Lösungen zum 35. Aufgabenblatt für MfI 3

1. Aufgabe :

   (a)

$$
\begin{aligned}
f(x,y) &= xy\\
g(x,y) &= x^2+y^2-1\\
&= 0\\
F(x,y,\lambda) &= xy-\lambda x^2-\lambda y^2+\lambda\\
\nabla F &= \mathbf{0}\\
\begin{pmatrix} y-2x\lambda\\ x-2y\lambda\\ -x^2-y^2+1\end{pmatrix} &= \mathbf{0}\\
\text{aus (1) :}&\\
\lambda &= \frac{y}{2x}\\
\text{aus (2) :}&\\
\lambda &= \frac{x}{2y}\\
\text{(1) und (2) :}&\\
y^2 &= x^2\\
\text{in (3) :}&\\
-x^2-x^2+1 &= 0\\
x^2 &= \frac{1}{2}\\
|x| &= \frac{1}{\sqrt{2}}\\
|y| &= \frac{1}{\sqrt{2}}\\
\frac{\partial g}{\partial x} &= 2x\\
\frac{\partial g}{\partial y} &= 2y\\
\frac{\partial F}{\partial x} &= y-2\lambda x\\
\frac{\partial^2 F}{\partial x^2} &= -2\lambda\\
\frac{\partial^2 F}{\partial x\partial y} &= 1\\
\frac{\partial F}{\partial y} &= x-2\lambda y
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F}{\partial y^2} &= -2\lambda \\
\frac{\partial^2 F}{\partial y \partial x} &= 1 \\
\det(H) &= \begin{vmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 1 \\ 2y & 1 & -2\lambda \end{vmatrix} \\
&= 8xy + 8x^2\lambda + 8y^2\lambda \\
&= 8(xy + \lambda)
\end{aligned}
$$

(x/y)	λ	$\det(H)$	Extremum
$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$\frac{1}{2}$	8	Maximum
$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$-\frac{1}{2}$	-8	Minimum
$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$-\frac{1}{2}$	-8	Minimum
$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\frac{1}{2}$	8	Maximum

(b)

$$
\begin{aligned}
f(x,y) &= x + y \\
g(x,y) &= x^{-2} + y^{-2} - a^2 \\
&= 0 \\
F(x,y,\lambda) &= x + y - \lambda x^{-2} - \lambda y^{-2} + \lambda a^2 \\
\nabla F &= \mathbf{0} \\
\begin{pmatrix} 1 + 2\lambda x^{-3} \\ 1 + 2\lambda y^{-3} \\ -x^{-2} - y^{-2} + a^2 \end{pmatrix} &= \mathbf{0} \\
\text{aus (1) :} & \\
\lambda &= -\frac{x^3}{2} \\
\text{aus (2) :} & \\
\lambda &= -\frac{y^3}{2} \\
\text{(1) und (2) :} & \\
y^3 &= x^3 \\
y &= x \\
\text{in (3) :} & \\
-x^{-2} - x^{-2} + a^2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
x^2 &= \frac{2}{a^2} \\
|x| &= \frac{\sqrt{2}}{|a|} \\
|y| &= \frac{\sqrt{2}}{|a|} \\
\frac{\partial g}{\partial x} &= -2x^{-3} \\
\frac{\partial g}{\partial y} &= -2y^{-3} \\
\frac{\partial^2 F}{\partial x^2} &= -6\lambda x^{-4} \\
\frac{\partial^2 F}{\partial x \partial y} &= 0 \\
\frac{\partial^2 F}{\partial y^2} &= -6\lambda y^{-4} \\
\frac{\partial^2 F}{\partial y \partial x} &= 0 \\
\det(H) &= \begin{vmatrix} 0 & -2x^{-3} & -2y^{-3} \\ -2x^{-3} & -6\lambda x^{-4} & 0 \\ -2y^{-3} & 0 & -6\lambda x^{-4} \end{vmatrix} \\
&= 24\lambda \frac{x^2+y^2}{x^6 y^6} \\
&= \frac{48\lambda}{x^{10}}
\end{aligned}
$$

(x/y)	λ	$\det(H)$	Extremum
$(-\frac{\sqrt{2}}{\lvert a\rvert}, -\frac{\sqrt{2}}{\lvert a\rvert})$	$\frac{\sqrt{2}}{\lvert a\rvert^3}$	$= \frac{3}{\sqrt{2}} \lvert a\rvert^7 > 0$	Maximum
$(\ \frac{\sqrt{2}}{\lvert a\rvert},\ \frac{\sqrt{2}}{\lvert a\rvert})$	$-\frac{\sqrt{2}}{\lvert a\rvert^3}$	$= -\frac{3}{\sqrt{2}} \lvert a\rvert^7 < 0$	Minimum

2. Aufgabe :

$$
\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, \mathrm{d}y\, \mathrm{d}x
$$

$$
\begin{aligned}
x &= r\cos(\varphi) \\
y &= r\sin(\varphi) \\
r &= \sqrt{x^2+y^2} \\
\mathrm{d}y\, \mathrm{d}x &= r\, \mathrm{d}r\, \mathrm{d}\varphi
\end{aligned}
$$

$$y = \sqrt{a^2 - x^2}$$

Viertelkreis im 1. Quadranten :

$$r \in [0, a]$$

$$\varphi \in \left[0, \frac{\pi}{2}\right]$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \int_0^a r^2 \,\mathrm{d}r \,\mathrm{d}\varphi
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{3} r^3\right]_0^a \mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{3} a^3 \,\mathrm{d}\varphi \\
&= \left[\frac{1}{3} a^3 \varphi\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{6} a^3
\end{aligned}$$

3. Aufgabe :

(a)

$$\begin{aligned}
|\Omega| &= 2^5 \\
&= 32
\end{aligned}$$

(b)

$$\begin{aligned}
|\Omega| &= 2^5 + 2^4 + 2^3 + 2^2 + 2^1 \\
&= 62
\end{aligned}$$

(c)

$$\begin{aligned}
P(A) &= \frac{|A|}{|\Omega|} \\
&= \frac{32}{62} \\
&= 0.51613
\end{aligned}$$