Applied Mathematics Letters 27 (2014) 43–48

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A study of isogeometric analysis for scalar convection–diffusion equations

Volker John [a,b,*], Liesel Schumacher [c]

[a] Weierstrass Institute for Applied Analysis and Stochastics, Leibniz Institute in Forschungsverbund Berlin e. V. (WIAS), Mohrenstr. 39, 10117 Berlin, Germany

[b] Free University of Berlin, Department of Mathematics and Computer Science, Arnimallee 6, 14195 Berlin, Germany

[c] Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany

ARTICLE INFO

Article history:
Received 3 July 2013
Received in revised form 15 August 2013
Accepted 16 August 2013

Keywords:
Isogeometric analysis
Streamline upwind Petrov–Galerkin stabilization
Hemker problem
Over- and undershoots
Sharpness of layers

ABSTRACT

Isogeometric analysis (IGA), in combination with the streamline upwind Petrov–Galerkin (SUPG) stabilization, is studied for the discretization of steady-state convection–diffusion equations. Numerical results obtained for the Hemker problem are compared with results computed with the SUPG finite element method of the same order. Using an appropriate parameterization for IGA, the computed solutions are much more accurate than those obtained with the finite element method, both in terms of the size of spurious oscillations and of the sharpness of layers.

1. Introduction

Scalar convection–diffusion equations model the transport of scalar quantities, like temperature, concentration, or salinity. In applications, the convective transport usually dominates the diffusive transport by several orders of magnitude. It is well known that in this situation so-called stabilized discretizations have to be used. There are many proposals of such stabilizations and an extensive numerical analysis concerning errors in Sobolev spaces exists; e.g., see the monograph [1]. However, in applications such errors are often of minor interest, but properties like (the size of) spurious oscillations or the smearing of layers are important. For example, an application is presented in [2] where spurious oscillations of a solution computed with a very popular stabilized discretization lead to blow-ups of the simulations. Several numerical studies were performed in recent years to investigate properties of stabilized discretizations, for the steady-state equation [3–5] as well as for the time-dependent equation [6,7]. It turned out that none of the proposed methods behaved satisfactory in all aspects and there is the urgent need to study further approaches and to improve available methods.

This note considers steady-state convection–diffusion equations. In the recent study [5], several stabilized discretizations were assessed at one of the currently most challenging benchmark problems, the so-called Hemker problem [8]. These discretizations included a finite volume scheme, the streamline-upwind Petrov–Galerkin (SUPG) finite element method (FEM), a spurious oscillations at layers diminishing (SOLD) FEM, a continuous interior penalty (CIP) FEM, a discontinuous Galerkin

* Corresponding author at: Weierstrass Institute for Applied Analysis and Stochastics, Leibniz Institute in Forschungsverbund Berlin e. V. (WIAS), Mohrenstr. 39, 10117 Berlin, Germany. Tel.: +49 30 20372 561.

E-mail addresses: volker.john@wias-berlin.de (V. John), schumach@mathematik.hu-berlin.de (L. Schumacher).

0893-9659/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.aml.2013.08.004

(DG) FEM, and a total variation diminishing (TVD) FEM. The evaluation of the results showed that all methods possess deficiencies, either with respect to the spurious oscillations, or the sharpness of layers, or efficiency.

Isogeometric analysis (IGA) is a rather new approach for discretizing partial differential equations [9]. The basic idea consists in using as basis functions of the discrete space the same functions that are used for the parameterization of (complex) domains. These functions are non-uniform rational B-splines (NURBS). Since the pioneering paper [9], many very promising simulations of applications with IGA, e.g., [10,11], and also contributions to the numerical analysis [12] have been published. However, there are only few studies of the potential of IGA for the solution of convection–diffusion equations [9, 12] and there is no comparison to other discretizations available.

This note supplements the assessment of stabilized discretizations in [5] with a study of a stabilized version of IGA. In particular, the results computed with the IGA for the Hemker problem will be compared with the results obtained with the SUPG FEM of the same order. The latter method was regarded in [5] to be the currently best performing discretization in the case that spurious oscillations can be tolerated in the numerical solutions.

2. IGA for the Hemker problem

2.1. The Hemker problem

The Hemker problem, introduced in [8], is considered to be currently one of the most challenging benchmark problems for steady-state convection–diffusion equations. It is given by

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u &= f \quad \text{in } \Omega, \\ u &= u_D \quad \text{on } \Gamma_D, \\ \varepsilon\nabla u\cdot\mathbf{n} &= 0 \quad \text{on } \Gamma_N, \end{aligned} \tag{1}$$

with $\mathbf{b} = (1,0)^T, f = 0$,

$$\Omega = \left\{(x,y) \mid -3 \le x \le 9, -3 \le y \le 3, x^2+y^2 > 1\right\},$$

and $\partial\Omega = \Gamma_D \cup \Gamma_N$. The Dirichlet boundary condition is prescribed by

$$u_D(x,y) = \begin{cases} 0, & \text{if } (x,y) \in \{-3\}\times[-3,3], \\ 1, & \text{if } x^2+y^2 = 1. \end{cases}$$

In the case $\varepsilon \ll \|\mathbf{b}\|_{L^\infty} = 1$, this problem is a simple model of a convection-dominated heat transfer from a hot column (circle). The solution of the Hemker problem possesses a boundary layer at the circle and two interior layers downwind the circle. It shows the birth of a boundary layer as discussed in [13].

Since the computational results of IGA and the SUPG FEM studied in [5] should be compared, $\varepsilon = 10^{-4}$ was chosen as in [5].

In [5], a number of quality measures for the discrete solutions were considered. We will use two of them which are especially of importance for applying a method in applications. One of them measures the size of the spurious oscillations and the other one measures the smearing of the layers. The solution of the Hemker problem takes values in $[0, 1]$. Let u_h be a discrete approximation of the solution, then the maximal undershoot is given by the minimal value of $u_h(x, y)$ and the maximal overshoot by the maximum of $u_h(x, y) - 1$. With respect to the second criterion, the layer width at $x = 4$ was considered, where the layer is defined by $\{y \mid 0.1 < u(4, y) < 0.9\}$. The reference value for the layer width given in [5] is 0.0723.

2.2. IGA combined with SUPG

The use of stabilized discretizations becomes necessary if important features of the solution cannot be resolved by the finite-dimensional approximation. Since for the Hemker problem the layers cannot be resolved on coarse and moderately fine meshes, it is clear that also the IGA has to be equipped with a stabilizing component. For the numerical studies presented below, the SUPG stabilization [14] was used, as it was already proposed in [9,12]. The SUPG approach is the most popular stabilization for finite element methods.

Denoting by u_h the discrete solution and let V_h be the space of (NURBS) test functions. The SUPG approach introduces to the standard Galerkin method the following additional term on the left-hand side of the discrete equation:

$$\sum_{K\in\mathcal{T}_h} (-\varepsilon\Delta u_h + \mathbf{b}\cdot\nabla u_h - f, \delta_K \mathbf{b}\cdot\nabla v_h)_K \quad \forall\, v_h \in V_h,$$

where $\mathcal{T}_h$ denotes the triangulation, $K \in \mathcal{T}_h$ are the mesh cells, and $(\cdot,\cdot)_K$ is the $L^2(K)$ inner product. The stabilization parameter is chosen to be [3]

$$\delta_K(x,y) = \frac{\tilde{h}_K}{2p\,|\mathbf{b}(x,y)|}\zeta\left(\mathrm{Pe}_K(x,y)\right), \tag{2}$$

with

$$\mathrm{Pe}_K(x,y) = \frac{|\mathbf{b}(x,y)|\,\tilde{h}_K}{2p\varepsilon}, \qquad \zeta(\alpha) = \coth\alpha - \frac{1}{\alpha},$$

where $\tilde{h}_K$ is the length of the mesh cell K in the direction of $\mathbf{b}$ (difference of largest x-coordinate and smallest x-coordinate of the vertices of K), p is the degree of the NURBS, and Pe_K the local Péclet number.

2.3. Parameterizations of the domain

IGA with NURBS has a tensor-product structure, i.e., the control points and the NURBS are defined on a square (for the simulations below on $(0,1)^2$) and the grid in Ω is obtained by a one-to-one mapping of this square to Ω. However, often it is hard to construct such a one-to-one mapping or this mapping has some bad properties. In this case, usually one uses a multi-patch parameterization of Ω by dividing Ω in pieces such that for each of them appropriate one-to-one mappings can be defined.

For the Hemker problem, we studied several parameterizations; see Fig. 1 for the meshes of the control points and the resulting meshes in Ω. In the first parameterization, called Ω_1, the parametric domain $(0,1)^2$ is bended around the circle. Therefore the parameterization resembles polar coordinates. However, downwind of the circle, the mesh is not aligned with the convection field. It is well known that such a situation may lead to considerable smearings in numerical solutions of convection–diffusion problems. The second parameterization, Ω_2, consists of two patches, where Ω is cut into two parts along the line $y = 0$. In the third parameterization Ω_3, also consisting of two patches, Ω was cut along the line $x = 0$. Finally, we considered a four patch parameterization Ω_4 by using both cut lines $x = 0$ and $y = 0$, where the parameterization of the individual patches was performed such that the grid becomes similar to the grid obtained with Ω_2. Of course, it is possible to parametrize the individual patches in Ω_4 also in such a way that the grid resembles the grid of Ω_3. For the sake of brevity, the results obtained with this approach will be only commented below, but no details will be presented.

Along the cut lines, continuity of the functions was always assured. We tested also the use of additional continuity of the gradient, which can be enforced here because of the simple form of the patches and the same degree of the NURBS on all patches. However, we could observe only a very small influence on the computational results. For the sake of brevity, Section 3 presents only the results obtained with continuous functions at the cutlines.

2.4. Further aspects of IGA

For the implementation of the Dirichlet boundary condition, the boundary function u_D is approximated by a NURBS $w_h = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} w_h^{i,j} N_{i,j}$. Here, the basis functions $N_{i,j}$ are the same as for the parameterization of the domain and $n^{(1)}, n^{(2)}$ are the number of control points in the x- and y-direction, respectively. In the case of the Hemker problem, the Dirichlet data are constant and therefore the enforcement of the boundary condition is straightforward. It is realized by setting the values of the control points along the Dirichlet boundary to 0 and 1, respectively. All other values of w_h are set to be 0. Due to the fact that NURBS interpolate endpoints exactly, one obtains an exact representation of the Dirichlet data.

To take the boundary condition into account in the simulations, the boundary function is inserted into the system such that the corresponding entries of the right-hand side become

$$\tilde{b}_{k_1 l_1} = b_{k_1 l_1} - \sum_{k_2=0}^{n^{(1)}} \sum_{l_2=0}^{n^{(2)}} w_h^{k_2,l_2} a_{k_1 l_1, k_2 l_2},$$

where $a_{k_1 l_1, k_2 l_2}$ denotes an entry of the stiffness matrix. With this new right-hand side, a solution with homogeneous Dirichlet condition is computed. Finally, the boundary function w_h is added to this solution to obtain a solution satisfying the original Dirichlet boundary condition.

For computing the under- and overshoots, the numerical solutions were evaluated at the mesh points.

The computation of the thickness of the interior layer at $x = 4$ is straightforward for the parameterizations Ω_2 and Ω_4, since a knot line of the parametric domain is mapped exactly onto $x = 4$ in the physical domain. For the other two parameterizations, the inverse of the line $x = 4$ is unknown and the points along this line were found by trial-and-error. Along the line $x = 4$, the solutions were evaluated at 10 001 points for Ω_2 and Ω_4, at 10 332 points for Ω_1, and at 10 454 points for Ω_3.

3. Numerical studies

Numerical simulations were performed with NURBS of degree two. The obtained results will be compared with the solutions for the SUPG discretization with second order finite elements from [5]. For this discretization, properly aligned meshes were used and the stabilization parameter is defined as given in (2), with $p = 2$ and $\tilde{h}_K$ is computed to be (an approximation of) the length of the mesh cell in the direction of the convection; see [3] for details of computing this approximation. The comparison of the results is based on the functionals described at the end of Section 2.1.

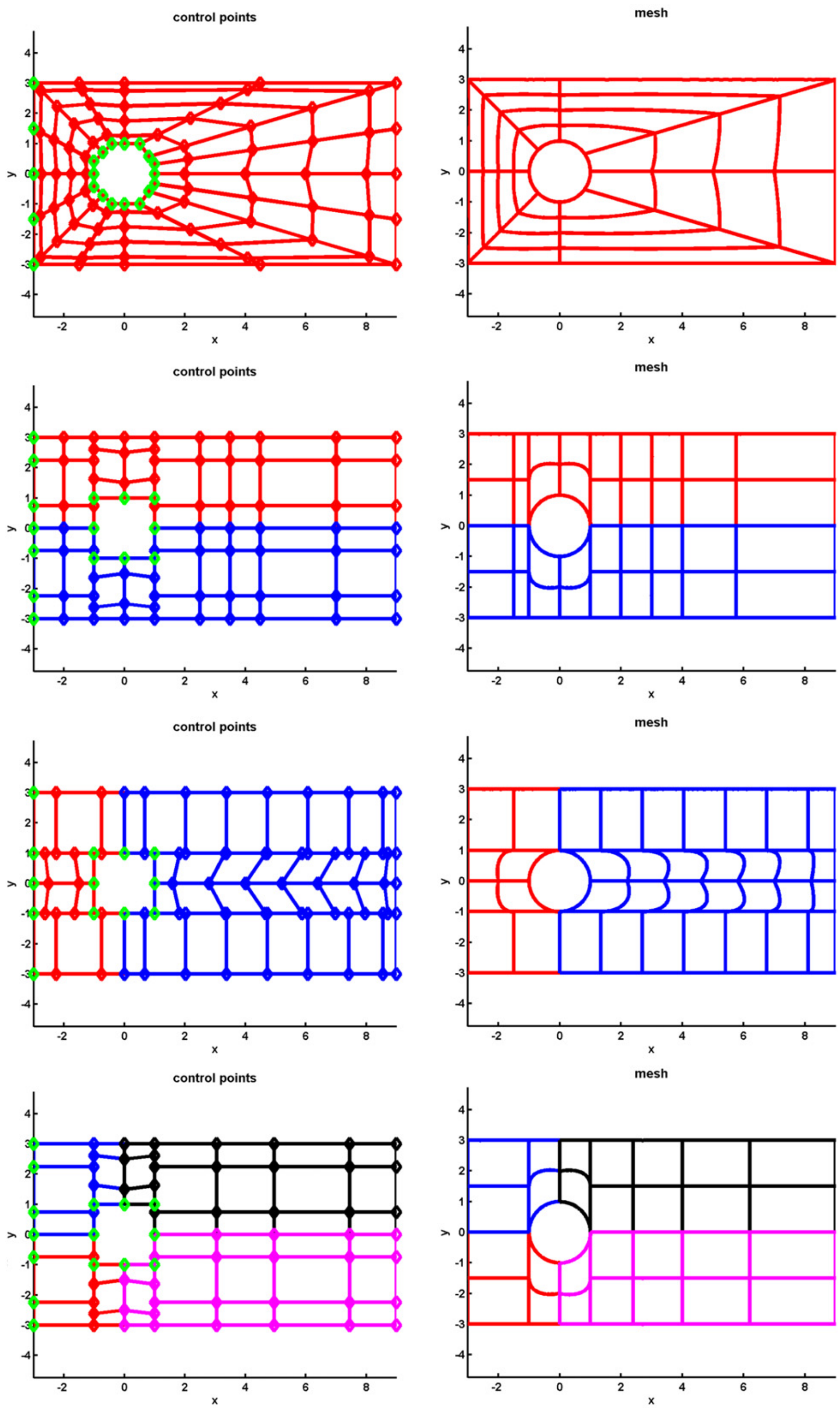

Fig. 1. Parameterizations $\Omega_1, \ldots, \Omega_4$ (top to bottom): control points (left) and resulting meshes on the coarsest level for Ω (right).

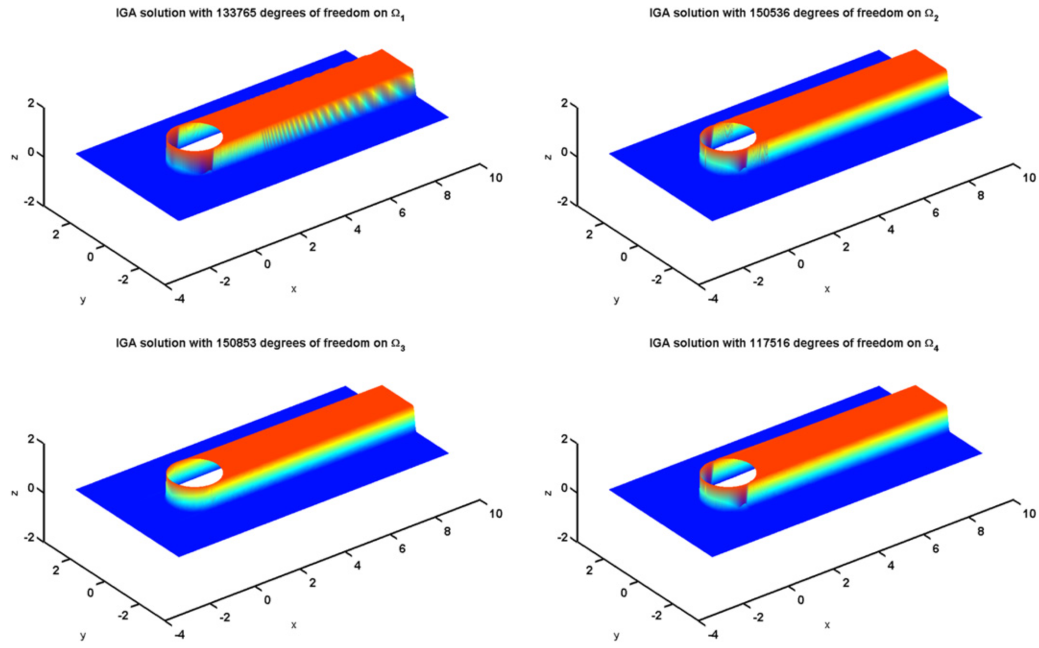

Fig. 2. Solutions of the Hemker problem computed with IGA for the different parameterizations.

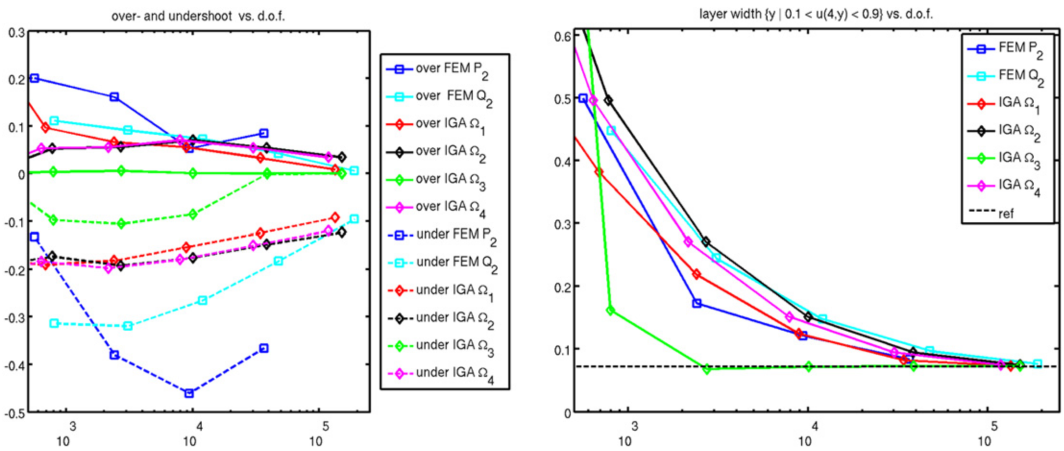

Fig. 3. Over- and undershoots (left), layer width at $x = 4$ and $y = 1$ (right).

The solutions obtained for the different parameterizations are depicted in Fig. 2. The transport of the Dirichlet data $u_D = 1$ at the circular hole in the direction of the convection can be clearly seen. However, already in these representations of the numerical solutions one can observe some differences. At the starting point of the interior layer at $(0, -1)$, some oscillations can be seen for Ω_1, Ω_2, and Ω_4. The interior layer of the solution on Ω_1 does not appear to be planar, as it is the case for the other parameterizations. We think that the non-aligned mesh is the reason for this behavior.

The over- and undershoots of the numerical solutions are presented in Fig. 3, left picture. It can be observed that with respect to the undershoots, the IGA solutions behave considerably better than the FEM solutions. In particular, the solution on Ω_3 shows comparably small undershoots. Concerning the overshoots, the IGA solutions are not worse than the FEM solutions and again the solution on Ω_3 is clearly better than the other solutions. The results on Ω_2 and Ω_4 are similar. Likewise, we observed that on a four-patch parameterization where the grid resembles the grid of Ω_3, very similar solutions as for Ω_3 were computed.

Fig. 3, right picture, shows the layer width of the computed solutions. Again, the solution obtained with Ω_3 is more accurate than the other numerical solutions. With all other parameterizations, one gets similar results, which are also comparable with the results from the finite element simulations.

A reason for the very accurate results computed on Ω_3 is, in our opinion, the existence of grid lines at $y = -1$ and $y = 1$, which are located in the positions of the interior layers. Another reason might be the use of a double knot in the definition of these two grid lines, which results in a reduced smoothness of the basis functions at these lines. It was already observed (for elliptic problems) in [15] that reducing the smoothness of NURBS at singularities might lead to an improvement of the accuracy of the computed results.

In applications, another important issue is the efficiency of the methods. An appropriate measure is of course computing time. However, the use of this measure is not meaningful in our studies since the IGA was implemented in MATLAB and the finite element simulations were performed with a research code written in C++; see [5]. Both approaches, the IGA and the FEM with SUPG stabilization, require the solution of one linear system of equations on each mesh. Thus, a certain measure for the numerical costs gives the sparsity of the matrix. For the simulations presented in this section, the average number of matrix entries per degree of freedom was 11.4 for the P_2 FEM, 15.8 for the Q_2 FEM, and 24.8 for the IGA, independently of the parameterization. In this measure, the matrices of the IGA are about twice as dense as the matrices for the P_2 FEM and 1.5 times as dense as the matrices for the Q_2 FEM. We think that these factors can be considered as lower bounds for the increase of the computational cost of (second order) IGA compared with (second order) FEM if sparse direct solvers are applied. If iterative solvers are used, also the condition number of the matrices is important. Since the condition number depends strongly on the concrete mesh and different meshes are used in our simulations, only a rough comparison is possible. We could observe that the condition number for the matrices from FEM and IGA was of the same order of magnitude for meshes with a similar number of degrees of freedom.

4. Summary

A study of IGA for steady-state convection–diffusion equations was presented in this note. IGA was combined with the SUPG stabilization and the simulations were performed for the Hemker problem. It was found that the results obtained with IGA depend considerably on the used parameterization of the domain and on the individual patches. For an appropriate parameterization, here Ω_3, the solutions computed with a second order IGA were clearly more accurate than the solutions obtained with a second order SUPG FEM, i.e., the spurious oscillations were considerably reduced and the layers were sharper. However, IGA does generally not remove spurious oscillations from the discrete solutions.

Since the SUPG FEM was considered in [5] to be the currently best available discretization in the case that spurious oscillations can be tolerated in the numerical solutions, this note shows the large potential of IGA (with SUPG) for the simulation of scalar convection–diffusion equations. The obtained results provide a strong motivation for further studies of IGA for this class of equations. Topics of forthcoming studies should include time-dependent problems, problems in three dimensions, the incorporation of more complicated boundary conditions, and general guidelines on how to choose appropriate parameterizations of the domain.

References

[1] Hans-Görg Roos, Martin Stynes, Lutz Tobiska, Robust Numerical Methods for Singularly Perturbed Differential Equations, second ed., in: Springer Series in Computational Mathematics, vol. 24, Springer-Verlag, Berlin, 2008.

[2] Volker John, Teodora Mitkova, Michael Roland, Kai Sundmacher, Lutz Tobiska, Andreas Voigt, Simulations of population balance systems with one internal coordinate using finite element methods, Chem. Eng. Sci. 64 (4) (2009) 733–741.

[3] Volker John, Petr Knobloch, On spurious oscillations at layers diminishing (SOLD) methods for convection–diffusion equations. I. A review, Comput. Methods Appl. Mech. Engrg. 196 (17–20) (2007) 2197–2215.

[4] Volker John, Petr Knobloch, On spurious oscillations at layers diminishing (SOLD) methods for convection–diffusion equations. II. Analysis for P_1 and Q_1 finite elements, Comput. Methods Appl. Mech. Engrg. 197 (21–24) (2008) 1997–2014.

[5] Matthias Augustin, Alfonso Caiazzo, André Fiebach, Jürgen Fuhrmann, Volker John, Alexander Linke, Rudolf Umla, An assessment of discretizations for convection-dominated convection–diffusion equations, Comput. Methods Appl. Mech. Engrg. 200 (47–48) (2011) 3395–3409.

[6] Volker John, Ellen Schmeyer, Finite element methods for time-dependent convection–diffusion–reaction equations with small diffusion, Comput. Methods Appl. Mech. Engrg. 198 (3–4) (2008) 475–494.

[7] Volker John, Julia Novo, On (essentially) non-oscillatory discretizations of evolutionary convection–diffusion equations, J. Comput. Phys. 231 (4) (2012) 1570–1586.

[8] P.W. Hemker, A singularly perturbed model problem for numerical computation, J. Comput. Appl. Math. 76 (1–2) (1996) 277–285.

[9] T.J.R. Hughes, J.A. Cottrell, Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, Comput. Methods Appl. Mech. Engrg. 194 (39–41) (2005) 4135–4195.

[10] Y. Bazilevs, V.M. Calo, T.J.R. Hughes, Y. Zhang, Isogeometric fluid–structure interaction: theory, algorithms, and computations, Comput. Mech. 43 (1) (2008) 3–37.

[11] Y. Bazilevs, M.-C. Hsu, M.A. Scott, Isogeomeric fluid–structure interaction analysis with emphasis on non-matching discretizations, and with application to wind turbines, Comput. Methods Appl. Mech. Engrg. 249–252 (2012) 28–41.

[12] Y. Bazilevs, L. Beirão da Veiga, J.A. Cottrell, T.J.R. Hughes, G. Sangalli, Isogeometric analysis: approximation, stability and error estimates for h-refined meshes, Math. Models Methods Appl. Sci. 16 (7) (2006) 1031–1090.

[13] J. Grasman, The birth of a boundary layer in an elliptic singular perturbation problem, in: Spectral Theory and Asymptotics of Differential Equations (Proc. Conf., Scheveningen, 1973), in: North-Holland Math. Studies, vol. 13, North-Holland, Amsterdam, 1974, pp. 175–179.

[14] T.J.R. Hughes, A. Brooks, A multidimensional upwind scheme with no crosswind diffusion, in: Finite Element Methods for Convection Dominated Flows (Papers, Winter Ann. Meeting Amer. Soc. Mech. Engrs., New York, 1979), in: AMD, vol. 34, Amer. Soc. Mech. Engrs. (ASME), New York, 1979, pp. 19–35.

[15] J.A. Cottrell, T.J.R. Hughes, A. Reali, Studies of refinement and continuity in isogeometric structural analysis, Comput. Methods Appl. Mech. Eng. 196 (41–44) (2007) 4160–4183.