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ERROR ANALYSIS OF THE SUPG FINITE ELEMENT DISCRETIZATION OF EVOLUTIONARY CONVECTION-DIFFUSION-REACTION EQUATIONS*

VOLKER JOHN† AND JULIA NOVO‡

Abstract. Conditions on the stabilization parameters are explored for different approaches in deriving error estimates for the streamline-upwind Petrov–Galerkin (SUPG) finite element stabilization of time-dependent convection-diffusion-reaction equations. Exemplarily, it is shown for the SUPG method combined with the backward Euler scheme that standard energy arguments lead to estimates for stabilization parameters that depend on the length of the time step. The stabilization vanishes in the time-continuous limit. However, based on numerical experience, this seems not to be the correct behavior. For this reason, the main focus of the paper consists in deriving estimates in which the stabilization parameters do not depend on the length of the time step. It is shown that such estimates can be obtained in the case of time-independent convection and reaction. An error estimate for the time-continuous case with the standard order of convergence is derived for stabilization parameters of the same form as they are optimal for the steady-state problem. Analogous estimates are obtained for the fully discrete case using the backward Euler method and the Crank–Nicolson scheme. Numerical studies support the analytical results.

Key words. evolutionary convection-diffusion-reaction equation, streamline-upwind Petrov–Galerkin (SUPG) finite element method, backward Euler scheme, Crank–Nicolson scheme, time-continuous problem, error analysis

AMS subject classifications. 65M12, 65M60

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1. Introduction.

Evolutionary convection-diffusion-reaction equations model the transport and reaction of species. In applications, typically the size of the diffusion is much smaller than the size of the convective term, and solutions develop sharp layers. In this case, it is well known that standard finite element methods perform poorly and exhibit nonphysical oscillations. Stabilization techniques are required in order to get physically sound numerical approximations. This paper studies one of the currently most popular finite element stabilizations, the streamline-upwind Petrov–Galerkin (SUPG) method that was introduced for steady-state equations in [9, 2]. In [10], the semidiscrete SUPG formulation was extended to the fully discrete space-time formulation using a discontinuous Galerkin method in time to treat linear convection-diffusion systems. Although time-space elements might be a natural setting to develop stabilized methods for evolutionary equations, effective algorithms for treating such problems are usually designed by separating temporal and spatial discretization. In particular, the increased cost with respect to the number of degrees of freedom for coupled time-space formulations is a significant drawback. Here, the approach of separating spatial and temporal discretization will be considered.

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†Weierstrass Institute for Applied Analysis and Stochastics (WIAS), Mohrenstr. 39, 10117 Berlin, Germany, and Department of Mathematics and Computer Science, Free University of Berlin, Arnimallee 6, 14195 Berlin, Germany (volker.john@wias-berlin.de).
‡Departamento de Matemáticas, Universidad Autónoma de Madrid, Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain (julia.novo@uam.cs). This author's research supported by the Spanish MEC under grants MTM2007-60528 and MTM2010-14919.

Concerning the numerical analysis of this approach, the transient convection equation without diffusive and reactive term is considered in [3]. It is shown that a finite element discretization in space coupled with the backward Euler, the Crank–Nicolson, or the second order backward differentiation formula in time leads to the classical error bound for the SUPG method in the L^2 norm (suboptimal by an order of one half) and to an optimal error bound in the norm of the material derivative. The results are obtained under certain regularity conditions on the data and with stabilization parameters that depend only on the mesh size in the space variable, i.e., $\delta = \mathcal{O}(h)$, where δ is the SUPG stabilization parameter. However, an optimal error bound for the error in the streamline derivative is not obtained. If the data are not sufficiently smooth or if the velocity field is nonsolenoidal, then the bound for the backward Euler method is valid under the condition $\delta^2 = \mathcal{O}(k)$, and the bound for the Crank–Nicolson scheme holds for $\delta = \mathcal{O}(k)$, where k is the length of the time step. On the other hand, as before, optimal error bounds in space require taking $\delta = \mathcal{O}(h)$. An analogous hypothesis for δ, i.e., $\delta = \mathcal{O}(k)$, was also applied in [15], where a Galerkin least-squares method in space coupled with a θ-scheme in time is analyzed. The analysis of [15] excludes the case $\theta = 1/2$. Finally, the stability of the SUPG finite element method for transient convection-diffusion equations is studied in [1]. However, as it is shown already in [3], the coercivity result of [1] leads to suboptimal global estimates in time.

Numerical studies of the SUPG method, together with a discussion on relations to other stabilized finite element methods, can be found in [6]. In [12, 13], the SUPG method was compared comprehensively with other stabilized finite element methods. The approach in these studies was as follows: (1) discretize the equation in time, (2) consider the equation in each discrete time as a steady-state convection-diffusion-reaction equation, and (3) discretize this equation in space with a stabilized method and apply a parameter choice that is appropriate for this type of steady-state equation. This methodology leads to a parameter that is (in the notation of formula (3.1) below) proportional to the length of the time step; see formulae (8) and (11) in [12]. The numerical results with this approach show large spurious oscillations compared with those of other methods. Such oscillations can be observed also if the SUPG method with this parameter choice is used in coupled systems coming from applications, as in [11].

Altogether, the numerical results obtained so far are not at all satisfactory. We think that a reason for this is the choice of the stabilization parameters depending on the length of the time step. In the case of a uniform time-space grid, i.e., $k \sim h$, this choice gives the size of the stabilization parameters as it is known from the steady-state case. However, approaching the time-continuous limit, i.e., using an anisotropic grid in time-space, the SUPG stabilization vanishes and one reverts to the Galerkin method that is unstable for the convection-dominated case; see also [8] for the same opinion. Very small time steps might be necessary if, for example, problems with very fast reactions are simulated. In addition, it is desirable to have balanced temporal and spatial errors. This balance is usually given by the orders of the methods. However, additional conditions coming from the coupling of the length of the time step and the mesh width via the stabilization parameters might contradict the balances given by these orders. In [8], another approach for deriving the fully discrete equation is considered: (1) discretize the equation in space with a stabilized method, (2) choose standard stabilization parameters for this equation, and (3) discretize the equation in time. Because the temporal discretization is performed after the choice of the stabilization parameters, these parameters cannot depend on the time step. Numerical studies in [8] show that this approach leads to much more stable results for small time steps compared with the approach from [12, 13]. In addition, another parameter choice is proposed in [8] that, for example, does not depend on the length of the time step if a steady-state solution is approached, the so-called element-vector-based parameter.

The goal of the present paper consists in exploring the conditions on the stabilization parameters for different approaches in the numerical analysis for deriving error estimates. In particular, error estimates that do not lead to a dependency of the stabilization parameters on the length of the time step are of interest. To the best of our knowledge, error estimates of this kind for the SUPG method applied to evolutionary convection-diffusion-reaction equations are not yet available. The main difficulty in the analysis of the method comes from the fact that the time derivative has to enter the stabilization term in order to ensure consistency. This adds a nonsymmetric term that cannot be easily bounded using standard energy arguments with stabilization parameters that do not depend on the length of the time step.

In the first part of the paper, exemplarily the backward Euler scheme is considered as temporal discretization. This is just for illustrating the difficulties in the standard analysis in the most simple setting. In sections 3 and 4, stability bounds and error estimates are derived based on energy arguments. Two different ways to argue lead to error estimates under the conditions $\delta = \mathcal{O}(k)$ and $\delta = \mathcal{O}(k^{1/2}h)$, respectively. These conditions arise in the stability bounds from the stabilization term with the discretization of the time derivative. In both choices, the stabilization parameters tend to zero on a fixed spatial grid as the length of the time step approaches zero. As discussed above, this seems not to be the correct choice. This is also seen in numerical studies, e.g., in Example 6.2 below. Altogether, the limit of the time-continuous case could not be treated so far satisfactorily by standard energy arguments.

To obtain some insight in the time-continuous case, section 5 studies a special problem where the convection field and the reaction do not depend on time and the SUPG method is applied on a uniform grid. The stabilization parameters are chosen to be the same on all mesh cells, depending only on the coefficients of the equation and on h: $\delta = \mathcal{O}(h)$. Under certain regularity assumptions on the solution and extending the analysis of [3], an error estimate for the L^2 norm and the norm of the material derivative is derived with the standard order of convergence. In the next step, based on this result, an estimate for the error in the norm of the streamline derivative is proven with the same order of convergence. To the best of our knowledge, this is the first result that proves standard order of convergence for the SUPG method applied to evolutionary convection-diffusion-reaction equations with a parameter choice that is essentially the same as in the steady-state case. A similar result was obtained in [7] for subgrid viscosity stabilizations of Galerkin approximations. Then the analysis is extended to the fully discrete case using the backward Euler method and the Crank–Nicolson scheme as time integrators, respectively.

Section 6 presents some numerical studies. In a first example, the derived error estimates are supported. Then a rotating body problem is studied for the P_1 finite element, on a given spatial grid, and for a very small length of the time step. The results show clearly that in this situation a choice of the stabilization parameter independently of the length of the time step has to be preferred. The paper concludes in section 7 with a summary of the results and an outlook on open questions.

2. The SUPG method and preliminaries of the analysis. Throughout this paper, standard notations are used for Lebesgue and Sobolev spaces. Generic constants that do not depend on the mesh width or the length of the time step are denoted by C.

A linear time-dependent convection-diffusion-reaction equation is given by

$$\begin{aligned} u_t - \varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f && \text{in } (0,T]\times\Omega,\\ u &= 0 && \text{on } [0,T]\times\partial\Omega,\\ u(0,\mathbf{x}) &= u_0(\mathbf{x}) && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded open domain in $\mathbb{R}^d$, $d\in\{1,2,3\}$, with boundary $\partial\Omega$, $\mathbf{b}(t,\mathbf{x})$ and $c(t,\mathbf{x})$ are given functions, $\varepsilon > 0$ is a constant diffusion coefficient, $u_0(\mathbf{x})$ are given initial data, and T is a given final time. For simplicity, the case that Ω is a convex polygonal or polyhedral domain is considered. In the following, it is assumed that there is a constant $\mu_0 > 0$ such that

$$0 < \mu_0 \le \mu(t,\mathbf{x}) = \left(c - \frac{1}{2}\nabla\cdot\mathbf{b}\right)(t,\mathbf{x}) \quad \forall\ (t,\mathbf{x})\in[0,T]\times\Omega. \tag{2.2}$$

This is a standard assumption in the analysis of equations of type (2.1); see [16].

Let $V = H_0^1(\Omega)$. A variational form of (2.1) reads as follows: Find $u\ :\ (0,T]\to V$ such that

$$(u_t, v) + (\varepsilon\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u + cu, v) = (f, v) \quad \forall\ v\in V, \tag{2.3}$$

and $u(0,\mathbf{x}) = u_0(\mathbf{x})$. Here, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)^d$, $d\in\{1,2,3\}$. In numerical simulations, V is replaced by a finite-dimensional (finite element) space $V_{h,r}$, where h indicates the fineness of the underlying triangulation $\mathcal{T}_h$ and $r\in\mathbb{N}$ the degree of the local polynomials. This paper considers the case of a conforming finite element method, i.e., $V_{h,r}\subset V$. The time-continuous finite element problem aims to find a function $u_h\in V_{h,r}$ that fulfills a problem of form (2.3) for all test functions from $V_{h,r}$ with an appropriate approximation of $u_0(\mathbf{x})$ at the initial time.

Using some temporal discretization, a finite element Galerkin method for solving (2.3) is obtained. It is well known that in the case of small diffusion, in particular compared with the convection, the Galerkin method is unstable and leads to solutions that are globally polluted with huge spurious oscillations. A stabilization of the Galerkin method becomes necessary. Probably the most popular stabilized finite element method is the SUPG method. This residual-based method adds artificial diffusion along the streamlines of the solution. It has the following form (time-continuous case): Find $u_h\ :\ (0,T]\to V_{h,r}$ such that

$$\begin{aligned} &(u_{h,t}, v_h) + a_{\mathrm{SUPG}}(u_h, v_h) + \sum_{K\in\mathcal{T}_h}\delta_K(u_{h,t}, \mathbf{b}\cdot\nabla v_h)_K\\ &\quad = (f, v_h) + \sum_{K\in\mathcal{T}_h}\delta_K(f, \mathbf{b}\cdot\nabla v_h)_K \quad \forall\ v_h\in V_{h,r}, \end{aligned}$$

with $u_h(0,\mathbf{x})$ being an appropriate approximation of $u_0(\mathbf{x})$ and

$$\begin{aligned} a_{\mathrm{SUPG}}(u_h, v_h) &= \varepsilon(\nabla u_h, \nabla v_h) + (\mathbf{b}\cdot\nabla u_h, v_h) + (cu_h, v_h)\\ &\quad + \sum_{K\in\mathcal{T}_h}\delta_K(-\varepsilon\Delta u_h + \mathbf{b}\cdot\nabla u_h + cu_h, \mathbf{b}\cdot\nabla v_h)_K. \end{aligned} \tag{2.4}$$

Here, $K\in\mathcal{T}_h$ denotes the mesh cells of the triangulation, $(\cdot,\cdot)_K$ the inner product in $L^2(K)$, and $\{\delta_K\}$ the local parameters which have to be chosen appropriately.

Next, preliminaries for the analysis are introduced. The elliptic projection $\pi_h : V \to V_{h,r}$ is defined by $(\nabla(u-\pi_h u), \nabla v_h) = 0$ for all $v_h \in V_{h,r}$. Note that the functions of $V_{h,r}$ do not depend on time. Hence, it follows that

$$(\pi_h u)_t = \pi_h(u_t) = \pi_h u_t. \tag{2.5}$$

Assuming that the meshes are quasi-uniform, the following inverse inequality holds for each $v_h \in V_{h,r}$ (see, e.g., [4, Theorem 3.2.6]):

$$\|v_h\|_{W^{m,q}(K)} \le c_{\text{inv}} h_K^{l-m-d\left(\frac{1}{q'}-\frac{1}{q}\right)} \|v_h\|_{W^{l,q'}(K)}, \tag{2.6}$$

where $0 \le l \le m \le 1$, $1 \le q' \le q \le \infty$, h_K is the size (diameter) of the mesh cell $K \in \mathcal{T}_h$, and $\|\cdot\|_{W^{m,q}(K)}$ is the norm in $W^{m,q}(K)$. The following interpolation error estimate for $u \in V \cap H^{r+1}(\Omega)$ is well known [5, 18]:

$$\|u - \pi_h u\|_0 + h\|u - \pi_h u\|_1 \le Ch^{r+1}\|u\|_{r+1}, \tag{2.7}$$

where $\|\cdot\|_r$ denotes the norm in $H^r(\Omega)$ with $H^0(\Omega) = L^2(\Omega)$. In particular, stability estimates for $u \in H_0^1(\Omega)$ of the following form can be derived:

$$\|\pi_h u\|_0 \le \|u - \pi_h u\|_0 + \|u\|_0 \le Ch\|u\|_1 + \|u\|_0 \le C\|u\|_1. \tag{2.8}$$

It is assumed that the space $V_{h,r}$ satisfies the following local approximation property: For each $u \in V \cap H^{r+1}(\Omega)$ there exists a $\hat{u}_h \in V_{h,r}$ such that

$$\|u - \hat{u}_h\|_{0,K} + h_K\|\nabla(u - \hat{u}_h)\|_{0,K} + h_K^2\|\Delta(u - \hat{u}_h)\|_{0,K} \le Ch_K^{r+1}\|u\|_{r+1,K} \tag{2.9}$$

for all $K \in \mathcal{T}_h$. For example, this property is given for Lagrange finite elements on mesh cells which allow an affine transform to a reference mesh cell.

LEMMA 2.1. *With* (2.9) *it follows for all* $u \in V \cap H^{r+1}(\Omega)$ *that*

$$\sum_{K\in\mathcal{T}_h} \|\Delta(u - \pi_h u)\|_{0,K}^2 \le Ch^{2r-2}\|u\|_{r+1}^2. \tag{2.10}$$

Proof. The result is easily obtained using the triangle inequality, the local approximation property (2.9), the inverse inequality (2.6), the quasi-uniformity of the mesh, and the interpolation error estimate (2.7). ∎

The coercivity of the bilinear form $a_{\text{SUPG}}(\cdot,\cdot)$ under the condition that the parameters $\{\delta_K\}$ are appropriately bounded from above is a well-known result; see, e.g., [17, Lemma 10.3].

LEMMA 2.2 (coercivity of $a_{\text{SUPG}}(\cdot,\cdot)$). *Let* (2.2) *be satisfied. If the SUPG parameters are chosen such that*

$$\delta_K \le \frac{\mu_0}{2\|c\|_{K,\infty}^2}, \quad \delta_K \le \frac{h_K^2}{2\varepsilon c_{\text{inv}}^2}, \tag{2.11}$$

then the bilinear form $a_{\text{SUPG}}(\cdot,\cdot)$ *associated with the SUPG method satisfies*

$$a_{\text{SUPG}}(u_h, u_h) \ge \frac{1}{2}\|u_h\|_{\text{SUPG}}^2 \tag{2.12}$$

with

$$\|u_h\|_{\text{SUPG}} := \left(\varepsilon\|\nabla u_h\|_0^2 + \sum_{K\in\mathcal{T}_h} \delta_K\|\mathbf{b}\cdot\nabla u_h\|_{0,K}^2 + \|\mu^{1/2}u_h\|_0^2\right)^{1/2}.$$

For linear finite elements, the condition $\delta_K \le h_K^2/(2\varepsilon c_{\text{inv}}^2)$ *can be omitted.*

3. Stability for stabilization parameters depending on the length of the time step. This section studies a fully discrete method for solving (2.3). Besides the finite element SUPG discretization (2.4), the temporal derivative is approximated with the backward Euler scheme. For this simple setting, standard energy arguments are applied to derive stability bounds. It turns out that this analysis proposes parameter choices in the SUPG method that depend on the length of the time step.

Consider the case of a fixed time step $k = \Delta t$. The fully discrete solution at time $t_n = nk$ will be denoted by U_h^n. The backward Euler/SUPG method reads as follows: For $n = 1, 2, \ldots$ find $U_h^n \in V_{h,r}$ such that

$$\begin{aligned} &\left(\frac{U_h^n - U_h^{n-1}}{k}, \varphi\right) + \varepsilon(\nabla U_h^n, \nabla\varphi) + (\mathbf{b}\cdot\nabla U_h^n, \varphi) + (cU_h^n, \varphi) = (f^n, \varphi) \\ &+ \sum_{K\in\mathcal{T}_h} \delta_K \left(f^n - \left(\frac{U_h^n - U_h^{n-1}}{k}\right) + \varepsilon\Delta U_h^n - \mathbf{b}\cdot\nabla U_h^n - cU_h^n, \mathbf{b}\cdot\nabla\varphi\right)_K \end{aligned} \tag{3.1}$$

for all $\varphi \in V_{h,r}$ and $U_h^0(\mathbf{x}) = u_h(0, \mathbf{x})$. Method (3.1) can be written equivalently in the form

$$\begin{aligned} (U_h^n - U_h^{n-1}, \varphi) + k a_{\mathrm{SUPG}}(U_h^n, \varphi) &= k(f^n, \varphi) + k\sum_{K\in\mathcal{T}_h} \delta_K (f^n, \mathbf{b}\cdot\nabla\varphi)_K \\ &\quad - \sum_{K\in\mathcal{T}_h} \delta_K (U_h^n - U_h^{n-1}, \mathbf{b}\cdot\nabla\varphi)_K. \end{aligned} \tag{3.2}$$

THEOREM 3.1 (stability, stabilization parameters proportional to the length of the time step). *Let* (2.2) *and* (2.11) *be fulfilled. With the additional condition*

$$\delta_K \le \frac{k}{4} \quad \forall\ K \in \mathcal{T}_h, \tag{3.3}$$

the solution of (3.1) *satisfies at* $t_n = nk$

$$\|U_h^n\|_0^2 + \frac{k}{2}\sum_{j=1}^n \|U_h^j\|_{\mathrm{SUPG}}^2 \le \|U_h^0\|_0^2 + k\left(\frac{2}{\mu_0} + k\right)\sum_{j=1}^n \|f^j\|_0^2.$$

Proof. The proof starts in the usual way by setting $\varphi = U_h^n$. This gives, with (3.2),

$$\begin{aligned} (U_h^n - U_h^{n-1}, U_h^n) + k a_{\mathrm{SUPG}}(U_h^n, U_h^n) &= k(f^n, U_h^n) + k\sum_{K\in\mathcal{T}_h} \delta_K (f^n, \mathbf{b}\cdot\nabla U_h^n)_K \\ &\quad - \sum_{K\in\mathcal{T}_h} \delta_K (U_h^n - U_h^{n-1}, \mathbf{b}\cdot\nabla U_h^n)_K. \end{aligned}$$

From $(U_h^n - U_h^{n-1}, U_h^n) = \frac{1}{2}\left(\|U_h^n\|_0^2 - \|U_h^{n-1}\|_0^2 + \|U_h^n - U_h^{n-1}\|_0^2\right)$ it follows that, with (2.12),

$$\begin{aligned} &\frac{1}{2}\left(\|U_h^n\|_0^2 - \|U_h^{n-1}\|_0^2 + \|U_h^n - U_h^{n-1}\|_0^2\right) + \frac{k}{2}\|U_h^n\|_{\mathrm{SUPG}}^2 \\ &\quad \le |k(f^n, U_h^n)| + \left|k\sum_{K\in\mathcal{T}_h} \delta_K (f^n, \mathbf{b}\cdot\nabla U_h^n)_K\right| \\ &\qquad + \left|\sum_{K\in\mathcal{T}_h} \delta_K (U_h^n - U_h^{n-1}, \mathbf{b}\cdot\nabla U_h^n)_K\right|. \end{aligned} \tag{3.4}$$

The first two terms on the right-hand side are estimated using the Cauchy–Schwarz inequality and Young's inequality:

$$|k(f^n, U_h^n)| \le k \left\| \frac{f^n}{\mu^{1/2}} \right\|_0^2 + \frac{k}{4} \|\mu^{1/2} U_h^n\|_0^2 \le \frac{k}{\mu_0} \|f^n\|_0^2 + \frac{k}{4} \|\mu^{1/2} U_h^n\|_0^2,$$

$$\left| k \sum_{K\in\mathcal{T}_h} \delta_K (f^n, \mathbf{b}\cdot\nabla U_h^n)_K \right| \le 2k \sum_{K\in\mathcal{T}_h} \delta_K \|f^n\|_{0,K}^2 + \frac{k}{8} \sum_{K\in\mathcal{T}_h} \delta_K \|\mathbf{b}\cdot\nabla U_h^n\|_{0,K}^2.$$

The estimate of the last term on the right-hand side of (3.4) uses condition (3.3) on the stabilization parameters:

$$\begin{aligned} &\left| \sum_{K\in\mathcal{T}_h} \delta_K (U_h^n - U_h^{n-1}, \mathbf{b}\cdot\nabla U_h^n)_K \right| \\ &\le \frac{2}{k} \sum_{K\in\mathcal{T}_h} \delta_K \|U_h^n - U_h^{n-1}\|_{0,K}^2 + \frac{k}{8} \sum_{K\in\mathcal{T}_h} \delta_K \|\mathbf{b}\cdot\nabla U_h^n\|_{0,K}^2 \\ &\le \frac{1}{2} \|U_h^n - U_h^{n-1}\|_0^2 + \frac{k}{8} \sum_{K\in\mathcal{T}_h} \delta_K \|\mathbf{b}\cdot\nabla U_h^n\|_{0,K}^2. \end{aligned}$$

Inserting all estimates leads to

$$\|U_h^n\|_0^2 + \frac{k}{2} \|U_h^n\|_{\mathrm{SUPG}}^2 \le \|U_h^{n-1}\|_0^2 + \frac{2k}{\mu_0} \|f^n\|_0^2 + 4k \sum_{K\in\mathcal{T}_h} \delta_K \|f^n\|_{0,K}^2. \tag{3.5}$$

Summation of the time steps $j = 1, \ldots, n$ and using once more condition (3.3) gives the statement of the theorem. □

Note that $k \sum_{j=1}^n \|U_h^j\|_{\mathrm{SUPG}}^2$ is an approximation of $\|U_h\|_{L^2(0,T;\mathrm{SUPG})}^2$ by a Riemann sum using as node in the quadrature rule always the right end of the time intervals.

Theorem 3.1 covers the case that the stabilization parameter is proportional to the length of the time step. On a fixed spatial grid, the stabilization becomes small for small time steps and vanishes in the time-continuous limit. This behavior does not seem to be correct; see the discussion in the introduction. The desired situation in the convection-dominated regime, $\delta_K \sim h_K$, is obtained if spatial and temporal mesh width are proportional, $h \sim k$. Note that when the mesh width and the time step are of the same order, the parameter choice of [12, 13] leads also to $\delta \sim k \sim h$.

Theorem 3.2 (stability, stabilization parameters proportional to some function of the length of the time step). *Let* (2.2) *and* (2.11) *be fulfilled. With the choice*

$$\delta_K = \frac{\sigma(k) h_K}{\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}} \quad \textit{with} \quad 0 < \sigma(k) \le \frac{1}{4} \quad \forall\, K \in \mathcal{T}_h, \tag{3.6}$$

where $\sigma(k)$ *is a function to be specified later, the solution of* (3.1) *satisfies at* $t_n = nk$

$$\begin{aligned} &\|U_h^n\|_0^2 + \frac{k}{2} \sum_{j=1}^n \|U_h^j\|_{\mathrm{SUPG}}^2 \\ &\le (1 + 2\sigma^2(k))^n \left[\|U_h^0\|_0^2 + 2k \sum_{j=1}^n \left(\frac{1}{\mu_0} \|f^j\|_0^2 + \sum_{K\in\mathcal{T}_h} \delta_K \|f^j\|_{0,K}^2 \right) \right]. \end{aligned} \tag{3.7}$$

Proof. The proof starts exactly as the proof of Theorem 3.1 until estimate (3.4) is reached. The first two terms on the right-hand side of (3.4) are estimated also in the same way as in the proof of Theorem 3.1:

$$|k(f^n, U_h^n)| \leq \frac{k}{\mu_0}\|f^n\|_0^2 + \frac{k}{4}\|\mu^{1/2}U_h^n\|_0^2,$$

$$\left| k \sum_{K\in\mathcal{T}_h} \delta_K (f^n, \mathbf{b}\cdot\nabla U_h^n)_K \right| \leq k \sum_{K\in\mathcal{T}_h} \delta_K \|f^n\|_{0,K}^2 + \frac{k}{4}\sum_{K\in\mathcal{T}_h} \delta_K \|\mathbf{b}\cdot\nabla U_h^n\|_{0,K}^2.$$

The last term on the right-hand side of (3.4) will now not be absorbed into $\frac{k}{2}\|U_h^n\|_{\mathrm{SUPG}}$. It is estimated by using the inverse inequality (2.6) and Young's inequality:

$$\begin{aligned}
&\left| \sum_{K\in\mathcal{T}_h} \delta_K (U_h^n - U_h^{n-1}, \mathbf{b}\cdot\nabla U_h^n)_K \right| \\
&\leq \sum_{K\in\mathcal{T}_h} \delta_K \frac{\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}{h_K} \|U_h^n - U_h^{n-1}\|_{0,K}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h} \delta_K \|\mathbf{b}\|_{\infty,K} \|U_h^n - U_h^{n-1}\|_{0,K} \|\nabla U_h^{n-1}\|_{0,K} \\
&\leq \sum_{K\in\mathcal{T}_h} \delta_K \frac{\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}{h_K} \|U_h^n - U_h^{n-1}\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} \frac{1}{4}\|U_h^n - U_h^{n-1}\|_{0,K}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h} \delta_K^2 \|\mathbf{b}\|_{\infty,K}^2 \|\nabla U_h^{n-1}\|_{0,K}^2 \\
&\leq \sum_{K\in\mathcal{T}_h} \left(\delta_K \frac{\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}{h_K} + \frac{1}{4} \right) \|U_h^n - U_h^{n-1}\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} \delta_K^2 \frac{\|\mathbf{b}\|_{\infty,K}^2 c_{\mathrm{inv}}^2}{h_K^2} \|U_h^{n-1}\|_{0,K}^2.
\end{aligned}$$

The first term can be absorbed into the left-hand side of (3.4) if

$$\delta_K \frac{\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}{h_K} + \frac{1}{4} \leq \frac{1}{2} \implies \delta_K \leq \frac{h_K}{4\|\mathbf{b}\|_{\infty,K} c_{\mathrm{inv}}}.$$

Set the stabilization parameter as in (3.6); then it follows that

$$\left| \sum_{K\in\mathcal{T}_h} \delta_K (U_h^n - U_h^{n-1}, \mathbf{b}\cdot\nabla U_h^n)_K \right| \leq \frac{1}{2}\|U_h^n - U_h^{n-1}\|_0^2 + \sigma^2(k)\|U_h^{n-1}\|_0^2.$$

Collecting all estimates leads to the recursion

$$\|U_h^n\|_0^2 + \frac{k}{2}\|U_h^n\|_{\mathrm{SUPG}}^2 \leq (1+2\sigma^2(k))\|U_h^{n-1}\|_0^2 + \frac{2k}{\mu_0}\|f^n\|_0^2 + 2k\sum_{K\in\mathcal{T}_h} \delta_K \|f^n\|_{0,K}^2. \tag{3.8}$$

Now, one obtains by induction

$$\begin{aligned}
&\|U_h^n\|_0^2 + \frac{k}{2}\|U_h^n\|_{\mathrm{SUPG}}^2 \\
&\leq (1+2\sigma^2(k))^n \|U_h^0\|_0^2 + 2k\sum_{j=1}^n \left(1+2\sigma^2(k)\right)^{n-j} \left(\frac{\|f^j\|_0^2}{\mu_0} + \sum_{K\in\mathcal{T}_h} \delta_K \|f^j\|_{0,K}^2 \right) \\
&\leq (1+2\sigma^2(k))^n \left[\|U_h^0\|_0^2 + 2k\sum_{j=1}^n \left(\frac{\|f^j\|_0^2}{\mu_0} + \sum_{K\in\mathcal{T}_h} \delta_K \|f^j\|_{0,K}^2 \right) \right]. \qquad (3.9)
\end{aligned}$$

Summation of (3.8) gives

$$\|U_h^n\|_0^2 + \frac{k}{2}\sum_{j=1}^n \|U_h^j\|_{\mathrm{SUPG}}^2 \le 2\sigma^2(k)\sum_{j=1}^{n-1}\|U_h^j\|_0^2 + (1+2\sigma^2(k))\|U_h^0\|_0^2 + 2k\sum_{j=1}^n\left(\frac{\|f^j\|_0^2}{\mu_0} + \sum_{K\in\mathcal{T}_h}\delta_K\|f^j\|_{0,K}^2\right).$$

Inserting (3.9) and applying some estimates for the sake of simplifying the representation leads to

$$\begin{aligned}
&\|U_h^n\|_0^2 + \frac{k}{2}\sum_{j=1}^n \|U_h^j\|_{\mathrm{SUPG}}^2 \\
&\le \left(2\sigma^2(k)\frac{(1+2\sigma^2(k))^n - (1+2\sigma^2(k))}{1+2\sigma^2(k)-1} + 1 + 2\sigma^2(k)\right)\|U_h^0\|_0^2 \\
&\quad + 2k\left[\left(1+2\sigma^2(k)\sum_{j=1}^{n-1}(1+2\sigma^2(k))^j\right)\sum_{j=1}^n\left(\frac{\|f^j\|_0^2}{\mu_0} + \sum_{K\in\mathcal{T}_h}\delta_K\|f^j\|_{0,K}^2\right)\right] \\
&\le (1+2\sigma^2(k))^n\left[\|U_h^0\|_0^2 + 2k\sum_{j=1}^n\left(\frac{\|f^j\|_0^2}{\mu_0} + \sum_{K\in\mathcal{T}_h}\delta_K\|f^j\|_{0,K}^2\right)\right]. \qquad \square
\end{aligned}$$

Consider a finite time interval $[0,T]$ and a fixed length of the time step. Then Theorem 3.2 gives stability with the desired stability parameter (in the convection-dominated regime) $\delta_K = \mathcal{O}(h_K)$ without a coupling of the mesh width to the time step by choosing $\sigma(k) = \text{const} \le 1/4$. However, the stability bound blows up for $\sigma(k) = \text{const}$ in the time-continuous limit $k \to 0$. Given a length of the time step k, the number of time steps to solve the equation in $[0,T]$ is $n = T/k$. The stability estimate will not blow up for $k \to 0$ if $(1+\sigma^2(k))^{1/k}$ is bounded uniformly. A possible choice is $\sigma(k) = \delta_0\sqrt{k}$, leading to the stabilization parameter

$$\delta_K = \delta_0\frac{\sqrt{k}h_K}{\|\mathbf{b}\|_{\infty,K}c_{\mathrm{inv}}}, \tag{3.10}$$

where δ_0 has to be chosen such that $\delta_0\sqrt{k} \le 1/4$. For fixed h and sufficiently small k, the parameter from (3.10) is larger than the parameter from (3.3).

4. Error estimates for stabilization parameters depending on the length of the time step. For the following error analysis, it is assumed that all functions are sufficiently regular. Summaries of these assumptions are given below in Theorem 4.1. The error analysis for (3.1) starts by decomposing the error into an interpolation error and the difference of the interpolation and the solution

$$U_h^n - u(t_n) = (U_h^n - \pi_h u(t_n)) + (\pi_h u(t_n) - u(t_n)).$$

The interpolation error can be estimated with (2.7). For brevity, denote $\pi_h^n u := \pi_h u(t_n)$ and $e_h^n = U_h^n - \pi_h u(t_n)$. Straightforward calculations yield the following error equation:

$$\begin{aligned}
&(e_h^n - e_h^{n-1},\varphi) + k a_{\mathrm{SUPG}}(e_h^n,\varphi) = k(\tilde{T}_{\mathrm{zero}}^n,\varphi) + k(T_{\mathrm{conv}}^n,\varphi) \\
&+ k\sum_{K\in\mathcal{T}_h}\delta_K(\tilde{T}_{\mathrm{stab},K}^n, \mathbf{b}\cdot\nabla\varphi)_K - \sum_{K\in\mathcal{T}_h}\delta_K(e_h^n - e_h^{n-1}, \mathbf{b}\cdot\nabla\varphi)_K,
\end{aligned}$$

with

$$\tilde{T}^n_{\text{zero}} = (u_t(t_n) - \pi^n_h u_t) + c\,(u(t_n) - \pi^n_h u) + \left(\pi^n_h u_t - \frac{\pi^n_h u - \pi^{n-1}_h u}{k}\right),$$
$$T^n_{\text{conv}} = \mathbf{b}\cdot\nabla(u(t_n) - \pi^n_h u),$$
$$\tilde{T}^n_{\text{stab},K} = \left(\tilde{T}^n_{\text{zero}} + T^n_{\text{conv}} + \varepsilon\Delta(\pi^n_h u - u(t_n))\right)\big|_K.$$

Using integration by parts and assuming $\delta_K > 0$, the convective term can be distributed to the term with the zeroth order derivatives (with respect to space) and the stabilization term

$$(T^n_{\text{conv}}, \varphi) = -((\nabla\cdot\mathbf{b})(\pi^n_h u - u(t_n)), \varphi) - \sum_{K\in\mathcal{T}_h} \delta_K \left(\frac{\pi^n_h u - u(t_n)}{\delta_K}, \mathbf{b}\cdot\nabla\varphi\right)_K.$$

Redefining the zeroth order and the stabilization term

$$T^n_{\text{zero}} = \tilde{T}^n_{\text{zero}} - (\nabla\cdot\mathbf{b})(\pi^n_h u - u(t_n)), \quad T^n_{\text{stab},K} = \tilde{T}^n_{\text{stab},K} - \frac{\pi^n_h u - u(t_n)}{\delta_K}$$

leads to the error equation

$$(4.1)\quad \begin{aligned}(e^n_h - e^{n-1}_h, \varphi) + k a_{\text{SUPG}}(e^n_h, \varphi) &= k(T^n_{\text{zero}}, \varphi) + k\sum_{K\in\mathcal{T}_h} \delta_K (T^n_{\text{stab},K}, \mathbf{b}\cdot\nabla\varphi)_K \\ &\quad - \sum_{K\in\mathcal{T}_h} \delta_K (e^n_h - e^{n-1}_h, \mathbf{b}\cdot\nabla\varphi)_K.\end{aligned}$$

This error equation is similar to (3.1); only the arguments on the first two terms on the right-hand side are not the same.

Deriving error estimates from (4.1) starts essentially in the same way as the derivation of the stability bounds. After this, the arising terms have to be bounded by norms of the solution of the continuous equation (2.3). Since the stability bounds derived in Theorems 3.1 and 3.2 are similar, the detailed analysis for the error estimates is presented here only for the case that was considered in Theorem 3.1.

For applying the techniques of the stability estimate to (4.1), only the last two terms cannot be combined in the summation of the analogue to (3.5). One gets

$$(4.2)\quad \|e^n_h\|_0^2 + \frac{k}{2}\sum_{j=1}^n \|e^j_h\|^2_{\text{SUPG}} \le \|e^0_h\|_0^2 + \frac{2k}{\mu_0}\sum_{j=1}^n \|T^j_{\text{zero}}\|_0^2 + 4k\sum_{j=1}^n \sum_{K\in\mathcal{T}_h} \delta_K \|T^j_{\text{stab},K}\|^2_{0,K}.$$

Using the triangle inequality and (2.7), one obtains

$$\begin{aligned}\|T^j_{\text{zero}}\|_0^2 \le Ch^{2r+2}\Big(&\|u_t(t_j)\|^2_{r+1} + \|c\|^2_{L^\infty(0,T;L^\infty)}\|u(t_j)\|^2_{r+1} \\ &+ \|\nabla\cdot\mathbf{b}\|^2_{L^\infty(0,T;L^\infty)}\|u(t_j)\|^2_{r+1}\Big) + C\left\|\pi_h u_t(t_j) - \frac{\pi^j_h u - \pi^{j-1}_h u}{k}\right\|_0^2.\end{aligned}$$

The last term is in essence the approximation error of $u_t(t_j)$ by a backward finite difference; hence an estimate of $\mathcal{O}(k)$ can be expected. The derivation of this estimate uses Taylor's formula with remainder in integral form, the application of (2.5), the Cauchy–Schwarz inequality, and the stability estimate (2.8):

$$\begin{aligned}\left\|\pi_h u_t(t_j) - \frac{\pi_h^j u - \pi_h^{j-1} u}{k}\right\|_0^2 &= \frac{1}{k^2}\left\|\int_{t_{j-1}}^{t_j} (t-t_{j-1})\pi_h u_{tt}\, dt\right\|_0^2 \\ &\le \frac{1}{k^2}\left(\left(\int_{t_{j-1}}^{t_j} (t-t_{j-1})^2\, dt\right)^{1/2}\left(\int_{t_{j-1}}^{t_j}\|\pi_h u_{tt}\|_0^2\, dt\right)^{1/2}\right)^2 \\ &\le Ck\int_{t_{j-1}}^{t_j}\|u_{tt}\|_1^2\, dt = Ck\|u_{tt}\|^2_{L^2(t_{j-1},t_j;H^1)}. \end{aligned} \tag{4.3}$$

Summation over the time steps, taking into account that the number of time steps n is inversely proportional to the length of the time step, and assuming that all norms are uniformly (in time) bounded, gives

$$k\sum_{j=1}^n \|T_{\text{zero}}^j\|_0^2 \le Cknh^{2r+2} + Ck^2\|u_{tt}\|^2_{L^2(0,t_n;H^1)} \le C\left(h^{2r+2}+k^2\right).$$

The estimate of the first term can be applied, in combination with (2.10), to obtain an estimate for the second term on the right-hand side of (4.2):

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\delta_K\|T^j_{\text{stab},K}\|^2_{0,K} \le C\left(\max_{K\in\mathcal{T}_h}\delta_K\right)&\Big(h^{2r+2}\left(\|u_t(t_j)\|^2_{r+1}+\|u(t_j)\|^2_{r+1}\right) \\ &+k\|u_{tt}\|^2_{L^2(t_{j-1},t_j;H^1)} + \|\mathbf{b}\|^2_{L^\infty(0,T;L^\infty)}h^{2r}\|u(t_j)\|^2_{r+1} \\ &+\varepsilon^2h^{2r-2}\|u(t_j)\|^2_{r+1}\Big) + C\left(\min_{K\in\mathcal{T}_h}\delta_K\right)^{-1}h^{2r+2}\|u(t_j)\|^2_{r+1}.\end{aligned}$$

Hence,

$$\begin{aligned}k\sum_{j=1}^n\sum_{K\in\mathcal{T}_h}\delta_K\|T^j_{\text{stab},K}\|^2_{0,K} \le C\Bigg(&\left(\max_{K\in\mathcal{T}_h}\delta_K\right)\left(h^{2r+2}+k^2+h^{2r}+\varepsilon^2h^{2r-2}\right) \\ &+\left(\min_{K\in\mathcal{T}_h}\delta_K\right)^{-1}h^{2r+2}\Bigg).\end{aligned}$$

Inserting all estimates into (4.2) and applying the triangle inequality leads to the following error estimates.

THEOREM 4.1 (error estimates for the stabilization parameter obeying (3.3)). *Suppose $\mathbf{b}\in L^\infty(0,T;(L^\infty)^d)$, $\nabla\cdot\mathbf{b}, c\in L^\infty(0,T;L^\infty)$ for the coefficients in (2.3), and $u, u_t\in L^\infty(0,T;H^{r+1})$, $u_{tt}\in L^2(0,T;H^1)$ for the solution of (2.3). Let the stabilization parameters $\{\delta_K\}$ fulfill (2.11), (3.3) and $\delta_K>0$ for all $K\in\mathcal{T}_h$. Denote $\delta=\max_{\in\mathcal{T}_h}\delta_K$. Then the error $U_h^n-u(t_n)$ satisfies*

$$\begin{aligned}\|U_h^n-u(t_n)\|_0 \le C\Bigg[&h^{r+1}+k+h^{r-1}\delta^{1/2}\left(h^2+h+\varepsilon\right) \\ &+\frac{h^{r+1}}{\left(\min_{K\in\mathcal{T}_h}\delta_K\right)^{1/2}} + \|\pi_h u_0 - U_h^0\|_0\Bigg]\end{aligned} \tag{4.4}$$

and

$$\left(k\sum_{j=1}^{n}\|U_h^j-u(t_j)\|_{\mathrm{SUPG}}^2\right)^{1/2} \le C\left[h^r(\varepsilon^{1/2}+\delta^{1/2}+h)+k+h^{r-1}\delta^{1/2}\left(h^2+h+\varepsilon\right)\right.$$

$$\left. + \frac{h^{r+1}}{\left(\min_{K\in\mathcal{T}_h}\delta_K\right)^{1/2}} + \|\pi_h u_0 - U_h^0\|_0\right], \tag{4.5}$$

where the constants C *depend on* $u, u_t, u_{tt}, \mathbf{b}, \nabla\cdot\mathbf{b}$*, and* c*.*

Applying the analysis of Theorem 3.2 to estimate (4.1) and using (3.7) leads essentially to (4.2), only with an additional factor of $(1+2\sigma^2(k))^n$ on the right-hand side. The same analysis as in the proof of Theorem 4.1 gives error estimates of the form (4.4) and (4.5) with this additional factor.

An alternative approach, without integration by parts of T_{conv}^n, leads to a suboptimal estimate with respect to space of order h^r. However, δ_K does not appear in the denominator of some term in this estimate. Hence, this suboptimal estimate shows that the error is bounded even as $\delta_K \to 0$.

5. Error analysis with stabilization parameters not depending on the length of the time step. The numerical analysis presented so far is valid only if, for a constant mesh and a small time step, the stabilization parameters are sufficiently small. In the time-continuous limit, the SUPG stabilization even vanishes. As discussed in the introduction and as demonstrated in the numerical studies, Example 6.2, we think that this is not the correct asymptotic of the stabilization parameters. In this section, optimal error estimates are derived with stabilization parameters proportional to the mesh width. In the first part, the time-continuous problem is considered, while the second and third parts extend the analysis to fully discrete cases using the backward Euler and the Crank–Nicolson schemes, respectively.

5.1. The time-continuous case. In the first step, an error estimate for the material derivative is derived, Lemma 5.1. The analysis of this step uses some ideas from [3], such as the application of a special test function to obtain (5.9). Extensions of the analysis from [3] were necessary to include diffusion and reaction and the case of a nonsolenoidal vector field $\mathbf{b}$. Based on the estimate for the material derivative, an error estimate for the streamline derivative is proven in a second step.

Consider problem (2.1) with the following assumptions:

- $\mathbf{b}_t(t,\mathbf{x})=\mathbf{0}$, $c_t(t,\mathbf{x})=0$, i.e., $\mathbf{b}=\mathbf{b}(\mathbf{x})$, $c=c(\mathbf{x})$, and $\mu=\mu(\mathbf{x})$;
- the mesh is uniform with mesh width h;
- the stabilization parameters are the same for all mesh cells, i.e., $\delta_K=\delta$.

Furthermore, it is assumed that all functions are sufficiently smooth such that all norms appearing below are well defined. In addition, only the convection-dominated regime is considered; i.e., it is assumed that $\varepsilon \le h$. Then the stabilization parameter is set to be

$$\delta = \min\left\{\frac{h}{4c_{\mathrm{inv}}\|\mathbf{b}\|_\infty}\min\left\{\frac{1}{2},\frac{\mu_0^{1/2}}{4},\frac{\mu_0}{4\|c\|_\infty},\frac{\mu_0^{1/2}}{2\|c\|_\infty},\frac{\mu_0^{1/2}}{\|c\|_\infty^{1/2}},\|\mathbf{b}\|_\infty^{1/2}\right\},1\right\}. \tag{5.1}$$

Hence, the stabilization parameter is proportional to the mesh width and is bounded from above by a constant. In addition, it is assumed that $\varepsilon \le \delta$.

Consider a finite time interval $[0,T]$ and let $t\in[0,T]$. In the analysis of this section, a formally steady-state problem derived from (2.1) is used. Let $\Pi_h u(t)\in V_{h,r}$

be the solution of

$$a_{\mathrm{SUPG}}(\Pi_h u(t), v_h) = (f(t) - u_t(t), v_h) + \delta(f(t) - u_t(t), \mathbf{b}\cdot\nabla v_h) \quad \forall\, v_h \in V_{h,r}. \tag{5.2}$$

The corresponding continuous equation is solved by $u(t)$. Hence, first the Galerkin orthogonality of the SUPG method gives

$$a_{\mathrm{SUPG}}(\Pi_h u(t), v_h) = a_{\mathrm{SUPG}}(u(t), v_h) \quad \forall\, v_h \in V_{h,r}. \tag{5.3}$$

Second, error estimates of the form

$$\|u(t) - \Pi_h u(t)\|_{\mathrm{SUPG}} \le C h^{r+1/2} \|u(t)\|_{r+1}, \quad t \in [0,T], \tag{5.4}$$

can be proven; see [16]. A direct calculation, using the linearity of the equation and the time-independency of convection, reaction, and the test functions, shows

$$(\Pi_h u(t))_t = \Pi_h(u_t(t)) = \Pi_h u_t. \tag{5.5}$$

For brevity, the dependency on time will be omitted in the notation.

Let $u_h : (0,T] \to V_{h,r}$ be the finite element solution of the continuous-in-time SUPG method:

$$(u_{h,t}, v_h) + a_{\mathrm{SUPG}}(u_h, v_h) = (f, v_h) + \delta(f - u_{h,t}, \mathbf{b}\cdot\nabla v_h) \quad \forall\, v_h \in V_{h,r} \tag{5.6}$$

with $u_h(0)$ given.

For the error analysis, the following norms in $V_{h,r}$ are introduced:

$$\|v_h\|_{\mathbf{b}} := \left(\|v_h\|_0^2 + \delta^2 \|\mathbf{b}\cdot\nabla v_h\|_0^2\right)^{1/2}, \quad \|v_h\|_{\mathrm{mat}} := \delta^{1/2}\|v_{h,t} + \mathbf{b}\cdot\nabla v_h\|_0.$$

The expression in the second norm is the material derivative. Note that $\|\cdot\|_{\mathbf{b}}$ is equivalent to the L^2 norm, since by using the inverse inequality and the definition (5.1) of the stabilization parameter, one obtains

$$\|v_h\|_0 \le \|v_h\|_{\mathbf{b}} \le \left(\|v_h\|_0^2 + \delta^2\|\mathbf{b}\|_\infty^2 c_{\mathrm{inv}}^2 h^{-2}\|v_h\|_0^2\right)^{1/2} \le \frac{\sqrt{17}}{4}\|v_h\|_0.$$

Denote the error between the continuous-in-time finite element solution and the solution of the steady-state problem by $e_h = u_h - \Pi_h u$ and let $T_{\mathrm{tr}} = u_t - \Pi_h u_t$ be the truncation error. An error equation is obtain by subtracting (5.2) from (5.6):

$$(e_{h,t}, v_h) + a_{\mathrm{SUPG}}(e_h, v_h) = (T_{\mathrm{tr}}, v_h) + \delta(T_{\mathrm{tr}}, \mathbf{b}\cdot\nabla v_h) - \delta(e_{h,t}, \mathbf{b}\cdot\nabla v_h) \quad \forall\, v_h \in V_{h,r}. \tag{5.7}$$

Setting $v_h = e_h$ in (5.7) gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|e_h\|_0^2 + \varepsilon\|\nabla e_h\|_0^2 + \delta\|\mathbf{b}\cdot\nabla e_h\|_0^2 + \|\mu^{1/2}e_h\|_0^2 + \delta(e_{h,t}, \mathbf{b}\cdot\nabla e_h) \\ &\quad = (T_{\mathrm{tr}}, e_h + \delta\mathbf{b}\cdot\nabla e_h) - \delta(ce_h, \mathbf{b}\cdot\nabla e_h) + \sum_{K\in\mathcal{T}_h}\delta\varepsilon(\Delta e_h, \mathbf{b}\cdot\nabla e_h)_K. \end{aligned} \tag{5.8}$$

Analogously, one obtains for $v_h = e_{h,t}$ in (5.7)

$$\begin{aligned} &\|e_{h,t}\|_0^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|\nabla e_h\|_0^2 + (\mathbf{b}\cdot\nabla e_h, e_{h,t}) + \frac{\delta}{2}\frac{d}{dt}\|\mathbf{b}\cdot\nabla e_h\|_0^2 + \frac{1}{2}\frac{d}{dt}\|c^{1/2}e_h\|_0^2 \\ &\quad = (T_{\mathrm{tr}}, (e_h + \delta\mathbf{b}\cdot\nabla e_h)_t) - \delta(ce_h, \mathbf{b}\cdot\nabla e_{h,t}) + \sum_{K\in\mathcal{T}_h}\delta\varepsilon(\Delta e_h, \mathbf{b}\cdot\nabla e_{h,t})_K \\ &\qquad - \delta(e_{h,t}, \mathbf{b}\cdot\nabla e_{h,t}). \end{aligned} \tag{5.9}$$

The addition of δ times (5.9) to (5.8) leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e_h\|_{\mathbf{b}}^2 + \varepsilon\|\nabla e_h\|_0^2 + \|\mu^{1/2}e_h\|_0^2 + \|e_h\|_{\mathrm{mat}}^2 + \frac{\varepsilon\delta}{2}\frac{d}{dt}\|\nabla e_h\|_0^2 + \frac{\delta}{2}\frac{d}{dt}\|c^{1/2}e_h\|_0^2 \\
&\quad = (T_{\mathrm{tr}}, e_h + \delta\mathbf{b}\cdot\nabla e_h) + \delta(T_{\mathrm{tr}}, (e_h + \delta\mathbf{b}\cdot\nabla e_h)_t) \\
&\qquad - \delta(ce_h, \mathbf{b}\cdot\nabla e_h) - \delta^2(ce_h, \mathbf{b}\cdot\nabla e_{h,t}) + \sum_{K\in\mathcal{T}_h}\delta\varepsilon(\Delta e_h, \mathbf{b}\cdot\nabla e_h)_K \\
&\qquad + \sum_{K\in\mathcal{T}_h}\delta^2\varepsilon(\Delta e_h, \mathbf{b}\cdot\nabla e_{h,t})_K - \delta^2(e_{h,t}, \mathbf{b}\cdot\nabla e_{h,t}),
\end{aligned}
\tag{5.10}
$$

where the definition of $\|\cdot\|_{\mathbf{b}}$ and $\delta\left(\|e_{h,t}\|_0^2 + 2(e_{h,t}, \mathbf{b}\cdot\nabla e_h) + \|\mathbf{b}\cdot\nabla e_h\|_0^2\right) = \|e_h\|_{\mathrm{mat}}^2$ have been used. Now, the goal consists in estimating all terms on the right-hand side of (5.10) and absorbing the terms with e_h and $e_{h,t}$ into the left-hand side. Note that all terms with $e_{h,t}$ have to be absorbed into the norm of the material derivative since this is the only term on the left-hand side which includes the temporal derivative.

The estimate of all terms on the right-hand side of (5.10) starts with the Cauchy–Schwarz inequality. Observe that

$$
\begin{aligned}
\|e_h + \delta\mathbf{b}\cdot\nabla e_h\|_0 &\le \|e_h\|_0 + \delta\|\mathbf{b}\|_\infty c_{\mathrm{inv}} h^{-1}\|e_h\|_0 \\
&\le \mu_0^{-1/2}\|\mu^{1/2}e_h\|_0 + \mu_0^{-1/2}\delta\|\mathbf{b}\|_\infty c_{\mathrm{inv}} h^{-1}\|\mu^{1/2}e_h\|_0.
\end{aligned}
$$

Assuming (5.1) gives for the first term on the right-hand side of (5.10), together with the application of Young's inequality,

$$
\begin{aligned}
(T_{\mathrm{tr}}, e_h + \delta\mathbf{b}\cdot\nabla e_h) &\le \|T_{\mathrm{tr}}\|_0\mu_0^{-1/2}\|\mu^{1/2}e_h\|_0 + \frac{1}{8}\|T_{\mathrm{tr}}\|_0\|\mu^{1/2}e_h\|_0 \\
&\le \left(4\mu_0^{-1} + \frac{1}{16}\right)\|T_{\mathrm{tr}}\|_0^2 + \frac{1}{8}\|\mu^{1/2}e_h\|_0^2.
\end{aligned}
\tag{5.11}
$$

For estimating the second term on the right-hand side of (5.10), a norm of the temporal derivative of the error has to be bounded. With (5.1), one obtains

$$
\begin{aligned}
\delta\|e_{h,t} + \delta\mathbf{b}\cdot\nabla e_{h,t}\|_0 &\le \delta(\|e_{h,t}\|_0 + \delta\|\mathbf{b}\|_\infty c_{\mathrm{inv}} h^{-1}\|e_{h,t}\|_0) \le 2\delta\|e_{h,t}\|_0 \\
&\le 2\delta(\|e_{h,t} + \mathbf{b}\cdot\nabla e_h\|_0 + \|\mathbf{b}\cdot\nabla e_h\|_0) \\
&\le 2\delta^{1/2}\|e_h\|_{\mathrm{mat}} + 2\delta\|\mathbf{b}\|_\infty c_{\mathrm{inv}} h^{-1}\mu_0^{-1/2}\|\mu^{1/2}e_h\|_0 \\
&\le 2\delta^{1/2}\|e_h\|_{\mathrm{mat}} + \frac{1}{8}\|\mu^{1/2}e_h\|_0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\delta(T_{\mathrm{tr}}, (e_h + \delta\mathbf{b}\cdot\nabla e_h)_t) &\le \|T_{\mathrm{tr}}\|_0\left(2\delta^{1/2}\|e_h\|_{\mathrm{mat}} + \frac{1}{8}\|\mu^{1/2}e_h\|_0\right) \\
&\le \left(16\delta + \frac{1}{16}\right)\|T_{\mathrm{tr}}\|_0^2 + \frac{1}{16}\|e_h\|_{\mathrm{mat}}^2 + \frac{1}{16}\|\mu^{1/2}e_h\|_0^2.
\end{aligned}
\tag{5.12}
$$

The third term on the right-hand side of (5.10) is estimated simply by

$$
\delta(ce_h, \mathbf{b}\cdot\nabla e_h) \le \delta\|c\|_\infty\|\mathbf{b}\|_\infty c_{\mathrm{inv}} h^{-1}\mu_0^{-1}\|\mu^{1/2}e_h\|_0^2 \le \frac{1}{16}\|\mu^{1/2}e_h\|_0^2,
\tag{5.13}
$$

where (5.1) has been used. Reasoning in the same way for the fourth term gives

$$\begin{aligned}\delta^2(ce_h, \mathbf{b}\cdot\nabla e_{h,t}) &\le \delta^2\|c\|_\infty\|\mathbf{b}\|_\infty c_{\text{inv}}h^{-1}\mu_0^{-1/2}\|\mu^{1/2}e_h\|_0\|e_{h,t}\|_0\\ &\le \delta^{3/2}\|c\|_\infty\|\mathbf{b}\|_\infty c_{\text{inv}}h^{-1}\mu_0^{-1/2}\|\mu^{1/2}e_h\|_0\|e_h\|_{\text{mat}}\\ &\quad + \delta^2\|c\|_\infty\|\mathbf{b}\|_\infty^2 c_{\text{inv}}^2 h^{-2}\mu_0^{-1}\|\mu^{1/2}e_h\|_0^2.\end{aligned}$$

Assuming (5.1) yields

$$\begin{aligned}\delta^2(ce_h, \mathbf{b}\cdot\nabla e_{h,t}) &\le \frac{\delta^{1/2}}{8}\|\mu^{1/2}e_h\|_0\|e_h\|_{\text{mat}} + \frac{1}{16}\|\mu^{1/2}e_h\|_0^2\\ &\le \left(\frac{\delta}{16}+\frac{1}{16}\right)\|\mu^{1/2}e_h\|_0^2 + \frac{1}{16}\|e_h\|_{\text{mat}}^2\\ &\le \frac{1}{8}\|\mu^{1/2}e_h\|_0^2 + \frac{1}{16}\|e_h\|_{\text{mat}}^2. \qquad (5.14)\end{aligned}$$

For the fifth term on the right-hand side of (5.10), one obtains

$$\sum_{K\in\mathcal{T}_h}\delta\varepsilon(\Delta e_h, \mathbf{b}\cdot\nabla e_h)_K \le \sum_{K\in\mathcal{T}_h}\delta\varepsilon c_{\text{inv}}h^{-1}\|\mathbf{b}\|_{\infty,K}\|\nabla e_h\|_{0,K}^2 \le \frac{\varepsilon}{2}\|\nabla e_h\|_0^2 \qquad (5.15)$$

if (5.1) is fulfilled. The estimate of the sixth term starts as follows:

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\delta^2\varepsilon(\Delta e_h, \mathbf{b}\cdot\nabla e_{h,t})_K &\le \sum_{K\in\mathcal{T}_h}\delta^2\varepsilon c_{\text{inv}}^2h^{-2}\|\mathbf{b}\|_{\infty,K}\|\nabla e_h\|_{0,K}\|e_{h,t}\|_{0,K}\\ &\le \sum_{K\in\mathcal{T}_h}\delta^{3/2}\varepsilon c_{\text{inv}}^2h^{-2}\|\mathbf{b}\|_{\infty,K}\|\nabla e_h\|_{0,K}\|e_h\|_{\text{mat},K}\\ &\quad + \sum_{K\in\mathcal{T}_h}\delta^2\varepsilon c_{\text{inv}}^2h^{-2}\|\mathbf{b}\|_{\infty,K}\|\nabla e_h\|_{0,K}\|\mathbf{b}\cdot\nabla e_h\|_{0,K}\\ &\le \sum_{K\in\mathcal{T}_h}\delta^{3/2}\varepsilon c_{\text{inv}}^2h^{-2}\|\mathbf{b}\|_{\infty,K}\|\nabla e_h\|_{0,K}\|e_h\|_{\text{mat},K}\\ &\quad + \sum_{K\in\mathcal{T}_h}\delta^2\varepsilon c_{\text{inv}}^2h^{-2}\|\mathbf{b}\|_{\infty,K}^2\|\nabla e_h\|_{0,K}^2.\end{aligned}$$

Now, assuming (5.1) and $\varepsilon\le\delta$ leads to

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\delta^2\varepsilon(\Delta e_h, \mathbf{b}\cdot\nabla e_{h,t})_K\\ &\le \sum_{K\in\mathcal{T}_h}\frac{\varepsilon\delta^{-1/2}}{16}\|\nabla e_h\|_{0,K}\|e_h\|_{\text{mat},K} + \sum_{K\in\mathcal{T}_h}\frac{\varepsilon}{16}\|\nabla e_h\|_{0,K}^2\\ &\le \frac{\varepsilon^2}{32\delta}\|\nabla e_h\|_0^2 + \frac{\varepsilon}{32}\|e_h\|_{\text{mat}}^2 + \frac{\varepsilon}{16}\|\nabla e_h\|_0^2 \le \frac{3}{32}\varepsilon\|\nabla e_h\|_0^2 + \frac{1}{32}\|e_h\|_{\text{mat}}^2. \qquad (5.16)\end{aligned}$$

Finally, the last term on the right-hand side of (5.10) is estimated as follows:

$$\begin{aligned}\delta^2(e_{h,t}, \mathbf{b}\cdot\nabla e_{h,t}) &\le \delta^2\|\mathbf{b}\|_\infty h^{-1}c_{\text{inv}}\|e_{h,t}\|_0^2\\ &\le 2\delta^2\|\mathbf{b}\|_\infty h^{-1}c_{\text{inv}}(\|e_{h,t}+\mathbf{b}\cdot\nabla e_h\|_0^2 + \|\mathbf{b}\cdot\nabla e_h\|_0^2)\\ &\le 2\delta\|\mathbf{b}\|_\infty h^{-1}c_{\text{inv}}\|e_h\|_{\text{mat}}^2 + 2\delta^2\|\mathbf{b}\|_\infty^2 h^{-2}c_{\text{inv}}^2\mu_0^{-1}\|\mu^{1/2}e_h\|_0^2.\end{aligned}$$

Assuming (5.1) gives

$$\delta^2(e_{h,t}, \mathbf{b}\cdot\nabla e_{h,t}) \le \frac{1}{4}\|e_h\|_{\mathrm{mat}}^2 + \frac{1}{8}\|\mu^{1/2}e_h\|_0^2. \tag{5.17}$$

Inserting now (5.11)–(5.17) into (5.10) leads to

$$\frac{1}{2}\frac{d}{dt}\|e_h\|_{\mathbf{b}}^2 + \frac{13}{32}\varepsilon\|\nabla e_h\|_0^2 + \frac{1}{2}\|\mu^{1/2}e_h\|_0^2 + \frac{19}{32}\|e_h\|_{\mathrm{mat}}^2 + \frac{\varepsilon\delta}{2}\frac{d}{dt}\|\nabla e_h\|_0^2 + \frac{\delta}{2}\frac{d}{dt}\|c^{1/2}e_h\|_0^2 \le C\|T_{\mathrm{tr}}\|_0^2,$$

where $C = 4\mu_0^{-1} + 16\delta + 1/8$ and the conditions on the stabilization parameter are given in (5.1). Redefining C, this inequality can be written in the form

$$\begin{aligned}&\frac{d}{dt}\|e_h\|_{\mathbf{b}}^2 + \varepsilon\|\nabla e_h\|_0^2 + \|\mu^{1/2}e_h\|_0^2 + \|e_h\|_{\mathrm{mat}}^2\\ &\qquad + \varepsilon\delta\frac{d}{dt}\|\nabla e_h\|_0^2 + \delta\frac{d}{dt}\|c^{1/2}e_h\|_0^2 \le C\|T_{\mathrm{tr}}\|_0^2.\end{aligned} \tag{5.18}$$

Integration in $(0,t)$ yields

$$\begin{aligned}&\|e_h(t)\|_{\mathbf{b}}^2 + \varepsilon\|\nabla e_h\|_{L^2(0,t;L^2)}^2 + \|\mu^{1/2}e_h\|_{L^2(0,t;L^2)}^2 + \|e_h\|_{L^2(0,t;\mathrm{mat})}^2\\ &\quad + \varepsilon\delta\|\nabla e_h(t)\|_0^2 + \delta\|(c^{1/2}e_h)(t)\|_0^2\\ &\le \|e_h(0)\|_{\mathbf{b}}^2 + \varepsilon\delta\|\nabla e_h(0)\|_0^2 + \delta\|(c^{1/2}e_h)(0)\|_0^2 + C\int_0^t \|T_{\mathrm{tr}}\|_0^2\,d\tau.\end{aligned} \tag{5.19}$$

Now, the terms on the right-hand side of (5.19) have to be bounded. One obtains

$$\begin{aligned}&\|e_h(0)\|_{\mathbf{b}}^2 + \varepsilon\delta\|\nabla e_h(0)\|_0^2 + \delta\|(c^{1/2}e_h)(0)\|_0^2\\ &\le \left(\frac{17}{16} + \frac{\varepsilon\delta c_{\mathrm{inv}}^2}{h^2} + \delta\|c\|_\infty\right)\|e_h(0)\|_0^2 \le C\|e_h(0)\|_0^2\end{aligned} \tag{5.20}$$

since $\varepsilon \le h$ is assumed.

The next step of the error analysis uses the fact that convection and reaction do not depend on time. Hence (5.2) can be differentiated with respect to time. Using (5.5), one obtains steady-state SUPG problems for $\Pi_h u_t(t)$ with corresponding error estimates of type (5.4):

$$\|T_{\mathrm{tr}}(t)\|_{\mathrm{SUPG}} \le Ch^{r+1/2}\|u_t(t)\|_{r+1} \implies \|T_{\mathrm{tr}}(t)\|_0 \le C\frac{h^{r+1/2}}{\mu_0^{1/2}}\|u_t(t)\|_{r+1},\ t\in[0,T]. \tag{5.21}$$

Summarizing all constant into a generic constant, the following lemma is proven.

Lemma 5.1 (error estimate for the time-continuous case involving the material derivative). *Let $t \le T < \infty$ and let $u_t \in L^2(0;T;H^{r+1}(\Omega))$. Then the error $e_h = u_h - \Pi_h u$ satisfies*

$$\begin{aligned}&\|e_h(t)\|_{\mathbf{b}} + \left(\varepsilon\|\nabla e_h\|_{L^2(0,t;L^2)}^2 + \|e_h\|_{L^2(0,t;\mathrm{mat})}^2 + \|\mu^{1/2}e_h\|_{L^2(0,t;L^2)}^2\right)^{1/2}\\ &\qquad + \delta^{1/2}\left(\varepsilon^{1/2}\|\nabla e_h(t)\|_0 + \|(c^{1/2}e_h)(t)\|_0\right)\\ &\le C\left(\|e_h(0)\|_0 + h^{r+1/2}\|u_t\|_{L^2(0,t;H^{r+1})}\right),\end{aligned} \tag{5.22}$$

where C depends on the coefficients of the problem and on c_{inv}.

An estimate for $u - u_h$ is now obtained by applying the triangle inequality and using (5.4) for estimating the terms with $u - \Pi_h u$.

In the second step, an estimate with the stronger SUPG norm $\delta\|\mathbf{b}\cdot\nabla e_h\|_{L^2(0,t;L^2)}$ instead of $\|e_h\|_{L^2(0,t;\mathrm{mat})}$ is derived. To this end, insert once more $v_h = e_h$ into the error equation (5.7) and apply a standard analysis by using the coercivity (2.12):

$$\frac{1}{2}\frac{d}{dt}\|e_h\|_0^2 + \frac{1}{2}\|e_h\|_{\mathrm{SUPG}}^2 \le \frac{\|T_{\mathrm{tr}}\|_0^2}{\mu_0} + \frac{\|\mu^{1/2}e_h\|_0^2}{4} + 2\delta\|T_{\mathrm{tr}}\|_0^2 + 2\delta\|e_{h,t}\|_0^2 + \delta\frac{\|\mathbf{b}\cdot\nabla e_h\|_0^2}{4}.$$

The second and the last terms can be absorbed into the left-hand side. The first and the third are estimated by (5.21). Integration in $(0,t)$ gives

$$\|e_h(t)\|_0^2 + \|e_h\|_{L^2(0,t;\mathrm{SUPG})}^2 \le \|e_h(0)\|_0^2 + C\left(h^{2r+1}\|u_t\|_{L^2(0,t;H^{r+1})}^2 + \delta\|e_{h,t}\|_{L^2(0,t;L^2)}^2\right). \quad (5.23)$$

The estimate of the last term uses once more the fact that convection and reaction are functions independent of time. Hence, (5.2) and (5.6) can be differentiated with respect to time, giving the same type of equations. Now, the error analysis for $e_{h,t}$ can be carried out in the same way as the analysis for e_h which led to (5.22). One obtains

$$\delta\|e_{h,t}\|_{L^2(0,t;L^2)}^2 \le \frac{\delta}{\mu^0}\|\mu^{1/2}e_{h,t}\|_{L^2(0,t;L^2)}^2 \le C\delta\Big(\|e_{h,t}(0)\|_0^2 + h^{2r+1}\|u_{tt}\|_{L^2(0,t;H^{r+1})}^2\Big). \quad (5.24)$$

Next, $\delta\|e_{h,t}(0)\|_0^2$ has to be bounded in terms of $e_h(0)$ and $T_{\mathrm{tr}}(0)$ since it is not clear how to control $e_{h,t}(0)$ by an appropriate choice of $u_h(0)$. To this end, $e_{h,t}(t)$ is inserted into the error equation (5.7) leading to

$$\|e_{h,t}\|_0^2 = -a_{\mathrm{SUPG}}(e_h, e_{h,t}) + (T_{\mathrm{tr}}, e_{h,t} + \delta\mathbf{b}\cdot\nabla e_{h,t}) - \delta(e_{h,t}, \mathbf{b}\cdot\nabla e_{h,t}). \quad (5.25)$$

Applying the Cauchy–Schwarz inequality and the inverse inequality, using the assumption $\varepsilon \le h$ and (5.1) yields

$$\begin{aligned} &a_{\mathrm{SUPG}}(u_h, v_h) \\ &\le \left(\frac{\varepsilon c_{\mathrm{inv}}}{h}\|\nabla u_h\|_0 + \|\mathbf{b}\cdot\nabla u_h\|_0 + \|c\|_\infty^{1/2}\|c^{1/2}u_h\|_0 + \frac{\varepsilon\delta c_{\mathrm{inv}}^2\|\mathbf{b}\|_\infty}{h^2}\|\nabla u_h\|_0\right. \\ &\quad \left. + \frac{\delta c_{\mathrm{inv}}\|\mathbf{b}\|_\infty}{h}\|\mathbf{b}\cdot\nabla u_h\|_0 + \frac{\delta c_{\mathrm{inv}}\|\mathbf{b}\|_\infty\|c\|_\infty^{1/2}}{h}\|c^{1/2}u_h\|_0\right)\|v_h\|_0 \\ &\le C\left(\|\nabla u_h\|_0 + \|\mathbf{b}\cdot\nabla u_h\|_0 + \|c^{1/2}u_h\|_0\right)\|v_h\|_0, \end{aligned}$$

where C depends on c_{inv} and the coefficients of the problem. Applying this estimate to (5.25) and using (5.1) gives

$$\|e_{h,t}\|_0^2 \le C\left(\|\nabla e_h\|_0 + \|\mathbf{b}\cdot\nabla e_h\|_0 + \|c^{1/2}e_h\|_0 + \|T_{\mathrm{tr}}\|_0\right)\|e_{h,t}\|_0 + \frac{1}{8}\|e_{h,t}\|_0^2.$$

From this follows, with (5.21),

$$\delta\|e_{h,t}\|_0^2 \le C\delta\left(\|\nabla e_h\|_0^2 + \|\mathbf{b}\cdot\nabla e_h\|_0^2 + \|c^{1/2}e_h\|_0^2 + h^{2r+1}\|u_t(t)\|_{r+1}^2\right).$$

Using this estimate for $t = 0$ in (5.24), inserting then (5.24) into (5.23), and applying the triangle inequality leads to the following error estimate.

THEOREM 5.2 (error estimate for the time-continuous case involving the SUPG norm). *Let $t \le T < \infty$, let $u_t(t) \in H^{r+1}(\Omega)$ for all $t \in [0,T]$, and let $u, u_t, u_{tt} \in L^2(0;T;H^{r+1}(\Omega))$. Then the error estimate*

$$\begin{aligned}
&\|(u-u_h)(t)\|_0 + \|u-u_h\|_{L^2(0,t;\mathrm{SUPG})}\\
&\le C\left[\|e_h(0)\|_0 + \delta^{1/2}\left(\|\nabla e_h(0)\|_0 + \|(\mathbf{b}\cdot\nabla e_h)(0)\|_0 + \|(c^{1/2}e_h)(0)\|_0\right)\right]\\
&\quad + Ch^{r+1/2}\left[\|u(t)\|_{r+1} + \delta^{1/2}\|u_t(0)\|_{r+1} + \|u\|_{L^2(0,t;H^{r+1})} + \|u_t\|_{L^2(0,t;H^{r+1})}\right.\\
&\qquad\left. + \delta^{1/2}\|u_{tt}\|_{L^2(0,t;H^{r+1})}\right]
\end{aligned} \tag{5.26}$$

holds. The constants depend on the coefficients of the problem and on c_{inv}.

Choosing the initial finite element solution $u_h(0)$ in the way that $u_h(0)$ solves

$$\begin{aligned}
a_{\mathrm{SUPG}}(u_h(0), v_h) &= (f(0) - u_t(0), v_h) + \delta(f(0) - u_t(0), \mathbf{b}\cdot\nabla v_h)\\
&= (-\varepsilon\Delta u_0 + \mathbf{b}\cdot\nabla u_0 + cu_0, v_h + \delta\mathbf{b}\cdot\nabla v_h) \quad \forall\, v_h \in V_{h,r}
\end{aligned}$$

leads to $e_h(0) = 0$ such that all terms with $e_h(0)$ vanish in (5.26).

5.2. Temporal discretization with the backward Euler scheme. The analysis of the backward Euler scheme (3.1) is based on the same ideas that were used in the time-continuous case. For this reason, only its most important steps are presented.

Let $e_h^n = u_h^n - \Pi_h u(t_n)$, $e_{h,\tau}^n = (e_h^n - e_h^{n-1})/k$ for $n \ge 1$, and

$$T_{\mathrm{tr,eul}}^n = (u_t(t_n) - \Pi_h u_t(t_n)) + \left(\Pi_h u_t(t_n) - \frac{\Pi_h u(t_n) - \Pi_h u(t_{n-1})}{k}\right).$$

A straightforward calculation, using (5.3), gives the error equation

$$(e_{h,\tau}^n, v_h) + a_{\mathrm{SUPG}}(e_h^n, v_h) = (T_{\mathrm{tr,eul}}^n, v_h) + \delta(T_{\mathrm{tr,eul}}^n, \mathbf{b}\cdot\nabla v_h) - \delta(e_{h,\tau}^n, \mathbf{b}\cdot\nabla v_h) \tag{5.27}$$

for all $v_h \in V_{h,r}$. Taking first the inner product with e_h^n and then with $\delta e_{h,\tau}^n$, one obtains

$$\begin{aligned}
&(e_{h,\tau}^n, e_h^n) + \delta^2(\mathbf{b}\cdot\nabla e_{h,\tau}^n, \mathbf{b}\cdot\nabla e_h^n) + \varepsilon\|\nabla e_h^n\|_0^2 + \|\mu^{1/2}e_h^n\|_0^2 + \|e_h^n\|_{\mathrm{mat,eul}}^2\\
&\quad + \varepsilon\delta(\nabla e_h^n, \nabla e_{h,\tau}^n) + \delta(ce_h^n, e_{h,\tau}^n)\\
&= (T_{\mathrm{tr,eul}}^n, e_h^n + \delta\mathbf{b}\cdot\nabla e_h^n) + \delta(T_{\mathrm{tr,eul}}^n, (e_h^n + \delta\mathbf{b}\cdot\nabla e_h^n)_{,\tau})\\
&\quad - \delta(ce_h^n, \mathbf{b}\cdot\nabla e_h^n) - \delta^2(ce_h^n, \mathbf{b}\cdot\nabla e_{h,\tau}^n)\\
&\quad + \sum_{K\in\mathcal{T}_h}\delta\varepsilon(\Delta e_h^n, \mathbf{b}\cdot\nabla e_h^n)_K + \sum_{K\in\mathcal{T}_h}\delta^2\varepsilon(\Delta e_h^n, \mathbf{b}\cdot\nabla e_{h,\tau}^n)_K\\
&\quad - \delta^2(e_{h,\tau}^n, \mathbf{b}\cdot\nabla e_{h,\tau}^n),
\end{aligned} \tag{5.28}$$

where the material derivative for the Euler scheme is defined by

$$\|v_h\|_{\mathrm{mat,eul}} := \delta^{1/2}\|v_{h,\tau} + \mathbf{b}\cdot\nabla v_h\|_0.$$

The right-hand side of (5.28) has principally the same form as the right-hand side of (5.10). The second term of (5.28) can be estimated in an analogous way as (5.12), with the material derivative replaced by its discrete counterpart. Applying the same techniques as for the time-continuous case, one arrives at the discrete version of (5.18):

$$(e_{h,\tau}^n, e_h^n) + \delta^2(\mathbf{b}\cdot\nabla e_{h,\tau}^n, \mathbf{b}\cdot\nabla e_h^n) + \varepsilon\|\nabla e_h^n\|_0^2 + \|\mu^{1/2}e_h^n\|_0^2 + \|e_h^n\|_{\mathrm{mat,eul}}^2 + \varepsilon\delta(\nabla e_h^n, \nabla e_{h,\tau}^n) + \delta(ce_h^n, e_{h,\tau}^n) \le C\|T_{\mathrm{tr,eul}}^n\|_0^2.$$

Note that $(e_{h,\tau}^n, e_h^n) = \frac{1}{2k}\left(\|e_h^n\|_0^2 - \|e_h^{n-1}\|_0^2 + \|e_h^n - e_h^{n-1}\|_0^2\right)$ and similar expressions hold for all other inner products on the left-hand side. Summation of the discrete times $j = 1,\ldots,n$ and neglecting terms of the form $\|e_h^j - e_h^{j-1}\|_0^2$ on the left-hand side gives

$$\|e_h^n\|_{\mathbf{b}}^2 + \sum_{j=1}^n k\left(\varepsilon\|\nabla e_h^j\|_0^2 + \|\mu^{1/2}e_h^j\|_0^2 + \|e_h^j\|_{\mathrm{mat,eul}}^2\right) + \varepsilon\delta\|\nabla e_h^n\|_0^2 + \delta\|c^{1/2}e_h^n\|_0^2 \le \|e_h^0\|_{\mathbf{b}}^2 + \varepsilon\delta\|\nabla e_h^0\|_0^2 + \delta\|c^{1/2}e_h^0\|_0^2 + C\sum_{j=1}^n k\|T_{\mathrm{tr,eul}}^j\|_0^2. \tag{5.29}$$

As in the time-continuous case, the errors at the initial time can be bounded by

$$\|e_h^0\|_{\mathbf{b}}^2 + \varepsilon\delta\|\nabla e_h^0\|_0^2 + \delta\|c^{1/2}e_h^0\|_0^2 \le C\|e_h^0\|_0^2.$$

Using (5.4) and reasoning as in (4.3) yields

$$\begin{aligned}
&\sum_{j=1}^n k\|T_{\mathrm{tr,eul}}^j\|_0^2\\
&\le 2\sum_{j=1}^n k\left(\|u_t(t_n) - \Pi_h u_t(t_n)\|_0^2 + \left\|\Pi_h u_t(t_n) - \frac{\Pi_h u(t_n) - \Pi_h u(t_{n-1})}{k}\right\|_0^2\right)\\
&\le C\left(knh^{2r+1}\max_{0\le s\le n}\|u_t(t_s)\|_{r+1}^2 + k^2\int_{t_0}^{t_n}\|\Pi_h u_{tt}\|_0^2\,dt\right) \qquad (5.30)\\
&\le C\left(knh^{2r+1}\|u_t\|_{L^\infty(0,t_n;H^{r+1})}^2 + k^2\|\Pi_h u_{tt}\|_{L^2(0,t_n;L^2)}^2\right).
\end{aligned}$$

Inserting the last two estimates into (5.29) gives an estimate for the material derivative of the error.

The convergence analysis for the error in the SUPG norm starts by taking $v_h = e_h^n$ in (5.27) and applying estimates similar to those in the time-continuous case. Absorbing terms leads to an estimate of the form (5.23):

$$\|e_h^n\|_0^2 + k\sum_{j=1}^n \|e_h^j\|_{\mathrm{SUPG}}^2 \le C\left(\|e_h^0\|_0^2 + \sum_{j=1}^n k\|T_{\mathrm{tr,eul}}^j\|_0^2 + \sum_{j=1}^n k\delta\|e_{h,\tau}^j\|_0^2\right).$$

The second term on the right-hand side is estimated in (5.30). Set $z_h^n = e_{h,\tau}^n$ and assume for simplicity $e_h^0 = 0$. Subtracting (5.27) for $n-1$ from (5.27) for n, as $n \ge 2$, and dividing by k gives for $n \ge 2$

$$(z_{h,\tau}^n, v_h) + a_{\mathrm{SUPG}}(z_h^n, v_h) = (T_{\mathrm{tr,eul},\tau}^n, v_h) + \delta(T_{\mathrm{tr,eul},\tau}^n, \mathbf{b}\cdot\nabla v_h) - \delta(z_{h,\tau}^n, \mathbf{b}\cdot\nabla v_h)$$

for all $v_h \in V_{h,r}$, where $T^n_{\mathrm{tr,eul},\tau} = (T^n_{\mathrm{tr,eul}} - T^{n-1}_{\mathrm{tr,eul}})/k$. Then the same error analysis with respect to the material derivative as for e_h^n can be applied to z_h^n, leading to the analogue of (5.29):

$$\mu_0 \sum_{j=1}^n k\delta \|z_h^j\|_0^2 \le \mu_0 k\delta \|z_h^1\|_0^2 + \sum_{j=2}^n k\delta \|\mu^{1/2} z_h^j\|_0^2 \le C\delta \|z_h^1\|_0^2 + C\sum_{j=2}^n k\delta \|T^j_{\mathrm{tr,eul},\tau}\|_0^2,$$

where a contribution to the first term on the right-hand side comes also from the error analysis for the material derivative. For the truncation error one uses the decomposition

$$T^j_{\mathrm{tr,eul},\tau} = \left(\frac{u_t(t_j) - u_t(t_{j-1})}{k} - u_{tt}(t_j)\right) + (u_{tt}(t_j) - \Pi_h u_{tt}(t_j)) \\ + \left(\Pi_h u_{tt}(t_j) - \frac{\Pi_h u(t_j) - 2\Pi_h u(t_{j-1}) + \Pi_h u(t_{j-2})}{k^2}\right).$$

The first and second terms can be bounded using the technique applied in (4.3) and the estimate (5.4). The third term is an $\mathcal{O}(k)$ approximation to $\Pi_h u_{tt}(t_j)$ such that an analogue estimate to (4.3) can be performed. One obtains

$$\sum_{j=2}^n k\delta \|T^j_{\mathrm{tr,eul},\tau}\|_0^2 \le C\delta\Big[knh^{2r+1} \max_{0\le s\le n} \|u_{tt}(t_s)\|_{r+1}^2 \\ + k^2 \left(\|u_{ttt}\|^2_{L^2(0,t_n;L^2)} + \|\Pi_h u_{ttt}\|^2_{L^2(0,t_n;L^2)}\right)\Big].$$

Now, it remains to bound $\delta\|z_h^1\|_0^2$. To this end, the same approach can be used as in the time-continuous framework to bound $\|e_{h,t}(0)\|_0$. One gets

$$\delta \|z_h^1\|_0^2 \le C\delta \left(\frac{\varepsilon^2}{h^2}\|\nabla e_h^1\|_0^2 + \|\mathbf{b}\cdot\nabla e_h^1\|_0^2 + \|c^{1/2} e_h^1\|_0^2 + \|T^1_{\mathrm{tr,eul}}\|_0^2\right) \\ \le C\left(\frac{\delta\varepsilon}{h^2}\varepsilon\|\nabla e_h^1\|_0^2 + \delta\|\mathbf{b}\cdot\nabla e_h^1\|_0^2 + \frac{\delta\|c\|_\infty}{\mu_0}\|\mu^{1/2} e_h^1\|_0^2 + \delta\|T^1_{\mathrm{tr,eul}}\|_0^2\right).$$

In addition to the time-continuous case, the first three terms on the right-hand side have to be estimated. Taking $n = 1$ and $v_h = e_h^1$ in (5.27), and recalling $e_h^0 = 0$, standard estimates lead to

$$\|e_h^1\|^2_{\mathrm{SUPG}} \le \frac{1}{k}\|e_h^1\|_0^2 + \|e_h^1\|^2_{\mathrm{SUPG}} \le C\|T^1_{\mathrm{tr,eul}}\|_0^2.$$

A direct estimate gives $\|T^1_{\mathrm{tr,eul}}\|_0^2 \le C(h^{2r+1} + k^2)$, where the constant depends on $\|\Pi_h u_{tt}(t_1)\|_0$ and $\|u_t(t_1)\|_{r+1}$. Now, the assumption $\varepsilon \le h$ and the definition (5.1) of the stabilization parameter lead to $\delta\|z_h^1\|_0^2 \le C\left(h^{2r+1} + k^2\right)$.

THEOREM 5.3 (error estimate for the backward Euler scheme involving the SUPG norm). *Let* $t_n = T < \infty$, *let* $u, u_t \in L^\infty(0,T;H^{r+1}(\Omega))$, *and let* $\Pi_h u_{tt}, u_{ttt}, \Pi_h u_{ttt} \in L^2(0;T;L^2(\Omega))$. *Then the error estimate*

$$\|(u-u_h)(t_n)\|_0^2 + k\sum_{j=1}^n \|(u-u_h)(t_j)\|^2_{\mathrm{SUPG}} \le C\left(h^{2r+1} + k^2\right) \tag{5.31}$$

holds with the stabilization parameter defined in (5.1). *The constant in* (5.31) *depends on norms of* u *and* $\Pi_h u$ *in the spaces given above, on the coefficients of the problem* (2.1), *and on* c_{inv}.

Applying the triangle inequality and (5.4), the constant can be expressed in norms of time derivatives of u instead of $\Pi_h u$.

5.3. Temporal discretization with the Crank–Nicolson scheme. The Crank–Nicolson/SUPG scheme has the following form: For $n = 1, 2, \ldots$ find $U_h^n \in V_{h,r}$ such that

$$(U_h^n - U_h^{n-1}, \varphi) + k a_{\mathrm{SUPG}}\left(\frac{U_h^n + U_h^{n-1}}{2}, \varphi\right) = k\left(\frac{f^n + f^{n-1}}{2}, \varphi\right)$$
$$+ \sum_{K\in\mathcal{T}_h} \delta_K \left(\frac{f^n + f^{n-1}}{2}, \mathbf{b}\cdot\nabla\varphi\right)_K - \sum_{K\in\mathcal{T}_h} \delta_K (U_h^n - U_h^{n-1}, \mathbf{b}\cdot\nabla\varphi)_K.$$

The error analysis for this method is for the most part similar to that of the backward Euler scheme. Using the Galerkin orthogonality (5.3), one can derive the error equation

$$(e_{h,\tau}^n, v_h) + a_{\mathrm{SUPG}}(e_h^{n,*}, v_h) = (T_{\mathrm{tr,CN}}^n, v_h) + \delta(T_{\mathrm{tr,CN}}^n, \mathbf{b}\cdot\nabla v_h) - \delta(e_{h,\tau}^n, \mathbf{b}\cdot\nabla v_h) \tag{5.32}$$

for all $v_h \in V_{h,r}$, where $e_h^{n,*} = (e_h^n + e_h^{n-1})/2$, and the truncation error is

$$T_{\mathrm{tr,CN}}^n = \frac{1}{2}\left((u_t(t_n) - \Pi_h u_t(t_n)) + (u_t(t_{n-1}) - \Pi_h u_t(t_{n-1}))\right)$$
$$+ \frac{1}{2}\left(\Pi_h u_t(t_n) + \Pi_h u_t(t_{n-1})\right) - \frac{\Pi_h u(t_n) - \Pi_h u(t_{n-1})}{k}.$$

Taking first $v_h = e_h^{n,*}$ in (5.32), then $v_h = \delta e_{h,\tau}^n$, and adding both equations yields

$$(e_{h,\tau}^n, e_h^{n,*}) + \delta^2(\mathbf{b}\cdot\nabla e_{h,\tau}^n, \mathbf{b}\cdot\nabla e_h^{n,*}) + \varepsilon\|\nabla e_h^{n,*}\|_0^2 + \|\mu^{1/2} e_h^{n,*}\|_0^2 + \|e_h^{n,*}\|_{\mathrm{mat,CN}}^2$$
$$+ \varepsilon\delta(\nabla e_h^{n,*}, \nabla e_{h,\tau}^n) + \delta(c e_h^{n,*}, e_{h,\tau}^n)$$
$$= (T_{\mathrm{tr,CN}}^n, e_h^{n,*} + \delta\mathbf{b}\cdot\nabla e_h^{n,*}) + \delta(T_{\mathrm{tr,CN}}^n, (e_h^n + \delta\mathbf{b}\cdot\nabla e_h^n)_\tau)$$
$$- \delta(c e_h^{n,*}, \mathbf{b}\cdot\nabla e_h^{n,*}) - \delta^2(c e_h^{n,*}, \mathbf{b}\cdot\nabla e_{h,\tau}^n)$$
$$+ \sum_{K\in\mathcal{T}_h} \delta\varepsilon(\Delta e_h^{n,*}, \mathbf{b}\cdot\nabla e_h^{n,*})_K + \sum_{K\in\mathcal{T}_h} \delta^2\varepsilon(\Delta e_h^{n,*}, \mathbf{b}\cdot\nabla e_{h,\tau}^n)_K - \delta^2(e_{h,\tau}^n, \mathbf{b}\cdot\nabla e_{h,\tau}^n),$$

where $\|e_h^{n,*}\|_{\mathrm{mat,CN}} := \delta^{1/2}\|e_{h,\tau}^n + \mathbf{b}\cdot\nabla e_h^{n,*}\|_0$. This equation has the same form as (5.28), with $e_h^{n,*}$ instead of e_h^n. Thus, using analogous estimates gives

$$(e_{h,\tau}^n, e_h^{n,*}) + \delta^2(\mathbf{b}\cdot\nabla e_{h,\tau}^n, \mathbf{b}\cdot\nabla e_h^{n,*}) + \varepsilon\|\nabla e_h^{n,*}\|_0^2 + \|\mu^{1/2} e_h^{n,*}\|_0^2 + \|e_h^{n,*}\|_{\mathrm{mat,CN}}^2$$
$$+ \varepsilon\delta(\nabla e_h^{n,*}, \nabla e_{h,\tau}^n) + \delta(c e_h^{n,*}, e_{h,\tau}^n) \le C\|T_{\mathrm{tr,CN}}^n\|_0^2.$$

Note that

$$(\nabla e_{h,\tau}^n, \nabla e_h^{n,*}) = \frac{1}{2k}\left(\|\nabla e_h^n\|_0^2 - \|\nabla e_h^{n-1}\|_0^2\right), \tag{5.33}$$

and similarly for the other inner products. Summation of the time steps $j = 1, \ldots, n$ gives

$$\|e_h^n\|_{\mathbf{b}}^2 + \sum_{j=1}^n k\left(\varepsilon\|\nabla e_h^{j,*}\|_0^2 + \|\mu^{1/2} e_h^{j,*}\|_0^2 + \|e_h^{j,*}\|_{\mathrm{mat,CN}}^2\right)$$
$$+ \varepsilon\delta\|\nabla e_h^n\|_0^2 + \delta\|c^{1/2} e_h^n\|_0^2$$
$$\le \|e_h^0\|_{\mathbf{b}}^2 + \varepsilon\delta\|\nabla e_h^0\|_0^2 + \delta\|c^{1/2} e_h^0\|_0^2 + C\sum_{j=1}^n k\|T_{\mathrm{tr,CN}}^j\|_0^2. \tag{5.34}$$

The initial error is bounded as in (5.20). For the truncation error, the first part is estimated with (5.4):

$$\begin{aligned}&\frac{1}{2}\left\|(u_t(t_n)-\Pi_h u_t(t_n))+(u_t(t_{n-1})-\Pi_h u_t(t_{n-1}))\right\|_0^2\\&\le Ch^{2r+1}\left(\|u_t(t_n)\|_{r+1}^2+\|u_t(t_{n-1})\|_{r+1}^2\right).\end{aligned}$$

For the second term, one obtains with a direct calculation, using integration by parts, and with the Cauchy–Schwarz inequality

$$\begin{aligned}&\left\|\frac{\Pi_h u_t(t_n)+\Pi_h u_t(t_{n-1})}{2}-\frac{\Pi_h u(t_n)-\Pi_h u(t_{n-1})}{k}\right\|_0^2\\&=\frac{1}{k^2}\left\|\int_{t_{n-1}}^{t_n}\frac{1}{2}(t_n-t)(t-t_{n-1})\Pi_h u_{ttt}\,dt\right\|_0^2\\&\le\frac{1}{k^2}\left(\left(\int_{t_{n-1}}^{t_n}\frac{1}{4}(t_n-t)^2(t-t_{n-1})^2\,dt\right)^{1/2}\left(\int_{t_{n-1}}^{t_n}\|\Pi_h u_{ttt}\|_0^2\,dt\right)^{1/2}\right)^2\\&\le Ck^3\int_{t_{n-1}}^{t_n}\|\Pi_h u_{ttt}\|_0^2\,dt.\end{aligned}$$

Altogether, one gets

$$\sum_{j=1}^n k\|T_{\mathrm{tr,CN}}^j\|_0^2\le Cknh^{2r+1}\max_{0\le s\le n}\|u_t(t_s)\|_{r+1}^2+Ck^4\|\Pi_h u_{ttt}\|_{L^2(0,t_n;L^2)}^2. \tag{5.35}$$

Inserting this estimate into (5.34) leads to an error estimate involving the material derivative.

Setting $v_h=e_h^{n,*}$ in (5.32) using the coercivity (2.12), a relation of form (5.33), and the absorption of terms similar to the time-continuous case gives

$$\|e_h^n\|_0^2+k\sum_{j=1}^n\|e_h^{j,*}\|_{\mathrm{SUPG}}^2\le C\left(\|e_h^0\|_0^2+\sum_{j=1}^n k\|T_{\mathrm{tr,CN}}^j\|_0^2+\sum_{j=1}^n k\delta\|e_{h,\tau}^j\|_0^2\right).$$

The second term on the right-hand side is estimated in (5.35), and it remains to estimate the third term. Setting $z_h^n=e_{h,\tau}^n$, assuming for simplicity $e_h^0=0$, and reasoning in the same manner as for the backward Euler scheme leads to

$$(z_{h,\tau}^n,v_h)+a_{\mathrm{SUPG}}(z_h^{n,*},v_h)=(T_{\mathrm{tr,CN},\tau}^n,v_h)+\delta(T_{\mathrm{tr,CN},\tau}^n,\mathbf{b}\cdot\nabla v_h)-\delta(z_{h,\tau}^n,\mathbf{b}\cdot\nabla v_h)$$

for all $v_h\in V_{h,r}$, where $T_{\mathrm{tr,CN},\tau}^n=(T_{\mathrm{tr,CN}}^n-T_{\mathrm{tr,CN}}^{n-1})/k$ and $n\ge 2$. In the same way as for e_h^n, an error estimate for z_h^n is derived. In contrast to the backward Euler scheme, the terms $\|\mu^{1/2}z_h^{j,*}\|_0^2$ are on the left-hand side of this estimate and not $\|\mu^{1/2}z_h^j\|_0^2$; see (5.34). Hence, different terms are needed to continue the estimate. Using the equivalence of $\|\cdot\|_{\mathbf{b}}$ and the L^2 norm and $T=kn$ give

$$\sum_{j=1}^n k\delta\|z_h^j\|_0^2\le k\delta\|z_h^1\|_0^2+C\sum_{j=2}^n k\delta\|z_h^j\|_{\mathbf{b}}^2\le CT\left(\delta\|z_h^1\|_0^2+\sum_{j=2}^n k\delta\|T_{\mathrm{tr,CN},\tau}^j\|_0^2\right).$$

The estimate of the terms in the last line proceeds in the same manner as for the backward Euler scheme. It is

$$T^n_{\mathrm{tr,CN},\tau} = \left(\frac{u_t(t_n)-u_t(t_{n-2})}{2k} - u_{tt}(t_{n-1})\right) + (u_{tt}(t_{n-1}) - \Pi_h u_{tt}(t_{n-1})) + \left(\Pi_h u_{tt}(t_{n-1}) - \frac{\Pi_h u(t_n) - 2\Pi_h u(t_{n-1}) + \Pi_h u(t_{n-2})}{k^2}\right).$$

Similarly to (4.3), it can be shown that the first and third terms are of order k^2 and that the factor depends on $\|u_{tttt}\|_{L^2(t_{n-2},t_n;L^2)}$ and $\|\Pi_h u_{tttt}\|_{L^2(t_{n-2},t_n;L^2)}$, respectively. The second term is estimated with (5.4) to be of order $h^{r+1/2}$ and the constant depends on $\|u_{tt}\|_{L^2(t_{n-2},t_n;H^{r+1})}$. Finally, $\delta\|z_h^1\|_0^2 \le C(h^{2r+1}+k^4)$ is proven analogously as for the backward Euler scheme, where C depends on $\|\Pi_h u_{ttt}(t_1)\|_0$ and $\|u_t(t_1)\|_{r+1}$.

Theorem 5.4 (error estimate for Crank–Nicolson scheme involving the SUPG norm). *Let* $t_n \le T < \infty$, *let* $u, u_t \in L^\infty(0,T;H^{r+1}(\Omega))$, *let* $u_{tt} \in L^2(0,T;H^{r+1}(\Omega))$, *and let* $\Pi_h u_{ttt}, u_{tttt}, \Pi_h u_{tttt} \in L^2(0;T;L^2(\Omega))$. *Then the error estimate*

$$\|(u-u_h)(t_n)\|_0^2 + k\sum_{j=1}^n \left\|\frac{u(t_j)+u(t_{j-1})}{2} - \frac{u_h(t_j)+u_h(t_{j-1})}{2}\right\|^2_{\mathrm{SUPG}} \le C\left(h^{2r+1}+k^4\right) \tag{5.36}$$

holds with the stabilization parameter defined in (5.1). *The constant in* (5.36) *depends on the coefficients of the problem* (2.1), *on norms of* u *and* $\Pi_h u$ *in the spaces given above, on* c_{inv}, *and linearly on* T.

6. Numerical studies. Two examples will be presented in the numerical studies. The first one, possessing a given smooth solution, serves as support for the orders of convergence that were proven in the previous sections. The second example is the well-known rotating body problem from [14]. It demonstrates the superiority of the parameter choice from section 5 compared with the choices from sections 3 and 4 for small time steps on a fixed, rather coarse, spatial mesh.

Example 6.1 (smooth solution). Consider (2.3) with $\Omega = (0,1)^2$, $T = 1$, different values of ε, $\mathbf{b} = (1,-1)$, $c = 1$, and the right-hand side chosen such that

$$u(t,x,y) = e^{\sin(2\pi t)}\sin(2\pi x)\sin(2\pi y)$$

is the solution of (2.3). The simulations were performed with $\varepsilon = 10^{-8}$ in the convection-dominated regime and with $\varepsilon = 1$ in the diffusion-dominated regime. Uniform triangular grids were used with the coarsest grid (level 0) obtained by dividing the unit square with a diagonal from $(0,0)$ to $(1,1)$. The mesh width h was defined by dividing the diameters of the mesh cells by $\sqrt{2}$. To prevent superconvergence, the convection field was chosen such that it is not parallel to any grid line.

Consider the error estimates (4.4) and (4.5). First, optimal scalings of the mesh width h and the length of the time step k were derived from these estimates. Then the error estimates lead to only one asymptotic order of convergence that serves as a criterion. The stabilization parameters for the estimates under the assumptions of Theorem 4.1 were set to be $\delta_K = \delta = k/4$ according to condition (3.3). In the convection-dominated regime, $\varepsilon \ll h$, the terms $\mathcal{O}(k)$ and $\mathcal{O}(h^{r+1}\delta^{-1/2}) = \mathcal{O}(h^{r+1}k^{-1/2})$ have to be balanced to obtain an optimal L^2-error estimate (4.4). This leads to the scaling $k = \mathcal{O}(h^{2(r+1)/3})$. The same reasoning applies for the SUPG error (4.5). If the final time $T = 1$ was not obtained exactly with the chosen time steps, the simulations were stopped at the first discrete time larger than T. In the diffusion-dominated regime, $h \le \varepsilon$, the terms $\mathcal{O}(k)$, $\mathcal{O}(k^{1/2}h^{r-1}\varepsilon)$, and $\mathcal{O}(h^{r+1}k^{-1/2})$ need to be balanced. This leads to $k = \mathcal{O}(h^{2(r+1)/3})$ or $k = (h^2/\varepsilon)$. If $h \ll \varepsilon$, the second scaling gives a better order of convergence for $r = 1$ (piecewise linear elements). Note, in this case, that $\delta = k = \mathcal{O}(h^2/\varepsilon)$ is a standard choice of the stabilization parameter in the diffusion-dominated regime for steady-state problems. For $r = 2$, both scalings are essentially the same, and for $r \ge 3$ the first scaling leads to a higher order of convergence. For the SUPG estimate, the same terms have to be balanced. In addition, the order of convergence is bounded by the term $\mathcal{O}(\varepsilon^{1/2}h^r)$, such that for $h \ll \varepsilon$ only first order convergence can be expected for $r = 1$. Figure 6.1 presents the orders of convergence for the P_1, P_2, and P_3 finite element. It can be seen that all orders match the predictions from the analysis.

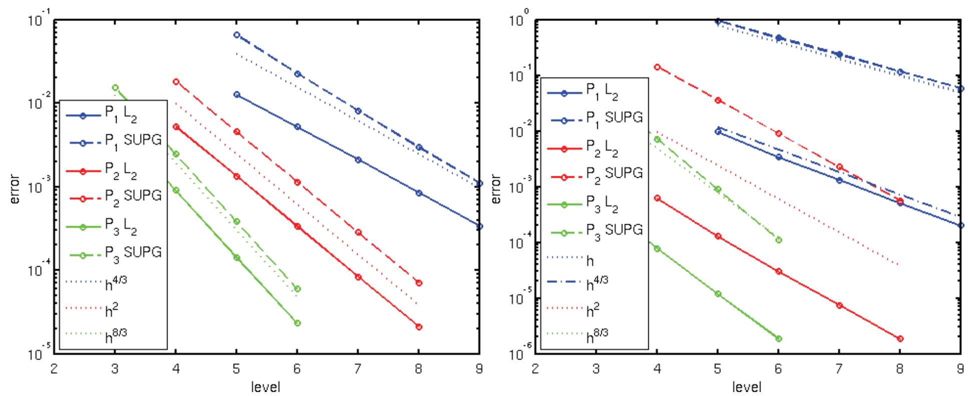

FIG. 6.1. *Example* 6.1*, orders of convergence for the estimates* (4.4) *and* (4.5)*; left: convection-dominated regime; right: diffusion-dominated regime.*

For the estimates which can be obtained with $\delta_K = \sqrt{k}h_K/(4\|\mathbf{b}\|_2)$, one can also observe the expected orders of convergence. For brevity, this is not presented in detail.

Next, estimate (5.26) for the time-continuous case is considered, from which one can expect convergence for the L^2 norm and the $L^2(0,T;\mathrm{SUPG})$ norm of order $r + 1/2$ for P_r finite elements if the initial condition is sufficiently well approximated. The simulations were performed with a very small time step, $k = 10^{-6}$. As initial condition, the Lagrange interpolant of $u(0,x,y)$ was used. The results are presented in Figure 6.2. The observed order of convergence in the L^2 norm is even higher than the prediction from the analysis.

As an example for the order of convergence obtained for a fully discrete scheme, estimate (5.36) of the Crank–Nicolson scheme is considered. The equilibration of the terms on the right-hand side of (5.36) gives the length of the time step $k = \mathcal{O}(h^{r/2+1/4})$, and the expected order of convergence is $r + 1/2$. This order can be well observed in the numerical results in the right panel of Figure 6.2.

Example 6.2 (rotating body problem). This problem was already studied numerically for finite element discretizations of convection-diffusion equations in [12]. Here, exactly the same setting is used. The aim of this example is to illustrate that the choice of the stabilization parameter $\delta_K = \mathcal{O}(h_K)$ from section 5 is much better than the choices $\delta_K = \mathcal{O}(k)$, $\delta_K = \mathcal{O}(k^{1/2}h_K)$ from sections 3 and 4 in the presence of very small time steps.

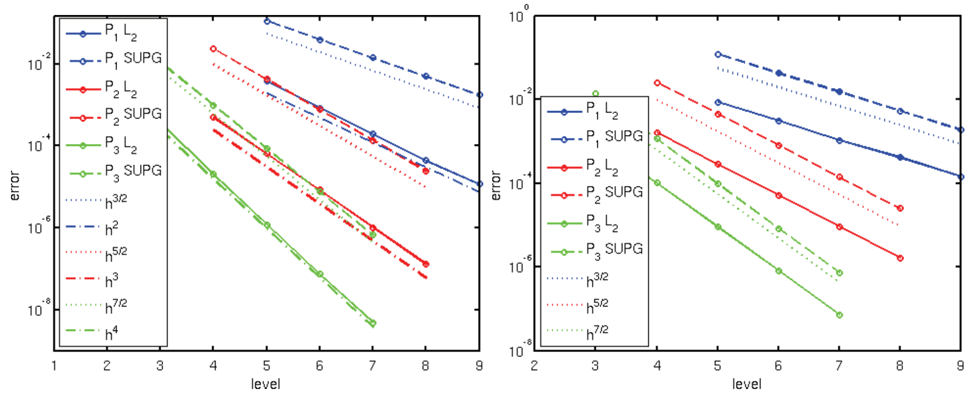

FIG. 6.2. *Example* 6.1. *Left: orders of convergence for the estimate* (5.26)*; right: orders of convergence for the estimate* (5.36) *for* $k = h^{r/2+1/4}$*; both panels are for the convection-dominated regime.*

Let $\Omega = (0,1)^2$, $\varepsilon = 10^{-20}$, $\mathbf{b} = (0.5 - y, x - 0.5)^T$, and $c = f = 0$. The initial condition, consisting of three disjoint bodies, is presented in Figure 6.3; see [12] for a precise definition of the bodies. The rotation of the bodies occurs counterclockwise. A full revolution takes $t = 6.28 \approx 2\pi$. With the extremely small diffusion, the solution after one revolution is essentially the same as the initial condition. Homogeneous Dirichlet boundary conditions were imposed.

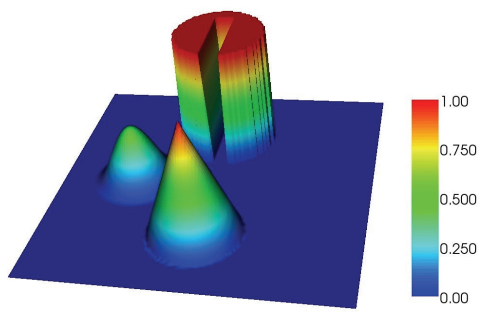

FIG. 6.3. *Example* 6.2*, initial condition and ideal solution after one rotation.*

In the simulations, a uniform grid consisting of 128×128 triangles was used. This leads to 16,641 degrees of freedom for the P_1 finite element method, including Dirichlet nodes. For brevity, only results obtained with the backward Euler scheme will be presented in detail. Computational studies were performed for the Galerkin finite element method ($\delta_K = 0$ for all mesh cells), the choice of the stabilization parameter from [12, formulae (8) and (11)], which results in $\delta_K = k$ for the backward Euler scheme, the choice $\delta_K = \sqrt{k} h_K/4$ and $\delta_K = h_K/4$. Analogously to [12], a measure for the under- and overshoots is given by

$$\mathrm{var}(t) := \max_{(x,y)\in\Omega} u_h(t;x,y) - \min_{(x,y)\in\Omega} u_h(t;x,y),$$

with the optimal value $\mathrm{var}(t) = 1$ for all t.

The backward Euler scheme with $k = 10^{-6}$ is considered. It can be observed that by far the best result was obtained with $\delta_K = h_K/4$; see Figures 6.4 and 6.5. However,

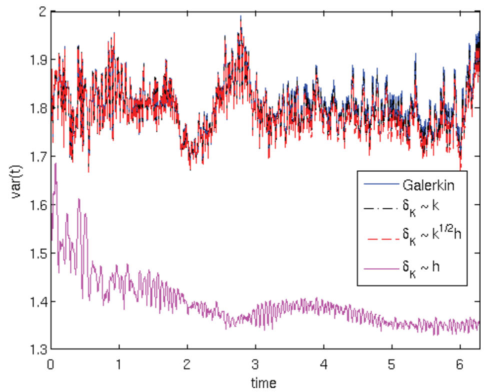

FIG. 6.4. *Example* 6.2, *under- and overshoots measured with* var(t), *backward Euler scheme,* $k = 10^{-6}$.

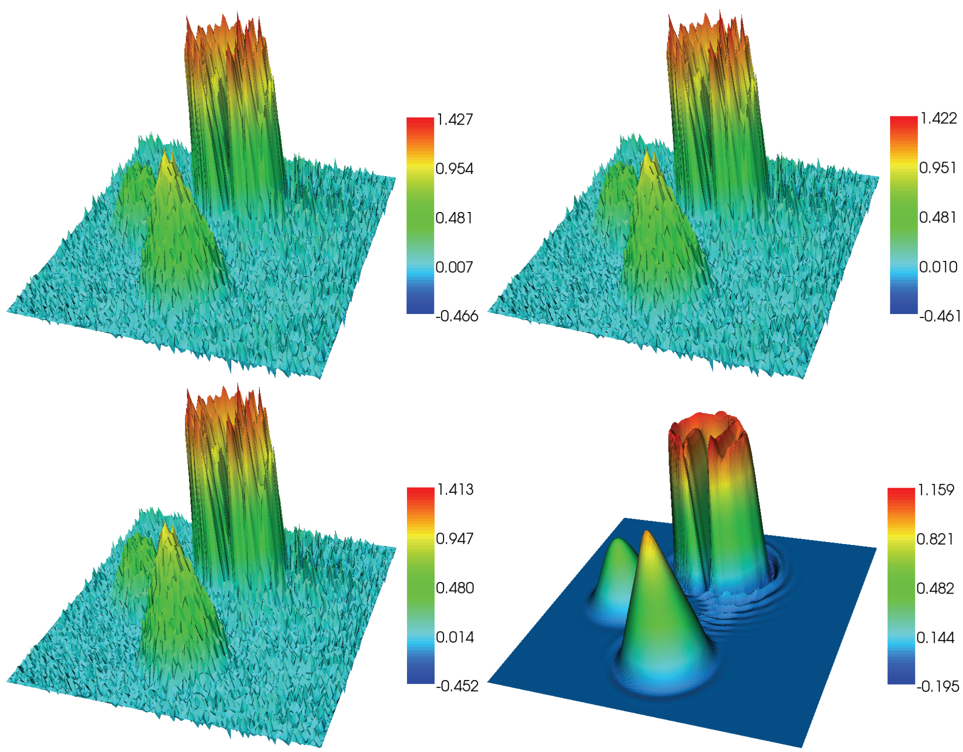

FIG. 6.5. *Example* 6.2, *computed solutions after one revolution with the backward Euler scheme and* $k = 10^{-6}$*: Galerkin finite element method (top left), SUPG with* $\delta_K = \mathcal{O}(k)$ *(top right), SUPG with* $\delta_K = \mathcal{O}(\sqrt{k}h_K)$ *(bottom left), SUPG with* $\delta_K = \mathcal{O}(h_K)$ *(bottom right).*

the computed solution with these parameters possesses still nonnegligible spurious oscillations. Using the stabilization parameters from the analysis of sections 3 and 4 leads for very small time steps on a fixed spatial grid to results similar to those for the Galerkin finite element method. Only a slight damping of the spurious oscillations can be observed; see the ranges of the finite element solutions in Figure 6.5.

7. Summary and outlook. This paper studied different ways to obtain error estimates for the SUPG finite element method applied to evolutionary convection-diffusion-reaction equations.

Exemplarily for the backward Euler scheme, it was shown that standard energy arguments for the fully discrete problem yield error estimates under conditions that couple the choice of the stabilization parameters to the length of the time step. In particular, the SUPG stabilization vanishes in the time-continuous limit. Numerical evidence shows that this is not the correct behavior.

For this reason, the special case of convection and reaction being independent of time was considered. On uniform grids, optimal error estimates could be proven with the choice of the stabilization parameter $\delta = \mathcal{O}(h)$ in the convection-dominated regime. The proofs were given for the time-continuous case and the fully discrete method with the backward Euler scheme as well as with the Crank–Nicolson scheme.

Analysis of the general time-continuous problem, with time-dependent coefficients and using nonuniform meshes, is open. An extension of the analysis from section 5 seems to be hard, since this analysis uses several times that the original equation can be differentiated with respect to time yielding essentially the same equation. Answering the question of whether the general time-continuous problem allows at all estimates with stabilization parameters independent of the length of the time step requires further research.

With respect to the usage of the SUPG finite element method in time-dependent convection-diffusion-reaction equations, the results of section 5, Example 6.2, and other numerical studies from the literature strongly suggest defining the stabilization parameters in the convection-dominated regime in the classical way by $\delta_K = \mathcal{O}(h_K)$, despite the open analysis for the general time-continuous problem.

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