \phi, D^{\boldsymbol{\alpha}} u \rangle \quad \forall\ u \in C_0^\infty(\Omega),$$

$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\alpha_j \ge 0, j = 1, \ldots, d$, $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_d$. □

*Remark 3.18. On derivatives of distributions.*

- Each distribution has derivatives in the sense of distributions of arbitrary order.
- If the derivative in the sense of distributions $D^{\boldsymbol{\alpha}} u(\boldsymbol{x})$ with $u \in L^1_{\mathrm{loc}}(\Omega)$ is a regular distribution, then also the weak derivative of $u(\boldsymbol{x})$ exists and both derivatives are identified.

□

[6] Paul Adrien Maurice Dirac (1902 – 1984)

## 3.3 Lebesgue Spaces and Sobolev Spaces

*Remark 3.19. On the spaces $L^p(\Omega)$.* These spaces were introduced in Definition 3.5.

- The elements of $L^p(\Omega)$ are, strictly speaking, equivalence classes of functions that are different only on a set of Lebesgue measure zero.
- The spaces $L^p(\Omega)$ are Banach[7] spaces (complete normed spaces). A space $X$ is complete, if each so-called Cauchy sequence $\{u_n\}_{n=0}^\infty \in X$, i.e., for all $\varepsilon > 0$ there is an index $n_0(\varepsilon)$ such that for all $i, j > n_0(\varepsilon)$

$$\|u_i - u_j\|_X < \varepsilon,$$

  converges and the limit is an element of $X$.
- The space $L^2(\Omega)$ becomes a Hilbert[8] spaces with the inner product

$$(f, g) = \int_\Omega f(\boldsymbol{x})g(\boldsymbol{x})\; d\boldsymbol{x}, \quad \|f\|_{L^2} = (f, f)^{1/2}, \quad f, g \in L^2(\Omega).$$

- The dual space of a space $X$ is the space of all bounded linear functionals defined on $X$. Let $\Omega$ be a domain with sufficiently smooth boundary $\Gamma$ and consider the Lebesgue space $L^p(\Omega)$, $p \in [1, \infty]$, then

$$\begin{aligned} (L^p(\Omega))' &= L^q(\Omega) \;\text{ with }\; p, q \in (1, \infty),\; \frac{1}{p} + \frac{1}{q} = 1, \\ \left(L^1(\Omega)\right)' &= L^\infty(\Omega), \\ (L^\infty(\Omega))' &\neq L^1(\Omega), \end{aligned}$$

  where the prime symbolizes the dual space. The spaces $L^1(\Omega)$, $L^\infty(\Omega)$ are not reflexive, i.e., the dual space of the dual space is not the original space again.

□

**Definition 3.20. Sobolev[9] spaces.** Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, then the Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \;:\; D^{\boldsymbol{\alpha}} u \in L^p(\Omega)\; \forall\, \boldsymbol{\alpha} \text{ with } |\boldsymbol{\alpha}| \le k\}.$$

This space is equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\boldsymbol{\alpha}| \le k} \|D^{\boldsymbol{\alpha}} u\|_{L^p(\Omega)}. \tag{3.7}$$

□

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[7] Stefan Banach (1892 – 1945)

[8] David Hilbert (1862 – 1943)

[9] Sergei Lvovich Sobolev (1908 – 1989)

*Remark 3.21. On the spaces $W^{k,p}(\Omega)$.*

- Definition 3.20 has the following meaning. From $u \in L^p(\Omega)$, $p \in [1,\infty)$, it follows in particular that $u \in L^1_{\mathrm{loc}}(\Omega)$, such that $u$ defines (represents) a distribution. Then, all derivatives $D^{\boldsymbol{\alpha}}u$ exist in the sense of distributions. The statement $D^{\boldsymbol{\alpha}}u \in L^p(\Omega)$ means that the distribution $D^{\boldsymbol{\alpha}}u \in (C_0^\infty(\Omega))'$ can be represented by a function from $L^p(\Omega)$.
- One can add elements from $W^{k,p}(\Omega)$ and one can multiply them with real numbers. The result is again a function from $W^{k,p}(\Omega)$. With this property, the space $W^{k,p}(\Omega)$ becomes a vector space (linear space). It is straightforward to check that (3.7) is a norm. (*exercise*)
- It is $D^{\boldsymbol{\alpha}}u(\boldsymbol{x}) = u(\boldsymbol{x})$ for $\boldsymbol{\alpha} = (0,\ldots,0)$ and $W^{0,p}(\Omega) = L^p(\Omega)$.
- The spaces $W^{k,p}(\Omega)$ are Banach spaces.
- Sobolev spaces have for $p \in [1,\infty)$ a countable basis $\{\varphi_n(\boldsymbol{x})\}_{n=1}^\infty$ (Schauder[10] basis), i.e., each element $u(\boldsymbol{x})$ can be written in the form

$$u(\boldsymbol{x}) = \sum_{n=1}^\infty u_n\varphi_n(\boldsymbol{x}), \quad u_n \in \mathbb{R},\ n = 1,2,\ldots\ .$$

- Sobolev spaces are reflexive for $p \in (1,\infty)$.
- The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$, see (Gilbarg & Trudinger, 1983, p. 154). Under a certain condition on the smoothness of the boundary of a bounded domain $\Omega$, one can show that $C_0^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$, $p \in [1,\infty)$, with respect to the norm (3.7), e.g., (Adams, 1975, Thm. 3.18). With this property, one can characterize the Sobolev spaces $W^{k,p}(\Omega)$ as completion of the functions from $C_0^\infty(\Omega)$ with respect to the norm (3.7). It follows that $C^k(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$, $p \in [1,\infty)$.
- The Sobolev space $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space with the inner product

$$(u,v)_{H^k(\Omega)} = \sum_{|\boldsymbol{\alpha}|\le k} \int_\Omega D^{\boldsymbol{\alpha}}u(\boldsymbol{x}) D^{\boldsymbol{\alpha}}v(\boldsymbol{x})\ d\boldsymbol{x}$$

and the induced norm $\|u\|_{H^k(\Omega)} = (u,u)_{H^k(\Omega)}^{1/2}$.

□

**Definition 3.22. The space $W_0^{k,p}(\Omega)$.** The Sobolev space $W_0^{k,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in the norm of $W^{k,p}(\Omega)$

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}}.$$

□

[10] Juliusz Pawel Schauder (1899 – 1943)

## 3.4 The Trace of a Function from a Sobolev Space

*Remark 3.23. Motivation.* This class considers boundary value problems for partial differential equations. In the theory of weak or variational solutions, the solution of the partial differential equation is searched in an appropriate Sobolev space. Then, for the boundary value problem, this solution has to satisfy the boundary condition. However, since the boundary of a domain is a manifold of dimension $(d-1)$, and consequently it has Lebesgue measure zero, one has to clarify how a function from a Sobolev space is defined on this manifold. This definition will be presented in this section. □

**Definition 3.24. Lipschitz boundary, Lipschitz domain, (Grisvard, 1985, Def. 1.2.1.1).** Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, then $\Omega$ is called Lipschitz[11] domain, respectively the boundary $\Gamma$ of $\Omega$ is called Lipschitz boundary, if for every $\boldsymbol{x} \in \Gamma$ there exists a neighborhood $U$ of $\boldsymbol{x}$ in $\mathbb{R}^d$ and new orthogonal coordinates $(y_1, \ldots, y_d)$ such that

1) $U$ is a hypercube in the new coordinates

$$U = \{(y_1, \ldots, y_d) \ : \ -a_i < y_i < a_i, \ i = 1, \ldots, d\} .$$

2) There exists a Lipschitz continuous function $\phi$, defined in

$$U' = \{(y_1, \ldots, y_{d-1}) \ : \ -a_i < y_i < a_i, \ i = 1, \ldots, d-1\} ,$$

such that

$$\begin{aligned} |\phi(\boldsymbol{y}')| &\le \frac{a_n}{2} \text{ for every } \boldsymbol{y}' = (y_1, \ldots, y_{d-1}) \in U', \\ \Omega \cap U &= \{\boldsymbol{y} = (\boldsymbol{y}', y_n) \in V \ : \ y_n < \phi(\boldsymbol{y}')\} , \\ \Gamma \cap U &= \{\boldsymbol{y} = (\boldsymbol{y}', y_n) \in V \ : \ y_n = \phi(\boldsymbol{y}')\} . \end{aligned}$$

□

*Remark 3.25. Lipschitz boundary.*
- In a neighborhood of $\boldsymbol{y}$, $\Omega$ is below the graph of $\phi$ und the boundary $\Gamma$ is the graph of $\phi$.
- The domain $\Omega$ is not on both sides of the boundary at any point of $\Gamma$.
- The outer normal vector is defined almost everywhere at the boundary and it is almost everywhere continuous.

□

*Example 3.26. On Lipschitz domains.*
- Domains with Lipschitz boundary are, for example, balls or polygonal domains in two dimensions where the domain is always on one side of the boundary.

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[11] Rudolf Otto Sigismund Lipschitz (1832 – 1903)

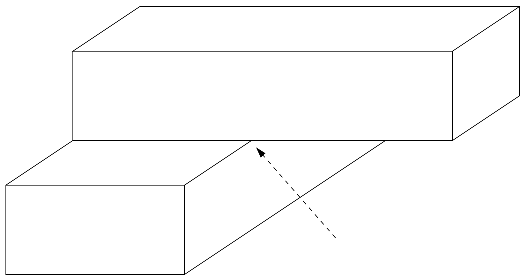

**Fig. 3.2** Polyhedral domain in three dimensions that is not Lipschitz continuous (at the corner where the arrow points to).

- A domain that is not a Lipschitz domain is a circle with a slit
$$\Omega = \{(x,y) \ : \ x^2+y^2<1\} \setminus \{(x,y) \ : \ x \ge 0, y=0\}.$$
At the slit, the domain is on both sides of the boundary.
- In three dimension, a polyhedral domain is not not necessarily a Lipschitz domain. For instance, if the domain is build of two bricks that are laying on each other like in Figure 3.2, then the boundary is not Lipschitz continuous where the edge of one brick meets the edge of the other brick.

$\square$

**Theorem 3.27. Trace theorem.** *Let $\Omega \subset \mathbb{R}^d$, $d \ge 2$, with a Lipschitz boundary. Then, there is exactly one linear and continuous operator $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$, $p \in [1,\infty)$, that gives for functions $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ the classical boundary values*
$$\gamma u(\boldsymbol{x}) = u(\boldsymbol{x}), \ \boldsymbol{x} \in \Gamma, \ \forall\, u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega),$$
*i.e., $\gamma u(\boldsymbol{x}) = u(\boldsymbol{x})|_{\boldsymbol{x}\in\Gamma}$.*

*Proof.* The proof can be found in the literature, e.g., in Adams (1975); Adams & Fournier (2003). $\blacksquare$

*Remark 3.28. On the trace.*
- The operator $\gamma$ is called trace or trace operator.
- By definition of the trace, one gets for $u \in C(\overline{\Omega})$ the classical boundary values. By the density of $C^\infty(\overline{\Omega}) \subset C(\overline{\Omega})$ in $W^{1,p}(\Omega)$ for domains with smooth boundary that for all $u \in W^{1,p}(\Omega)$ there is a sequence $\{u_n\}_{n=1}^\infty \in C^\infty(\overline{\Omega})$ with $u_n \to u$ in $W^{1,p}(\Omega)$. Then, the trace of $u$ is defined to be $\gamma u = \lim_{n\to\infty}(\gamma u_n)$.
- Since a linear and continuous operator is bounded, there is a constant $C > 0$ with
$$\|\gamma u\|_{L^p(\Gamma)} \le C\,\|u\|_{W^{1,p}(\Omega)} \ \forall\, u \in W^{1,p}(\Omega)$$

or

$$\|\gamma\|_{\mathcal{L}(W^{1,p}(\Omega),L^p(\Gamma))} \le C.$$

- It is

$$\begin{aligned} \gamma u(\boldsymbol{x}) &= 0 \quad \forall\, u \in W_0^{1,p}(\Omega), \\ \gamma D^{\boldsymbol{\alpha}} u(\boldsymbol{x}) &= 0 \quad \forall\, u \in W_0^{k,p}(\Omega), |\boldsymbol{\alpha}| \le k-1. \end{aligned} \tag{3.8}$$

□

## 3.5 Sobolev Spaces with Non-Integer and Negative Exponents

*Remark 3.29. Motivation.* Sobolev spaces with non-integer and negative exponents are important in the theory of variational formulations of partial differential equations.

Let $\Omega \subset \mathbb{R}^d$ be a domain and $p \in (1,\infty)$ with $p^{-1} + q^{-1} = 1$. □

**Definition 3.30. The space $W^{-k,q}(\Omega)$.** The space $W^{-k,q}(\Omega), k \in \mathbb{N} \cup \{0\}$, contains distributions that are defined on $W^{k,p}(\Omega)$

$$W^{-k,q}(\Omega) = \left\{ \varphi \in (C_0^\infty(\Omega))' \ : \ \|\varphi\|_{W^{-k,q}(\Omega)} < \infty \right\}$$

with

$$\|\varphi\|_{W^{-k,q}(\Omega)} = \sup_{u \in C_0^\infty(\Omega), u \ne 0} \frac{\langle \varphi, u \rangle}{\|u\|_{W^{k,p}(\Omega)}}.$$

□

*Remark 3.31. On the spaces $W^{-k,p}(\Omega)$.*

- It is $W^{-k,q}(\Omega) = \left[ W_0^{k,p}(\Omega) \right]'$, i.e., $W^{-k,q}(\Omega)$ can be identified with the dual space of $W_0^{k,p}(\Omega)$. In particular, it is $H^{-1}(\Omega) = \left( H_0^1(\Omega) \right)'$.
- It is

$$\ldots \subset W^{2,p}(\Omega) \subset W^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,q}(\Omega) \subset W^{-2,q}(\Omega) \ldots$$

□

**Definition 3.32. Sobolev–Slobodeckij space.** Let $s \in \mathbb{R}$, then the Sobolev–Slobodeckij[12] or Sobolev space $H^s(\Omega)$ is defined as follows:

- $s \in \mathbb{Z}$. $H^s(\Omega) = W^{s,2}(\Omega)$.
- $s > 0$ with $s = k + \sigma$, $k \in \mathbb{N} \cup \{0\}$, $\sigma \in (0,1)$. The space $H^s(\Omega)$ contains all functions $u$ for which the following norm is finite:

[12] L. N. Slobodeckij

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^k(\Omega)} + |u|^2_{k+\sigma},$$

with

$$(u,v)_{H^s(\Omega)} = (u,v)_{H^k} + (u,v)_{k+\sigma}, \quad |u|^2_{k+\sigma} = (u,u)_{k+\sigma},$$

and

$$(u,v)_{k+\sigma} = \sum_{|\boldsymbol{\alpha}|=k} \int_\Omega \int_\Omega \frac{(D^{\boldsymbol{\alpha}} u(\boldsymbol{x}) - D^{\boldsymbol{\alpha}} u(\boldsymbol{y}))\,(D^{\boldsymbol{\alpha}} v(\boldsymbol{x}) - D^{\boldsymbol{\alpha}} v(\boldsymbol{y}))}{\|\boldsymbol{x}-\boldsymbol{y}\|_2^{d+2\sigma}}\, d\boldsymbol{x} d\boldsymbol{y},$$

- $s < 0$. $H^s(\Omega) = \left[H_0^{-s}(\Omega)\right]'$ with $H_0^{-s}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^{-s}(\Omega)}}$.

□

## 3.6 Theorem on Equivalent Norms

**Definition 3.33. Equivalent norms.** Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the linear space $X$ are said to be equivalent if there are constants $C_1$ and $C_2$ such that

$$C_1 \|u\|_1 \le \|u\|_2 \le C_2 \|u\|_1 \ \ \forall\, u \in X.$$

□

*Remark 3.34. On equivalent norms.*

- Many important properties, like continuity or convergence, do not change if an equivalent norm is considered.
- In finite-dimensional spaces, all norms are equivalent.

□

**Theorem 3.35. Equivalent norms in $W^{k,p}(\Omega)$ (Smirnow, 1967, § 114, Satz 3).** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\Gamma$, $p \in [1,\infty]$, and $k \in \mathbb{N}$. Let $\{f_i\}_{i=1}^l$ be a system of functionals with the following properties:*

*1) $f_i\ :\ W^{k,p}(\Omega) \to \mathbb{R}_+ \cup \{0\}$ is a seminorm,*

*2) boundedness: $\exists C_i > 0$ with $0 \le f_i(v) \le C_i \|v\|_{W^{k,p}(\Omega)}$, $\forall\, v \in W^{k,p}(\Omega)$,*

*3) $f_i$ is a norm on the polynomials of degree $k-1$, i.e., if for $v \in P_{k-1} = \left\{\sum_{|\boldsymbol{\alpha}|\le k-1} C_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}\right\}$, it holds that $f_i(v) = 0$, $i = 1,\ldots,l$, then it is $v \equiv 0$.*

*Then, the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ defined in (3.7) and the norm*

$$\|u\|'_{W^{k,p}(\Omega)} := \left(\sum_{i=1}^{l} f_i^p(u) + |u|^p_{W^{k,p}(\Omega)}\right)^{1/p} \quad \textit{with}$$

$$|u|_{W^{k,p}(\Omega)} = \left(\sum_{|\boldsymbol{\alpha}|=k} \int_\Omega |D^{\boldsymbol{\alpha}} u(\boldsymbol{x})|^p \ d\boldsymbol{x}\right)^{1/p}$$

*are equivalent.*

*Remark 3.36. On seminorms.* For a seminorm $f_i(\cdot)$, one cannot conclude from $f_i(v) = 0$ that $v = 0$. The third assumptions however states, that this conclusion can be drawn for all polynomials up to a certain degree. □

*Example 3.37. Equivalent norms in Sobolev spaces.*

- The following norms are equivalent to the standard norm (3.7) in $W^{1,p}(\Omega)$:

$$\text{a) } \|u\|'_{W^{1,p}(\Omega)} = \left(\left|\int_\Omega u \ d\boldsymbol{x}\right|^p + |u|^p_{W^{1,p}(\Omega)}\right)^{1/p},$$

$$\text{b) } \|u\|'_{W^{1,p}(\Omega)} = \left(\left|\int_\Gamma u \ d\boldsymbol{s}\right|^p + |u|^p_{W^{1,p}(\Omega)}\right)^{1/p},$$

$$\text{c) } \|u\|'_{W^{1,p}(\Omega)} = \left(\int_\Gamma |u|^p \ d\boldsymbol{s} + |u|^p_{W^{1,p}(\Omega)}\right)^{1/p}.$$

- In $W^{k,p}(\Omega)$, it is

$$\|u\|'_{W^{k,p}(\Omega)} = \left(\sum_{i=0}^{k-1} \int_\Gamma \left|\frac{\partial^i u}{\partial \boldsymbol{n}^i}\right|^p \ d\boldsymbol{s} + |u|^p_{W^{k,p}(\Omega)}\right)^{1/p}$$

  equivalent to the standard norm. Here, $\boldsymbol{n}$ denotes the outer normal on $\Gamma$ with $\|\boldsymbol{n}\|_2 = 1$.
- In the case $W_0^{k,p}(\Omega)$, one does not need the regularity of the boundary. It is

$$\|u\|'_{W_0^{k,p}(\Omega)} = |u|_{W^{k,p}(\Omega)},$$

  i.e., in the spaces $W_0^{k,p}(\Omega)$ the standard seminorm is equivalent to the standard norm.
  In particular, it is for $u \in H_0^1(\Omega)$ $(k = 1, p = 2)$

$$C_1 \|u\|_{H^1(\Omega)} \le \|\nabla u\|_{L^2(\Omega)} \le C_2 \|u\|_{H^1(\Omega)}.$$

  It follows that there is a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \le C \|\nabla u\|_{L^2(\Omega)} \quad \forall\, u \in H_0^1(\Omega). \tag{3.9}$$

□

## 3.7 Some Inequalities in Sobolev Spaces

*Remark 3.38. Motivation.* This section presents a generalization of the last part of Example 3.37. It will be shown that for inequalities of type (3.9), it is not necessary that the trace vanishes on the complete boundary.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\Gamma$ and let $\Gamma_1 \subset \Gamma$ with $\mathrm{meas}_{\mathbb{R}^{d-1}}(\Gamma_1) = \int_{\Gamma_1} d\boldsymbol{s} > 0$.

One considers the space

$$
\begin{aligned}
V_0 &= \left\{ v \in W^{1,p}(\Omega) \ : \ v|_{\Gamma_1} = 0 \right\} \subset W^{1,p}(\Omega) \text{ if } \Gamma_1 \subsetneq \Gamma, \\
V_0 &= W_0^{1,p}(\Omega) \text{ if } \Gamma_1 = \Gamma,
\end{aligned}
$$

with $p \in [1, \infty)$. □

**Lemma 3.39. Friedrichs[13] inequality, Poincaré[14] inequality, Poincaré–Friedrichs inequality.** *Let $p \in [1,\infty)$ and $\mathrm{meas}_{\mathbb{R}^{d-1}}(\Gamma_1) > 0$. Then, it is for all $u \in V_0$*

$$
\int_\Omega |u(\boldsymbol{x})|^p \ d\boldsymbol{x} \le C_P \int_\Omega \|\nabla u(\boldsymbol{x})\|_2^p \ d\boldsymbol{x}, \tag{3.10}
$$

*where $\|\cdot\|_2$ is the Euclidean vector norm.*

*Proof.* The inequality will be proved with the theorem on equivalent norms, Theorem 3.35. Let $f_1(u) \ : \ W^{1,p}(\Omega) \to \mathbb{R}_+ \cup \{0\}$ with

$$
f_1(u) = \left( \int_{\Gamma_1} |u(\boldsymbol{s})|^p \ d\boldsymbol{s} \right)^{1/p}.
$$

This functional has the following properties:

1) $f_1(u)$ is a seminorm.
2) It is bounded, since

$$
\begin{aligned}
0 \le f_1(u) &= \left( \int_{\Gamma_1} |u(\boldsymbol{s})|^p \ d\boldsymbol{s} \right)^{1/p} \le \left( \int_{\Gamma} |u(\boldsymbol{s})|^p \ d\boldsymbol{s} \right)^{1/p} \\
&= \|u\|_{L^p(\Gamma)} = \|\gamma u\|_{L^p(\Gamma)} \le C \|u\|_{W^{1,p}(\Omega)}.
\end{aligned}
$$

   The last estimate follows from the continuity of the trace operator.
3) Let $v \in P_0$, i.e., $v$ is a constant. Then, one obtains from

$$
0 = f_1(v) = \left( \int_{\Gamma_1} |v(\boldsymbol{s})|^p \ d\boldsymbol{s} \right)^{1/p} = |v| \left( \mathrm{meas}_{\mathbb{R}^{d-1}}(\Gamma_1) \right)^{1/p},
$$

   that $|v| = 0$.

Hence, all assumptions of Theorem 3.35 are satisfied. That means, there are two constants $C_1$ and $C_2$ with

---

[13] Kurt Otto Friedrichs (1901 – 1982)

[14] Henri Poincaré (1854 – 1912)

$$C_1 \underbrace{\left( \int_{\Gamma_1} |u(\boldsymbol{s})|^p \ d\boldsymbol{s} + \int_\Omega \|\nabla u(\boldsymbol{x})\|_2^p \ d\boldsymbol{x} \right)^{1/p}}_{\|u\|'_{W^{1,p}(\Omega)}} \le \|u\|_{W^{1,p}(\Omega)} \le C_2 \|u\|'_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$. In particular, it follows that

$$\int_\Omega |u(\boldsymbol{x})|^p \ d\boldsymbol{x} + \int_\Omega \|\nabla u(\boldsymbol{x})\|_2^p \ d\boldsymbol{x} \le C_2^p \left( \int_{\Gamma_1} |u(\boldsymbol{s})|^p \ d\boldsymbol{s} + \int_\Omega \|\nabla u(\boldsymbol{x})\|_2^p \ d\boldsymbol{x} \right)$$

or, neglecting the non-negative term on the left-hand side,

$$\int_\Omega |u(\boldsymbol{x})|^p \ d\boldsymbol{x} \le C_P \left( \int_{\Gamma_1} |u(\boldsymbol{s})|^p \ d\boldsymbol{s} + \int_\Omega \|\nabla u(\boldsymbol{x})\|_2^p \ d\boldsymbol{x} \right)$$

with $C_P = C_2^p$. Since $u \in V_0$ vanishes on $\Gamma_1$, the statement of the lemma is proved. ■

*Remark 3.40. On the Poincaré–Friedrichs inequality.* In the space $V_0$ becomes $|\cdot|_{W^{1,p}}$ a norm that is equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$. The classical Poincaré–Friedrichs inequality is given for $\Gamma_1 = \Gamma$ and $p = 2$

$$\|u\|_{L^2(\Omega)} \le C_P \|\nabla u\|_{L^2(\Omega)} \ \ \forall\, u \in H_0^1(\Omega),$$

where the constant depends only on the diameter of the domain $\Omega$, e.g., see (Galdi, 2011, Theorem II.5.1). □

## 3.8 The Gaussian Theorem

*Remark 3.41. Motivation.* The Gaussian theorem is the generalization of the integration by parts from calculus. This operation is very important for the theory of weak or variational solutions of partial differential equations. One has to study, under which conditions on the regularity of the domain and of the functions it is well defined. □

**Theorem 3.42. Gaussian theorem.** *Let $\Omega \subset \mathbb{R}^d, d \ge 2$, be a bounded domain with Lipschitz boundary $\Gamma$. Then, the following identity holds for all $u \in W^{1,1}(\Omega)$*

$$\int_\Omega \partial_i u(\boldsymbol{x}) \ d\boldsymbol{x} = \int_\Gamma u(\boldsymbol{s}) \boldsymbol{n}_i(\boldsymbol{s}) \ d\boldsymbol{s}, \tag{3.11}$$

*where $\boldsymbol{n}$ is the unit outer normal vector on $\Gamma$.*

*Proof.* It is referred to the literature. ■

**Corollary 3.43. Vector field.** *Let the conditions of Theorem 3.42 on the domain $\Omega$ be satisfied and let $\boldsymbol{u} \in \left(W^{1,1}(\Omega)\right)^d$ be a vector field. Then, it is*

$$\int_\Omega \nabla \cdot \boldsymbol{u}(\boldsymbol{x}) \ d\boldsymbol{x} = \int_\Gamma \boldsymbol{u}(\boldsymbol{s}) \cdot \boldsymbol{n}(\boldsymbol{s}) \ d\boldsymbol{s}.$$

*Proof.* The statement follows by adding (3.11) from $i = 1$ to $i = d$. ■

**Corollary 3.44. Integration by parts.** *Let the conditions of Theorem 3.42 on the domain $\Omega$ be satisfied. Consider $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ with $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, it is*

$$\int_\Omega \partial_i u(\boldsymbol{x}) v(\boldsymbol{x})\ d\boldsymbol{x} = \int_\Gamma u(\boldsymbol{s}) v(\boldsymbol{s}) \boldsymbol{n}_i(\boldsymbol{s})\ d\boldsymbol{s} - \int_\Omega u(\boldsymbol{x}) \partial_i v(\boldsymbol{x})\ d\boldsymbol{x}.$$

*Proof.* *exercise.* ■

**Corollary 3.45. First Green[15]'s formula.** *Let the conditions of Theorem 3.42 on the domain $\Omega$ be satisfied, then it is*

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\ d\boldsymbol{x} = \int_\Gamma \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{s}) v(\boldsymbol{s})\ d\boldsymbol{s} - \int_\Omega \Delta u(\boldsymbol{x}) v(\boldsymbol{x})\ d\boldsymbol{x}$$

*for all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$.*

*Proof.* From the definition of the Sobolev spaces, it follows that the integrals are well defined. Now, the proof follows the proof of Corollary 3.44, where one has to sum over the components and one has to take $\partial_i v$ instead of $v$. ■

*Remark 3.46. On the first Green's formula.* The first Green's formula is the formula of integrating by parts once. The boundary integral can be equivalently written in the form

$$\int_\Gamma \nabla u(\boldsymbol{s}) \cdot \boldsymbol{n}(\boldsymbol{s}) v(\boldsymbol{s})\ d\boldsymbol{s}.$$

The formula of integrating by parts twice is called second Green's formula. □

**Corollary 3.47. Second Green's formula.** *Let the conditions of Theorem 3.42 on the domain $\Omega$ be satisfied, then one has*

$$\int_\Omega \big(\Delta u(\boldsymbol{x}) v(\boldsymbol{x}) - \Delta v(\boldsymbol{x}) u(\boldsymbol{x})\big)\ d\boldsymbol{x} = \int_\Gamma \left(\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{s}) v(\boldsymbol{s}) - \frac{\partial v}{\partial \boldsymbol{n}}(\boldsymbol{s}) u(\boldsymbol{s})\right)\ d\boldsymbol{s}$$

*for all $u, v \in H^2(\Omega)$.*

## 3.9 Sobolev Imbedding Theorems

*Remark 3.48. Motivation.* This section studies the question which (Sobolev) spaces are subspaces of other Sobolev spaces. With this property, called

[15] Georg Green (1793 – 1841)

imbedding, it is possible to estimate the norm of a function in the subspace by the norm in the larger space, compare (3.12). □

**Lemma 3.49. Imbedding of Sobolev spaces with same integration power $p$ and different orders of the derivative.** *Let $\Omega \subset \mathbb{R}^d$ be a domain, $p \in [1, \infty]$, and $k \le m$, then it is $W^{m,p}(\Omega) \subset W^{k,p}(\Omega)$.*

*Proof.* The statement of this lemma follows directly from the definition of Sobolev spaces, see Definition 3.20. ■

**Lemma 3.50. Imbedding of Sobolev spaces with the same order of the derivative $k$ and different integration powers.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $k \ge 0$, and $p, q \in [1, \infty]$ with $q > p$. Then, it is $W^{k,q}(\Omega) \subset W^{k,p}(\Omega)$.*

*Proof.* *exercise.* ■

*Remark 3.51. Imbedding of Sobolev spaces with the same order of the derivative $k$ and the same integration power $p$ in imbedded domains.* Let $\Omega \subset \mathbb{R}^d$ be a domain with sufficiently smooth boundary $\Gamma$, $k \ge 0$, and $p \in [1, \infty]$. Then, there is a map $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$, the so-called (simple) extension, with

- $Ev|_\Omega = v$,
- $\|Ev\|_{W^{k,p}(\mathbb{R}^d)} \le C \|v\|_{W^{k,p}(\Omega)}$, with $C > 0$ independent of $v$,

e.g., see (Adams, 1975, Chapter IV) for details. Likewise, the natural restriction $e : W^{k,p}(\mathbb{R}^d) \to W^{k,p}(\Omega)$ can be defined and it is $\|ev\|_{W^{k,p}(\Omega)} \le \|v\|_{W^{k,p}(\mathbb{R}^d)}$. □

**Theorem 3.52. A Sobolev inequality.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\Gamma$, $k \ge 0$, and $p \in [1, \infty)$ with*

$$\begin{aligned} k &\ge d && \text{for } p = 1, \\ k &> d/p && \text{for } p > 1. \end{aligned}$$

*Then, there is a constant $C$ such that for all $u \in W^{k,p}(\Omega)$, it follows that $u \in C_B(\Omega)$, where*

$$C_B(\Omega) = \{v \in C(\Omega) \ : \ v \text{ is bounded}\},$$

*and it is*

$$\|u\|_{C_B(\Omega)} = \|u\|_{L^\infty(\Omega)} \le C \|u\|_{W^{k,p}(\Omega)}. \tag{3.12}$$

*Proof.* See literature, e.g., Adams (1975); Adams & Fournier (2003). ■

*Remark 3.53. On the Sobolev inequality.*

- The Sobolev inequality states that each function with sufficiently many weak derivatives (the number depends on the dimension of $\Omega$ and the integration power) can be considered as a continuous and bounded function in $\Omega$, i.e., there is such a representative in the equivalence class where this function belongs to. One says that $W^{k,p}(\Omega)$ is imbedded in $C_B(\Omega)$.

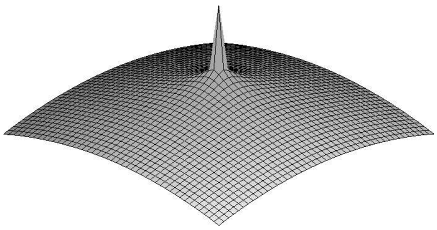

**Fig. 3.3** The function $f(\boldsymbol{x})$ of Example 3.55 for $d = 2$.

- It is
$$C\left(\overline{\Omega}\right) \subsetneq C_B(\Omega) \subsetneq C(\Omega).$$
Consider $\Omega = (0,1)$ and $f_1(x) = 1/x$ and $f_2(x) = \sin(1/x)$. Then, $f_1 \in C(\Omega)$, $f_1 \notin C_B(\Omega)$ and $f_2 \in C_B(\Omega)$, $f_2 \notin C(\overline{\Omega})$.
- Of course, it is possible to apply this theorem to weak derivatives of functions. Then, one obtains imbeddings like $W^{k,p}(\Omega) \to C_B^s(\Omega)$ for $(k-s)p > d, p > 1$. A comprehensive overview on imbeddings can be found in Adams (1975); Adams & Fournier (2003).

□

*Example 3.54. $H^1(\Omega)$ in one dimension.* Let $d = 1$ and $\Omega$ be a bounded interval. Then, each function from $H^1(\Omega)$ $(k = 1, p = 2)$ is continuous and bounded in $\Omega$. □

*Example 3.55. $H^1(\Omega)$ in higher dimensions.* The functions from $H^1(\Omega)$ are in general not continuous for $d \geq 2$. This property will be shown with the following example.

Let $\Omega = \{\boldsymbol{x} \in \mathbb{R}^d \ : \ \|\boldsymbol{x}\|_2 < 1/2\}$ and $f(\boldsymbol{x}) = \ln|\ln\|\boldsymbol{x}\|_2|$, see Figure 3.3. For $\|\boldsymbol{x}\|_2 < 1/2$, it is $|\ln\|\boldsymbol{x}\|_2| = -\ln\|\boldsymbol{x}\|_2$ and one gets for $\boldsymbol{x} \neq \boldsymbol{0}$

$$\partial_i f(\boldsymbol{x}) = -\frac{1}{\ln\|\boldsymbol{x}\|_2}\frac{1}{\|\boldsymbol{x}\|_2}\frac{x_i}{\|\boldsymbol{x}\|_2} = -\frac{x_i}{\|\boldsymbol{x}\|_2^2\ln\|\boldsymbol{x}\|_2}.$$

For $p \leq d$, one obtains

$$\left|\frac{\partial f}{\partial x_i}(\boldsymbol{x})\right|^p = \underbrace{\left|\frac{x_i}{\|\boldsymbol{x}\|_2}\right|^p}_{\leq 1}\underbrace{\left|\frac{1}{\|\boldsymbol{x}\|_2\ln\|\boldsymbol{x}\|_2}\right|^p}_{\geq e} \leq \left|\frac{1}{\|\boldsymbol{x}\|_2\ln\|\boldsymbol{x}\|_2}\right|^d.$$

The estimate of the second factor can be obtained, e.g., with a discussion of the curve. Using now spherical coordinates, $\rho = e^{-t}$ and $S^{d-1}$ is the unit sphere, yields

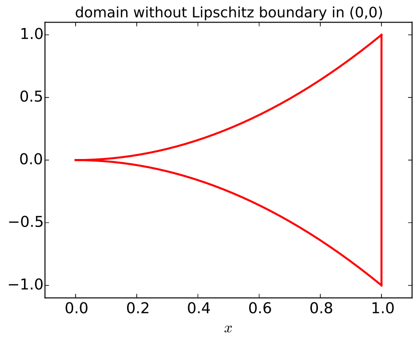


**Fig. 3.4** Domain of Example 3.56.

$$\int_\Omega |\partial_i f(\boldsymbol{x})|^p \, d\boldsymbol{x} \le \int_\Omega \frac{d\boldsymbol{x}}{\|\boldsymbol{x}\|_2^d \left|\ln \|\boldsymbol{x}\|_2\right|^d} = \int_{S^{d-1}} \int_0^{1/2} \frac{\rho^{d-1}}{\rho^d |\ln \rho|^d} \, d\rho d\omega$$
$$= \operatorname{meas}\left(S^{d-1}\right) \int_0^{1/2} \frac{d\rho}{\rho |\ln \rho|^d} = -\operatorname{meas}\left(S^{d-1}\right) \int_\infty^{\ln 2} \frac{dt}{t^d} < \infty,$$

because of $d \ge 2$.

It follows that $\partial_i f \in L^p(\Omega)$ with $p \le d$. Analogously, one proves that $f \in L^p(\Omega)$ with $p \le d$. Altogether, one has $f \in W^{1,p}(\Omega)$ with $p \le d$. However, it is $f \notin L^\infty(\Omega)$ and consequently $f \notin C_B(\Omega)$. This example shows that the condition $k > d/p$ for $p > 1$ is sharp.

In particular, it was proved for $p = 2$ that from $f \in H^1(\Omega)$ in general it does not follow that $f \in C(\Omega)$. □

*Example 3.56. The assumption of a Lipschitz boundary.* Also the assumption that $\Omega$ is a Lipschitz domain is of importance.

Consider $\Omega = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 1, \ |y| < x^r, r > 1\}$, see Figure 3.4 for $r = 2$. The boundary is not Lipschitz in $(0, 0)$.

For $u(x, y) = x^{-\varepsilon/p}$ with $0 < \varepsilon \le r$, it is

$$\partial_x u = x^{-\varepsilon/p-1} \left(-\frac{\varepsilon}{p}\right) = C(\varepsilon, p) x^{-\varepsilon/p-1}, \ \partial_y u = 0.$$

Using the same notation for the constant, which might take different values at different occasions, it follows that

$$\begin{aligned}
\sum_{|\boldsymbol{\alpha}|=1} \int_{\Omega} |D^{\boldsymbol{\alpha}} u(x,y)|^p \; dxdy &= C(\varepsilon,p) \int_{\Omega} x^{-\varepsilon-p} \; dxdy \\
&= C(\varepsilon,p) \int_0^1 x^{-\varepsilon-p} \left( \int_{-x^r}^{x^r} \; dy \right) dx \\
&= C(\varepsilon,p) \int_0^1 x^{-\varepsilon-p+r} \; dx.
\end{aligned}$$

This value is finite for $-\varepsilon - p + r > -1$ or for $p < 1 + r - \varepsilon$, respectively. If one chooses $r \geq \varepsilon > 0$, then it is $u \in W^{1,p}(\Omega)$. But for $\varepsilon > 0$, the function $u(\boldsymbol{x})$ is not bounded in $\Omega$, i.e., $u \notin L^{\infty}(\Omega)$ and consequently $u \notin C_B(\Omega)$.

The unbounded values of the function are compensated in the integration by the fact that the neighborhood of the singular point $(0,0)$ possesses a small measure. $\square$