On the Choice of Parameters in Stabilization Methods for Convection–Diffusion Equations

V. John and P. Knobloch

Abstract A popular finite element approach for the numerical solution of convection–diffusion equations is the streamline upwind/Petrov–Galerkin (SUPG) method. Unfortunately, in the convection–dominated regime, the SUPG solution often contains spurious oscillations along sharp layers. A possible remedy is to introduce an additional artificial diffusion term in the SUPG discretization. We call such approaches spurious oscillations at layers diminishing (SOLD) methods. The properties of the SOLD methods are significantly influenced by the choice of the respective stabilization parameter which determines the amount of the artificial diffusion. The aim of this paper is to discuss various definitions of these stabilization parameters.

1 Introduction

This paper is devoted to the numerical solution of the steady scalar convection–diffusion equation

$$-\varepsilon\Delta u+\mathbf{b}\cdot\nabla u=f \quad \text{in } \Omega, \qquad\qquad u=u_b \quad \text{on } \partial\Omega. \tag{1}$$

We assume that Ω is a bounded domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$, $\varepsilon>0$ is the constant diffusivity, $\mathbf{b}=(b_1,b_2)$, f and u_b are given functions and u is an unknown scalar quantity, e.g., temperature or concentration.

It is well known that the standard Galerkin finite element discretization of (1) loses its stability if convection strongly dominates diffusion. Therefore, various

Volker John
Universität des Saarlandes, Fachbereich 6.1 – Mathematik, Postfach 15 11 50, 66041 Saarbrücken, Germany, e-mail: john@math.uni-sb.de

Petr Knobloch
Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: knobloch@karlin.mff.cuni.cz

stabilized finite element methods have been developed for the numerical solution of (1). A widely used approach is the streamline upwind/Petrov–Galerkin (SUPG) method proposed in [1]. Denoting by W_h a finite element space approximating the Sobolev space $H^1(\Omega)$, by $u_{bh} \in W_h$ a function whose trace approximates u_b and setting $V_h = W_h \cap H_0^1(\Omega)$, the SUPG method reads:

Find $u_h \in W_h$ such that $u_h - u_{bh} \in V_h$ and

$$\varepsilon\,(\nabla u_h, \nabla v_h) + (\mathbf{b}\cdot\nabla u_h, v_h) + (R_h(u_h), \tau\,\mathbf{b}\cdot\nabla v_h) = (f, v_h) \qquad \forall\, v_h \in V_h\,. \tag{2}$$

Here $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$, $R_h(u) = -\varepsilon\,\Delta\,u + \mathbf{b}\cdot\nabla\,u - f$ is the residual (defined elementwise) and τ is a nonnegative stabilization parameter, see Section 2.

Unfortunately, the SUPG method is not monotone and hence a discrete solution satisfying (2) usually contains spurious oscillations along sharp layers. A possible remedy is to add a suitable artificial diffusion term to the left–hand side of the SUPG discretization (2). We call such approaches spurious oscillations at layers diminishing (SOLD) methods, see the review paper [8]. There are three basic types of SOLD terms and they add either isotropic artificial diffusion or crosswind artificial diffusion to the SUPG method (2) or they are based on so–called edge stabilizations. These three types of SOLD terms are respectively defined by

$$(\widetilde{\varepsilon}\,\nabla u_h, \nabla v_h)\,, \tag{3}$$

$$(\widetilde{\varepsilon}\,\mathbf{b}^\perp\cdot\nabla u_h, \mathbf{b}^\perp\cdot\nabla v_h) \qquad \text{with} \qquad \mathbf{b}^\perp = \frac{(-b_2, b_1)}{|\mathbf{b}|}\,, \tag{4}$$

$$\sum_{K\in\mathscr{T}_h} \int_{\partial K} \widetilde{\varepsilon}|_K\,\mathrm{sign}\left(\frac{\partial u_h}{\partial \mathbf{t}_{\partial K}}\right) \frac{\partial v_h}{\partial \mathbf{t}_{\partial K}}\,\mathrm{d}\sigma\,, \tag{5}$$

where $\mathscr{T}_h = \{K\}$ is a triangulation of Ω satisfying the usual compatibility assumptions and $\mathbf{t}_{\partial K}$ is a tangent vector to the boundary ∂K of K. The parameter $\widetilde{\varepsilon}$, which determines the amount of the artificial diffusion added to the SUPG method, is nonnegative and usually depends on u_h. Thus, the resulting methods are nonlinear although the original problem (1) is linear.

Comparative numerical studies of a large number of SOLD methods can be found in, e.g., [6, 7, 8, 9]. It was observed that there are large differences between the SOLD methods. In some cases, many SOLD methods were able to significantly improve the SUPG solution and to provide a discrete solution with negligible spurious oscillations and without an excessive smearing of layers. However, it was not possible to identify a method which could be preferred in all the test cases. The aim of the present paper is to discuss the definitions of the parameters $\widetilde{\varepsilon}$ for those SOLD methods which achieved high rankings in the mentioned numerical studies.

The paper is organized in the following way. In the next section, we present the definitions of the parameter $\widetilde{\varepsilon}$ for several promising SOLD methods. The main part of the paper is Section 3 where we discuss the optimality of these definitions of $\widetilde{\varepsilon}$ for three academic tests problems. We finish the paper by our conclusions in Section 4.

2 Definitions of the Stabilization Parameters

In this section, we present various choices of the parameters in the stabilization terms in (2)–(5). Generally, the parameters should depend on the approximation properties of the finite element space W_h. For simplicity, throughout this paper, we restrict ourselves to spaces

$$W_h = \{v \in H^1(\Omega);\ v|_K \in R(K)\ \ \forall\, K \in \mathscr{T}_h\},$$

where $R(K) = P_1(K)$ if K is a triangle and $R(K) = Q_1(K)$ if K is a rectangle. We assume that the triangulation $\mathscr{T}_h$ consists either of triangles or of rectangles.

The choice of the SUPG parameter τ in (2) may dramatically influence the accuracy of the discrete solution and therefore it has been a subject of an extensive research over the last three decades, see, e.g., the review in [8]. Unfortunately, a general optimal definition of τ is still not known. In our computations, we define τ, on any element $K \in \mathscr{T}_h$, by the formula

$$\tau|_K = \frac{h_K}{2\,|\mathbf{b}|}\left(\coth Pe_K - \frac{1}{Pe_K}\right) \qquad \text{with} \qquad Pe_K = \frac{|\mathbf{b}|\,h_K}{2\,\varepsilon}, \tag{6}$$

where h_K is the element diameter in the direction of the convection vector $\mathbf{b}$, $|\mathbf{b}|$ is the Euclidean norm of $\mathbf{b}$ and Pe_K is the local Péclet number. We refer to [8] for various justifications of this formula. Note that, generally, the parameters h_K, Pe_K and $\tau|_K$ are functions of the points $\mathbf{x} \in K$.

According to the criteria and tests in [6, 7, 8], one of the best choices of $\widetilde{\varepsilon}$ in (3) is to set

$$\widetilde{\varepsilon} = \max\left\{0, \frac{\tau\,|\mathbf{b}|\,|R_h(u_h)|}{|\nabla u_h|} - \tau\,\frac{|R_h(u_h)|^2}{|\nabla u_h|^2}\right\}, \tag{7}$$

as proposed in [4]. Here and in the following, we always assume that $\widetilde{\varepsilon} = 0$ if the denominator of a formula defining $\widetilde{\varepsilon}$ vanishes. In case (4), we suggested in [8] to set, on any $K \in \mathscr{T}_h$,

$$\widetilde{\varepsilon}|_K = \max\left\{0, \eta\,\frac{\operatorname{diam}(K)\,|R_h(u_h)|}{2\,|\nabla u_h|} - \varepsilon\right\}, \tag{8}$$

where $\operatorname{diam}(K)$ is the diameter of K and η is a suitable constant, for which the value $\eta \approx 0.7$ was recommended in [5]. The relation (8) is a slight modification of a formula proposed in [5]. Another promising variant of (4) tested in [6, 7, 8, 9] is defined by

$$\widetilde{\varepsilon} = \frac{\tau\,|\mathbf{b}|\,|R_h(u_h)|}{|\nabla u_h|}\,\frac{|\mathbf{b}|\,|\nabla u_h|}{|\mathbf{b}|\,|\nabla u_h| + |R_h(u_h)|}\,. \tag{9}$$

This choice of $\widetilde{\varepsilon}$ was proposed in [8] as a simplification of a formula from [2]. For the edge stabilization term (5), acceptable results were computed with

$$\widetilde{\varepsilon}|_K = C\,|K|\,|(R_h(u_h)|_K)| \qquad \forall\, K \in \mathscr{T}_h, \tag{10}$$

where $|K|$ is the area of K and C is a nonnegative constant. Let us mention that, to achieve convergence of the nonlinear iterative process, the sign operator in (5) is regularized by replacing it by the hyperbolic tangent as recommended in [3].

If convection strongly dominates diffusion in Ω and hence the local Péclet numbers Pe_K are very large, the parameter τ defined in (6) satisfies $\tau|_K = h_K/(2\,|\mathbf{b}|)$ for any $K \in \mathscr{T}_h$. Then we have in (7) and (9)

$$\frac{\tau\,|\mathbf{b}|\,|R_h(u_h)|}{|\nabla u_h|} \approx \frac{h_K\,|R_h(u_h)|}{2\,|\nabla u_h|}\,.$$

Hence, in the definitions of $\widetilde{\varepsilon}$ in (7)–(9), an important role is played by a term of the type $h\,|R_h(u_h)|/|\nabla u_h|$. Moreover, in view of (10), the edge stabilization term (5) can be written in the form

$$\sum_{K\in\mathscr{T}_h} |K| \int_{\partial K} C\, \frac{\big|R_h(u_h)|_K\big|}{\left|\frac{\partial u_h}{\partial \mathbf{t}_{\partial K}}\right|}\, \frac{\partial u_h}{\partial \mathbf{t}_{\partial K}}\, \frac{\partial v_h}{\partial \mathbf{t}_{\partial K}}\, \mathrm{d}\sigma\,,$$

which is an expression of a similar structure as the SOLD terms (3) and (4) with $\widetilde{\varepsilon}$ defined by (7)–(9). Thus, we observe the interesting fact that all three types of SOLD terms with the above described definitions of the parameter $\widetilde{\varepsilon}$ are similar although the formulas for $\widetilde{\varepsilon}$ were derived using completely different arguments.

3 Optimal Choice of Stabilization Parameters for Model Problems

In this section, we shall discuss the optimality of the parameters $\widetilde{\varepsilon}$ introduced in the previous section for three model problems whose solutions possess characteristic features of solutions of (1). We shall confine ourselves to the two types of triangulations depicted in Fig. 1. To characterize these triangulations, we shall use the notion '$N_1 \times N_2$ mesh' where N_1 and N_2 are the numbers of vertices in the horizontal and vertical directions, respectively. The corresponding mesh widths will be denoted by h_1 and h_2, i.e., $h_1 = 1/(N_1-1)$ and $h_2 = 1/(N_2-1)$.

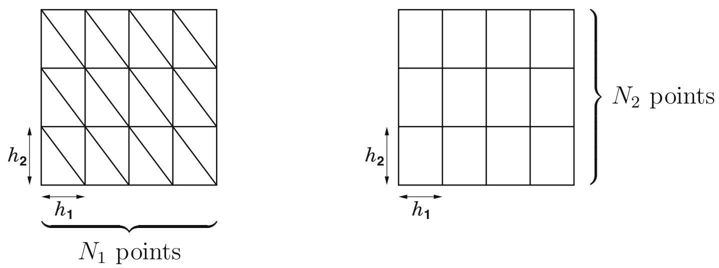

Fig. 1 Triangulations used in Section 3.

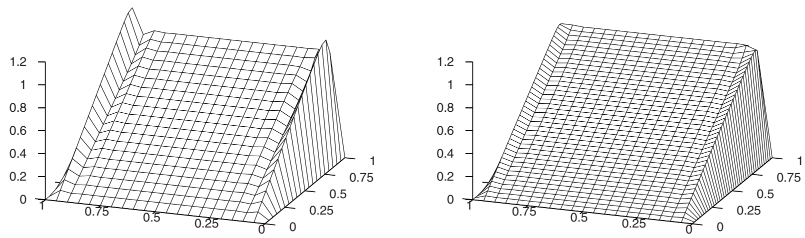

Fig. 2 Example 1, P_1 finite element: SUPG solution on a 21×21 mesh (left) and SOLD solution defined using (4) and (9) on a 41×21 mesh (right).

The analysis below will include the consideration of moderately anisotropic grids. Using such grids might not be reasonable for the considered examples since these grids are not adapted to the layers of the solution. However, convection–diffusion equations are often just a part of a coupled system of equations. For such problems, an adaptation of the grid is performed rather with respect to other equations in the system, for instance with respect to the Navier–Stokes equations in fluid flow applications. Nevertheless, the SOLD methods still should provide satisfactory results.

Example 1 (*Solution with parabolic and exponential boundary layers*). We consider the convection–diffusion equation (1) with $\Omega = (0,1)^2$, $\varepsilon = 10^{-8}$, $\mathbf{b} = (1,0)^T$, $f = 1$, and $u_b = 0$. The solution $u(x,y)$ of this problem possesses an exponential boundary layer at $x = 1$ and parabolic (characteristic) boundary layers at $y = 0$ and $y = 1$. Outside the layers, the solution $u(x,y)$ is very close to x.

For this special example, the stabilization parameter τ given in (6) leads to a nodally exact SUPG solution outside the parabolic layers. However, there are strong oscillations at the parabolic layers, see Fig. 2.

Let us consider a SOLD discretization of (1) with the isotropic SOLD term (3) or the crosswind SOLD term (4) and with $\widetilde{\varepsilon}$ defined by (8). In the triangular case, it is easy to show that η equal to

$$\eta_{opt} = \frac{2\,h_2}{3\sqrt{h_1^2 + h_2^2}} \tag{11}$$

is optimal for $\varepsilon \to 0$ with respect to the parabolic layers. Indeed, for $\eta = \eta_{opt}$ the discrete solution is nodally exact outside the exponential boundary layer whereas, for $\eta > \eta_{opt}$, the parabolic boundary layers are smeared and, for $\eta < \eta_{opt}$, spurious oscillations along the parabolic boundary layers appear. Moreover, for the nodally exact solution with $\varepsilon \to 0$, the SUPG term $(R_h(u_h), \tau\,\mathbf{b}\cdot\nabla v_h)$ vanishes outside the exponential boundary layer which shows that the optimal value of η does not depend on the definition of the SUPG stabilization parameter τ. In the quadrilateral case, it is not possible to derive a simple formula for η_{opt} but numerical results suggest that the optimal values of η do not differ much from (11).

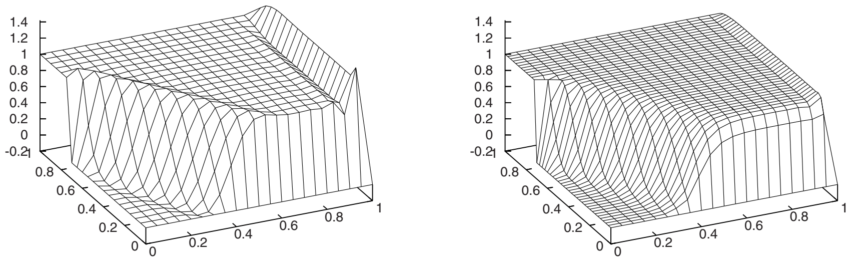

Fig. 3 Example 2, Q_1 finite element: SUPG solution on a 21×21 mesh (left) and SOLD solution defined using (4) and (8) with $\eta = 0.7$ on a 21×41 mesh (right).

Since $\eta_{opt} < 2/3$, spurious oscillations should not appear for the value $\eta \approx 0.7$ recommended in [5]. On the other hand, if we consider $\widetilde{\varepsilon}$ defined by (7) or (9), a comparison of these relations with the formula (8) reveals that spurious oscillations in the discrete solution should be expected for $h_K = h_1 < \eta_{opt}\,\mathrm{diam}(K)$, i.e., for $h_1/h_2 < 2/3$, as it is demonstrated in Fig. 2.

For the edge stabilization term (5) and both the P_1 and Q_1 finite elements, it is easy to derive that the optimal value of C in (10) is $1/6$. However, in practice, the discrete solution slightly differs from the nodally exact solution at the parabolic boundary layers due to the regularization of the sign operator. Moreover, in contrast with the above SOLD methods, the discrete solution is significantly smeared along the exponential boundary layer. A sharp approximation of this layer requires to set $C = 0$ in this region.

The above considerations show that satisfactory numerical results can be obtained generally only using the isotropic or crosswind SOLD term with $\widetilde{\varepsilon}$ defined by (8) or using the edge stabilization (5) with $\widetilde{\varepsilon}$ defined by (10).

Example 2 (*Solution with interior layer and exponential boundary layers*). We consider the convection–diffusion equation (1) with $\Omega = (0,1)^2$, $\varepsilon = 10^{-8}$, $\mathbf{b} = (\cos(-\pi/3), \sin(-\pi/3))^T$, $f = 0$, and

$$u_b(x,y) = \begin{cases} 0 & \text{for } x = 1 \text{ or } y \le 0.7, \\ 1 & \text{else.} \end{cases}$$

The solution possesses an interior (characteristic) layer in the direction of the convection starting at $(0, 0.7)$. On the boundary $x = 1$ and on the right part of the boundary $y = 0$, exponential layers are developed.

We shall assume that $h_1 b_2 + h_2 b_1 < 0$. Then, for both the P_1 and Q_1 finite elements, the SUPG solution of Example 2 contains oscillations along the interior layer and along the boundary layer at $x = 1$. However, there are no oscillations along the boundary layer at $y = 0$ and this layer is not smeared, see Fig. 3.

For a SOLD discretization of (1) with the isotropic SOLD term (3) or the crosswind SOLD term (4) and with $\widetilde{\varepsilon}$ defined by (8), it is easy to derive optimal values of η such that, for $\varepsilon \to 0$, the discrete solution is nodally exact away from the interior

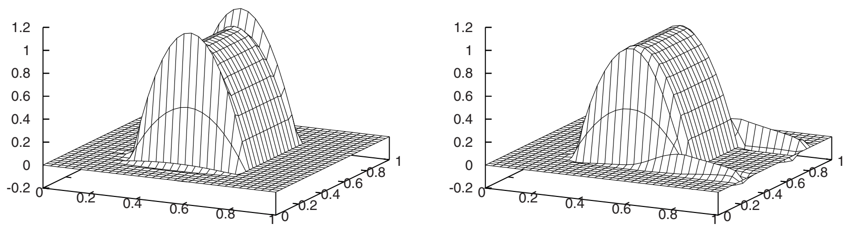

Fig. 4 Example 3, P_1 finite element, 33×33 mesh: SUPG solution (left) and SOLD solution defined using (3) and (7) (right).

layer. First, it is clear that, along the boundary layer at $y = 0$, the optimal choice of η is $\eta = 0$. For the boundary layer at $x = 1$, the optimal values are

$$\eta_{opt}^{isotropic} = \frac{h_1 b_2 + h_2 b_1}{\sqrt{h_1^2 + h_2^2}\, b_2}, \qquad \eta_{opt}^{crosswind} = \frac{(h_1 b_2 + h_2 b_1)|\mathbf{b}|^2}{\sqrt{h_1^2 + h_2^2}\, b_2^3}.$$

These formulas hold for both the P_1 and the Q_1 finite elements. One can see that the optimal choice of η depends not only on the aspect ratio of the elements of the triangulation but also on the direction of the convection vector $\mathbf{b}$. For $\mathbf{b}$ of Example 2 and for $h_1 = 2h_2$, we obtain $\eta_{opt}^{crosswind} \approx 0.85$ and hence we have to expect spurious oscillations for the recommended value $\eta \approx 0.7$. This is really the case as Fig. 3 shows. For $\widetilde{\varepsilon}$ defined by (7) and (9) the oscillations at $x = 1$ are even much larger and, moreover, there are nonnegligible oscillations at the beginning of the interior layer. For the edge stabilization term (5) and both the P_1 and Q_1 finite elements, the optimal value of C in (10) is $C_{opt} = (h_1 b_2 + h_2 b_1)/(4 h_1 b_2)$ along $x = 1$.

The above discussion supports our conclusion to Example 1 and shows that it is in general not sufficient to consider constant values of η and C.

Example 3 (*Solution with two interior layers*). We consider the convection–diffusion equation (1) with $\Omega = (0,1)^2$, $\varepsilon = 10^{-8}$, $\mathbf{b} = (1,0)^T$, $u_b = 0$, and

$$f(x,y) = \begin{cases} 16(1-2x) & \text{for } (x,y) \in [0.25, 0.75]^2, \\ 0 & \text{else.} \end{cases}$$

The solution $u(x,y)$ possesses two interior (characteristic) layers at $(0.25, 0.75) \times \{0.25\}$ and $(0.25, 0.75) \times \{0.75\}$. In $(0.25, 0.75)^2$, it is very close to the quadratic function $(4x-1)(3-4x)$.

As expected, the SUPG solution of Example 3 possesses spurious oscillations along the interior layers, see Fig. 4. Applying any of the SOLD methods discussed above, the spurious oscillations present in the SUPG solution are significantly suppressed, however, the solution is wrong in the region $(0.75, 1) \times (0, 1)$, see Fig. 4. This behaviour is the same for both the P_1 and Q_1 finite elements. Thus, Example 3 represents a problem for which all the SOLD methods described in Section 2 fail.

4 Conclusions

This paper was devoted to the numerical solution of convection–diffusion equations using SOLD methods. It was demonstrated that SOLD methods without user–chosen parameters are in general not able to remove the spurious oscillations of the solution obtained with the SUPG discretization. For the two studied methods involving a parameter, values of the parameter could be given in two examples such that the spurious oscillations were almost removed. The parameter has to be generally non–constant and depends on the mesh and the data of the problem. Therefore, for more complicated problems, it is not clear how suitable parameters can be found. Moreover, an example was presented for which none of the investigated methods provided a qualitatively correct discrete solution. Consequently, we have to conclude that it is in general completely open how to obtain oscillation–free solutions using the considered classes of methods.

Acknowledgements The work of P. Knobloch is a part of the research project MSM 0021620839 financed by MSMT and it was partly supported by the Grant Agency of the Czech Republic under the grant No. 201/08/0012.

References

1. Brooks, A., Hughes, T.: Streamline upwind/Petrov–Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier–Stokes equations. Comput. Methods Appl. Mech. Engrg. **32**, 199–259 (1982)
2. Burman, E., Ern, A.: Nonlinear diffusion and discrete maximum principle for stabilized Galerkin approximations of the convection–diffusion–reaction equation. Comput. Methods Appl. Mech. Engrg. **191**, 3833–3855 (2002)
3. Burman, E., Hansbo, P.: Edge stabilization for Galerkin approximations of convection–diffusion–reaction problems. Comput. Methods Appl. Mech. Engrg. **193**, 1437–1453 (2004)
4. do Carmo, E., Galeão, A.: Feedback Petrov–Galerkin methods for convection–dominated problems. Comput. Methods Appl. Mech. Engrg. **88**, 1–16 (1991)
5. Codina, R.: A discontinuity–capturing crosswind–dissipation for the finite element solution of the convection–diffusion equation. Comput. Methods Appl. Mech. Engrg. **110**, 325–342 (1993)
6. John, V., Knobloch, P.: A computational comparison of methods diminishing spurious oscillations in finite element solutions of convection–diffusion equations. In: J. Chleboun, K. Segeth, T. Vejchodský (eds.) Proceedings of the International Conference Programs and Algorithms of Numerical Mathematics 13, pp. 122–136. Academy of Science of the Czech Republic (2006)
7. John, V., Knobloch, P.: On discontinuity–capturing methods for convection–diffusion equations. In: A. Bermúdez de Castro, D. Gómez, P. Quintela, P. Salgado (eds.) Numerical Mathematics and Advanced Applications, Proceedings of ENUMATH 2005, pp. 336–344. Springer–Verlag, Berlin (2006)
8. John, V., Knobloch, P.: On spurious oscillations at layers diminishing (SOLD) methods for convection–diffusion equations: Part I – A review. Comput. Methods Appl. Mech. Engrg. **196**, 2197–2215 (2007)
9. John, V., Knobloch, P.: On the performance of SOLD methods for convection–diffusion problems with interior layers. Int. J. Computing Science and Mathematics **1**, 245–258 (2007)