1

The Navier–Stokes Equations

Remark 1.1. Basic principles and variables. The basic equations of fluid dynamics are called Navier–Stokes equations. In the case of an isothermal flow, a flow at constant temperature, they represent two physical conservation laws – the conservation of mass and the conservation of linear momentum. There are various ways for deriving these equations. Here, the classical one of continuum mechanics will be used. Let the flow variables be

- $\rho(t, \mathbf{x})$: density $[kg/m^3]$,
- $\mathbf{v}(t, \mathbf{x})$: velocity $[m/s]$,
- $P(t, \mathbf{x})$: pressure $[N/m^2]$,

which are assumed to be sufficiently smooth functions in the time interval $[0, T]$ and the domain $\Omega \subset \mathbb{R}^3$. □

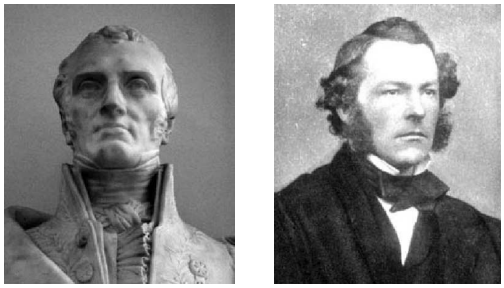

Fig. 1.1. Left: Claude Louis Marie Henri Navier (1785 – 1836), right: George Gabriel Stokes (1819 – 1903).

1.1 The Conservation of Mass

Remark 1.2. General conservation law. Let V be an arbitrary open volume in Ω with sufficiently smooth surface ∂V which is constant in time and with mass

$$m(t) = \int_V \rho(t, \mathbf{x})\ d\mathbf{x},\ [kg].$$

If mass in V is conserved, the rate of change of mass in V must be equal to the flux of mass $\rho\mathbf{v}(t, \mathbf{x})\ [kg/(m^2 s)]$ across the boundary ∂V of V

$$\frac{d}{dt}m(t) = \frac{d}{dt}\int_V \rho(t, \mathbf{x})\ d\mathbf{x} = -\int_{\partial V} (\rho\mathbf{v})\,(t, \mathbf{s}) \cdot \mathbf{n}(\mathbf{s})\ d\mathbf{s}, \tag{1.1}$$

where $\mathbf{n}(\mathbf{s})$ is the outward pointing unit normal on $\mathbf{s} \in \partial V$. Since all functions are assumed to be sufficiently smooth, the divergence theorem can be applied (integration by parts), which gives

$$\int_V \nabla \cdot (\rho\mathbf{v})\,(t, \mathbf{x})\ d\mathbf{x} = \int_{\partial V} (\rho\mathbf{v})\,(t, \mathbf{s}) \cdot \mathbf{n}(\mathbf{s})\ d\mathbf{s}.$$

Inserting this identity into (1.1) leads to

$$\int_V \left(\frac{\partial \rho}{\partial t}(t, \mathbf{x}) + \nabla \cdot (\rho\mathbf{v})\,(t, \mathbf{x})\right)\ d\mathbf{x} = 0.$$

Since V is an arbitrary volume, it follows

$$\left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v})\right)(t, \mathbf{x}) = 0 \quad \forall\ t \in (0, T],\ \mathbf{x} \in \Omega. \tag{1.2}$$

This is the first equation of mathematical fluid dynamics, which is called continuity equation. □

Remark 1.3. Incompressible, homogeneous fluid. If the fluid is incompressible and homogeneous, i.e., composed of one fluid only, then $\rho(t, \mathbf{x}) = \rho_0 > 0$ and (1.2) reduces to

$$(\partial_x v_1 + \partial_y v_2 + \partial_z v_3)\,(t, \mathbf{x}) = \nabla \cdot \mathbf{v}(t, \mathbf{x}) = 0 \quad \forall\ t \in (0, T],\ \mathbf{x} \in \Omega, \tag{1.3}$$

where

$$\mathbf{v}(t, \mathbf{x}) = \begin{pmatrix} v_1(t, \mathbf{x}) \\ v_2(t, \mathbf{x}) \\ v_3(t, \mathbf{x}) \end{pmatrix}.$$

Thus, the conservation of mass for an incompressible, homogeneous fluid imposes a constraint on the velocity only. □

1.2 The Conservation of Linear Momentum

Remark 1.4. Newton's second law of motion. The conservation of linear momentum is the formulation of Newton's second law of motion

$$\text{net force} = \text{mass} \times \text{acceleration}$$

for flows. It states that the rate of change of the linear momentum must be equal to the net force acting on a collection of fluid particles. □

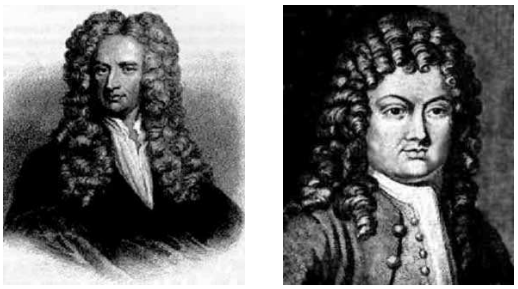

Fig. 1.2. Left: Isaac Newton (1642 – 1727), right: Brook Taylor (1685 – 1731).

Remark 1.5. Acceleration. Consider a fluid particle at time t and position $\mathbf{x}$ with velocity $\mathbf{v}(t,\mathbf{x})$ and a small time interval Δt. A linear extrapolation of the particle path gives that the particle has at time $t+\Delta t$ the position $\mathbf{x}+\Delta t\mathbf{v}$, see Figure 1.3. The acceleration of the particle is, using again a linear approximation (first order Taylor series expansion in time),

$$\begin{aligned}
\frac{d\mathbf{v}}{dt}(t,\mathbf{x}) &= \lim_{\Delta t\to 0} \frac{\mathbf{v}\left(t+\Delta t, \mathbf{x}+\Delta t\mathbf{v}(t,\mathbf{x})\right)-\mathbf{v}(t,\mathbf{x})}{\Delta t} \\
&\approx \lim_{\Delta t\to 0} \frac{\mathbf{v}(t,\mathbf{x})+\Delta t\mathbf{v}_t(t,\mathbf{x})+\Delta t\left(\mathbf{v}(t,\mathbf{x})\cdot\nabla\right)\mathbf{v}(t,\mathbf{x})-\mathbf{v}(t,\mathbf{x})}{\Delta t} \\
&= \frac{\partial \mathbf{v}}{\partial t}(t,\mathbf{x})+\left(\mathbf{v}(t,\mathbf{x})\cdot\nabla\right)\mathbf{v}(t,\mathbf{x}) \\
&= \mathbf{v}_t(t,\mathbf{x})+\left(\mathbf{v}(t,\mathbf{x})\cdot\nabla\right)\mathbf{v}(t,\mathbf{x}).
\end{aligned}$$

In the linear Taylor series approximation with respect of the second argument, the increment, which is needed in the Taylor expansion, is $\Delta t\mathbf{v}(t,\mathbf{x})$ and the derivative has to be taken with respect to $\mathbf{x}$ for each component of the velocity

$$\Delta t \begin{pmatrix} \mathbf{v}(t,\mathbf{x})\cdot\nabla v_1(t,\mathbf{x}) \\ \mathbf{v}(t,\mathbf{x})\cdot\nabla v_2(t,\mathbf{x}) \\ \mathbf{v}(t,\mathbf{x})\cdot\nabla v_3(t,\mathbf{x}) \end{pmatrix} = \Delta t \begin{pmatrix} v_1\partial_x v_1+v_2\partial_y v_1+v_3\partial_z v_1 \\ v_1\partial_x v_2+v_2\partial_y v_2+v_3\partial_z v_2 \\ v_1\partial_x v_3+v_2\partial_y v_3+v_3\partial_z v_3 \end{pmatrix}, \tag{1.4}$$

where the explicit dependency on t and $\mathbf{x}$ has been neglected in the right term of (1.4) for clarity of presentation. In the usual notation $(\mathbf{v} \cdot \nabla)\mathbf{v}$, one thinks of $\mathbf{v} \cdot \nabla = v_1\partial_x + v_2\partial_y + v_3\partial_z$ acting on each component of $\mathbf{v}$. This expression is the same as (1.4) without Δt. In the literature, one often finds the notation $\mathbf{v} \cdot \nabla\mathbf{v}$. The gradient of the velocity is a tensor with the components

$$(\nabla\mathbf{v})_{ij} = \partial_j v_i = \frac{\partial v_i}{\partial x_j}, \quad i, j = 1, 2, 3.$$

Altogether, the first order approximation is used as model for 'mass $\times$ acceleration' in an arbitrary volume V

$$\int_V \rho(t, \mathbf{x}) \left(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}\right)(t, \mathbf{x}) \; d\mathbf{x} \quad [N].$$

This expression must be balanced by the net forces acting on V which are composed of external (body) forces and internal forces. □

Remark 1.6. External forces. External forces include, e.g., gravity, buoyancy and electromagnetic forces (in liquid metals). These forces are collected in a body force term

$$\int_V \mathbf{F}(t, \mathbf{x}) \; d\mathbf{x}, \quad \mathbf{F} \; : \; [N/m^3].$$

□

Remark 1.7. Internal forces, Cauchy's principle and the stress tensor. Internal forces are forces which a fluid exerts on itself in trying to get out of its own way. These include pressure and viscous drag that a fluid element exerts on the adjacent element. The internal forces of a fluid are contact forces, i.e., they act on the surface of the fluid element V. Let $\mathbf{t}$ $[N/m^2]$ denote this internal force vector, which is called Cauchy stress vector or torsion vector, then the contribution of the internal forces on V is

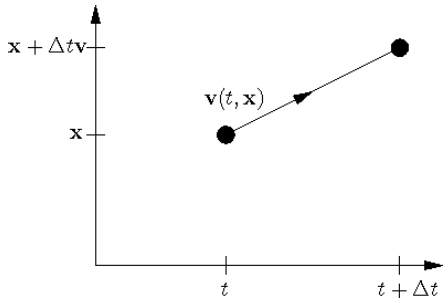

Fig. 1.3. Movement of a fluid particle.

$$\int_{\partial V} \mathbf{t}(t,\mathbf{s})\ d\mathbf{s}.$$

Thus, the equation for the conservation of linear momentum is, for an arbitrary constant-in-time volume V,

$$\int_V \rho(t,\mathbf{x})\left(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\right)(t,\mathbf{x})\ d\mathbf{x} = \int_V \mathbf{F}(t,\mathbf{x})\ d\mathbf{x} + \int_{\partial V} \mathbf{t}(t,\mathbf{s})\ d\mathbf{s}. \quad (1.5)$$

The right hand side of (1.5) describes the net force acting on and inside V. Now, a detailed description of the internal forces represented by $\mathbf{t}(t,\mathbf{s})$ is necessary.

The foundation of continuum mechanics is the stress principle of Cauchy. The idea of Cauchy on internal contact forces was that on any (imaginary) plane on ∂V there is a force that depends (geometrically) only on the orientation of the plane. Thus, $\mathbf{t} = \mathbf{t}(\mathbf{n})$, where $\mathbf{n}$ is the unit normal vector of the imaginary plane which points outward of V. It can be shown that, if linear momentum is conserved, $\mathbf{t}$ is a linear function of $\mathbf{n}$, i.e.,

$$\mathbf{t} = \mathbb{S}\mathbf{n}, \quad (1.6)$$

where $\mathbb{S}(t,\mathbf{x})$ $[N/m^2]$ is a 3×3-matrix (tensor) which is called stress tensor. The stress tensor represents all internal forces of the flow. Inserting (1.6) into the term representing the internal forces in (1.5) and applying the divergence theorem give

$$\int_{\partial V} \mathbf{t}(t,\mathbf{s})\ d\mathbf{s} = \int_V \nabla\cdot\mathbb{S}(t,\mathbf{x})\ d\mathbf{x},$$

where the divergence of a matrix (tensor) is defined row-wise

$$\nabla\cdot A = \begin{pmatrix} (a_{11})_x + (a_{12})_y + (a_{13})_z \\ (a_{21})_x + (a_{22})_y + (a_{23})_z \\ (a_{31})_x + (a_{32})_y + (a_{33})_z \end{pmatrix} = \begin{pmatrix} \partial_x a_{11} + \partial_y a_{12} + \partial_z a_{13} \\ \partial_x a_{21} + \partial_y a_{22} + \partial_z a_{23} \\ \partial_x a_{31} + \partial_y a_{32} + \partial_z a_{33} \end{pmatrix}.$$

Since (1.5) holds for every volume V, it follows

$$\rho\left(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \nabla\cdot\mathbb{S} + \mathbf{F} \quad \forall\ t\in(0,T],\ \mathbf{x}\in\Omega. \quad (1.7)$$

This is the momentum equation. □

Remark 1.8. Symmetry of the stress tensor. Let V be an arbitrary volume with sufficiently smooth boundary ∂V and let the net force given by the right hand side of (1.5). The torque in V with respect to the origin $\mathbf{0}$ of the coordinate system is then defined by

$$\mathbf{M_0} = \int_V \mathbf{r}\times\mathbf{F}\ d\mathbf{x} + \int_{\partial V} \mathbf{r}\times(\mathbb{S}\mathbf{n})\ d\mathbf{s} \quad [N\,m]. \quad (1.8)$$


Fig. 1.4. Left: Augustin Louis Cauchy (1789 – 1857), right: Robert Hooke (1635 – 1703).

Here, $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ is the vector pointing from $\mathbf{0}$ to a point $\mathbf{x} \in \overline{V}$. A straightforward calculation shows

$$\mathbf{r} \times (\mathbb{S}\mathbf{n}) = (\mathbf{r} \times \mathbb{S}_{*1} \ \mathbf{r} \times \mathbb{S}_{*2} \ \mathbf{r} \times \mathbb{S}_{*3})\,\mathbf{n},$$

where $\mathbb{S}_{*i}$ is the i-th column of $\mathbb{S}$ and $(\cdot)$ denotes here the matrix. Inserting this expression into (1.8) and applying integration by parts leads to

$$\begin{aligned}\mathbf{M_0} &= \int_V \mathbf{r} \times \mathbf{F}\ d\mathbf{x} + \int_V \nabla \cdot ((\mathbf{r} \times \mathbb{S}_{*1} \ \mathbf{r} \times \mathbb{S}_{*2} \ \mathbf{r} \times \mathbb{S}_{*3}))\, d\mathbf{x} \qquad (1.9)\\ &= \int_V \mathbf{r} \times (\mathbf{F} + \nabla \cdot \mathbb{S})\, d\mathbf{x} + \int_V \partial_x\mathbf{r} \times \mathbb{S}_{*1} + \partial_y\mathbf{r} \times \mathbb{S}_{*2} + \partial_z\mathbf{r} \times \mathbb{S}_{*3}\ d\mathbf{x}.\end{aligned}$$

Consider now a fluid in equilibrium state, i.e. the net forces acting on this fluid are zero. Hence, the right hand side of (1.7) vanishes and so the first integral of (1.9). In addition, equilibrium requires in particular that $\mathbf{M_0} = \mathbf{0}$. Thus, from (1.9) follows

$$\mathbf{0} = \int_V \partial_x\mathbf{r} \times \mathbb{S}_{*1} + \partial_y\mathbf{r} \times \mathbb{S}_{*2} + \partial_z\mathbf{r} \times \mathbb{S}_{*3}\ d\mathbf{x}. \qquad (1.10)$$

Using now

$$\partial_x\mathbf{r} = \lim_{\Delta x_1 \to 0} \frac{(x_1 + \Delta x_1)\mathbf{e}_1 - x_1\mathbf{e}_1}{\Delta x_1} = \mathbf{e}_1,$$

$\partial_y\mathbf{r} = \mathbf{e}_2$, $\partial_z\mathbf{r} = \mathbf{e}_3$, and inserting these equations into (1.10) leads finally to

$$\mathbf{0} = \int_V \begin{pmatrix} \mathbb{S}_{32} - \mathbb{S}_{23} \\ \mathbb{S}_{13} - \mathbb{S}_{31} \\ \mathbb{S}_{21} - \mathbb{S}_{12} \end{pmatrix} (t, \mathbf{x})\ d\mathbf{x}.$$

for an arbitrary volume V. It follows that $\mathbb{S}$ has to be symmetric, $\mathbb{S} = \mathbb{S}^T$, and $\mathbb{S}$ possesses six unknown components. □

Remark 1.9. Decomposition of the stress tensor. To model the stress tensor in the basis variables introduced in Remark 1.1, the stress tensor is decomposed into

$$\mathbb{S} = \mathbb{V} + P\mathbb{I}.$$

Here, $\mathbb{V}$ $[N/m^2]$ is the so-called viscous stress tensor, representing the forces coming from the friction of the particles, and P $[N/m^2]$ is the pressure, describing the forces acting on the surface of each fluid volume V. The viscous stress tensor will be modeled in terms of the velocity, see Remark 1.11. □

Remark 1.10. The pressure. The pressure P acts on a surface only normal to that surface of a fluid volume and it is directed into the volume V. Thus, the total force exerted by the pressure which is acting on the volume is

$$-\int_{\partial V} P\mathbf{n}\, d\mathbf{s} = -\int_V \nabla P\, d\mathbf{x} = -\int_V \nabla\cdot(P\mathbb{I})\, d\mathbf{x}.$$

□

Remark 1.11. The viscous stress tensor. Friction between fluid particles can only occur if the particles move with different velocities. For this reason, the viscous stress tensor depends on the gradient of the velocity. For the reason of symmetry, Remark 1.8, it depends on the symmetric part of the gradient, the so-called velocity deformation tensor

$$\mathbb{D}(\mathbf{v}) = \frac{\nabla\mathbf{v} + (\nabla\mathbf{v})^T}{2}.$$

If the velocity gradients are not too large, one can assume that the dependency is linear. This leads to the model

$$\mathbb{V} = 2\mu\mathbb{D}(\mathbf{v}) + \left(\zeta - \frac{2\mu}{3}\right)(\nabla\cdot\mathbf{v})\,\mathbb{I}, \tag{1.11}$$

where μ $[kg/(m\ s)]$ and ζ $[kg/(m\ s)]$ are the first and second order viscosities of the fluid. The viscosity μ is also called dynamic or shear viscosity. The law (1.11) is the analog for fluids of Hooke's law for solids. □

Remark 1.12. Newtonian fluids. The linear relation (1.11) is only an approximation for a real fluid. In general, the relation will be non-linear. Only for small stresses, a linear approximation of the general stress-deformation relation can be used. The first scientist to postulate a linear stress-deformation relation was Newton. For this reason, a fluid satisfying assumption (1.11) is called Newtonian fluid. More general relations than (1.11) exist, however they are less well understood. □

Remark 1.13. Normal and shear stresses, trace of the stress tensor. The diagonal components of the stress tensor $\mathbb{S}_{11}, \mathbb{S}_{22}, \mathbb{S}_{33}$ are called normal stresses and the off-diagonal components shear stresses .

The trace of the stress tensor is the sum of the normal stresses

$$\begin{aligned}\operatorname{tr}(\mathbb{S}) &= \mathbb{S}_{11} + \mathbb{S}_{22} + \mathbb{S}_{33} \\ &= \mu(\partial_x \mathbf{v}_1 + \partial_y \mathbf{v}_2 + \partial_z \mathbf{v}_3) + 3\left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \mathbf{v}) + 3P \\ &= (3\zeta - \mu)\,\nabla \cdot \mathbf{v} + 3P.\end{aligned}$$

For incompressible flows, it follows

$$P(t, \mathbf{x}) = \frac{1}{3}\left(\mathbb{S}_{11} + \mathbb{S}_{22} + \mathbb{S}_{33}\right)(t, \mathbf{x}).$$

□

Remark 1.14. The Navier–Stokes equations with dimensions. Now, the pressure part of the stress tensor and the ansatz (1.11) of the viscous stress tensor can be inserted into (1.7) giving the general Navier–Stokes equations (including the conservation of mass)

$$\begin{aligned}\rho\left(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) \\ -2\nabla \cdot (\mu \mathbb{D}(\mathbf{v})) - \nabla \cdot \left(\left(\zeta - \tfrac{2\mu}{3}\right)\nabla \cdot \mathbf{v}\mathbb{I}\right) + \nabla P = \mathbf{F} \text{ in } (0,T] \times \Omega, \\ \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \text{ in } (0,T] \times \Omega.\end{aligned}$$

If the fluid is incompressible and homogeneous, such that μ is a constant, the Navier–Stokes equations simplify to

$$\begin{aligned}\mathbf{v}_t - 2\nu\nabla \cdot \mathbb{D}(\mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla\frac{P}{\rho_0} = \frac{\mathbf{F}}{\rho_0} \text{ in } (0,T] \times \Omega, \\ \nabla \cdot \mathbf{v} = 0 \text{ in } (0,T] \times \Omega.\end{aligned} \tag{1.12}$$

Here, $\nu = \mu/\rho_0$ $[m^2/s]$ is the kinematic viscosity of the fluid. □

1.3 The Dimensionless Navier–Stokes Equations

Remark 1.15. Characteristic scales. Mathematical analysis and numerical simulations are based on dimensionless equations. The functions in system (1.12) are not dimensionless. To derive a dimensionless equations, the quantities

- L $[m]$ – a characteristic length scale of the flow problem,
- U $[m/s]$ – a characteristic velocity scale of the flow problem,
- T^* $[s]$ – a characteristic time scale of the flow problem,

are introduced. Let $(t', \mathbf{x}')$ $[s, m]$ be the old variables. □

Remark 1.16. The Navier–Stokes equations in dimensionless form. Applying the transform of variables

$$\mathbf{x} = \frac{\mathbf{x}'}{L}, \quad \mathbf{u} = \frac{\mathbf{v}}{U}, \quad t = \frac{t'}{T^*},$$

one obtains from (1.12) and a rescaling

$$\begin{aligned} \frac{L}{UT^*}\mathbf{u}_t - \frac{2\nu}{UL}\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\frac{P}{\rho_0 U^2} &= \frac{L}{\rho_0 U^2}\mathbf{F} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \qquad\quad \text{in } (0,T]\times\Omega, \end{aligned}$$

where all derivatives are with respect to the new variables. Defining

$$p = \frac{P}{\rho_0 U^2}, \quad Re = \frac{UL}{\nu}, \quad St = \frac{L}{UT^*}, \quad \mathbf{f} = \frac{L}{\rho_0 U^2}\mathbf{F}, \tag{1.13}$$

the Navier–Stokes equations in dimensionless form

$$\begin{aligned} St\mathbf{u}_t - \frac{2}{Re}\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } (0,T]\times\Omega \end{aligned} \tag{1.14}$$

are obtained. The constant Re is called Reynolds number and the constant St Strouhal number. These numbers allow the classification and comparison of different flows. □


Fig. 1.5. Left: Osborne Reynolds (1842 – 1912), right: Čeněk Strouhal (1850 – 1923).

Remark 1.17. Simplified notation of the dimensionless Navier–Stokes equations. To simplify the notations, one uses the characteristic quantities $L = 1\ m$, $U = 1\ m/s$, and $T^* = L/U = 1\ s$ such that (1.14) simplifies to

$$\begin{aligned} \mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \text{ in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \text{ in } (0,T]\times\Omega, \end{aligned} \tag{1.15}$$

with the dimensionless viscosity $\nu = Re^{-1}$.

This transform and the resulting equations (1.15) will be the basis equations for the mathematical analysis and the numerical simulation of the Navier–Stokes equations. There are two important difficulties for the mathematical analysis and the numerical simulation of the Navier–Stokes equations:

- the coupling of velocity and pressure,
- the non-linearity of the convective term.

Additionally, difficulties for the numerical simulation occur if

- the convective term dominates the viscous term, i.e. ν is small.

□

Remark 1.18. Different forms of terms in (1.15). With the help of the divergence constraint, i.e. the second equation in (1.15), the viscous and the convective term of the Navier–Stokes equations can be reformulated equivalently.

Assume, $\mathbf{u}$ is sufficiently smooth with $\nabla \cdot \mathbf{u} = 0$. Then

$$\nabla \cdot (\nabla \mathbf{u}) = \Delta \mathbf{u}, \quad \nabla \cdot \left(\nabla \mathbf{u}^T\right) = \begin{pmatrix} (\nabla \cdot \mathbf{u})_x \\ (\nabla \cdot \mathbf{u})_y \\ (\nabla \cdot \mathbf{u})_z \end{pmatrix} = \mathbf{0}.$$

Thus, the viscous term becomes

$$-2\nu \nabla \cdot \mathbb{D}(\mathbf{u}) = -\nu \Delta \mathbf{u}.$$

For the convective term, ones uses the identity (product rule)

$$\nabla \cdot \left(\mathbf{u}\mathbf{v}^T\right) = (\nabla \cdot \mathbf{v})\mathbf{u} + (\mathbf{v} \cdot \nabla)\mathbf{u}.$$

In the case $\mathbf{v} = \mathbf{u}$ with $\nabla \cdot \mathbf{u} = 0$, it follows

$$(\mathbf{u} \cdot \nabla)\,\mathbf{u} = \nabla \cdot \left(\mathbf{u}\mathbf{u}^T\right).$$

Note that different forms of the terms are in general not longer equivalent for a discretization of the Navier–Stokes equations since the discrete velocity field is in general not divergence-free. □

Remark 1.19. Two-dimensional Navier–Stokes equations. Even if real flows occur only in three dimensions, the consideration of the Navier–Stokes equations (1.15) in two dimensions is also of interest. E.g., there are applications where the flow is constant in the third direction and it behaves virtually two-dimensional. □

Remark 1.20. Special cases of incompressible flow models.

- In a stationary flow, the velocity and the pressure do not change in time. Hence $\mathbf{u}_t = \mathbf{0}$ and these flows are modelled by the so-called stationary or steady-state Navier–Stokes equations

$$\begin{aligned} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega. \end{aligned} \tag{1.16}$$

  A necessary condition for the stationarity of a flow field is that the data of the problem, i.e. the right hand side and the boundary conditions, see Sect. 1.4, are time-independent. But this condition is not sufficient.

- If in a stationary flow the viscous transport dominates the convective transport, i.e. if the fluid flows very slowly, the non-linear convective term of the Navier–Stokes equations (1.16) can be neglected. This gives a linear equation, the so-called Stokes equations

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega. \end{aligned} \tag{1.17}$$

  Here, the momentum equation was divided by ν, defining a new pressure and a new right hand side.
- In the numerical analysis, often the so-called Oseen equations are considered. Given a divergence-free flow field $\mathbf{u}_0$, the Oseen equations are a system of linear equations of the form

$$\begin{aligned} -\nu\Delta \mathbf{u} + (\mathbf{u}_0 \cdot \nabla)\mathbf{u} + \nabla p + c\mathbf{u} &= \mathbf{f} \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega, \end{aligned} \tag{1.18}$$

  with a real number $c \geq 0$.

□

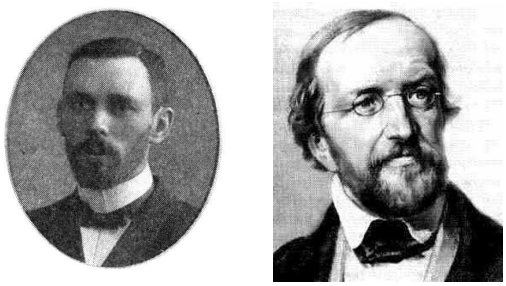

Fig. 1.6. Left: Carl Wilhelm Oseen (1879 – 1944), right: Johann Peter Gustav Lejeune Dirichlet (1805 – 1859).

1.4 Initial and Boundary Conditions

Remark 1.21. General. The Navier–Stokes equations (1.15) are a first order partial differential equation with respect to time and a second order partial differential equation with respect to space. Thus, they have to be equipped with an initial condition at $t = 0$ and with boundary conditions on the boundary $\Gamma = \partial\Omega$ of Ω, if Ω is a bounded domain. There are several kinds of boundary conditions which can be prescribed for incompressible flows. Of course, a

compatibility condition should be fulfilled between the boundary conditions of the initial velocity field and the limit of the prescribed boundary conditions for $t \to 0, t > 0$. □

Remark 1.22. Initial condition. Concerning the initial condition, an initial velocity field, which has to be divergence-free, is prescribed at $t = 0$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \text{with} \quad \nabla \cdot \mathbf{u}_0 = 0 \quad \text{in} \quad \Omega.$$

□

Remark 1.23. Dirichlet boundary conditions, no-slip boundary conditions. An often used boundary conditions describes the velocity field on a part of the boundary

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{g}(t, \mathbf{x}) \quad \text{in} \quad (0, T] \times \Gamma_{\text{diri}} \subset \Gamma.$$

This boundary condition is called Dirichlet boundary condition. It models in particular prescribed inflows into Ω and outflows from Ω. In the special case $\mathbf{g}(t, \mathbf{x}) = \mathbf{0}$ in $(0, T] \times \Gamma_{\text{diri}}$, this boundary condition is called no-slip boundary condition. Let $\mathbf{n}$ be the unit normal vector in $\mathbf{x} \in \Gamma_{\text{nosl}} \subset \Gamma_{\text{diri}}$ and $\{\mathbf{t}_1, \mathbf{t}_2\}$ unit tangential vectors such that $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$ is an orthonormal system of vectors. Then, the no-slip boundary condition can be decomposed into three parts:

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{0} \iff \mathbf{u}(t, \mathbf{x}) \cdot \mathbf{n} = 0, \ \mathbf{u}(t, \mathbf{x}) \cdot \mathbf{t}_1 = 0, \quad \mathbf{u}(t, \mathbf{x}) \cdot \mathbf{t}_2 = 0$$

in $\mathbf{x} \in \Gamma_{\text{nosl}}$. The condition $\mathbf{u}(t, \mathbf{x}) \cdot \mathbf{n} = 0$ states that the fluid does not penetrate the wall. The other two conditions describe that the fluid does not slip along the wall.

If Dirichlet boundary conditions are prescribed on the whole boundary of Ω, the pressure is determined only up to an additive constant. An additional condition for fixing the constant has to be introduced, e.g., that the integral mean value of the pressure should vanish

$$\int_\Omega p(t, \mathbf{x}) \, d\mathbf{x} = 0 \quad t \in (0, T].$$

□

Remark 1.24. Free slip boundary conditions, slip with friction boundary conditions. The free slip boundary condition is applied on boundaries without friction. It has the form

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= g \quad \text{in } (0, T] \times \Gamma_{\text{slip}} \subset \Gamma, \\ \mathbf{n}^T \mathbb{S} \mathbf{t}_k &= 0 \quad \text{in } (0, T] \times \Gamma_{\text{slip}}, \quad 1 \le k \le d - 1. \end{aligned} \tag{1.19}$$

There is no penetration through the wall if $g = 0$ on Γ_{slip}.

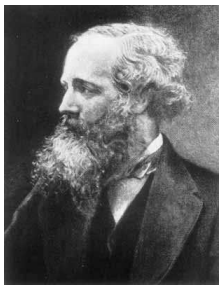

Fig. 1.7. James Clerk Maxwell (1831 – 1879)

The slip with linear friction and no penetration boundary condition has the form

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{in } (0,T] \times \Gamma_{\text{slfr}} \subset \Gamma, \\ \mathbf{u} \cdot \mathbf{t}_k + \beta^{-1} \mathbf{n}^T \mathbb{S} \mathbf{t}_k &= 0 \quad \text{in } (0,T] \times \Gamma_{\text{slfr}}, \quad 1 \leq k \leq d-1. \end{aligned} \tag{1.20}$$

This boundary condition states that the fluid does not penetrate the wall and it slips along the wall whereas it loses energy. The loss of energy is given by the friction parameter β. In the limit case $\beta^{-1} \to 0$, the no-slip condition is recovered and in the limit case $\beta^{-1} \to \infty$ the free slip condition. Slip with friction boundary conditions were studied already by Maxwell [Max79] and Navier [Nav23]. The difficulty in the application of this boundary condition consists in the determination of the friction parameter β, which might depend, e.g., on the local flow field and on the roughness of the wall.

Since $\mathbf{n}$ and $\mathbf{t}_k$ are orthogonal vectors, the values of the pressure do not play any role in the boundary conditions (1.19) and (1.20). Hence, an additional condition for the pressure is needed to fix the additive constant. □

Remark 1.25. Outflow or do-nothing boundary conditions. For numerical simulations, the so-called outflow boundary condition or do-nothing boundary condition

$$\mathbb{S}\mathbf{n} = \mathbf{0} \quad \text{in} \quad (0,T] \times \Gamma_{\text{outf}} \subset \Gamma$$

is often applied. This boundary condition models that the normal stress, which is equal to the Cauchy stress vector (1.6), vanishes on the boundary part Γ_{outf}. The do-nothing boundary condition is used in flow problems, e.g., where no other outflow boundary condition is available. □

Remark 1.26. Conditions for an infinite domain, periodic boundary conditions. The case $\Omega = \mathbb{R}^3$ is also considered in analytical and numerical studies of the Navier–Stokes equations. There are two situations in this case. In the first one, the decay of the velocity field as $\|\mathbf{x}\|_2 \to \infty$ is prescribed. The second situation consists of periodic boundary conditions. These boundary

conditions do not posses any physical meaning. They are used to simulate an infinite extension of Ω in one or more directions. Let, e.g., this direction be $\mathbf{e}_i$. It is assumed that the flow is periodic in this direction with the length l of the period. In computations, e.g., the cube $\Omega = (0, l)^d$ is used and the periodic boundary conditions are given by

$$\mathbf{u}(t, \mathbf{x} + l\mathbf{e}_i) = \mathbf{u}(t, \mathbf{x}) \quad \forall\ (t, \mathbf{x}) \in (0, T] \times \Gamma.$$

From the point of view of the finite computational domain, all appearing functions have to be extended periodically in the periodic direction to return to the original problem.

The use of space periodic boundary conditions may also facilitate analytical investigations, see Temam [Tem95, p. 4]. □