Chapter 3
The Stokes Equations

Remark 3.1. Motivation. The Stokes equations model the simplest incompressible flow problems. These problems are steady-state and the convective term can be neglected. Hence, the arising model is linear. Thus, the only difficulty which remains from the problems mentioned in Remark 1.10 is the coupling of velocity and pressure.

The analysis of the Stokes equations and of finite element discretizations of these equations introduces already important techniques which are used also in the analysis of more complicated problems. □

3.1 The Continuous Equations

Remark 3.2. The Stokes equations. Consider a stationary flow, i.e., $\partial_t \boldsymbol{u} = \mathbf{0}$. If the flow is in addition very slow, i.e., the Reynolds number is very small, then the viscous term $Re^{-1}\Delta\boldsymbol{u}$ dominates the convective term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ and the convective term can be neglected. The resulting momentum equation can be scaled by the Reynolds number, defining a new pressure and right-hand side in this way. One obtains the so-called Stokes equations

$$\begin{aligned} -\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \text{ in } \Omega, \end{aligned} \tag{3.1}$$

that have to be equipped with appropriate boundary conditions.

The theory from Sections 2.1 and 2.2 will be applied here to study system (3.1). For simplicity of presentation, the Stokes equations will be considered with homogeneous Dirichlet boundary conditions $\boldsymbol{u} = \mathbf{0}$ on Γ. □

Remark 3.3. The weak form of Stokes equations. The weak form of the Stokes equations equipped with homogeneous Dirichlet boundary conditions is obtained in the usual way by multiplying the equations with test functions, integrating these equations on Ω, and applying integration by parts to transfer derivatives from the solution to the test functions. One obtains the following problem: Given $\boldsymbol{f} \in H^{-1}(\Omega)$, find $(\boldsymbol{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} && \forall\, \boldsymbol{v} \in H_0^1(\Omega), \\ -(\nabla \cdot \boldsymbol{u}, q) &= 0 && \forall\, q \in L_0^2(\Omega). \end{aligned} \tag{3.2}$$

This weak form can be cast into the framework of the abstract linear saddle point problem by setting $V = H_0^1(\Omega)$ and $Q = L_0^2(\Omega)$ in (2.4), equipped with the norms $\|\cdot\|_V = |\cdot|_{H^1(\Omega)}$ and $\|\cdot\|_Q = \|\cdot\|_{L^2(\Omega)}$,

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \quad b(\boldsymbol{v}, q) = -(\nabla \cdot \boldsymbol{v}, q), \quad r = 0.$$

An equivalent formulation of (3.2) is as follows: Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V', V} \quad \forall\, (\boldsymbol{v}, q) \in V \times Q. \tag{3.3}$$

If (3.2) holds, one gets (3.3) by subtracting the second equation from the first equation in (3.2). In turn, if (3.3) is valid, then the individual equations of (3.2) are obtained by considering in (3.3) the sets $\{(\boldsymbol{v}, 0)\}$ and $\{(\boldsymbol{0}, q)\}$ as test functions.

Let $V_0 = V_{\mathrm{div}}$, the space of weakly divergence-free functions defined in (2.23). Then, the associated problem to (3.2), which corresponds to (2.12), is: Find $\boldsymbol{u} \in V_{\mathrm{div}}$ such that

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V', V} \quad \forall\, \boldsymbol{v} \in V_{\mathrm{div}}. \tag{3.4}$$

□

Lemma 3.4. The norm of the bilinear form $a(\cdot,\cdot)$. *For the bilinear form $a(\cdot,\cdot)$ associated with the Stokes problem it holds $\|a\| = 1$.*

Proof. One gets with the Cauchy–Schwarz inequality (A.10)

$$\begin{aligned} \|a\| &= \sup_{\boldsymbol{v}, \boldsymbol{w} \in V \setminus \{\boldsymbol{0}\}} \frac{(\nabla \boldsymbol{v}, \nabla \boldsymbol{w})}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}} \\ &\le \sup_{\boldsymbol{v}, \boldsymbol{w} \in V \setminus \{\boldsymbol{0}\}} \frac{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}} = 1. \end{aligned}$$

Choosing $\boldsymbol{v} = \boldsymbol{w}$ shows that the supremum is not smaller than 1. ■

Theorem 3.5. Existence and uniqueness of a solution of the Stokes equations. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \in \{2, 3\}$, with a Lipschitz continuous boundary Γ and let $\boldsymbol{f} \in H^{-1}(\Omega)$. Then, there exists a unique pair $(\boldsymbol{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ that solves (3.2).*

Proof. The bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition, see Theorem 2.37. In addition, the bilinear form $a(\cdot,\cdot)$ is V_{div}-elliptic since

$$a(\boldsymbol{v}, \boldsymbol{v}) = |\boldsymbol{v}|_{H^1(\Omega)}^2 = \|\boldsymbol{v}\|_V^2 \quad \forall\, \boldsymbol{v} \in V \supset V_{\mathrm{div}}. \tag{3.5}$$

The statement of the theorem follows now from Lemma 2.14. ■

Lemma 3.6. Stability of the solution. *Let the conditions of Theorem 3.5 be given. Then, the solution* $(\boldsymbol{u}, p)$ *of the Stokes problem* (3.2) *depends continuously on the right-hand side*

$$\|\boldsymbol{u}\|_V = \|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \leq \|\boldsymbol{f}\|_{H^{-1}(\Omega)}\,, \tag{3.6}$$

$$\|p\|_Q = \|p\|_{L^2(\Omega)} \leq \frac{2}{\beta_{\text{is}}} \|\boldsymbol{f}\|_{H^{-1}(\Omega)}\,. \tag{3.7}$$

If $\boldsymbol{f} \in L^2(\Omega)$*, then it holds*

$$\|\boldsymbol{u}\|_V \leq C \|P_{\text{helm}} \boldsymbol{f}\|_{L^2(\Omega)}\,, \tag{3.8}$$

where the constant comes from the Poincaré inequality (A.12).

Proof. Inserting $\boldsymbol{u}$ as test function in (3.4) gives

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{u}) = \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{V',V}.$$

Applying the dual estimate yields

$$\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} \leq \|\boldsymbol{f}\|_{H^{-1}(\Omega)} \|\nabla \boldsymbol{u}\|_{L^2(\Omega)}\,.$$

In the case $\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} = 0$, the estimate for the velocity is trivially true. Otherwise, division with $\|\nabla \boldsymbol{u}\|_{L^2(\Omega)}$ leads to (3.6).

For the estimate of the pressure, the inf-sup condition (2.30), the equation (3.2), the estimate of the dual pairing, and the Cauchy–Schwarz inequality (A.10) are utilized

$$\begin{aligned}
\|p\|_{L^2(\Omega)} &\leq \frac{1}{\beta_{\text{is}}} \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{-(\nabla \cdot \boldsymbol{v}, p)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} \\
&= \frac{1}{\beta_{\text{is}}} \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V',V} - (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} \\
&\leq \frac{1}{\beta_{\text{is}}} \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} \\
&= \frac{1}{\beta_{\text{is}}} \left(\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \right).
\end{aligned}$$

Inserting now the stability estimate (3.6) for the velocity gives the estimate (3.7) for the pressure.

Let $\boldsymbol{f} \in L^2(\Omega)$, then the application of Helmholtz decomposition (2.74)

$$\boldsymbol{f} = P_{\text{helm}} \boldsymbol{f} + \nabla r,$$

using the Helmholtz projection from Definition 2.81, integration by parts, $\boldsymbol{u} \in V_{\text{div}}$, the Cauchy–Schwarz inequality, and Poincaré inequality yields

$$\begin{aligned}
\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} &= (P_{\text{helm}} \boldsymbol{f}, \boldsymbol{u}) + (\nabla r, \boldsymbol{u}) = (P_{\text{helm}} \boldsymbol{f}, \boldsymbol{u}) \\
&\leq \|P_{\text{helm}} \boldsymbol{f}\|_{L^2(\Omega)} \|\boldsymbol{u}\|_{L^2(\Omega)} \leq C \|P_{\text{helm}} \boldsymbol{f}\|_{L^2(\Omega)} \|\nabla \boldsymbol{u}\|_{L^2(\Omega)}\,.
\end{aligned}$$

Division by $\|\nabla \boldsymbol{u}\|_{L^2(\Omega)}$ proves the last statement (3.8) of the lemma. ■

Remark 3.7. On the implication of the inf-sup condition. It can be shown with a straightforward calculation that the inf-sup condition guarantees the uniqueness of the pressure.

Since the weak form of the homogeneous Stokes equations

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) &= 0 \quad \forall\, \boldsymbol{v} \in V, \\ b(\boldsymbol{u},q) &= 0 \quad \forall\, q \in Q, \end{aligned} \tag{3.9}$$

is a linear system, one has to show for uniqueness of a solution that $\boldsymbol{f} = \boldsymbol{0}$ implies $\boldsymbol{u} = \boldsymbol{0}$ and $p = 0$. Assume there is a solution $(\boldsymbol{u}, p) \in V \times Q$ of (3.9). Taking $(\boldsymbol{u}, p)$ as test functions gives

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{u}) + b(\boldsymbol{u},p) &= 0, \\ b(\boldsymbol{u},p) &= 0. \end{aligned}$$

Including the second equation in the first one leads to $a(\boldsymbol{u},\boldsymbol{u}) = 0$. The ellipticity of $a(\cdot,\cdot)$ in V, see (3.5), implies $\boldsymbol{u} = \boldsymbol{0}$. Up to this point, the inf-sup condition was not needed.

Considering now the first equation of (3.9) with $\boldsymbol{u} = \boldsymbol{0}$ gives

$$b(\boldsymbol{v},p) = 0 \quad \forall\, \boldsymbol{v} \in V.$$

It follows that

$$\sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{b(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_V} = 0.$$

The inf-sup condition written in the form (2.29) implies now immediately $p = 0$ in $L_0^2(\Omega)$ since $\beta_{\mathrm{is}} > 0$. □

Remark 3.8. On the balance of different parts of the source term in the momentum equation. Let $\boldsymbol{f} \in L^2(\Omega)$. According to Theorem 2.80, the right-hand side admits a Helmholtz decomposition of the form $\boldsymbol{f} = P_{\mathrm{helm}}\boldsymbol{f} + \nabla r$, with $P_{\mathrm{helm}}\boldsymbol{f} \in H_{\mathrm{div}}(\Omega)$ and $\nabla r \in (H_{\mathrm{div}}(\Omega))^{\perp}$. Inserting this decomposition in the momentum balance of the Stokes equations (3.1) yields

$$-\,\Delta\boldsymbol{u} + \nabla p = P_{\mathrm{helm}}\boldsymbol{f} + \nabla r. \tag{3.10}$$

Assume that $\Delta\boldsymbol{u}, \nabla p \in L^2(\Omega)$, then it is even $\Delta\boldsymbol{u} \in H_{\mathrm{div}}(\Omega)$. This property can be checked by a direct calculation, utilizing that weak derivatives can be always interchanged, see (Evans, 2010, Section 5.2.3, Theorem 1), and using $\nabla\cdot\boldsymbol{u} = 0$

$$\begin{aligned} \nabla\cdot\Delta\boldsymbol{u} &= \partial_x\left(\partial_{xx}u_1 + \partial_{yy}u_1\right) + \partial_y\left(\partial_{xx}u_2 + \partial_{yy}u_2\right) \\ &= \partial_{xx}\left(\partial_x u_1 + \partial_y u_2\right) + \partial_{yy}\left(\partial_x u_1 + \partial_y u_2\right) = 0, \end{aligned}$$

and analogously in three dimensions.

Consider now not only the momentum balance but the complete boundary value problem. Inserting the decomposition of $\boldsymbol{f}$ in the weak formulation

(3.2) and assuming that the solution is sufficiently smooth gives, applying integration by parts,

$$-(\Delta \boldsymbol{u}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) = (P_{\text{helm}} \boldsymbol{f}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, r) \quad \forall\, \boldsymbol{v} \in V. \tag{3.11}$$

It can be seen that the irrotational forces ∇r are balanced completely by the pressure. Already the stability estimate (3.8) shows that irrotational forces do not possess an impact on the velocity. Altogether, one finds that if $\boldsymbol{f}$ is changed to $\boldsymbol{f} + \nabla r$, then the pressure solution of the Stokes equations changes to $p+r$. Likewise, divergence-free forces are balanced by the velocity, more precisely by $-\Delta \boldsymbol{u}$. Thus, there is a separate balance of irrotational and divergence-free forces.

Note that for the integration by parts leading to (3.11) the respective boundary integrals have to vanish. This situation is given if the test space is $V = H_0^1(\Omega)$, i.e., if the Stokes equations are equipped with Dirichlet conditions on the whole boundary. There is not such a clear separation of the impact of different kinds of the forces in the case of other boundary conditions, in particular, in the case of boundary conditions which involve also the pressure. □

3.2 Finite Element Error Analysis

Remark 3.9. The finite element formulation. Let V^h be a velocity finite element space and let Q^h be a pressure finite element space. The finite element discretization of the Stokes equations (3.2) reads as follows: Let $\boldsymbol{f}$ be given, find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{aligned} a^h\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) + b^h\left(\boldsymbol{v}^h, p^h\right) &= \sum_{K \in \mathcal{T}^h} \int_K \left(\boldsymbol{f} \boldsymbol{v}^h\right)(\boldsymbol{x})\, d\boldsymbol{x} \;\forall\, \boldsymbol{v}^h \in V^h, \\ b^h\left(\boldsymbol{u}^h, q^h\right) &= 0 \qquad\qquad\qquad\qquad \forall\, q^h \in Q^h, \end{aligned} \tag{3.12}$$

with

$$a^h\left(\boldsymbol{v}^h, \boldsymbol{w}^h\right) = \sum_{K \in \mathcal{T}^h} \left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{w}^h\right)_K, \quad b^h\left(\boldsymbol{v}^h, q^h\right) = -\sum_{K \in \mathcal{T}^h} \left(\nabla \cdot \boldsymbol{v}^h, q^h\right)_K, \tag{3.13}$$

and the right-hand side $\boldsymbol{f}$ is assumed to be sufficiently smooth such that the right-hand side of the momentum equation in (3.12) is well defined. □

Theorem 3.10. Existence and uniqueness of a solution of the finite element Stokes equations. *Let* $(a^h(\cdot,\cdot))^{1/2}$ *define a norm in* V^h *and let the discrete inf-sup condition* (2.32) *hold. Then,* (3.12) *has a unique solution.*

Proof. If $(a^h(\cdot,\cdot))^{1/2}$ is a norm in V^h, then this bilinear form is V^h-elliptic and in particular V^h_{div}-elliptic. Then, the proof is performed analogously to the proof of Theorem 3.5. ■

Remark 3.11. Goal and approach. The goal of the finite element error analysis consists in getting information on the order of convergence of the finite element solution to the solution of the weak problem in norms of interest. To this end, families of triangulations $\{\mathcal{T}^h\}$ with corresponding finite element spaces $\{V^h \times Q^h\}$ are considered. A general approach of obtaining finite element error estimates consists in the following steps, e.g., see the proof of Theorem 3.17:

- derive an equation or an inequality for the considered norm of the error,
- modify the equation or the inequality in such a way that approximation errors to the finite element spaces appear,
- estimate the considered norm of the error by constants times best approximation errors.

In the second step, one has usually to add and subtract terms in a clever way. The best approximation errors are independent of the considered problem. They can be estimated by interpolation errors. Interpolation error estimates are known from the general theory of finite element methods, see Appendix C. □

3.2.1 Conforming Inf-Sup Stable Pairs of Finite Element Spaces

Remark 3.12. Conforming inf-sup stable finite element spaces. This section studies conforming inf-sup stable finite element spaces, i.e., besides the discrete inf-sup condition (2.32), it holds $V^h \subset V$ and $Q^h \subset Q$. The bilinear forms are identical to the forms of the continuous problem, i.e., it is $a^h(\cdot,\cdot) = a(\cdot,\cdot)$ and $b^h(\cdot,\cdot) = b(\cdot,\cdot)$. □

Corollary 3.13. Unique solvability of the finite element problem. *Let V^h and Q^h be conforming finite element spaces that satisfy the discrete inf-sup condition* (2.32). *Then, the finite element problem* (3.12) *has a unique solution.*

Proof. Based on Theorem 3.10, it is sufficient to show that $(a(\cdot,\cdot))^{1/2}$ defines a norm in V^h. This property is given, because $(a(\cdot,\cdot))^{1/2}$ defines a norm in V, see (3.5), it defines also a norm in every subspace $V^h \subset V$. ■

Lemma 3.14. Stability of the finite element solution. *Let $V^h \times Q^h$ be a pair of inf-sup stable finite element spaces. Then, the solution of* (3.12) *fulfills*

$$\left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le \|\boldsymbol{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta^h_{\mathrm{is}}} \|\boldsymbol{f}\|_{H^{-1}(\Omega)}.$$

Proof. The proof follows the lines of the proof of Lemma 3.6. ■

Remark 3.15. Reduction to a problem in the space of discretely divergence-free functions. From the abstract theory of linear saddle point problems, Section 2.1, Remark 2.11, it follows that the finite element approximation of the velocity can be computed by solving the following problem: Find $\boldsymbol{u}^h \in V_{\mathrm{div}}^h$ such that

$$a\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) = \left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle_{V',V} \quad \forall\, \boldsymbol{v}^h \in V_{\mathrm{div}}^h. \tag{3.14}$$

□

3.2.1.1 The case $\boldsymbol{V}_{\mathbf{div}}^{\boldsymbol{h}} \not\subset \boldsymbol{V}_{\mathbf{div}}$

Remark 3.16. The case $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$. This case applies for most pairs of inf-sup stable finite element spaces. □

Theorem 3.17. Finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with polyhedral and Lipschitz continuous boundary and let $(\boldsymbol{u}, p) \in V \times Q$ be the unique solution of the Stokes problem* (3.2)*. Assume that this problem is discretized with inf-sup stable conforming finite element spaces $V^h \times Q^h$ and denote by $\boldsymbol{u}^h \in V_{\mathrm{div}}^h$ the velocity solution. Then, the following error estimate holds*

$$\left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le 2 \inf_{\boldsymbol{v}^h \in V_{\mathrm{div}}^h} \left\|\nabla(\boldsymbol{u} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} + \inf_{q^h \in Q^h} \left\|p - q^h\right\|_{L^2(\Omega)}. \tag{3.15}$$

Proof. The proof starts by formulating the error equation. Since $V_{\mathrm{div}}^h \subset V$, functions from V_{div}^h can be used as test function in the continuous Stokes equations (3.3), which is equivalent to (3.2). Using that the velocity solution of the continuous equation is weakly divergence-free, one obtains by subtracting (3.14) from (3.3)

$$\left(\nabla(\boldsymbol{u} - \boldsymbol{u}^h), \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p\right) = 0 \quad \forall\, \boldsymbol{v}^h \in V_{\mathrm{div}}^h. \tag{3.16}$$

The pressure term appears since in general $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$. Equation (3.16) is the error equation.

Now, the pressure term will be modified such that an approximation term with respect to the pressure is obtained. Observing that $\left(\nabla \cdot \boldsymbol{v}^h, q^h\right) = 0$ for all $q^h \in Q^h$ leads to

$$\left(\nabla(\boldsymbol{u} - \boldsymbol{u}^h), \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p - q^h\right) = 0 \quad \forall\, \boldsymbol{v}^h \in V_{\mathrm{div}}^h,\ \forall\, q^h \in Q^h. \tag{3.17}$$

In the next step, the approximation error for the velocity will be introduced. To this end, the error is decomposed into

$$\boldsymbol{u} - \boldsymbol{u}^h = \left(\boldsymbol{u} - I^h \boldsymbol{u}\right) - \left(\boldsymbol{u}^h - I^h \boldsymbol{u}\right) = \boldsymbol{\eta} - \boldsymbol{\phi}^h.$$

Here, $I^h \boldsymbol{u}$ denotes an arbitrary interpolant of $\boldsymbol{u}$ in V_{div}^h. Hence, $\boldsymbol{\eta}$ is just an approximation error which depends only on the finite element space V_{div}^h but not on the considered equation. The goal is to estimate $\boldsymbol{\phi}^h \in V^h_{\mathrm{div}}$ by approximation errors as well. To this end, this decomposition is inserted in (3.17) and the test function $\boldsymbol{v}^h = \boldsymbol{\phi}^h$ is chosen. It follows that

$$\left\|\nabla\boldsymbol{\phi}^h\right\|^2_{L^2(\Omega)} = \left(\nabla\boldsymbol{\phi}^h, \nabla\boldsymbol{\phi}^h\right) = \left(\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\phi}^h\right) - \left(\nabla\cdot\boldsymbol{\phi}^h, p - q^h\right) \quad \forall\, q^h \in Q^h. \tag{3.18}$$

Now, the terms on the right-hand side are estimated using the Cauchy–Schwarz inequality (A.10)

$$\left|\left(\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\phi}^h\right)\right| \le \|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)} \left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}$$

and, using in addition (2.26),

$$\begin{aligned} \left|-\left(\nabla\cdot\boldsymbol{\phi}^h, p - q^h\right)\right| &\le \left\|p - q^h\right\|_{L^2(\Omega)} \left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\ &\le \left\|p - q^h\right\|_{L^2(\Omega)} \left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}. \end{aligned} \tag{3.19}$$

Inserting these estimates in (3.18) and dividing by $\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \neq 0$ yields

$$\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \le \|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)} + \left\|p - q^h\right\|_{L^2(\Omega)}.$$

In the case that $\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} = 0$, this estimate trivially holds.

With the triangle inequality, it follows that

$$\begin{aligned} \left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} &\le \left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} + \|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)} \\ &\le 2\,\|\nabla\boldsymbol{\eta}\|_{L^2(\Omega)} + \left\|p - q^h\right\|_{L^2(\Omega)} \end{aligned}$$

for all $I^h\boldsymbol{u} \in V^h_{\mathrm{div}}$ and for all $q^h \in Q^h$, from what (3.15) follows. ■

Remark 3.18. On error estimates (3.15). The error in the V norm of the velocity is estimated with (3.15) by best approximation errors for both, the velocity and the pressure. The occurrence of the best approximation error for the velocity is natural, since the velocity finite element space has certainly an impact on the error. Inspecting the proof of Theorem 3.17, one finds that the reason for the appearance of the best approximation error for the pressure is the property $V^h_{\mathrm{div}} \not\subset V_{\mathrm{div}}$. Otherwise, the second term on the left-hand side of (3.3) would vanish and the pressure would have been eliminated from the estimate. The case $V^h_{\mathrm{div}} \subset V_{\mathrm{div}}$ will be studied in more detail in Section 3.2.1.2. □

Corollary 3.19. Finite element error estimate for the $L^2(\Omega)$ norm of the divergence of the velocity. *Let the assumptions of Theorem 3.17 be fulfilled, then*

$$\left\|\nabla\cdot\boldsymbol{u}^h\right\|_{L^2(\Omega)} \le 2 \inf_{\boldsymbol{v}^h \in V^h_{\mathrm{div}}} \left\|\nabla(\boldsymbol{u} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} + \inf_{q^h \in Q^h} \left\|p - q^h\right\|_{L^2(\Omega)}. \tag{3.20}$$

Proof. The estimate follows easily from $\nabla\cdot\boldsymbol{u} = 0$, (2.26), and the estimate (3.15)

$$\left\|\nabla\cdot\boldsymbol{u}^h\right\|_{L^2(\Omega)} = \left\|\nabla\cdot(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le \left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)}.$$

■

Theorem 3.20. Finite element error estimate for the $L^2(\Omega)$ norm of the pressure. *Let the assumption of Theorem 3.17 be satisfied. Then the following error estimate holds*

$$\begin{aligned}\left\|p - p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta_{\text{is}}^h} & \inf_{\boldsymbol{v}^h \in V_{\text{div}}^h} \left\|\nabla(\boldsymbol{u} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} \\ & + \left(1 + \frac{2}{\beta_{\text{is}}^h}\right) \inf_{q^h \in Q^h} \left\|p - q^h\right\|_{L^2(\Omega)} . \end{aligned} \tag{3.21}$$

Proof. Let $q^h \in Q^h$ be arbitrary, then one gets with the triangle inequality

$$\left\|p - p^h\right\|_{L^2(\Omega)} \le \left\|p - q^h\right\|_{L^2(\Omega)} + \left\|p^h - q^h\right\|_{L^2(\Omega)} .$$

Replacing the right-hand side of momentum equation of the finite element Stokes problem (3.12) by the left-hand side of the the momentum equation of the continuous Stokes problem (3.2) for $\boldsymbol{v}^h \in V^h$ gives

$$\begin{aligned} b\left(\boldsymbol{v}^h, p^h - q^h\right) &= -a\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) + \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle_{V',V} - b\left(\boldsymbol{v}^h, q^h\right) \\ &= a\left(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h\right) + b\left(\boldsymbol{v}^h, p - q^h\right) \quad \forall\, \boldsymbol{v}^h \in V^h, \forall\, q^h \in Q^h. \end{aligned}$$

With the discrete inf-sup condition (2.32), the Cauchy–Schwarz inequality (A.10), and (2.26), it follows now that

$$\begin{aligned} &\left\|p^h - q^h\right\|_{L^2(\Omega)} \\ &\le \frac{1}{\beta_{\text{is}}^h} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h, p^h - q^h\right)}{\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)}} \\ &= \frac{1}{\beta_{\text{is}}^h} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{a\left(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h\right) + b\left(\boldsymbol{v}^h, p - q^h\right)}{\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)}} \\ &\le \frac{1}{\beta_{\text{is}}^h} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{\left\|\nabla\left(\boldsymbol{u} - \boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} + \left\|p - q^h\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)}}{\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)}} \\ &= \frac{1}{\beta_{\text{is}}^h} \left(\left\|\nabla\left(\boldsymbol{u} - \boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} + \left\|p - q^h\right\|_{L^2(\Omega)}\right) \quad \forall\, q^h \in Q^h. \end{aligned}$$

Inserting the error estimate (3.15) for the velocity yields the error estimate for the pressure. ■

Remark 3.21. Error of the velocity in the $L^2(\Omega)$ norm. A simple error estimate of the velocity error in the $L^2(\Omega)$ can be obtained with the Poincaré inequality (A.12)

$$\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le C \left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)}$$

and the application of (3.15). However, the resulting estimate is not optimal. The derivation of an optimal error estimate for $\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)}$ requires additional assumptions and analytical tools. □

Remark 3.22. Regular dual Stokes problem. To obtain an estimate for the velocity error in $L^2(\Omega)$, the classical argument by Aubin (1967) and Nitsche

(1968) is applied. To this end, a dual problem of the Stokes equations has to be considered. The dual or adjoint operator is obtained by bringing the test function of the weak formulation of the Stokes equations (3.3) in the place of the solution and vice versa. Using the symmetry of the viscous term and changing position of terms, one gets from the left-hand side of (3.3)

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{u}) + (\nabla \cdot \boldsymbol{u}, q) - (\nabla \cdot \boldsymbol{v}, p).$$

Now, the test function is replaced by the solution of the dual problem $\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}}\right)$ and the solution $(\boldsymbol{u}, p)$ by the test function, leading to the following left-hand side of the weak form of the dual problem

$$\left(\nabla \boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \nabla \boldsymbol{v}\right) + \left(\nabla \cdot \boldsymbol{v}, \xi_{\hat{\boldsymbol{f}}}\right) - \left(\nabla \cdot \boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, q\right).$$

Using integration by parts, one gets the strong form of the dual Stokes problem for a velocity in V_{div}: For given $\hat{\boldsymbol{f}} \in L^2(\Omega)$, find $(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}}) \in V \times Q$ such that

$$\begin{aligned} -\Delta \boldsymbol{\phi}_{\hat{\boldsymbol{f}}} - \nabla \xi_{\hat{\boldsymbol{f}}} &= \hat{\boldsymbol{f}} \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{\phi}_{\hat{\boldsymbol{f}}} &= 0 \text{ in } \Omega. \end{aligned} \tag{3.22}$$

Note that in the general dual problem, the right-hand side of the divergence constraint might be different than zero.

Problem (3.22) is said to be regular, if the mapping

$$\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}}\right) \mapsto -\Delta \boldsymbol{\phi}_{\hat{\boldsymbol{f}}} - \nabla \xi_{\hat{\boldsymbol{f}}} \tag{3.23}$$

is an isomorphism from $\left(H^2(\Omega) \cap V\right) \times \left(H^1(\Omega) \cap Q\right)$ onto $L^2(\Omega)$. That means, in comparison with the Stokes equations, the higher regularity conditions $\boldsymbol{\phi}_{\hat{\boldsymbol{f}}} \in H^2(\Omega)$ and $\xi_{\hat{\boldsymbol{f}}} \in H^1(\Omega)$ are required. It can be proved that this regularity is given, e.g., for bounded convex polyhedral domains in two and three dimensions, see Kellogg & Osborn (1976); Dauge (1989). □

Theorem 3.23. Finite element error estimate for the $L^2(\Omega)$ norm of the velocity. *Let the assumption of Theorem 3.17 hold and let* $(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}})$ *be the solution of* (3.22). *Then, the* $L^2(\Omega)$ *error of the velocity can be estimated as follows*

$$\begin{aligned} \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le & \left(\left\|\nabla\left(\boldsymbol{u} - \boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} + \inf_{q^h \in Q^h} \left\|p - q^h\right\|_{L^2(\Omega)}\right) \\ & \times \sup_{\hat{\boldsymbol{f}} \in L^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{1}{\left\|\hat{\boldsymbol{f}}\right\|_{L^2(\Omega)}} \left[\inf_{\boldsymbol{\phi}^h \in V_{\text{div}}^h} \left\|\nabla\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}} - \boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)} \right. \\ & \left. + \inf_{r^h \in Q^h} \left\|\xi_{\hat{\boldsymbol{f}}} - r^h\right\|_{L^2(\Omega)}\right]. \end{aligned} \tag{3.24}$$

Proof. For interested students only, not presented in the class.

It is, by definition,

$$\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} = \sup_{\hat{\boldsymbol{f}}\in L^2(\Omega)\setminus\{\boldsymbol{0}\}} \frac{\left(\hat{\boldsymbol{f}},\boldsymbol{u}-\boldsymbol{u}^h\right)}{\left\|\hat{\boldsymbol{f}}\right\|_{L^2(\Omega)}}. \tag{3.25}$$

The weak form of the dual problem (3.22) is: Find $(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}},\xi_{\hat{\boldsymbol{f}}}) \in V\times Q$ such that

$$\begin{aligned}\left(\nabla\boldsymbol{v},\nabla\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}\right)+\left(\nabla\cdot\boldsymbol{v},\xi_{\hat{\boldsymbol{f}}}\right) &= \left(\hat{\boldsymbol{f}},\boldsymbol{v}\right) \quad \forall\,\boldsymbol{v}\in V,\\ \left(\nabla\cdot\boldsymbol{\phi}_{\hat{\boldsymbol{f}}},q\right) &= 0 \qquad\quad \forall\,q\in Q.\end{aligned}$$

Choosing $\boldsymbol{v}=\boldsymbol{u}-\boldsymbol{u}^h\in V$ gives for the numerator of (3.25)

$$\left(\hat{\boldsymbol{f}},\boldsymbol{u}-\boldsymbol{u}^h\right) = \left(\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\nabla\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}\right)+\left(\nabla\cdot\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\xi_{\hat{\boldsymbol{f}}}\right). \tag{3.26}$$

The aim consists now in adding terms to this equation such that approximation errors are obtained. For all $\boldsymbol{\phi}^h\in V_{\mathrm{div}}^h\subset V$ and $q^h\in Q^h$ it holds, using the weak form of the Stokes problem (3.2) and the finite element problem (3.14),

$$\begin{aligned}\left(\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\nabla\boldsymbol{\phi}^h\right) &= \left(\nabla\cdot\boldsymbol{\phi}^h,p\right)+\langle\boldsymbol{f},\boldsymbol{\phi}^h\rangle_{V',V}-\left(\nabla\cdot\boldsymbol{\phi}^h,p^h\right)-\langle\boldsymbol{f},\boldsymbol{\phi}^h\rangle_{V',V}\\ &= \left(\nabla\cdot\boldsymbol{\phi}^h,p\right)=\left(\nabla\cdot\boldsymbol{\phi}^h,p-q^h\right)\quad\forall\,q^h\in Q^h.\end{aligned} \tag{3.27}$$

In addition, it is

$$\left(\nabla\cdot\boldsymbol{\phi}_{\hat{\boldsymbol{f}}},p-q^h\right)=0\quad\forall\,q^h\in Q^h,$$

and

$$\left(\nabla\cdot\left(\boldsymbol{u}-\boldsymbol{u}^h\right),r^h\right)=0\quad\forall\,r^h\in Q^h,$$

since $Q^h\subset Q$. Inserting these terms in (3.26) leads to

$$\begin{aligned}\left(\hat{\boldsymbol{f}},\boldsymbol{u}-\boldsymbol{u}^h\right) &= \left(\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\nabla\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}-\boldsymbol{\phi}^h\right)\right)+\left(\nabla\cdot\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}-\boldsymbol{\phi}^h\right),p-q^h\right)\\ &\quad+\left(\nabla\cdot\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\xi_{\hat{\boldsymbol{f}}}-r^h\right)\end{aligned} \tag{3.28}$$

for all $\boldsymbol{\phi}^h\in V_{\mathrm{div}}^h$ and all $q^h,r^h\in Q^h$. The application of the Cauchy–Schwarz inequality (A.10) and of (2.26) yields

$$\begin{aligned}\left|\left(\hat{\boldsymbol{f}},\boldsymbol{u}-\boldsymbol{u}^h\right)\right| &\le \left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}-\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}\\ &\quad+\left\|p-q^h\right\|_{L^2(\Omega)}\left\|\nabla\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}-\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}\\ &\quad+\left\|\xi_{\hat{\boldsymbol{f}}}-r^h\right\|_{L^2(\Omega)}\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\\ &\le \left(\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}+\left\|p-q^h\right\|_{L^2(\Omega)}\right)\\ &\quad\times\left(\left\|\nabla\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}-\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}+\left\|\xi_{\hat{\boldsymbol{f}}}-r^h\right\|_{L^2(\Omega)}\right)\end{aligned} \tag{3.29}$$

for all $\boldsymbol{\phi}^h\in V_{\mathrm{div}}^h$ and all $q^h,r^h\in Q^h$. Taking the infimum of all approximation errors gives (3.24). ■

Remark 3.24. On the dependency of the error bounds on the discrete inf-sup constant. One can estimate the best approximation error with respect to V^h_{div} in (3.15), (3.20), (3.21), and (3.24) with (2.45) giving a term with $\left(\beta^h_{\mathrm{is}}\right)^{-1}$. The obtained estimates are worst case estimates.

There are finite element spaces where a local interpolation operator that preserves the discrete divergence can be constructed, see Girault & Scott (2003). Then, the constants in the velocity estimates depend on the inverse of local discrete inf-sup constants. In contrast to the error estimates with respect to the velocity, the error bound (3.21) for the pressure depends always on $\left(\beta^h_{\mathrm{is}}\right)^{-1}$.

In all cases, if the local inf-sup constants or β^h_{is} depend on the mesh width, then an optimal order of convergence cannot be expected. $\square$

Corollary 3.25. Finite element error estimates for conforming inf-sup stable pairs of finite element spaces. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with polyhedral and Lipschitz continuous boundary which is decomposed by a regular and quasi-uniform family of triangulations $\{\mathcal{T}^h\}$. Let $(\boldsymbol{u}, p)$ be the solution of the Stokes equations (3.2) with $\boldsymbol{u} \in H^{k+1}(\Omega) \cap V$ and $p \in H^k(\Omega) \cap Q$. Then for the inf-sup stable pairs of finite element spaces*

- $P_k^{\mathrm{bubble}}/P_k$, *$k = 1$ (MINI element),*
- P_k/P_{k-1}, Q_k/Q_{k-1}, *$k \ge 2$ (Taylor–Hood element),*
- $P_2^{\mathrm{bubble}}/P_1^{\mathrm{disc}}$, $P_3^{\mathrm{bubble}}/P_2^{\mathrm{disc}}$, $P_2^{\mathrm{BR}}/P_1^{\mathrm{disc}}$, $Q_k/P_{k-1}^{\mathrm{disc}}$, $k \ge 2$,

the following error estimates hold

$$\left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le Ch^k \left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\right), \tag{3.30}$$

$$\left\|\nabla \cdot \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le Ch^k \left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\right), \tag{3.31}$$

$$\left\|p - p^h\right\|_{L^2(\Omega)} \le Ch^k \left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\right). \tag{3.32}$$

If the dual Stokes problem (3.22) possesses a regular solution $(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}})$ in the sense of Remark 3.22, then it holds in addition

$$\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le Ch^{k+1} \left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\right). \tag{3.33}$$

The constants C depend either on the inverse of the discrete inf-sup constant β^h_{is} or on the inverse of local inf-sup constants, compare Remark 3.24.

Proof. The best approximation errors in estimates (3.15), (3.20), (3.21), and (3.24) can be estimated by interpolation errors, since an interpolation error cannot be lower than the best approximation error. Then, estimates (3.30) – (3.32) follow directly from interpolation error estimates for finite element spaces, see (2.45) for the velocity.

The additional power of h in estimate (3.33) comes from the application of the interpolation estimates (2.45) and for the pressure to the second factor of (3.24) and the regularity of $(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}, \xi_{\hat{\boldsymbol{f}}})$

$$\inf_{\boldsymbol{\phi}^h \in V^h_{\mathrm{div}}} \left\|\nabla\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}} - \boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)} + \inf_{r^h \in Q^h} \left\|\xi_{\hat{\boldsymbol{f}}} - r^h\right\|_{L^2(\Omega)}$$

$$\leq Ch\left(\left\|\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}\right\|_{H^2(\Omega)} + \left\|\xi_{\hat{\boldsymbol{f}}}\right\|_{H^1(\Omega)}\right). \tag{3.34}$$

Since (3.23) is an isomorphism from $\left(H^2(\Omega) \cap V\right) \times \left(H^1(\Omega) \cap Q\right)$ onto $L^2(\Omega)$ there is a constant C such that

$$\left\|\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}\right\|_{H^2(\Omega)} + \left\|\xi_{\hat{\boldsymbol{f}}}\right\|_{H^1(\Omega)} \leq C\left\|\hat{\boldsymbol{f}}\right\|_{L^2(\Omega)}. \tag{3.35}$$

Inserting (3.34) and (3.35) in (3.24) proves (3.33). ■

Example 3.26. A two-dimensional steady-state example with prescribed solution in the unit square and with homogeneous Dirichlet boundary conditions. Consider $\Omega = (0,1)^2$ and the stream function

$$\phi = 1000x^2(1-x)^4y^3(1-y)^2.$$

Then, the velocity field is defined by

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y\phi \\ -\partial_x\phi \end{pmatrix} = 1000\begin{pmatrix} x^2(1-x)^4y^2(1-y)(3-5y) \\ -2x(1-x)^3(1-3x)y^3(1-y)^2 \end{pmatrix}. \tag{3.36}$$

It follows, using the Theorem of Schwarz, that

$$\nabla\cdot\boldsymbol{u} = \partial_x u_1 + \partial_y u_2 = \partial_{xy}\phi - \partial_{yx}\phi = \partial_{xy}\phi - \partial_{xy}\phi = 0.$$

The equations will be equipped with Dirichlet boundary conditions on the whole boundary. It is $\boldsymbol{u} = \boldsymbol{0}$ on Γ.

Because of the Dirichlet boundary conditions on Γ, the pressure should be in $L_0^2(\Omega)$. This is the only essential requirement. The following pressure was chosen for the definition of this example

$$p = \pi^2(xy^3\cos(2\pi x^2y) - x^2y\sin(2\pi xy)) + \frac{1}{8}.$$

The stream function, velocity, and pressure are presented in Figure 3.1. It can be seen that the velocity field consists essentially of one big vortex.

Because of the homogeneous Dirichlet boundary conditions on Γ, this example fits into the framework of many results obtained in the numerical analysis of finite element methods. □

Example 3.27. Analytical example which supports the error estimates (3.30) – (3.33). Error estimates of form (3.30) – (3.33) are usually supported by using problems with a known (prescribed) solution and by measuring the errors to this solution on a sequence of subsequently refined grids. Here, Example 3.26 will be considered, which is defined on the unit square. Initial grids (level 0) of the form presented in Figure 3.2 were used in the simulations. On such unstructured or distorted grids, it is not likely to obtain superconvergence effects in the numerical results. The right-hand side was integrated with a quadrature rule of high order to reduce the influence of quadrature errors.

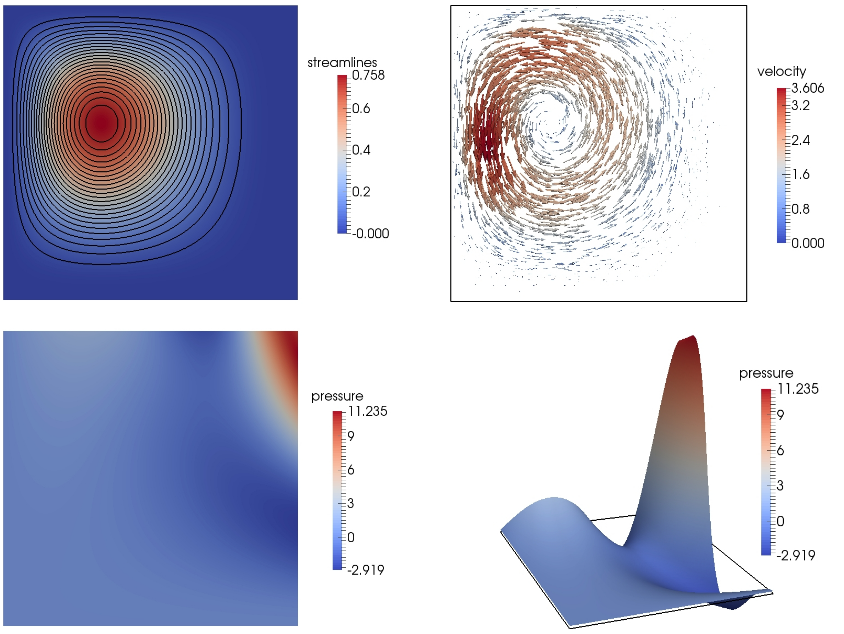

Fig. 3.1 Example 3.26. Stream function (top left) velocity (top right) and pressure (bottom). These plots are based on results obtained with numerical simulations.

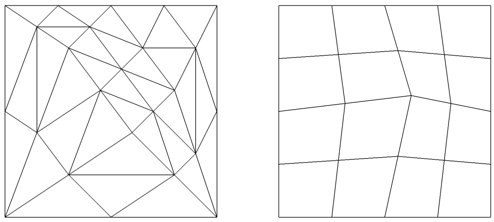

Fig. 3.2 Example 3.27. Initial grids (level 0).

The errors in different norms are presented in Figures 3.3 – 3.6[1]. It can be observed that in the most cases the orders of convergence predicted by the numerical analysis coincide with the orders of convergence in the numerical simulations. Generally, different discretizations of the same order show a similar accuracy. Only the pressure error obtained with $P_2^{\text{bubble}}/P_1^{\text{disc}}$, see Figure 3.5, is much higher than with all other discretizations with second order velocity and first order pressure.

For the P_1^{bubble}/P_1 pair of finite element spaces, the order of convergence for the $L^2(\Omega)$ error of the pressure is higher by 0.5 than predicted by the analysis. □

[1] 2D graphics were plotted with Matplotlib, Hunter (2007), `http://matplotlib.org`.

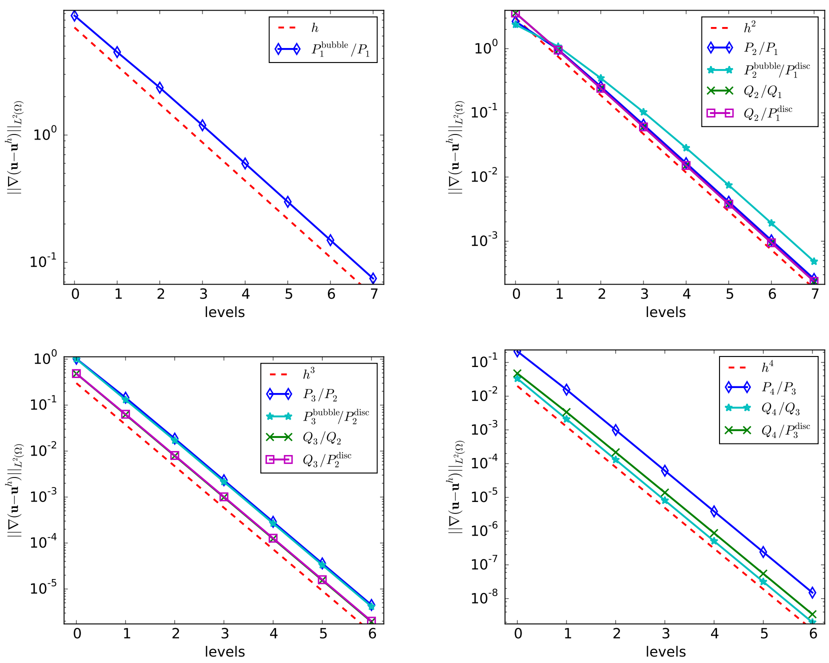

Fig. 3.3 Example 3.27. Convergence of the errors $\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)}$ for different discretizations with different orders k. In the top right and bottom left pictures, the magenta line is on top of the green line.

Remark 3.28. Scaled Stokes problem. A scaled Stokes problem of the form

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \ \text{ in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \ \text{ in } \Omega, \end{aligned} \tag{3.37}$$

$\nu > 0$, is sometimes of interest for academic purposes, since in (3.37), the viscous term is scaled in the same form as for the Navier–Stokes equations. Dividing the momentum equation in (3.37) by ν yields

$$\begin{aligned} -\Delta\boldsymbol{u} + \nabla\left(\frac{p}{\nu}\right) &= \frac{\boldsymbol{f}}{\nu} \ \text{ in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \ \text{ in } \Omega, \end{aligned}$$

which is of the same form as the unscaled version (3.1) with a new pressure and a new source term. Now, the finite element error analysis can be applied in the same way as presented in this section, leading to the estimates of form (3.15), (3.20), (3.21), and (3.24), where the pressure terms are scaled with ν^{-1}. Consequently, one obtains also estimates of the form (3.30) and (3.31) with ν^{-1} in front of $\|p\|_{H^k(\Omega)}$

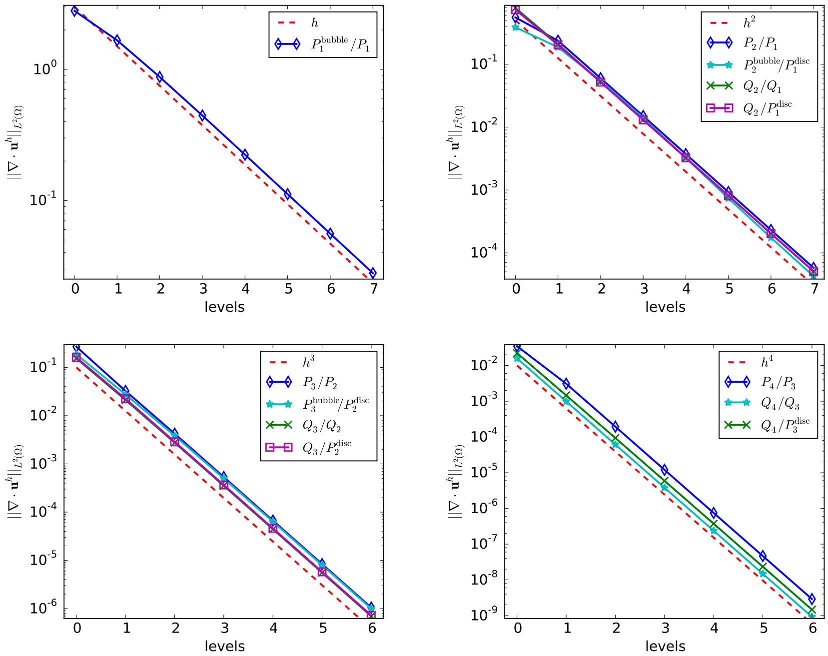

Fig. 3.4 Example 3.27. Convergence of the errors $\left\|\nabla\cdot\boldsymbol{u}^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k. In the top right and bottom left pictures, the magenta line is on top of the green line.

$$\begin{aligned}\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} &\le Ch^k\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+\nu^{-1}\|p\|_{H^k(\Omega)}\right), \\ \left\|\nabla\cdot\boldsymbol{u}^h\right\|_{L^2(\Omega)} &\le Ch^k\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+\nu^{-1}\|p\|_{H^k(\Omega)}\right).\end{aligned} \tag{3.38}$$

For the estimate of the $L^2(\Omega)$ error of the velocity, one considers again the dual Stokes problem (3.22). Since this problem is formulated with unit viscosity, neither $\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}$ nor $\xi_{\hat{\boldsymbol{f}}}$ nor $\hat{\boldsymbol{f}}$ depend on ν. The error estimate is performed in the same way as in the proof of Theorem 3.23. Instead of (3.27), one obtains

$$\left(\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\nabla\boldsymbol{\phi}^h\right)=\frac{1}{\nu}\left(\nabla\cdot\boldsymbol{\phi}^h,p-q^h\right)\quad\forall\, q^h\in Q^h.$$

Then, the middle term on the right-hand side of (3.28) is scaled with ν^{-1} and one gets the scaling ν^{-1} in (3.29) in front of $\left\|p-q^h\right\|_{L^2(\Omega)}$. Finally, the error estimate

$$\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}\le Ch^{k+1}\left(\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+\nu^{-1}\|p\|_{H^k(\Omega)}\right)$$

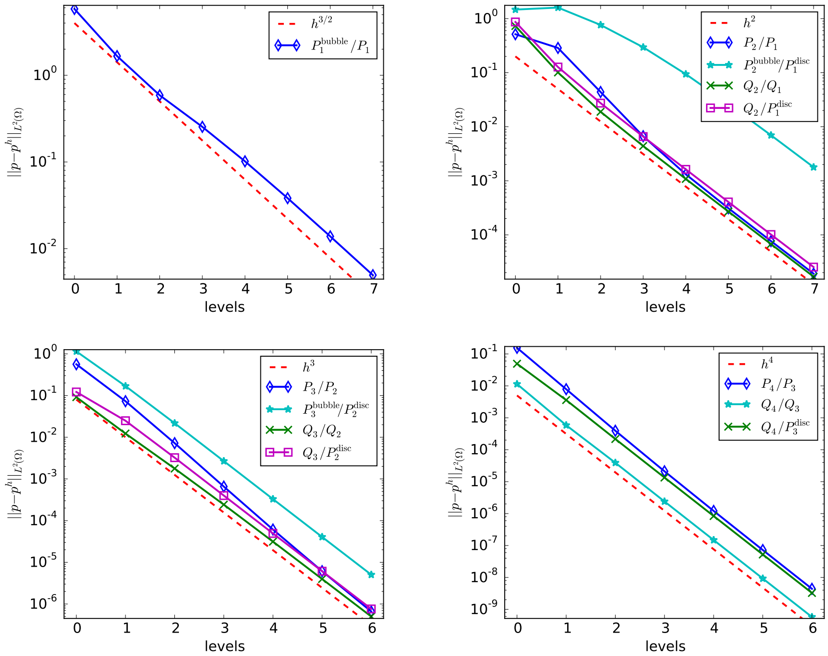

Fig. 3.5 Example 3.27. Convergence of the errors $\left\|p - p^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k.

$$\times \left(\left\|\boldsymbol{\phi}_{\hat{\boldsymbol{f}}}\right\|_{H^2(\Omega)} + \left\|\xi_{\hat{\boldsymbol{f}}}\right\|_{H^1(\Omega)} \right) \tag{3.39}$$

is derived.

Thus, for small values of ν, the term $\nu^{-1}\left\|p\right\|_{H^k(\Omega)}$ might dominate the right-hand side of all velocity error bounds.

In estimate (3.32) for the pressure, one has to scale also the term on the left-hand side with ν^{-1}. Rescaling this estimate leads to

$$\left\|p - p^h\right\|_{L^2(\Omega)} \le Ch^k \left(\nu \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \left\|p\right\|_{H^k(\Omega)} \right). \tag{3.40}$$

$\square$

Example 3.29. Scaled Stokes problem. Again, the problem defined in Example 3.26 is considered, see Example 3.27 for the simulations with the unscaled Stokes equations. From the estimates (3.38) and (3.39), one would expect that the velocity errors become large for small ν and then they scale linearly with ν^{-1}. In contrast, from (3.40) one has the expectation that the pressure error becomes large for large values of ν and then it scales linearly with ν.

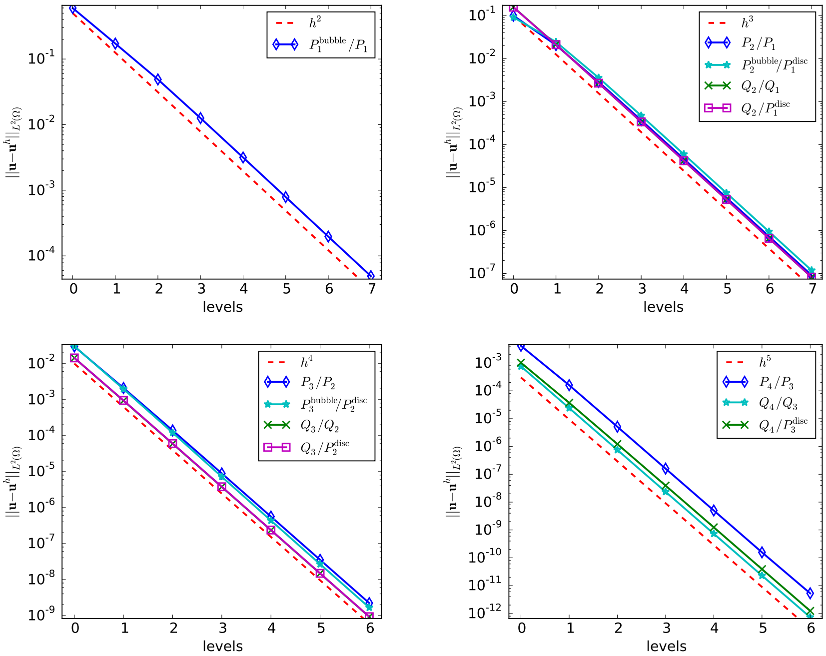

Fig. 3.6 Example 3.27. Convergence of the errors $\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k.

Representative results for the second order Taylor–Hood pair of finite element spaces P_2/P_1 on the unstructured grid from Figure 3.2 are presented in Figure 3.7. The dependency of the velocity errors on ν^{-1} and the pressure error on ν is clearly visible. On coarse grids, also the linear dependencies on ν^{-1} and ν, respectively, can be observed. However, one can also see a higher order of decrease for the curves with large errors until they reach the curves for which a dependency on the value of ν cannot be observed. This decrease is higher by half an order for the velocity errors and by one order for the $L^2(\Omega)$ error of the pressure. To the best of our knowledge, there is no explanation for this behavior so far. □

3.2.1.2 The case $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$

Remark 3.30. Pairs of finite element spaces with $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$. This section inspects the proofs of the error estimates from Section 3.2.1.1 under the condition that $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$. It turns out that some terms vanish. An important consequence is that the error estimates for the velocity do not depend any longer on the best approximation errors of the pressure finite element space.

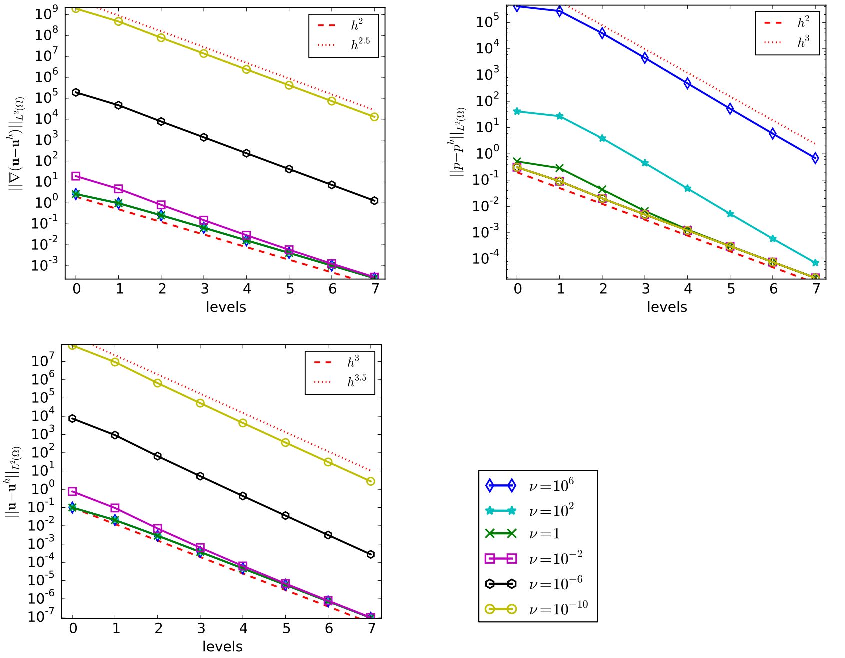

Fig. 3.7 Example 3.29. Convergence of the errors for the scaled Stokes problem and the P_2/P_1 pair of finite element spaces.

In addition, also a scaling of the viscous term as discussed in Remark 3.28 does not influence the velocity error estimates. The estimates in the case $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$ reflect the physics of the problem properly, in contrast to the estimates for the case $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$, compare Remark 3.8.

The most important pair of conforming inf-sup stable finite element spaces satisfying $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$ is the Scott–Vogelius pair of finite element spaces $P_k/P_{k-1}^{\mathrm{disc}}$, $k \ge d$, on special grids, see Remarks 2.76 and 2.77. $\square$

Corollary 3.31. Finite element error estimates for the velocity for inf-sup stable pairs of finite element spaces with $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$. *Let the assumptions of Theorem 3.17 be fulfilled and consider an inf-sup stable pair of finite element spaces with $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$, then*

$$\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le 2 \inf_{\boldsymbol{v}^h \in V_{\mathrm{div}}^h} \left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)} \tag{3.41}$$

and

$$\left\|\nabla \cdot \boldsymbol{u}^h\right\|_{L^2(\Omega)} = 0. \tag{3.42}$$

Proof. The proof of (3.41) is performed in the same way as the proof of Theorem 3.17. Inspecting this proof for pairs of spaces with $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$, one finds that (3.19) equals zero since $\left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} = 0$ for all $\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h$.

Property (3.42) follows directly from the definition of V_{div}. ■

Corollary 3.32. Finite element error estimate for the $L^2(\Omega)$ norm of the velocity for inf-sup stable pairs of finite element spaces with $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$. *Let the assumptions of Theorem 3.23 be fulfilled. If for an inf-sup stable pair of finite element spaces $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$, then*

$$\begin{aligned}\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} &\le \left\|\nabla\left(\boldsymbol{u} - \boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} \\ &\times \sup_{\hat{\boldsymbol{f}} \in L^2(\Omega)} \frac{1}{\left\|\hat{\boldsymbol{f}}\right\|_{L^2(\Omega)}} \inf_{\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h} \left\|\nabla\left(\boldsymbol{\phi}_{\hat{\boldsymbol{f}}} - \boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}. \end{aligned} \tag{3.43}$$

Proof. The proof proceeds in the same way as the proof of Theorem 3.23. In addition, one can use in (3.28) that $\nabla \cdot \left(\boldsymbol{\phi}^h - \boldsymbol{\phi}_{\hat{\boldsymbol{f}}}\right) = 0$ and $\nabla \cdot \left(\boldsymbol{u} - \boldsymbol{u}^h\right) = 0$ in the weak sense. ■

Remark 3.33. Pairs of finite element spaces with $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$. For pairs of finite element spaces with the property $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$, it follows from (3.41) and (3.42) that

$$\left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le Ch^k \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}, \tag{3.44}$$

$$\left\|\nabla \cdot \boldsymbol{u}^h\right\|_{L^2(\Omega)} = 0, \tag{3.45}$$

$$\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \le Ch^{k+1} \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}. \tag{3.46}$$

These estimates are in particular true for the Scott–Vogelius spaces $P_k/P_{k-1}^{\mathrm{disc}}$, $k \ge d$, on barycentric-refined grids. □

Remark 3.34. Scaled Stokes problem. Considering a scaled Stokes problem of the form (3.37), one finds that the scaling ν^{-1} does not affect the velocity error estimates, in contrast to pairs of finite element spaces with $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$, see Remark 3.28. □

Example 3.35. The Scott–Vogelius pair of finite element spaces P_2/P_1^{disc} on a barycentric-refined grid for a no-flow problem. In this example, the scaled Stokes problem (3.37) is considered in $\Omega = (0,1)^2$ and with the prescribed solution $\boldsymbol{u} = \boldsymbol{0}$ and the pressure

$$p(x,y) = 10\left((x-0.5)^3 y + (1-x)^2(y-0.5)^2 - \frac{1}{36}\right).$$

The Scott–Vogelius pair is known to satisfy the discrete inf-sup condition on barycentric-refined grids, see Remark 2.76. The grids were constructed as follows. The unit square was divided into two triangles by connecting

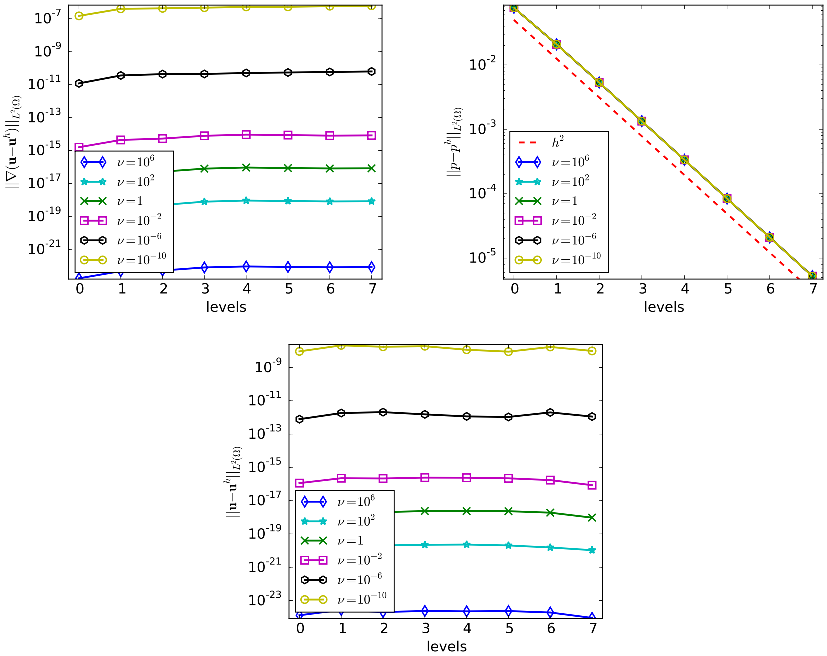

Fig. 3.8 Example 3.35. Convergence of the errors for the scaled Stokes problem and the P_2/P_1^{disc} pair of finite element spaces on a barycentric-refined grid.

the lower left and the upper right corner. This triangulation was uniformly refined once. Then a barycentric-refined grid as depicted in Figure 2.3 was created, giving level 0. After having simulated the problem on this grid, the barycentric refinements were removed, the grid was uniformly refined once more, and again a barycentric refinement was applied, leading to level 1. This process was continued.

The results are presented in Figure 3.8. The velocity error is always a small constant, independently of the value of ν. The increase of this constant is due to the increase of the condition number of the linear saddle point problems for small ν. Hence, this increase reflects round-off errors of the solver. Since the first term on the right-hand side of estimate (3.40) vanishes, one expects that the pressure error in $L^2(\Omega)$ is independent of ν, which can be observed very well in the computational results.

As comparison, results obtained with the Taylor–Hood finite element P_2/P_1 computed on the irregular grid from Figure 3.2 are depicted in Figure 3.9. Even for moderate values of ν, the discrete velocity field is far away from being a no-flow field. The dependency of the velocity errors on the viscosity can be clearly observed in this simple example. This result reflects once more the potential impact of the pressure on the error of the velocity for pairs

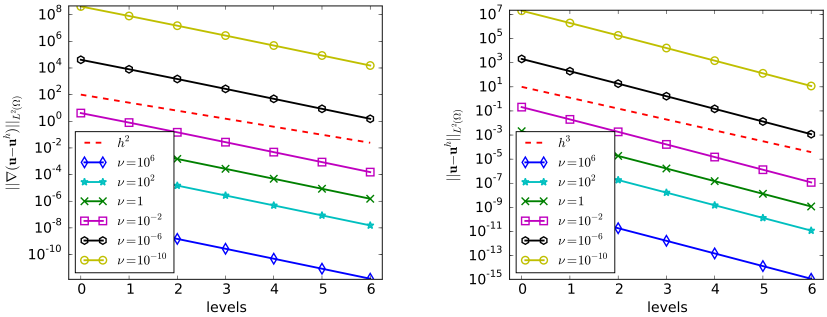

Fig. 3.9 Example 3.35. Convergence of the velocity errors for the scaled Stokes problem and the P_2/P_1 pair of finite element spaces.

of spaces that do not satisfy $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$. Note that the order of convergence of both velocity errors is higher by 0.5 than predicted by the analysis. □

3.3 Stabilized Finite Element Methods Circumventing the Discrete Inf-Sup Condition

Remark 3.36. Motivation. The application of inf-sup stable pairs of finite elements requires the use of different spaces for velocity and pressure. Moreover, it is not possible to use conforming spaces of lowest order for the discrete velocity, i.e., P_1 or Q_1 finite element spaces. However, software for solving incompressible flow problems contains often just one finite element space and then usually P_1 or Q_1. In such situations, it is necessary to modify the discrete problem such that the satisfaction of the discrete inf-sup condition (2.32) is not longer necessary. To this end, one has to remove the saddle point structure of the discrete problem, i.e., one has to remove the zero matrix block in the pressure-pressure coupling of the saddle point problem.

There are several proposals in the literature for defining appropriate pressure-pressure couplings, so-called stabilization terms. One class are so-called pseudo-compressibility methods. The strong form of such methods might look as follows:

- Pressure Stabilization Petrov–Galerkin (PSPG) method

$$-\nabla \cdot \boldsymbol{u} + \delta \Delta p = 0,$$

- penalty method

$$-\nabla \cdot \boldsymbol{u} - \delta p = 0,$$

- artificial compressibility method

$$-\nabla \cdot \boldsymbol{u} - \delta \partial_t p = 0,$$

where δ is some parameter which has to be chosen appropriately.

Note that the introduction of a pressure-pressure coupling perturbs the continuity equation and thus the conservation of mass. It is not possible that such a method leads to estimates for velocity errors without pressure terms in the error bounds. In addition, each stabilization term contains parameters. The asymptotic optimal choice of stabilization parameters can be determined often with results from numerical analysis, e.g., with conditions for the existence and uniqueness of a solution of the stabilized problem or from optimal error estimates. However, the user still has to choose concrete parameters in simulations and, depending on the parameter, different concrete choices of the same asymptotic type might sometimes lead to rather different numerical solutions.

Detailed information on pressure-stabilized methods can be found in the recent review paper John *et al.* (2020). $\square$