Chapter 5

The Two-Level Method

Remark 5.1 *The two-level method.* In this chapter, the two-level method or coarse grid correction scheme will be analyzed. The two-level method, whose principle was already introduced in Remark 4.6, has the following form:

- Smooth $A^h \mathbf{u}^h = \mathbf{f}^h$ on Ω^h with some steps of a simple iterative scheme. This procedure gives an approximation $\mathbf{v}^h$. Compute the residual $\mathbf{r}^h = \mathbf{f}^h - A^h \mathbf{v}^h$.
- Restrict the residual to the coarse grid Ω^{2h} using the restriction operator I_h^{2h} (weighted restriction for finite difference methods, canonical restriction for finite element methods).
- Solve the coarse grid equation

$$A^{2h} \mathbf{e}^{2h} = I_h^{2h} \left(\mathbf{r}^h \right) \tag{5.1}$$

  on Ω^{2h}.
- Prolongate $\mathbf{e}^{2h}$ to Ω^h using the prolongation operator I_{2h}^h.
- Update $\mathbf{v}^h := \mathbf{v}^h + I_{2h}^h \left(\mathbf{e}^{2h} \right)$.

After the update, one can apply once more some iterations with the smoother. This step is called post smoothing, whereas the first step of the two-level method is called pre smoothing. □

5.1 The Coarse Grid Problem

Remark 5.2 *The coarse grid system.* The two-level method still lacks a definition of the coarse grid matrix A^{2h}. This matrix should be a "Ω^{2h} version of the fine grid matrix A^h". Possible choices of A^{2h} will be discussed in this section. □

Remark 5.3 *Definition of the coarse grid matrix by using a discrete scheme on Ω^{2h}.* A straightforward approach consists in defining A^{2h} by applying a finite difference or finite element method to the differential operator on Ω^{2h}. □

Remark 5.4 *Definition of the coarse grid matrix by Galerkin projection.* Starting point for the derivation of an appropriate coarse grid matrix by the Galerkin projection is the residual equation

$$A^h \mathbf{e}^h = \mathbf{r}^h. \tag{5.2}$$

It will be assumed for the moment that $\mathbf{e}^h$ lies in the range of the prolongation operator I_{2h}^h. Then, there is a vector $\mathbf{e}^{2h}$ defined on the coarse grid such that

$$\mathbf{e}^h = I_{2h}^h \left(\mathbf{e}^{2h} \right).$$

Substituting this equation into (5.2) gives

$$A^h I_{2h}^h \left(\mathbf{e}^{2h}\right) = \mathbf{r}^h.$$

Applying now on both sides of this equation the restriction operator gives

$$I_h^{2h} A^h I_{2h}^h \left(\mathbf{e}^{2h}\right) = I_h^{2h} \mathbf{r}^h.$$

Comparing this definition with (5.1) leads to the definition

$$A^{2h} := I_h^{2h} A^h I_{2h}^h. \tag{5.3}$$

This definition of the coarse grid matrix is called Galerkin projection.

The derivation of (5.3) was based on the assumption that the error $\mathbf{e}^h$ is in the range of the prolongation. This property is in general not given. If it would be true, then an exact solution of the coarse grid equation would result in obtaining the solution of $A\mathbf{u}^h = \mathbf{f}^h$ with one step of the coarse grid correction scheme. Nevertheless, this derivation gives a motivation for defining A^{2h} in the form (5.3). □

Remark 5.5 *Matrix representation of the Galerkin projection.* For all operators on the right-hand side of (5.3), matrix representations are known, e.g., see (2.3), (4.3), and (4.5) for the case of the finite difference discretization. Using these representations, one obtains

$$
\begin{aligned}
A^{2h} &= \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & & 1 & 2 & 1 & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & 2 & 1 \end{pmatrix} \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \vdots & \vdots & \vdots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \\
&\quad \times \frac{1}{2}\begin{pmatrix} 1 & & & & \\ 2 & & & & \\ 1 & 1 & & & \\ & 2 & & & \\ & 1 & 1 & & \\ & & 2 & \ddots & \\ & & 1 & \ddots & \\ & & & & 1 \\ & & & & 2 \\ & & & & 1 \end{pmatrix} \\
&= \frac{1}{8h^2}\begin{pmatrix} 1 & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & & 1 & 2 & 1 & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & & & & \\ 2 & -1 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ -1 & 2 & -1 & & & & \\ & & & \ddots & & & \\ & & & & -1 & 2 & -1 \\ & & & & 0 & 0 & 0 \\ & & & & 0 & -1 & 2 \\ & & & & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{8h^2}\begin{pmatrix} 4 & -2 & & & \\ -2 & 4 & -2 & & \\ & \vdots & \vdots & \vdots & \\ & & -2 & 4 & -2 \\ & & & -2 & 4 \end{pmatrix} \\
&= \frac{1}{4h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \vdots & \vdots & \vdots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.
\end{aligned}
$$

This matrix has the form of the matrix (2.3) with h replaced by $2h$. Thus, in the case of the model problem, the matrix defined by the Galerkin projection (5.3) and the matrix (2.3) obtained by discretizing the differential operator on the coarse grid Ω^{2h} coincide.

In the finite element case, the matrices differ only by the factors in front of the parentheses: $1/2$, $1/h$, $1/2$, instead of $1/4$, $1/h^2$, $1/2$. Then, the final factor is $1/(2h)$ instead of $1/(4h^2)$. The factor $1/(2h)$ is exactly the factor of the finite element matrix on Ω^{2h}, see (2.4). That means, also in this case Galerkin projection and the discretization on Ω^{2h} coincide.

This connection of the Galerkin projection and of the discretized problem on Ω^{2h} does not hold in all cases (problems and discretizations), but it can be found often. $\square$

5.2 General Approach for Proving the Convergence of the Two-Level Method

Remark 5.6 *The iteration matrix of the two-level method.* For studying the convergence of the two-level method, one first has to find the iteration matrix $S_{2\text{lev}}$ of this scheme. For simplicity, only the case of pre smoothing is considered, but no post smoothing.

Let S_{sm} be the iteration matrix of the smoother. The approximation of the solution before the pre smoothing step is denoted by $\mathbf{v}^{(n)}$ and the result after the update will be $\mathbf{v}^{(n+1)}$. Applying ν pre smoothing steps, then it is known from (3.7) that

$$\mathbf{e}^{(\nu)} = S_{\text{sm}}^{\nu}\mathbf{e}^{(0)}, \quad \text{with } \mathbf{e}^{(0)} = \mathbf{u} - \mathbf{v}^{(n)},\ \mathbf{e}^{(\nu)} = \mathbf{u} - \mathbf{v}_{\nu}^{(n)}.$$

It follows that

$$\mathbf{v}_{\nu}^{(n)} = \mathbf{u} - S_{\text{sm}}^{\nu}\left(\mathbf{u} - \mathbf{v}^{(n)}\right),$$

where now $\mathbf{v}_{\nu}^{(n)}$ stands for $\mathbf{v}^h$ in the general description of the two-level method from Remark 5.1. It follows that

$$\mathbf{r} = \mathbf{f} - A^h\mathbf{v}_{\nu}^{(n)} = \mathbf{f} - A^h\mathbf{u} + A^h S_{\text{sm}}^{\nu}\left(\mathbf{u} - \mathbf{v}^{(n)}\right) = A^h S_{\text{sm}}^{\nu}\left(\mathbf{u} - \mathbf{v}^{(n)}\right).$$

Applying this formula in the two-level method from Remark 5.1, starting with the update step, one obtains

$$\begin{aligned}
\mathbf{v}^{(n+1)} &= \mathbf{v}_{\nu}^{(n)} + I_{2h}^{h}\left(\mathbf{e}^{2h}\right) \\
&= \mathbf{u} - S_{\text{sm}}^{\nu}\left(\mathbf{u} - \mathbf{v}^{(n)}\right) + I_{2h}^{h}\left(A^{2h}\right)^{-1} I_{h}^{2h}\mathbf{r} \\
&= S_{\text{sm}}^{\nu}\mathbf{v}^{(n)} + \left(I - S_{\text{sm}}^{\nu}\right)\left(A^h\right)^{-1}\mathbf{f} \\
&\quad + I_{2h}^{h}\left(A^{2h}\right)^{-1} I_{h}^{2h} A^h S_{\text{sm}}^{\nu}\left(\left(A^h\right)^{-1}\mathbf{f} - \mathbf{v}^{(n)}\right) \\
&= \left(I - I_{2h}^{h}\left(A^{2h}\right)^{-1} I_{h}^{2h} A^h\right) S_{\text{sm}}^{\nu}\mathbf{v}^{(n)} \qquad (5.4) \\
&\quad + \left(\left(I - S_{\text{sm}}^{\nu}\right) + I_{2h}^{h}\left(A^{2h}\right)^{-1} I_{h}^{2h} A^h S_{\text{sm}}^{\nu}\right)\left(A^h\right)^{-1}\mathbf{f}.
\end{aligned}$$

Hence, the iteration matrix of the two-level method is given by

$$S_{2\text{lev}} = \left(I - I_{2h}^{h}\left(A^{2h}\right)^{-1} I_{h}^{2h} A^h\right) S_{\text{sm}}^{\nu}. \qquad (5.5)$$

Inserting $\mathbf{u} = \left(A^h\right)^{-1}\mathbf{f}$ into the two-level method (5.4) shows that $\mathbf{u}$ is a fixed point, *exercise*. It follows that in the case this fixed point is the only fixed point and that the two-level method converges, then it converges to $\mathbf{u}$. □

Remark 5.7 *Goal of the convergence analysis.* From the course Numerical Mathematics II, Theorem 3.3 in the part on iterative solvers, it is known that a sufficient and necessary condition for the convergence of the fixed point iteration is that $\rho\left(S_{2\text{lev}}\right) < 1$. But the computation of $\rho\left(S_{2\text{lev}}\right)$ is rather complicated, even in simple situations. However, from linear algebra it is known that $\rho\left(S_{2\text{lev}}\right) \le |||S_{2\text{lev}}|||$ for induced matrix norms, e.g., the spectral norm. The goal of the convergence analysis will be to show that

$$|||S_{2\text{lev}}||| \le \rho < 1$$

independently of h. The analysis is based on a splitting of $S_{2\text{lev}}$ in the form

$$S_{2\text{lev}} = \left(\left(A^h\right)^{-1} - I_{2h}^{h}\left(A^{2h}\right)^{-1} I_{h}^{2h}\right) A^h S_{\text{sm}}^{\nu}.$$

It follows that

$$|||S_{2\text{lev}}||| \leq ||| \left(A^h\right)^{-1} - I_{2h}^h \left(A^{2h}\right)^{-1} I_h^{2h}||| \; |||A^h S_{\text{sm}}^\nu|||. \tag{5.6}$$

The first factor in (5.6) describes the effect of the coarse grid approximation. The second factor measures the efficiency of the smoothing step. The smaller the first factor is, the better is the coarse grid solution which approximates $\mathbf{e}^h$. Hence, the two essential components of the two-level method, the smoothing and the coarse grid correction, can be analyzed separately. □

Definition 5.8 Smoothing property. The matrix S_{sm} is said to possess the smoothing property, if there exist functions $\eta(\nu)$ and $\overline{\nu}(t)$, whose definition is independent of h, and a number $\alpha > 0$ such that

$$|||A^h S_{\text{sm}}^\nu||| \leq \eta(\nu) h^{-\alpha} \quad \text{for all } 1 \leq \nu \leq \overline{\nu}(h), \tag{5.7}$$

with $\eta(\nu) \to 0$ as $\nu \to \infty$ and $\overline{\nu}(h) = \infty$ or $\overline{\nu}(h) \to \infty$ as $h \to 0$. □

Remark 5.9 *On the smoothing property.* The smoothing property does not necessarily mean that the smoothing iteration is a convergent iteration. It is only required that the error is smoothed in a certain way using up to $\overline{\nu}(h)$ smoothing steps. In fact, there are examples where divergent iterative schemes are good smoothers. But in this course, only the case $\overline{\nu}(h) = \infty$ will be considered, i.e., the case of a convergent smoothing iteration. □

Definition 5.10 Approximation property. The approximation property holds if there is a constant C_a, which is independent of h, such that

$$||| \left(A^h\right)^{-1} - I_{2h}^h \left(A^{2h}\right)^{-1} I_h^{2h}||| \leq C_a h^\alpha \tag{5.8}$$

with the same α as in the smoothing property. □

Theorem 5.11 Convergence of the two-level method. *Suppose the smoothing property and the approximation property hold. Let $\rho > 0$ be a fixed number. If $\overline{\nu}(t) = \infty$ for all t, then there is a number $\underline{\nu}$ such that*

$$|||S_{2\text{lev}}||| \leq C_a \eta(\nu) \leq \rho, \tag{5.9}$$

whenever $\nu \geq \underline{\nu}$.

Proof: From (5.6) one obtains with the approximation property (5.8) and the smoothing property (5.7)

$$|||S_{2\text{lev}}||| \leq C_a h^\alpha \; \eta(\nu) h^{-\alpha} = C_a \eta(\nu).$$

Since $\eta(\nu) \to 0$ as $\nu \to \infty$, the right-hand side of this estimate is smaller than any given $\rho > 0$ if ν is sufficiently large, e.g., if $\nu \geq \underline{\nu}$. ■

Remark 5.12 *On the convergence theorem.* Note that the estimate $C_a\eta(\nu)$ is independent of h. The convergence theorems says that the two-level method converges with a rate that is independent of h if sufficiently many smoothing steps are applied. For many problems, one finds that only a few pre smoothing steps, i.e., 1 to 3, are sufficient for convergence. □

5.3 The Smoothing Property of the Damped Jacobi Iteration

Remark 5.13 *Contents of this section.* In this section, the smoothing property of the damped Jacobi iteration for the model problem will be proved. Therefore, one has to estimate $|||A^h S_{\text{jac},\omega}^\nu|||$, where now the spectral matrix norm $\left\|A^h S_{\text{jac},\omega}^\nu\right\|_2$ is considered. In the proof, one has to estimate a term of the form $\|B(I-B)^\nu\|_2$ for some symmetric positive definite matrix with $0 < B \le I$, i.e., it is for all eigenvalues λ of B that $\lambda \in (0,1]$. □

Lemma 5.14 Estimate for a symmetric positive definite matrix. *Let* $0 < B = B^T \le I$*, then*

$$\|B(I-B)^\nu\|_2 \le \eta_0(\nu)$$

with

$$\eta_0(\nu) = \frac{\nu^\nu}{(\nu+1)^{\nu+1}}, \quad \nu \in \mathbb{N}. \tag{5.10}$$

Proof: The matrix $B(I-B)^\nu$ is symmetric, *exercise*.

First, the eigenvalues of $B(I-B)^\nu$ will be computed. Let $\lambda \in \mathbb{R}_+$ be an eigenvalue of B. It will be shown that $\lambda(1-\lambda)^\nu$ is an eigenvalue of $B(I-B)^\nu$. The proof is performed by induction Let $\nu = 1$. Then, one has

$$B(I-B)\mathbf{x} = B\mathbf{x} - BB\mathbf{x} = \lambda\mathbf{x} - B\lambda\mathbf{x} = \lambda\mathbf{x} - \lambda^2\mathbf{x} = \lambda(1-\lambda)\mathbf{x}.$$

Thus, the statement of the theorem is true for $\nu = 1$. The induction step has the form

$$\begin{aligned} B(I-B)^\nu\mathbf{x} &= B(I-B)(I-B)^{\nu-1}\mathbf{x} = B(I-B)^{\nu-1}\mathbf{x} - BB(I-B)^{\nu-1}\mathbf{x} \\ &= \lambda(1-\lambda)^{\nu-1}\mathbf{x} - B\lambda(1-\lambda)^{\nu-1}\mathbf{x} = \lambda(1-\lambda)^{\nu-1}\mathbf{x} - \lambda^2(1-\lambda)^{\nu-1}\mathbf{x} \\ &= \left(\lambda - \lambda^2\right)(1-\lambda)^{\nu-1}\mathbf{x} = \lambda(1-\lambda)^\nu\mathbf{x}. \end{aligned}$$

Since $0 \le B \le I$, one has $0 < \lambda \le 1$. Then, it is obvious that

$$0 \le \lambda(1-\lambda)^\nu \le 1,$$

since both factors are between 0 and 1. Hence $B(I-B)^\nu$ is positive semi-definite. One gets, using the definition of the spectral norm, the symmetry of the matrix, the eigenvalue of the square of a matrix, and the nonnegativity of the eigenvalues,

$$\begin{aligned} \|B(I-B)^\nu\|_2 &= \left(\lambda_{\max}\left(\left(B(I-B)^\nu\right)^T B(I-B)^\nu\right)\right)^{1/2} \\ &= \left(\lambda_{\max}\left(\left(B(I-B)^\nu\right)^2\right)\right)^{1/2} \\ &= \left(\left(\lambda_{\max}\left(B(I-B)^\nu\right)\right)^2\right)^{1/2} \\ &= \lambda_{\max}\left(B(I-B)^\nu\right) \\ &= \max_{\lambda \text{ is eigenvalue of } B} \lambda(1-\lambda)^\nu. \end{aligned}$$

Thus, one has to maximize $\lambda(1-\lambda)^\nu$ for $\lambda \in [0,1]$ to get an upper bound for $\|B(I-B)^\nu\|_2$. This expression takes the value zero at the boundary of the interval and it is positive in the interior. Thus, one can compute the maximum with standard calculus

$$\frac{d}{d\lambda}\lambda(1-\lambda)^\nu = (1-\lambda)^\nu - \nu\lambda(1-\lambda)^{\nu-1} = 0.$$

This necessary conditions becomes

$$1 - \lambda - \nu\lambda = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{1+\nu}.$$

It follows that

$$\|B(I-B)^\nu\|_2 \le \frac{1}{1+\nu}\left(1-\frac{1}{1+\nu}\right)^\nu = \frac{\nu^\nu}{(1+\nu)^{1+\nu}}.$$

■

Remark 5.15 *Damped Jacobi method.* Now, the smoothing property of the damped Jacobi method can be proved. The iteration matrix of the damped Jacobi method for the model problem is given by, see also (3.11),

$$S_{\mathrm{jac},\omega} = I - \omega D^{-1}A^h, \quad \omega \in (0,1], \tag{5.11}$$

where $D^{-1}A^h$ is the same for the finite difference and the finite element method. □

Theorem 5.16 Smoothing property of the damped Jacobi method. *Let $S_{\mathrm{jac},\omega}$ be the iteration matrix of the damped Jacobi method given in* (5.11)*, let $\nu \ge 1$, $\nu \in \mathbb{N}$, and let $\omega \in (0, 1/2]$. Then it is*

$$\left\|A^h S^\nu_{\mathrm{jac},\omega}\right\|_2 \le \frac{2}{\omega h}\eta_0(\nu),$$

where $\eta_0(\nu)$ was defined in (5.10)*.*

Proof: The proof will be presented for the finite element method, it can be performed analogously for the finite difference method. For the finite element method, it is $D = 2I/h$. Hence, one gets

$$\begin{aligned}\left\|A^h S^\nu_{\mathrm{jac},\omega}\right\|_2 &= \left\|A^h\left(I-\omega D^{-1}A^h\right)^\nu\right\|_2 = \left\|A^h\left(I-\frac{\omega h}{2}A^h\right)^\nu\right\|_2 \\ &= \frac{2}{\omega h}\left\|\frac{\omega h}{2}A^h\left(I-\frac{\omega h}{2}A^h\right)^\nu\right\|_2.\end{aligned}$$

The matrix $B = \frac{\omega h}{2}A^h$ is symmetric and positive definite and its eigenvalues are, see (2.6),

$$\lambda\left(\frac{\omega h}{2}A^h\right) = \frac{\omega h}{2}\lambda\left(A^h\right) = \frac{\omega h}{2}\frac{4}{h}\sin^2\left(\frac{k\pi}{2N}\right) \le 2\omega \le 1$$

with the assumptions of the theorem. Hence $B \le I$ and Lemma 5.14 can be applied, which gives immediately the statement of the theorem. ■

Remark 5.17 *To the smoothing property.*

- The smooting property does not hold for the non-damped Jacobi method or the SOR method with relaxation parameter $\omega \ge \omega_{\mathrm{opt}}$, see (Hackbusch, 1994, p. 340).
- The bound $\eta_0(\nu)$ behaves like ν^{-1}, *exercise.* It follows that

$$\left\|A^h S^\nu_{\mathrm{jac},\omega}\right\|_2 \le \frac{2}{\omega h}\frac{1}{\nu}$$

and the smoothing rate is said to be linear, i.e., $\mathcal{O}\left(\nu^{-1}\right)$.

□

5.4 The Approximation Property

Remark 5.18 *Contents.* Proofs of the approximation property are not only of algebraic nature. They generally use properties of the underlying boundary value problem. Hence, results from the theory of partial differential equations, like error estimates, have to be applied. □

Remark 5.19 *Isomorphism between finite element spaces and Euclidean spaces.* There is a bijection between the functions in the finite element space V^h and the coefficients of the finite element functions in the space $\mathbb{R}^{n_h}$. This bijection is denoted by $P^h : \mathbb{R}^{n_h} \to V^h$, $\mathbf{v}^h \mapsto v^h$ with

$$v^h(x) = \sum_{i=1}^{n_h} v_i^h \varphi_i^h(x), \quad \mathbf{v}^h = \left(v_i^h\right).$$

If the Euclidean space $\mathbb{R}^{n_h}$ is equipped with the standard Euclidean norm, then the norm equivalence

$$C_0 h^{1/2} \left\|\mathbf{v}^h\right\|_2 \le \left\|P^h \mathbf{v}^h\right\|_{L^2((0,1))} \le C_1 h^{1/2} \left\|\mathbf{v}^h\right\|_2 \tag{5.12}$$

holds with constants that are independent of the mesh size, *exercise.*

There are commutation properties between the grid transfer operators and the bijektion. For instance, for a function given in V^{2h}, one gets the same result if one first applies the bisection to $\mathbb{R}^{n_{2h}}$ and then the interpolation to $\mathbb{R}^{n_h}$ or if one first applies the prolongation to V^h (imbedding) and then applies the bisection to $\mathbb{R}^{n_h}$, i.e.,

$$I_{2h}^h \left(P^{2h}\right)^{-1} v^{2h} = \left(P^h\right)^{-1} \mathcal{I}_{2h}^h v^{2h}, \tag{5.13}$$

where I_{2h}^h on the left-hand side is the matrix representation of the prolongation operator $\mathcal{I}_{2h}^h$ between the finite element spaces. Similarly, if the vector of coefficients is given on the fine grid, one can first apply the bijection and then the restriction or vice versa

$$\mathcal{I}_h^{2h} P^h \mathbf{v}^h = P^{2h} I_h^{2h} \mathbf{v}^h. \tag{5.14}$$

□

Theorem 5.20 Approximation property for the finite element discretization. *Let A^h be defined in* (2.4), *A^{2h} be defined by the Galerkin projection* (5.3), *the prolongation $\mathcal{I}_{2h}^h$ be defined in Example 4.9, and the restriction in Example 4.16. Assume that the boundary value problem* (2.1) *is 2-regular, then the approximation property*

$$\left\| \left(A^h\right)^{-1} - I_{2h}^h \left(A^{2h}\right)^{-1} I_h^{2h} \right\|_2 \le Ch$$

holds.

Proof: Using the definition of an operator norm, the left-hand side of the approximation property (5.8) can be rewritten in the form

$$\sup_{\mathbf{w}^h \in \mathbb{R}^{n_h}} \frac{\left\| \left(\left(A^h\right)^{-1} - I_{2h}^h \left(A^{2h}\right)^{-1} I_h^{2h} \right) \mathbf{w}^h \right\|_2}{\left\|\mathbf{w}^h\right\|_2} \tag{5.15}$$

Let $A^h \mathbf{z}^h = \mathbf{w}^h$, $A^{2h}\mathbf{z}^{2h} = I_h^{2h}\mathbf{w}^h$, then the numerator can be written as

$$\left\| \mathbf{z}^h - I_{2h}^h \mathbf{z}^{2h} \right\|_2. \tag{5.16}$$

By construction, $\mathbf{z}^h$ is the solution of a finite element problem on the fine grid and $\mathbf{z}^{2h}$ is the solution of almost the same problem on the coarse grid. The right-hand side of the coarse grid problem is the restriction of the right-hand side of the fine grid problem. Therefore, it is a straightforward idea to apply results that are known from finite element error analysis. Consider the finite element problems

$$\begin{aligned} \left(\left(u^h\right)', \left(\varphi^h\right)'\right) &= \left(P^h \mathbf{w}^h, \varphi^h\right) = \left(w^h, \varphi^h\right), \quad \forall\, \varphi^h \in V^h, \\ (7.5)\ \left(\left(u^{2h}\right)', \left(\varphi^{2h}\right)'\right) &= \left(w^h, \varphi^{2h}\right), \quad \forall\, \varphi^{2h} \in V^{2h}. \end{aligned}$$

Approximating the right-hand side of the first problem by the composite trapezoidal rule and using $\varphi_i^h(x_{i-1}) = \varphi_i^h(x_{i+1}) = 0$, $\varphi_i^h(x_i) = 1$, one gets

$$\begin{aligned}
&\int_{x_{i-1}}^{x_{i+1}} w^h(x)\varphi_i^h(x)\,dx \\
\approx\ & h\frac{w^h(x_{i-1})\varphi_i^h(x_{i-1}) + w^h(x_i)\varphi_i^h(x_i)}{2} + h\frac{w^h(x_i)\varphi_i^h(x_i) + w^h(x_{i+1})\varphi_i^h(x_{i+1})}{2} \\
=\ & hw^h(x_i) = hw_i.
\end{aligned}$$

This formula, which is exact for constant vectors $\mathbf{w}^h$, is the algebraic form of the right-hand side of the first problem $A^h\mathbf{u}^h = h\mathbf{w}^h$. With the definition of $\mathbf{z}^h$, one obtains

$$\mathbf{z}^h = \left(A^h\right)^{-1}\mathbf{w}^h = h^{-1}\mathbf{u}^h = h^{-1}\left(P^h\right)^{-1}u^h.$$

Using the commutation $P^{2h}I_h^{2h}\mathbf{w}^h = \mathcal{I}_h^{2h}P^h\mathbf{w}^h = \mathcal{I}_h^{2h}w^h$, see (5.14), the finite element function $z^{2h} = P^{2h}\mathbf{z}^{2h}$ is the solution of the coarse grid problem

$$\left(\left(z^{2h}\right)', \left(\varphi^{2h}\right)'\right) = \left(\mathcal{I}_h^{2h}w^h, \varphi^{2h}\right) = \left(w^h, \mathcal{I}_{2h}^h\varphi^{2h}\right), \quad \forall\, \varphi^{2h} \in V^{2h},$$

where the duality of prolongation and restriction was used, see Example 4.16. The canonical prolongation of φ^{2h} is the embedding, see Example 4.9, hence $\mathcal{I}_{2h}^h\varphi^{2h} = \varphi^{2h}$ and one obtains

$$\left(\left(z^{2h}\right)', \left(\varphi^{2h}\right)'\right) = \left(w^h, \varphi^{2h}\right), \quad \forall\, \varphi^{2h} \in V^{2h}.$$

With the same quadrature rule as on the fine grid, it follows that

$$\begin{aligned}
z^{2h} &= \left(P^{2h}\right)\mathbf{z}^{2h} = (2h)^{-1}u^{2h} \quad \Longrightarrow \\
I_{2h}^h\mathbf{z}^{2h} &= (2h)^{-1}I_{2h}^h\left(P^{2h}\right)^{-1}u^{2h} = (2h)^{-1}\left(P^h\right)^{-1}\mathcal{I}_{2h}^h u^{2h},
\end{aligned}$$

where (5.13) was used. Since $\mathcal{I}_{2h}^h$ is the identity, one gets that (5.16) can be written in the form

$$\left\|\mathbf{z}^h - I_{2h}^h\mathbf{z}^{2h}\right\|_2 = h^{-1}\left\|\left(P^h\right)^{-1}\left(u^h - u^{2h}\right)\right\|_2. \tag{5.17}$$

Since the norm equivalence (5.12) should be applied, the error $\left\|u^h - u^{2h}\right\|_{L^2((0,1))}$ will be estimated. Let $u \in H_0^1((0,1))$ be the solution of the variational problem

$$\left(u', \varphi'\right) = \left(w^h, \varphi\right) \quad \forall\, \varphi \in H_0^1((0,1)).$$

This problem is by assumption 2-regular, i.e., it is $u \in H^2((0,1))$ and it holds $\|u\|_{H^2((0,1))} \le c\left\|w^h\right\|_{L^2((0,1))}$. Then, it is known from Numerical Mathematics 3 that the error estimates

$$\left\|u - u^h\right\|_{L^2((0,1))} \le Ch^2\left\|w^h\right\|_{L^2((0,1))}, \quad \left\|u - u^{2h}\right\|_{L^2((0,1))} \le C\,(2h)^2\left\|w^h\right\|_{L^2((0,1))}$$

hold. Thus, one obtains with the triangle inequality

$$\left\|u^h - u^{2h}\right\|_{L^2((0,1))} \le \left\|u - u^h\right\|_{L^2((0,1))} + \left\|u - u^{2h}\right\|_{L^2((0,1))} \le Ch^2\left\|w^h\right\|_{L^2((0,1))}. \tag{5.18}$$

Finally, inserting (5.16), (5.17), (5.18) into (5.15) and using the norm equivalence

(5.12) gives

$$
\begin{aligned}
& \sup_{\mathbf{w}^h \in \mathbb{R}^{n_h}} \frac{\left\| \left(\left(A^h\right)^{-1} - I_{2h}^h \left(A^{2h}\right)^{-1} I_h^{2h} \right) \mathbf{w}^h \right\|_2}{\left\| \mathbf{w}^h \right\|_2} \\
= & \sup_{\mathbf{w}^h \in \mathbb{R}^{n_h}} \frac{\left\| \mathbf{z}^h - I_{2h}^h \mathbf{z}^{2h} \right\|_2}{\left\| \mathbf{w}^h \right\|_2} \\
= & \; Ch^{-1} \sup_{\mathbf{w}^h \in \mathbb{R}^{n_h}} \frac{\left\| \left(P^h\right)^{-1} \left(u^h - u^{2h}\right) \right\|_2}{\left\| \mathbf{w}^h \right\|_2} \\
\le & \; Ch^{-3/2} \sup_{\mathbf{w}^h \in \mathbb{R}^{n_h}} \frac{\left\| P^h \left(P^h\right)^{-1} \left(u^h - u^{2h}\right) \right\|_{L^2((0,1))}}{\left\| \mathbf{w}^h \right\|_2} \\
= & \; Ch^{-3/2} \sup_{\mathbf{w}^h \in \mathbb{R}^{n_h}} \frac{\left\| u^h - u^{2h} \right\|_{L^2((0,1))}}{\left\| \mathbf{w}^h \right\|_2} \\
\le & \; Ch^{1/2} \sup_{\mathbf{w}^h \in \mathbb{R}^{n_h}} \frac{\left\| w^h \right\|_{L^2((0,1))}}{\left\| \mathbf{w}^h \right\|_2} \\
\le & \; Ch \sup_{\mathbf{w}^h \in \mathbb{R}^{n_h}} \frac{\left\| \mathbf{w}^h \right\|_2}{\left\| \mathbf{w}^h \right\|_2} \\
= & \; Ch.
\end{aligned}
$$

■

Remark 5.21 *On the approximation property.*

- In the one-dimensional model problem, the assumption on the regularity are satisfied if the right-hand side $f(x)$ is sufficiently smooth. In multiple dimensions, one needs in addition conditions on the domain.
- The proof is literally the same in higher dimensions.

□

5.5 Summary

Remark 5.22 *Summary.* This chapter considered the convergence of the two-level method or coarse grid correction scheme. First, an appropriate coarse grid operator was defined. It was shown that the spectral radius of the iteration matrix of the two-level method can be bounded with a constant lower than 1, independently of the mesh width h, if

- the smoothing property holds and sufficiently many smoothing steps are performed,
- and if the approximation property holds.

Considering the model problem (2.1), the smoothing property for the damped Jacobi problem with $\omega \in (0, 1/2]$ was proved as well as the approximation property. □