Appendix A
Functional Analysis

Remark A.1. Motivation. The study of the existence and uniqueness of solutions of the Navier–Stokes equations as well as the finite element error analysis requires tools from functional analysis, in particular the use of function spaces, certain inequalities, and imbedding theorems. There will be no difference in the notation for functions spaces for scalar, vector-valued, and tensor-valued functions.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a domain, i.e., Ω is an open set. □

A.1 Metric Spaces, Banach Spaces, and Hilbert Spaces

Definition A.2. Metric space. Let $X \neq \emptyset$ be a set. A map $d : X \times X \to \mathbb{R}$ is called a metric on X if for all $x, y, z \in X$ it is
i) $d(x, y) = 0 \iff x = y$,
ii) symmetry: $d(x, y) = d(y, x)$,
iii) triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.
Then (X, d) is called a metric space. □

Definition A.3. Isometric metric space. Two metric spaces (X_1, d_1) and $X_2, d_2)$ are called isometric, if there is a surjective map $g : X_1 \to X_2$ such that for all $x, y \in X_1$ it is $d_1(x, y) = d_2(g(x), g(y))$. □

Definition A.4. Cauchy sequence, convergent sequence. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a metric space (X, d). It is called a Cauchy sequence if for each $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that

$$d(x_k, x_l) < \varepsilon \quad \forall\ k, l \geq N.$$

The sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$, denoted by $x_n \to x$, if

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

□

Definition A.5. Complete metric space. A metric space (X, d) is called complete, if each Cauchy sequence converges in X. That means, for each Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ there exists an element $x \in X$ such that $x_n \to x$. □

Definition A.6. Norm, triangle inequality, seminorm, normed space. Let X be a linear space over $\mathbb{R}$ (or $\mathbb{C}$). A mapping $\|\cdot\|_X : X \to \mathbb{R}$ is called a norm on X if
i) definiteness: $\|x\|_X = 0$ if and only if $x = 0$,
ii) homogeneity: $\|\alpha x\|_X = |\alpha| \, \|x\|_X$ for all $x \in X$, $\alpha \in \mathbb{R}$,
iii) the triangle inequality holds: $\|x + y\|_X \le \|x\|_X + \|y\|_X$ for all $x, y \in X$.

A mapping from X to $\mathbb{R}$ that satisfies only ii) and iii) is called a seminorm on X.

The space $(X, \|\cdot\|_X)$ is called normed space. □

Definition A.7. Equivalent norms. Two norms $\|\cdot\|_{X,1}$, $\|\cdot\|_{X,2}$ of a normed space X are called equivalent, if there are two positive constants C_1 and C_2 such that

$$C_1 \|x\|_{X,1} \le \|x\|_{X,2} \le C_2 \|x\|_{X,1} \quad \forall\, x \in X.$$

□

Remark A.8. On norms.

- All norms in finite-dimensional spaces are equivalent.
- A normed space $(X, \|\cdot\|_X)$ becomes a metric space with the induced metric

$$d(x_1, x_2) = \|x_1 - x_2\|_X, \quad x_1, x_2 \in X.$$

□

Definition A.9. Banach space. A normed space is called complete if it is a complete metric space with the induced metric. A complete normed space is called Banach space. □

Remark A.10. Compact set, precompact set. A subset Y of a normed space X is called compact if every sequence of elements in Y has a subsequence that converges in the norm of X to an element of Y. The set Y is called precompact if its closure $\overline{Y}$ is compact.

Compact sets are closed and bounded. The reverse statement is only true for finite-dimensional spaces. □

Definition A.11. Inner product, scalar product. Let X be a linear space over $\mathbb{R}$. A map $(\cdot,\cdot)_X : X \times X \to \mathbb{R}$ is called symmetric sesquilinear form if for all $x, y, z \in X$ and all $\alpha \in \mathbb{R}$ it holds that
i) symmetry: $(x, y)_X = (y, x)_X$,
ii) $(\alpha x, y)_X = \alpha(x, y)$,

iii) $(x, y + z) = (x, y) + (x, z)$.

The symmetric sesquilinear form $(\cdot, \cdot)_X$ is called positive semi-definite if for all $x \in X$ it is $(x, x)_X \geq 0$. A positive semi-definite symmetric sesquilinear form with

$$(x, x)_X = 0 \quad \Longleftrightarrow \quad x = 0$$

is called inner product or scalar product on X. □

Definition A.12. Induced norm, inner product space, Hilbert space. Let $(\cdot, \cdot)_X$ be an inner product on X, then $(X, (\cdot, \cdot)_X)$ is called pre Hilbert space. The inner product induces the norm

$$\|x\|_X = (x, x)_X^{1/2}$$

in X. A complete inner product space is called Hilbert space.

For simplicity of notation, the subscript at the inner product symbol will be neglected if the inner product is clear from the context. □

Lemma A.13. Cauchy–Schwarz inequality. *Let $(X, (\cdot, \cdot))$ be an inner product space, then it holds the so-called Cauchy–Schwarz inequality*

$$|(x, y)| \leq \|x\|_X \, \|y\|_X \quad \forall \, x, y \in X. \tag{A.1}$$

Example A.14. Cauchy–Schwarz inequality for sums. Consider $X = \mathbb{R}^n$ with the standard inner product for vectors, then one obtains with the triangle inequality and the Cauchy–Schwarz inequality (A.1)

$$\left|\sum_{i=1}^n x_i y_i\right| \leq \sum_{i=1}^n |x_i| \, |y_i| \leq \left(\sum_{i=1}^n x_i^2\right)^{1/2} \left(\sum_{i=1}^n y_i^2\right)^{1/2}, \tag{A.2}$$

for all $\underline{x} = (x_1, \ldots, x_n)^T, \underline{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$. □

Example A.15. Hölder inequality for sums. The Cauchy–Schwarz inequality (A.2) is a special case of the Hölder inequality

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |b_i|^q\right)^{1/q}, \quad 1 < p, q < \infty, \ \frac{1}{p} + \frac{1}{q} = 1. \tag{A.3}$$

The following inequalities for sums of non-negative real numbers hold:

$$\sum_{i=1}^n a_i \leq \left(\sum_{i=1}^n a_i^{1/p}\right)^p \leq n^{p/q} \sum_{i=1}^n a_i, \quad a_i \geq 0, \ p \in (1, \infty), \ \frac{1}{p} + \frac{1}{q} = 1. \tag{A.4}$$

The right inequality of (A.4) is just a consequence of (A.3). □

Definition A.16. Orthogonal elements, orthogonal complement of a subspace. Let X be a normed space endowed with an inner product $(\cdot,\cdot)$. Two elements $x, y \in X$ are said to be orthogonal if $(x,y) = 0$.

Let $Y \subset X$ be a subspace of X, then $Y^\perp = \{x \in X \ : \ (x,y) = 0 \text{ for all } y \in Y\}$ is the orthogonal complement of Y. □

Lemma A.17. Orthogonal complement is closed subspace. *Let $W \subset V$ be a subspace of a Hilbert space V. Then, $W^\perp$ is a closed subspace of V.*

Lemma A.18. Young's inequality. *Let $a, b \in \mathbb{R}$, $a, b \geq 0$, then the following inequality is called Young's inequality:*

$$ab \leq \frac{t}{p} a^p + \frac{t^{-q/p}}{q} b^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p, q < \infty, \quad t > 0. \tag{A.5}$$

Proof. The proof is based on the strict convexity of the exponential, which follows from the strict positivity of the second derivative. This property reads for $\alpha, \beta \in \mathbb{R}$ and p, q as in (A.5)

$$\exp\left(\frac{\alpha}{p} + \frac{\beta}{q}\right) \leq \frac{1}{p}\exp(\alpha) + \frac{1}{q}\exp(\beta).$$

Choosing $\alpha = \ln(ta^p)$ and $\beta = \ln\left(t^{-q/p} b^q\right)$ gives (A.5). ■

Lemma A.19. Estimate for a Rayleigh quotient. *Let $A \in \mathbb{R}^{m\times n}$ be a matrix, then it is*

$$\inf_{\underline{x}\in\mathbb{R}^n, \underline{x}\neq\underline{0}} \frac{\underline{x}^T A^T A \underline{x}}{\underline{x}^T \underline{x}} = \lambda_{\min}\left(A^T A\right),$$

where $\lambda_{\min}\left(A^T A\right)$ is the smallest eigenvalue of $A^T A$. The infimum is taken, i.e., it is even a minimum. The quotient on the left-hand side is called Rayleigh quotient.

Proof. The matrix $A^T A$ is symmetric and positive semi-definite. Hence, all eigenvalues are non-negative, the (normalized eigenvectors) $\{\underline{\phi}_i\}_{i=1}^n$ form a basis of $\mathbb{R}^n$, and they are mutually orthonormal. Let the eigenvalues be ordered such that

$$0 \leq \lambda_{\min}\left(A^T A\right) = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

Each vector $\underline{x} \in \mathbb{R}^n$ can be written in the form $\underline{x} = \sum_{i=1}^n x_i \underline{\phi}_i$. Using that the eigenvectors are orthonormal, it follows that $\underline{x}^T\underline{x} = \sum_{i=1}^n x_i^2$ and

$$\underline{x}^T A^T A \underline{x} = \underline{x}^T \sum_{i=1}^n x_i A^T A \underline{\phi}_i = \sum_{j=1}^n \sum_{i=1}^n \lambda_i x_j x_i \underline{\phi}_j \underline{\phi}_i = \sum_{i=1}^n \lambda_i x_i^2 \geq \lambda_{\min}\left(A^T A\right) \sum_{i=1}^n x_i^2.$$

Hence, one gets

$$\inf_{\underline{x}\in\mathbb{R}^n, \underline{x}\neq\underline{0}} \frac{\underline{x}^T A^T A \underline{x}}{\underline{x}^T \underline{x}} \geq \lambda_{\min}\left(A^T A\right).$$

Choosing $\underline{x} = x_1 \underline{\phi}_1$, $x_1 \neq 0$, leads to

$$\frac{\underline{x}^T A^T A \underline{x}}{\underline{x}^T \underline{x}} = \frac{\lambda_1 x_1^2}{x_1^2} = \lambda_1 = \lambda_{\min}\left(A^T A\right),$$

such that the equal sign holds. ■

A.2 Function Spaces

Definition A.20. Derivatives and multi-index. A multi-index $\boldsymbol{\alpha}$ is a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, \ldots, n$. Derivatives are denoted by

$$D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \quad \text{with} \quad |\boldsymbol{\alpha}| = \sum_{i=1}^{n} \alpha_i.$$

Low order derivatives are also denoted by subscripts, e.g.,

$$\partial_x u = \frac{\partial u}{\partial x}.$$

□

Definition A.21. Spaces of continuously differentiable functions $C^m(\Omega)$, $C^m(\overline{\Omega})$, and $C_B^m(\Omega)$. Let $m \in \mathbb{N} \cup \{0\}$, then the space of m-times continuously differentiable functions in Ω is denoted by

$$C^m(\Omega) = \big\{ f \ : \ f \text{ and all its derivatives up to order } m \text{ are continuous in } \Omega \big\}.$$

It is

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega).$$

The space $C^m(\overline{\Omega})$ for $m < \infty$ is defined by

$$C^m(\overline{\Omega}) = \big\{ f \ : \ f \in C^m(\Omega) \text{ and all derivatives can be extended continuously to } \overline{\Omega} \big\}.$$

One defines

$$C^\infty(\overline{\Omega}) = \bigcap_{m=0}^{\infty} C^m(\overline{\Omega}).$$

Finally, the following space is introduced

$$C_B^m(\Omega) = \big\{ f \ : \ f \in C^m(\Omega) \text{ and } f \text{ is bounded} \big\}. \tag{A.6}$$

□

Remark A.22. Spaces of continuously differentiable functions $C^m(\Omega)$, $C^m(\overline{\Omega})$, and $C_B^m(\Omega)$.

- If Ω is bounded, then $C^m(\overline{\Omega})$, equipped with the norm

$$\|f\|_{C^m(\overline{\Omega})} = \sum_{0\le|\boldsymbol{\alpha}|\le m} \max_{\boldsymbol{x}\in\overline{\Omega}} |D^{\boldsymbol{\alpha}} f(\boldsymbol{x})|\,,$$

is a Banach space.

- The space $C_B^m(\Omega)$ becomes a Banach space with the norm

$$\|f\|_{C_B^m(\Omega)} = \max_{0\le|\boldsymbol{\alpha}|\le m} \sup_{\boldsymbol{x}\in\Omega} |D^{\boldsymbol{\alpha}} f(\boldsymbol{x})|\,.$$

- It is

$$C^m(\overline{\Omega}) \subset C_B^m(\Omega) \subset C^m(\Omega).$$

Consider, e.g., $\Omega = (0,1)$ and $f(x) = \sin(1/x)$, then $f \in C_B(\Omega)$ but $f \notin C(\overline{\Omega})$.

□

Definition A.23. Support. Let $f \in C(\Omega)$, then

$$\operatorname{supp}(f) = \overline{\{\boldsymbol{x} \ : \ f(\boldsymbol{x}) \neq 0\}}$$

is the support of $f(\boldsymbol{x})$. The closure is taken with respect to $\mathbb{R}^d$. A function $f \in C(\Omega)$ is said to have a compact support, if the support of $f(\boldsymbol{x})$ is bounded in $\mathbb{R}^d$ and if $\operatorname{supp}(f) \subset \Omega$. □

Definition A.24. The space $C_0^m(\Omega)$. The space $C_0^m(\Omega)$ is given by

$$C_0^m(\Omega) = \{f \ : f \in C^m(\Omega) \text{ and } \operatorname{supp}(f) \text{ is compact in } \Omega\}.$$

In the literature, the space $C_0^\infty(\Omega)$ is often denoted by $\mathcal{D}(\Omega)$.

An important space for the study of the Navier–Stokes equations is

$$C_{0,\mathrm{div}}^\infty(\Omega) = \{\boldsymbol{f} \ : \ \boldsymbol{f} \in C_0^\infty(\Omega), \nabla\cdot\boldsymbol{f} = 0\}. \tag{A.7}$$

□

Definition A.25. The spaces $C^{m,\alpha}(\overline{\Omega})$, spaces of Hölder continuous functions. Let $M \in \mathbb{R}^d$, $d \in \{2,3\}$, be a set and let $\alpha \in (0,1]$. Then, the constant

$$|f|_{C^{0,\alpha}(M)} = \sup_{\boldsymbol{x}\neq\boldsymbol{y}\in M} \left\{ \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\alpha} \right\}$$

is called Hölder coefficient or Hölder constant. For $\alpha = 1$, it is usually called Lipschitz constant.

Let Ω be bounded. For $m \in \mathbb{N}\cup\{0\}$, the following spaces are defined

$$C^{m,\alpha}(\overline{\Omega}) = \left\{ f \in C^m(\overline{\Omega}) \ : \ \left|D^{\boldsymbol{\beta}} f\right|_{C^{0,\alpha}(\overline{\Omega})} < \infty, |\boldsymbol{\beta}| = m \right\}.$$

For $m = 0$, these spaces are called spaces of Hölder continuous functions and for $\alpha = 1$, space of Lipschitz continuous functions. □

Remark A.26. The spaces $C^{m,\alpha}(\overline{\Omega})$. The spaces $C^{m,\alpha}(\overline{\Omega})$ are Banach spaces if they are equipped with the norm

$$\|f\|_{C^{m,\alpha}(\overline{\Omega})} = \|f\|_{C^m(\overline{\Omega})} + \sum_{|\boldsymbol{\beta}|=m} [D^{\boldsymbol{\beta}} f]_{C^{0,\alpha}(\overline{\Omega})}.$$

□

Definition A.27. Spaces of (Lebesgue) integrable functions $L^p(\Omega)$. The Lebesgue spaces are defined by

$$L^p(\Omega) = \left\{ f \ : \ \int_\Omega |f(\boldsymbol{x})|^p \ d\boldsymbol{x} < \infty \right\}, \quad p \in [1, \infty),$$

where the integral is to be understood in the sense of Lebesgue. The space $L^\infty(\Omega)$ is the space of all functions that are bounded for almost all $\boldsymbol{x} \in \Omega$

$$L^\infty(\Omega) = \{ f \ : \ |f(\boldsymbol{x})| < \infty \text{ for almost all } \boldsymbol{x} \in \Omega \}.$$

□

Remark A.28. Lebesgue spaces.

- The space $L^p(\Omega)$ is a normed vector space with norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(\boldsymbol{x})|^p \ d\boldsymbol{x} \right)^{1/p}, \quad p \in [1, \infty).$$

- An important special case is $L^2(\Omega)$ since this space is a Hilbert space. The inner product $(f, g)_{L^2(\Omega)}$ of $L^2(\Omega)$ and the induced norm are given by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(\boldsymbol{x}) g(\boldsymbol{x}) \ d\boldsymbol{x}, \quad \|f\|_{L^2(\Omega)} = (f, f)^{1/2}_{L^2(\Omega)}.$$

- The space $L^\infty(\Omega)$ becomes a Banach space if it is equipped with the norm

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\boldsymbol{x} \in \Omega} |f(\boldsymbol{x})|,$$

  where $\operatorname{ess\,sup}_{\boldsymbol{x} \in \Omega}$ is the essential supremum.
- Let $|\Omega| < \infty$ and $1 \le p \le q \le \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_{L^p(\Omega)} \le \left(\int_\Omega d\boldsymbol{x} \right)^{1/p - 1/q} \|u\|_{L^q(\Omega)}, \tag{A.8}$$

  see (Adams, 1975, Theorem 2.8).

□

Example A.29. Cauchy–Schwarz inequality and Hölder's inequality. Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $p, q \in [1, \infty]$ and $1/p + 1/q = 1$. Then it is

$fg \in L^1(\Omega)$ and the Hölder inequality holds

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} . \tag{A.9}$$

For $p = q = 2$, this inequality is called Cauchy–Schwarz inequality

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} . \tag{A.10}$$

□

Definition A.30. Sobolev spaces $W^{k,p}(\Omega)$. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ consists of all integrable functions $f : \Omega \to \mathbb{R}$ such that for each multi-index $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le k$, the derivative $D^{\boldsymbol{\alpha}} f$ exists in the weak sense and it belongs to $L^p(\Omega)$. □

Remark A.31. Sobolev spaces.

- It is $L^p(\Omega) = W^{0,p}(\Omega)$.
- A norm in Sobolev spaces is defined by

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\boldsymbol{\alpha}| \le k} \|D^{\boldsymbol{\alpha}} f\|^p_{L^p(\Omega)}\right)^{1/p} & \text{if } p \in [1, \infty), \\ \sum_{|\boldsymbol{\alpha}| \le k} \operatorname{ess\,sup}_{\boldsymbol{x} \in \Omega} |D^{\boldsymbol{\alpha}} f| & \text{if } p = \infty. \end{cases}$$

  Sobolev spaces equipped with this norm are Banach spaces, e.g., see (Evans, 2010, p. 262).
- The Sobolev spaces for $p = 2$ are Hilbert spaces. They are often denoted by $W^{m,2}(\Omega) = H^m(\Omega)$ and they are equipped with the inner product

$$(f, g)_{H^k(\Omega)} = \sum_{|\boldsymbol{\alpha}| \le k} (D^{\boldsymbol{\alpha}} f, D^{\boldsymbol{\alpha}} g)_{L^2(\Omega)} .$$

- In particular, the Sobolev spaces of first order are important for the study of the Navier–Stokes equations

$$W^{1,p}(\Omega) = \left\{ f \ : \ \int_\Omega |f(\boldsymbol{x})|^p + |\nabla f(\boldsymbol{x})|^p \ d\boldsymbol{x} < \infty \right\}, \quad p \in [1, \infty),$$

  which are equipped with the norm

$$\|f\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |f(\boldsymbol{x})|^p + |\nabla f(\boldsymbol{x})|^p \ d\boldsymbol{x} \right)^{1/p}, \quad p \in [1, \infty).$$

- The definition of Sobolev spaces can be extended to $k \in \mathbb{R}$, e.g., see Adams (1975).

□

Definition A.32. Sobolev spaces $W_0^{k,p}(\Omega)$. The Sobolev spaces $W_0^{k,p}(\Omega)$ are defined by the closure of $C_0^\infty(\Omega)$ in the norm of $W^{k,p}(\Omega)$. □

Remark A.33. On the smoothness of the boundary. The Sobolev imbedding theorem requires that Ω has the so-called cone property or the strong local Lipschitz property. In the case that Ω is bounded, these assumptions reduce to the requirement that Ω has a locally Lipschitz boundary, (Adams, 1975, p. 67). That means, each point $\boldsymbol{x}$ on the boundary $\partial\Omega$ of Ω has a neighborhood $U_{\boldsymbol{x}}$ such the $\partial\Omega \cap U_{\boldsymbol{x}}$ is the graph of a Lipschitz continuous function. □

Theorem A.34. Trace theorem, *(Lions & Magenes, 1972, Theorem 9.4), (Galdi, 2011, Theorem II.4.1 for $m = 1$). Let Ω be a bounded domain with locally Lipschitz boundary $\partial\Omega$. Then, there is a bounded linear operator T : $W^{1,q}(\Omega) \to L^r(\partial\Omega)$, $q \in [1,\infty)$, such that*

i) $r \in [1, q(d-1)/(d-q)]$ if $q < d$ and $r \in [1,\infty)$ else,
ii) $Tf = f|_{\partial\Omega}$ if $f \in W^{1,q}(\Omega) \cap C\left(\overline{\Omega}\right)$,
iii) $\|Tf\|_{L^r(\partial\Omega)} \le C\,\|f\|_{W^{1,q}(\Omega)}$ for each $f \in W^{1,q}(\Omega)$, with the constant C depending only on q and Ω.

The mapping

$$H^s(\Omega) \to \prod_{j=0}^{s_0} H^{s-j-1/2}(\partial\Omega), \quad f \mapsto \left\{\frac{\partial^j f}{\partial \boldsymbol{n}^j},\ j = 0,1,\ldots,s_0\right\} \tag{A.11}$$

is continuous, where s_0 is the greatest integer such that $s_0 < s - 1/2$, and $\boldsymbol{n}$ is the outward pointing unit normal vector. The mapping is surjective and there exists a continuous right inverse.

Theorem A.35. Functions with vanishing trace, *(Galdi, 2011, Theorem II.4.2), (Evans, 2010, p. 273). Let the assumptions of Theorem A.34 be given. Then $f \in W_0^{1,p}(\Omega)$ if and only if $Tf = 0$ on $\partial\Omega$.*

Theorem A.36. Poincaré's inequality, Poincaré–Friedrichs' inequality, *(Galdi, 2011, Theorem II.5.1), (Gilbarg & Trudinger, 1983, p. 164). Let $f \in W_0^{1,p}(\Omega)$, then*

$$\|f\|_{L^p(\Omega)} \le \left(\frac{|\Omega|}{\omega_d}\right)^{1/d} \|\nabla f\|_{L^p(\Omega)} = C_{\mathrm{PF}}\,\|\nabla f\|_{L^p(\Omega)} \quad p \in [1,\infty), \tag{A.12}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Remark A.37. Poincaré's inequality. Poincaré's inequality (A.12) holds also for functions $v \in H^1(\Omega)$ with $v = 0$ on $\Gamma_0 \subset \Gamma$ with $|\Gamma_0| > 0$.

Poincaré's inequality stays valid for vector-valued functions $\boldsymbol{v}$ if Ω is bounded with a locally Lipschitz boundary, $\boldsymbol{v} \in W^{1,q}(\Omega)$, $1 \le q < \infty$, and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, see (Galdi, 1994, Section II.4). □

Theorem A.38. Density of continuous functions in Sobolev spaces, *(Gilbarg & Trudinger, 1983, p. 154). The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Remark A.39. Density of continuous functions in Sobolev spaces. For $C^\infty(\overline{\Omega})$ to be dense in $W^{k,p}(\Omega)$, one needs some smoothness assumptions on the boundary $\partial\Omega$, like $\partial\Omega$ is C^1 or the so-called segment property, e.g., see (Gilbarg & Trudinger, 1983, p. 155). This segment property follows from the strong local Lipschitz property, see (Adams, 1975, p. 67). □

Theorem A.40. Interpolation theorem for Sobolev spaces, *(Adams, 1975, Theorem 4.17). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a locally Lipschitz boundary and let $p \in [1,\infty)$. Then there exists a constant $C(m,p,\Omega)$ such that for $0 \le j \le m$ and any $u \in W^{m,p}(\Omega)$*

$$\|u\|_{W^{j,p}(\Omega)} \le C(m,p,\Omega) \|u\|_{W^{m,p}(\Omega)}^{j/m} \|u\|_{L^p(\Omega)}^{(m-j)/m}. \tag{A.13}$$

In addition, (A.13) *is valid for all $u \in W_0^{m,p}(\Omega)$ with a constant $C(m,p,d)$ independent of Ω.*

Remark A.41. Imbedding theorems. Imbedding theorems for Sobolev spaces are used frequently in the analysis of partial differential equations. The imbedding theorems state that all functions belonging to a certain space do belong also to another space and that the norm of the functions in the larger space can be estimated by the norm in the smaller space. Let V be a Banach space such that an imbedding $W^{m,p}(\Omega) \to V$ holds. Then, there is a constant C depending on Ω such that

$$\|v\|_V \le C \|v\|_{W^{m,p}(\Omega)}$$

for all functions $v \in W^{m,p}(\Omega)$. The validity of imbeddings depends on the dimension d of the domain Ω. The larger the dimension, the less imbeddings are valid, compare Example A.44. □

Theorem A.42. The Sobolev imbedding theorem, *(Adams, 1975, Theorem 5.4, Remark 5.5. (6), Theorem 6.2). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a locally Lipschitz boundary. Let j and m be non-negative integers and let p satisfy $1 \le p < \infty$.*

i) Let $mp < d$, then the imbedding

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega), \quad 1 \le q \le \frac{dp}{d-mp} \tag{A.14}$$

holds. In particular, it is

$$W^{m,p}(\Omega) \to L^q(\Omega), \quad 1 \le q \le \frac{dp}{d-mp}. \tag{A.15}$$

ii) Suppose $mp = d$. Then the imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega), \quad 1 \le q < \infty \tag{A.16}$$

is valid. If in addition $p = 1$, then this imbedding holds also for $q = \infty$

$$W^{d,1}(\Omega) \to L^{\infty}(\Omega). \tag{A.17}$$

and even

$$W^{d,1}(\Omega) \to C_B(\Omega),$$

see (A.6) *for the definition of latter space.*

iii) Suppose that $mp > d$, then the imbedding

$$W^{m,p}(\Omega) \to C_B(\Omega) \tag{A.18}$$

holds.

iv) Suppose $mp > d > (m-1)p$, then

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}\left(\overline{\Omega}\right) \quad \textit{for} \quad 0 < \lambda \le m - \frac{d}{p}. \tag{A.19}$$

v) Suppose $d = (m-1)p$, then

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}\left(\overline{\Omega}\right) \quad \textit{for} \quad 0 < \lambda < 1.$$

This imbedding holds for $\lambda = 1$ if $p = 1$ and $d = m - 1$.

vi) All imbeddings are true for arbitrary domains provided the W spaces undergoing the imbedding are replaced with the corresponding W_0 spaces.

vii) Rellich–Kondrachov theorem: The imbeddings (A.14) – (A.16) *are compact with the conditions on Ω stated at the beginning of the theorem, i.e., the imbedding operator is compact, see Definition A.63.*

Remark A.43. Spaces of continuous functions in $\overline{\Omega}$. Since the compact imbedding

$$C^{j,\lambda}\left(\overline{\Omega}\right) \to C^{j}\left(\overline{\Omega}\right) \quad j \ge 0, 0 < \lambda \le 1,$$

holds for bounded domains, (Adams, 1975, Theorem 1.31), one can derive from Theorem A.42, cases iv) and v), also imbeddings for $C^j\left(\overline{\Omega}\right)$: if $mp > d \ge (m-1)p$, $p \in [1, \infty)$, then

$$W^{j+m,p}(\Omega) \to C^{j}\left(\overline{\Omega}\right). \tag{A.20}$$

□

Example A.44. Important Sobolev imbeddings. Let $d = 2$. Then, it follows from (A.16) that

$$H^1(\Omega) = W^{1,2}(\Omega) \to L^q(\Omega), \quad q \in [1, \infty). \tag{A.21}$$

For $d = 3$, one gets with (A.15) that

$$H^1(\Omega) = W^{1,2}(\Omega) \to L^q(\Omega), \quad q \in [1,6]. \tag{A.22}$$

□

Remark A.45. Spaces of functions defined in space-time domains. Let X be any normed space introduced above that is equipped with the norm $\|\cdot\|_X$ and let (t_0, t_1) be a time interval. Then, the following function space on the space-time domain can be defined

$$L^p(t_0,t_1;X) = \left\{ f(t,\boldsymbol{x}) \ : \ \int_{t_0}^{t_1} \|f\|_X^p(\tau)\ d\tau < \infty \right\}, \quad p \in [1,\infty).$$

The norm of $L^p(t_0,t_1;X)$ is

$$\|f\|_{L^p(t_0,t_1;X)} = \left(\int_{t_0}^{t_1} \|f\|_X^p(\tau)\ d\tau \right)^{1/p}, \quad p \in [1,\infty).$$

The modifications for $p = \infty$ are the same as for the Lebesgue spaces. □

A.3 Some Definitions, Statements, and Theorems

Remark A.46. Convolution. The convolution of two scalar functions f and g is defined by

$$(f * g)(y) = \int_{\mathbb{R}} f(y-x)\, g(x)\ dx = \int_{\mathbb{R}} f(x)\, g(y-x)\ dx = (g * f)(y),$$

provided that the integrals exist for almost all $y \in \mathbb{R}$. □

Remark A.47. Fourier transform: definition and some properties. The Fourier transform of a scalar function f is defined by

$$\mathcal{F}(f)(y) = \int_{\mathbb{R}} f(x)\, e^{-ixy}\ dx \tag{A.23}$$

and the inverse Fourier transform of $F(y)$ by

$$\mathcal{F}^{-1}(F)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} F(y)\, e^{ixy}\ dy. \tag{A.24}$$

It holds

$$\mathcal{F}(f * g) = \mathcal{F}(f)\, \mathcal{F}(g), \quad \mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g). \tag{A.25}$$

If f is differentiable and $\lim_{|x|\to\infty} f(x) = 0$, integration by parts yields

$$y\mathcal{F}(f)(y) = -i\mathcal{F}(f')(y).$$

This formula implies the relations

$$\|\boldsymbol{y}\|_2^2 \mathcal{F}(\boldsymbol{f}) = -\mathcal{F}(\Delta \boldsymbol{f}), \tag{A.26}$$

$$\frac{1}{\|\boldsymbol{y}\|_2^2}\mathcal{F}(\boldsymbol{f}) = -\mathcal{F}\left(\Delta^{-1}(\boldsymbol{f})\right), \tag{A.27}$$

$$\frac{1}{1+c\|\boldsymbol{y}\|_2^2}\mathcal{F}(\boldsymbol{f}) = \mathcal{F}\left((I-c\Delta)^{-1}(\boldsymbol{f})\right). \tag{A.28}$$

The $L^r(\Omega)$ norm of $f * g$, $1 \le r \le \infty$, can be estimated by Young's inequality for convolutions (sometimes also called Hölder's inequality for convolutions), e.g., see (Hörmander, 1990, Section IV.4.5). Let $1 \le p, q \le \infty$, $p^{-1} + q^{-1} \ge 1$, and $r^{-1} = p^{-1} + q^{-1} - 1$. For $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, it is $f * g \in L^r(\mathbb{R}^d)$ and Young's inequality for convolution

$$\|f * g\|_{L^r(\Omega)} \le \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \tag{A.29}$$

holds. If $p^{-1} + q^{-1} - 1 = 0$, then $f * g \in L^r(\mathbb{R}^d)$ is continuous and bounded. □

Definition A.48. Absolutely continuous function. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called absolutely continuous if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint subintervals (x_k, y_k) of I satisfies $\sum_k (y_k - x_k) < \delta$, then $\sum_k |f(y_k) - f(x_k)| < \varepsilon$. □

Remark A.49. Absolutely continuous functions. Absolute continuity of a function is a stronger condition than continuity and even uniform continuity. On a compact interval $I = [a, b]$, absolute continuity of a function f is equivalent to the property that this function has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and it holds

$$f(t) = f(a) + \int_a^t f'(\tau)\, d\tau \quad \forall\, t \in [a, b]$$

(fundamental theorem of calculus). □

Theorem A.50. Local existence and uniqueness theorem of Carathéodory, *(Carathéodory, 1918, Kap. 11), (Kamke, 1944, p. 34), (Filippov, 1988, Section 1.1). Let $f_m(t, y_1, \ldots, y_n)$, $m = 1, \ldots, n$, be defined in*

$$\Omega_T = (t_0, t_0 + T) \times \Omega_y$$

with $\Omega_y = \{\boldsymbol{y} = (y_1, \ldots, y_n)^T : \|\boldsymbol{y} - \boldsymbol{y}_0\|_2 \le b\}$ for some $b > 0$, let the functions $f_1, \ldots, f_n$ for each fixed system $y_1, \ldots, y_n$ be measurable with respect to t, let for each fixed t the functions $f_1, \ldots, f_n$ be continuous with respect to $y_1, \ldots, y_n$, and let

$$|f_m(t, y_1, \ldots, y_n)| \leq F(t), \quad m = 1, \ldots, n,$$

where $F(t)$ is a Lebesgue integrable function in $(t_0, t_0 + T)$. Then there exists a system of absolutely continuous functions $y_1(t), \ldots, y_n(t)$ that satisfies for all t in some interval $[t_0, t_0 + a]$, $0 < a \leq T$,

$$y_m(t) = y_{0m} + \int_{t_0}^{t} f_m(s, y_1(s), \ldots, y_n(s))\ ds, \quad m = 1, \ldots, n. \tag{A.30}$$

At each point where the term in the integral is continuous, the functions satisfy the ordinary differential equation

$$\frac{d}{dt} y_m(t) = f_m(t, y_1, \ldots, y_n), \quad m = 1, \ldots, n. \tag{A.31}$$

If in addition for any two points $(t, \overline{y}_1, \ldots, \overline{y}_n), (t, \hat{y}_1, \ldots, \hat{y}_n) \in \Omega_T$ the Lipschitz condition

$$|f_m(t, \overline{y}_1, \ldots, \overline{y}_n) - f_m(t, \hat{y}_1(t), \ldots, \hat{y}_n(t))| \leq G(t) \sum_{l=1}^{n} |\overline{y}_l - \hat{y}_l(t)|,$$

$m = 1, \ldots, n$, with a Lebesgue integrable function $G(t)$ is satisfied, then there exists exactly one solution of (A.30) *in $[t_0, t_0 + a]$.*

Remark A.51. On Carathéodory's theorem. The theorem of Carathéodory is an extension of the theorem of Peano to ordinary differential equations of type (A.31) with discontinuous right-hand side. □

Remark A.52. On Gronwall's lemma. Gronwall's lemma is an important tool for the analysis and finite element analysis of time-dependent problems. Two versions of this lemma in the continuous setting will be given below, see Emmrich (1999) for complete proofs and a discussion of the differences of these versions. □

Lemma A.53. Gronwall's lemma in integral form. *Let $T \in \mathbb{R}^+ \cup \infty$, $f, g, \in L^\infty(0, T)$, and $\lambda \in L^1(0, T)$, $\lambda(t) \geq 0$ for almost all $t \in [0, T]$. Then*

$$f(t) \leq g(t) + \int_0^t \lambda(s) f(s)\ ds \quad \text{a.e. in } [0, T] \tag{A.32}$$

implies for almost all $t \in [0, T]$ that

$$f(t) \leq g(t) + \int_0^t \exp\left(\int_s^t \lambda(\tau)\ d\tau\right) \lambda(s) g(s)\ ds. \tag{A.33}$$

If $g \in W^{1,1}(0, T)$, it follows that

$$f(t) \leq \exp\left(\int_0^t \lambda(\tau)\ d\tau\right) \left(g(0) + \int_0^t \exp\left(-\int_0^s \lambda(\tau)\ d\tau\right)\right) g'(s)\ ds.$$

Moreover, if $g(t)$ is a monotonically increasing continuous function, it holds

$$f\left(t\right) \leq \exp\left(\int_0^t \lambda\left(\tau\right)\ d\tau\right) g\left(t\right). \tag{A.34}$$

Proof. For illustration, the derivation of (A.33) and (A.34) will be presented.
(A.33). Let

$$\tilde{f}(t) = \exp\left(-\int_0^t \lambda\left(\tau\right) d\tau\right) \int_0^t \lambda(s) f(s)\ ds,$$

then one obtains for almost all $t \in [0, T]$ with the product rule, the Leibniz integral rule, (A.32), and $\lambda(t) \geq 0$

$$\begin{aligned}
\tilde{f}'(t) &= \exp\left(-\int_0^t \lambda\left(\tau\right) d\tau\right)\left(-\lambda(t)\int_0^t \lambda(s) f(s)\ ds + \lambda(t) f(t)\right) \\
&\leq \exp\left(-\int_0^t \lambda\left(\tau\right) d\tau\right)\left(\lambda(t)\left(g(t) - f(t)\right) + \lambda(t) f(t)\right) \\
&= \exp\left(-\int_0^t \lambda\left(\tau\right) d\tau\right)\lambda(t) g(t).
\end{aligned}$$

Integration yields, using $\tilde{f}(0) = 0$,

$$\tilde{f}(t) \leq \int_0^t \exp\left(-\int_0^s \lambda\left(\tau\right) d\tau\right)\lambda(s) g(s)\ ds.$$

With (A.32), one obtains

$$\begin{aligned}
\exp\left(-\int_0^t \lambda\left(\tau\right) d\tau\right)\left(f(t) - g(t)\right) &\leq \exp\left(-\int_0^t \lambda\left(\tau\right) d\tau\right)\int_0^t \lambda\left(s\right) f\left(s\right)\ ds \\
&= \tilde{f}(t) \leq \int_0^t \exp\left(-\int_0^s \lambda\left(\tau\right) d\tau\right)\lambda(s) g(s)\ ds.
\end{aligned}$$

Multiplying this inequality with $\exp\left(\int_0^t \lambda\left(\tau\right) d\tau\right)$ and bringing $g(t)$ to the right-hand side gives (A.33).

(A.34). If $g(t)$ is a monotonically increasing continuous function, one gets from (A.33), using that $g(t)$ takes its largest value at the final time and that $\lambda(t) \geq 0$, and applying the fundamental theorem of calculus

$$\begin{aligned}
f(t) &\leq g(t)\left(1 + \int_0^t \exp\left(\int_s^t \lambda\left(\tau\right)\ d\tau\right)\lambda\left(s\right)\ ds\right) \\
&= g(t)\left(1 + \exp\left(\int_0^t \lambda\left(\tau\right)\ d\tau\right)\int_0^t \exp\left(-\int_0^s \lambda\left(\tau\right)\ d\tau\right)\lambda\left(s\right)\ ds\right) \\
&= g(t)\left(1 + \exp\left(\int_0^t \lambda\left(\tau\right)\ d\tau\right)\int_0^t \frac{d}{ds}\left(-\exp\left(-\int_0^s \lambda\left(\tau\right)\ d\tau\right)\right)\ ds\right) \\
&= g(t)\left(1 + \exp\left(\int_0^t \lambda\left(\tau\right)\ d\tau\right)\left(-\exp\left(-\int_0^t \lambda\left(\tau\right)\ d\tau\right) + 1\right)\right) \\
&= \exp\left(\int_0^t \lambda\left(\tau\right)\ d\tau\right) g(t).
\end{aligned}$$

■

Lemma A.54. Gronwall's lemma in differential form. *Let* $T \in \mathbb{R}^+ \cup \infty$, $f \in W^{1,1}(0,T)$, *and* $g, \lambda \in L^1(0,T)$. *Then*

$$f'(t) \le g(t) + \lambda(t) f(t) \quad \text{a.e. in } [0,T] \tag{A.35}$$

implies for almost all $t \in [0,T]$

$$f(t) \le \exp\left(\int_0^t \lambda(\tau)\, d\tau\right) f(0) + \int_0^t \exp\left(\int_s^t \lambda(\tau)\, d\tau\right) g(s)\, ds. \tag{A.36}$$

Proof. Defining

$$\tilde{f}(t) = \exp\left(-\int_0^t \lambda(\tau)\, d\tau\right) f(t) = \exp(-\Lambda(t)) f(t), \tag{A.37}$$

applying the chain rule, the Leibniz integral rule, and (A.35) gives

$$\begin{aligned}\tilde{f}'(t) &= -\Lambda'(t) \exp(-\Lambda(t)) f(t) + \exp(-\Lambda(t)) f'(t) = \exp(-\Lambda(t)) \big(f'(t) - \lambda(t) f(t)\big) \\ &\le \exp(-\Lambda(t)) g(t).\end{aligned}$$

Integration in $(0,t)$ and using (A.37) yields

$$\tilde{f}(t) - \tilde{f}(0) = \exp\left(-\int_0^t \lambda(\tau)\, d\tau\right) f(t) - f(0) \le \int_0^t \exp(-\Lambda(s)) g(s)\, ds.$$

Multiplication with $\exp\left(\int_0^t \lambda(\tau)\, d\tau\right)$ gives (A.36). ■

Lemma A.55. Variation of Gronwall's lemma in differential form. *Let* $T \in \mathbb{R}^+ \cup \infty$, $f \in W^{1,1}(0,T)$, $h, g, \lambda \in L^1(0,T)$, *and* $h(t), \lambda(t) \ge 0$ *a.e. in* $(0,T)$. *Then,*

$$f'(t) + h(t) \le g(t) + \lambda(t) f(t) \quad \text{a.e. in } [0,T] \tag{A.38}$$

implies for almost all $t \in [0,T]$

$$\begin{aligned} &f(t) + \int_0^t h(s)\, ds \\ &\le \exp\left(\int_0^t \lambda(\tau)\, d\tau\right) f(0) + \int_0^t \exp\left(\int_s^t \lambda(\tau)\, d\tau\right) g(s)\, ds. \end{aligned} \tag{A.39}$$

Moreover, if $g(t) \ge 0$ *a.e. in* $(0,T)$, *it holds*

$$f(t) + \int_0^t h(s)\, ds \le \exp\left(\int_0^t \lambda(\tau)\, d\tau\right) \left(f(0) + \int_0^t g(s)\, ds\right). \tag{A.40}$$

Proof. From (A.38), it follows a.e. in $[0,T]$ that

$$f'(s) - \lambda(s) f(s) + h(s) \le g(s).$$

The positivity of the exponential implies

$$\exp\left(-\int_0^s \lambda(\tau)\ d\tau\right)\left(f'(s) - \lambda(s) f(s) + h(s)\right) \le \exp\left(-\int_0^s \lambda(\tau)\ d\tau\right) g(s).$$

Integration on $(0,t) \subset [0,T]$ gives

$$\exp\left(-\int_0^t \lambda(\tau)\ d\tau\right) f(t) - f(0) + \int_0^t \exp\left(-\int_0^s \lambda(\tau)\ d\tau\right) h(s)\ ds$$
$$\le \int_0^t \exp\left(-\int_0^s \lambda(\tau)\ d\tau\right) g(s)\ ds. \tag{A.41}$$

Using the monotonicity of the exponential yields

$$\exp\left(-\int_0^t \lambda(\tau)\ d\tau\right)\int_0^t h(s)\ ds \le \int_0^t \exp\left(-\int_0^s \lambda(\tau)\ d\tau\right) h(s)\ ds.$$

Applying this inequality to bound the left-hand side of (A.41) from below and multiplication of the resulting inequality with $\exp\left(\int_0^t \lambda(\tau)\ d\tau\right)$ proves (A.39).

If g is non-negative, one obtains

$$\int_0^t \exp\left(\int_s^t \lambda(\tau)\ d\tau\right) g(s)\ ds \le \exp\left(\int_0^t \lambda(\tau)\ d\tau\right)\int_0^t g(s)\ ds,$$

from which (A.40) follows. ■

Lemma A.56. Discrete Gronwall's lemma, *(Heywood & Rannacher, 1990, Lemma 5.1). Let $k, B, a_n, b_n, c_n, \alpha_n$ be non-negative numbers for integers $n \ge 1$ and let the inequality*

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \le B + k\sum_{n=1}^{N+1} c_n + k\sum_{n=1}^{N+1} \alpha_n a_n \quad \textit{for } N \ge 0 \tag{A.42}$$

hold. If $k\alpha_n < 1$ for all $n = 1, \ldots, N+1$, then

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \le \exp\left(k\sum_{n=1}^{N+1}\frac{\alpha_n}{1-k\alpha_n}\right)\left(B + k\sum_{n=1}^{N+1} c_n\right) \quad \textit{for } N \ge 0. \tag{A.43}$$

If the inequality

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \le B + k\sum_{n=1}^{N+1} c_n + k\sum_{n=1}^{N} \alpha_n a_n \quad \textit{for } N \ge 0 \tag{A.44}$$

is given, then it holds

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \le \exp\left(k\sum_{n=1}^{N}\alpha_n\right)\left(B + k\sum_{n=1}^{N+1} c_n\right) \quad \textit{for } N \ge 0. \tag{A.45}$$

Definition A.57. Weak convergence and weak* convergence, (Yosida, 1995, p. 120, p. 125). A sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be weakly convergent if a finite limit $\lim_{n\to\infty} f(x_n)$ exists for each $f \in X'$, where X' is the (strong) dual space of X. If

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \forall\, f \in X',$$

then $\{x_n\}_{n=1}^{\infty}$ is called to be weakly convergent to x, in notation $x_n \rightharpoonup x$.

A sequence $\{f_n\}_{n=1}^{\infty}$ in the (strong) dual space X' of a linear space X is said to be weakly* convergent if a finite limit $\lim_{n\to\infty} f_n(x)$ exists for every $x \in X$. The sequence is said to converge weakly* to an element $f \in X'$ if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall\, x \in X,$$

in notation $f_n \overset{*}{\rightharpoonup} f$. □

Remark A.58. On the weak and weak convergence.*

- If the limit x or f exist, then the limit is unique, (Yosida, 1995, p. 120).
- If X is a reflexive Banach space and if $\{x_n\}_{n=1}^{\infty} \subset X$ is bounded, then there exists a subsequence $\{x_{n_l}\}_{l=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ and an element $x \in X$ such that $x_{n_l} \rightharpoonup x$, see (Evans, 2010, p. 723).
- If $\{f_n\}_{n=1}^{\infty} \subset X'$ is bounded in the dual X' of X and X is a separable Banach space, then there exists a weakly* convergent subsequence.

□

Definition A.59. Linear operator, range, kernel. Let X and Y be real Banach spaces. A mapping $A \,:\, X \to Y$ is a linear operator if

$$A\left(\alpha x_1 + \beta x_2\right) = \alpha A x_1 + \beta A x_2 \quad \forall\, x_1, x_2 \in X, \alpha, \beta \in \mathbb{R}.$$

The range or image of A is given by

$$\text{range}(A) = \{y \in Y \;:\; y = Ax \text{ for some } x \in X\}.$$

The kernel or the null space of A is defined by

$$\ker(A) = \{x \in X \;:\; Ax = 0\}.$$

□

Definition A.60. Bounded operator, continuous operator. An operator $A \,:\, X \to Y$, X, Y Banach spaces, is bounded if

$$\|A\| = \sup_{x\in X} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{x\in X, \|x\|_X \le 1} \|Ax\|_Y = \sup_{x\in X, \|x\|_X = 1} \|Ax\|_Y < \infty. \tag{A.46}$$

The operator A is continuous in $x_0 \in X$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$ with $\|x - x_0\|_X < \delta$, it follows that $\|Ax - Ax_0\|_Y < \varepsilon$. The operator A is called a continuous operator if A is continuous for all $x \in X$. □

Remark A.61. Equivalent definition of a continuous operator. The operator $A : X \to Y$, X, Y Banach spaces, is continuous in $x_0 \in X$ if and only if for all sequences $\{x_n\}_{n=1}^{\infty}$, $x_n \in X$, with $x_n \to x_0$ it holds that $Ax_n \to Ax_0$ in Y. □

Lemma A.62. Properties of bounded linear operators, *(Kolmogorov & Fomīn, 1975, §4.5.2, §4.5.3), (Yosida, 1995, p.43). Let X, Y be Banach spaces.*
i) A bounded linear operator $A : X \to Y$ is continuous.
ii) A continuous linear operator $A : X \to Y$ is bounded.
iii) The set

$$\mathcal{L}(X, Y) = \{A : A \textit{ is a bounded linear operator from } X \textit{ to } Y\}$$

is a Banach space endowed with the norm (A.46).

Definition A.63. Compact operator. An operator $A : X \to Y$ is compact, if $A(x)$ is precompact in Y for every bounded set $\tilde{X} \subset X$. □

Theorem A.64. Rank-nullity theorem. *Let $A : V \to W$ be a linear map between two finite-dimensional linear spaces V and W, then it holds*

$$\dim V = \dim(\ker(A)) + \dim(\mathrm{range}(A)).$$

Definition A.65. Linear functional. A (real) linear functional f defined on a Banach space X is a linear operator with range$(f) \subset \mathbb{R}$. □

Definition A.66. Bounded bilinear form, coercive bilinear form, V-elliptic bilinear form. Let $b(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bilinear form on the Banach space V. Then, it is bounded if

$$|b(u,v)| \le M \|u\|_V \|v\|_V \quad \forall\, u, v \in V, M > 0, \tag{A.47}$$

where the constant M is independent of u and v. The bilinear form is coercive or V-elliptic if

$$b(u,u) \ge m \|u\|_V^2 \quad \forall\, u \in V, m > 0, \tag{A.48}$$

where the constant m is independent of u. □

Remark A.67. Application to an inner product. Let V be a Hilbert space. Then the inner product $a(\cdot,\cdot)$ is a bounded and coercive bilinear form, since by the Cauchy–Schwarz inequality

$$|a(u,v)| \le \|u\|_V \|v\|_V \quad \forall\, u, v \in V,$$

and obviously $a(u,u) = \|u\|_V^2$. Hence, the constants can be chosen to be $M = 1$ and $m = 1$. □

Theorem A.68. Banach's fixed point theorem, *(Gilbarg & Trudinger, 1983, Theorem 5.1). A contraction mapping T in a Banach space B has a unique fixed point, that is there exists a unique solution $x \in B$ of the equation $Tx = x$.*

The statement holds true if B is replaced by any closed subset, see (Gilbarg & Trudinger, 1983, p. 74).

Theorem A.69. Brouwer's fixed point theorem, *(Gilbarg & Trudinger, 1983, Theorem 11.1). Let S be a compact convex set in a Banach space B and let T be a continuous mapping of S into itself. Then T has a fixed point, that is, $Tx = x$ for some $x \in S$.*

Theorem A.70. Theorem of Banach on the inverse operator, *(Kolmogorov & Fomīn, 1975, p. 225). Let $A : X \to Y$ be a bounded linear operator that defines a one-to-one mapping between the Banach spaces X and Y. Then, the inverse operator A^{-1} is bounded.*

Theorem A.71. Closed Range Theorem of Banach, *(Yosida, 1995, p. 205). Let X, Y be Banach spaces, let $A : X \to Y$ be a bounded linear operator, and let $A' : Y' \to X'$ be its dual. Then, the following statements are equivalent:*

i) $\mathrm{range}(A)$ *is closed in* Y,
ii) $\mathrm{range}(A) = \{y \in Y : \langle y', y\rangle_{Y',Y} = 0 \text{ for all } y' \in \ker(A')\}$,
iii) $\mathrm{range}(A')$ *is closed in* X',
iv) $\mathrm{range}(A') = \{x' \in X' : \langle x', x\rangle_{X',X} = 0 \text{ for all } x \in \ker(A)\}$.

Theorem A.72. Hahn–Banach Theorem, *(Yosida, 1995, Section IV.1), (Triebel, 1972, p. 67). Let X be a Banach space, let Y be a subspace of X, and let f be a bounded linear functional defined on Y. Then, there exists an extension g of f to X, where g is a linear functional with the same norm as f.*