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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 01

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. The following operators are defined for a scalar function u a and a vector-valued function $\mathbf{v} = (v_1, v_2, v_3)^T$ with the help of the nabla operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right)^T$$

- grad $u = \nabla u = (u_x, u_y, u_z)^T$,
- div $\mathbf{v} = \nabla \cdot \mathbf{v} = (v_1)_x + (v_2)_y + (v_3)_z$,
- rot $\mathbf{v} = \nabla \times \mathbf{v} = \begin{pmatrix} (v_3)_y - (v_2)_z \\ (v_1)_z - (v_3)_x \\ (v_2)_x - (v_1)_y \end{pmatrix}$.

Assuming that all considered functions are sufficiently smooth (sufficiently often differentiable), show the following identities

- $\nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} + u_{zz}$,
- $\nabla \cdot (\nabla \times \mathbf{v}) = 0$,
- $\nabla \times (\nabla u) = 0$,
- $\nabla \times (u\mathbf{v}) = u\,(\nabla \times \mathbf{v}) - (\mathbf{v} \times \nabla u)$,
- $\nabla \cdot (u\mathbf{v}) = \nabla u \cdot \mathbf{v} + u\,\nabla \cdot \mathbf{v}$.

2. Classify the following partial differential equations

$$\begin{aligned} u_{xx} + 2u_{xy} + 2u_{yy} + 4u_{yz} + 5u_{zz} + u_x + u_y &= 0, && (3d)\,, \\ e^z u_{xy} - u_{xx} - \log\left(x^2 + y^2 + z^2\right) &= 0, && (3d)\,, \\ u_{xx} + 4u_{xy} + 3u_{yy} + 3u_x - u_y + 2u &= 0, && (2d)\,, \\ au_{xx} + 2au_{xy} + au_{yy} + bu_x + cu_y + u &= 0, && (2d)\,. \end{aligned}$$

$2d$ – in two dimensions, $3d$ – in three dimensions.

3. Show that

$$u\,(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0\,(z)\, e^{\frac{-(x-z)^2}{4t}}\, dz$$

is a solution of the one-dimensional heat equation

$$\begin{aligned} u_t - u_{xx} &= 0, && x \in \mathbb{R}, t > 0, \\ u\,(0, x) &= u_0\,(x)\,, && x \in \mathbb{R}. \end{aligned}$$

It shall be assumed that $u_0(x)$ is sufficiently smooth.
Hint. To check the initial condition, assume that $u_0(x)$ can be expanded into a Fourier series

$$u_0(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \cos(n\omega x) + \beta_n \sin(n\omega x)\right)$$

with $\omega \in \mathbb{R}$ and use the following identities

$$\begin{aligned} \int_{-\infty}^{\infty} \sin(nz)\, e^{\frac{-(x-z)^2}{4t}}\, dz &= \sqrt{4\pi t}\, e^{-n^2 t} \sin(nx), \\ \int_{-\infty}^{\infty} \cos(nz)\, e^{\frac{-(x-z)^2}{4t}}\, dz &= \sqrt{4\pi t}\, e^{-n^2 t} \cos(nx), \qquad n \in \mathbb{R}. \end{aligned}$$

The exercise problems should be solved in groups of two or three students. They have to be submitted until **Tuesday, Apr. 16, 2013** either before or after one of the lectures or directly at the office of Mrs. Hardering.