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A streamline–diffusion method for nonconforming finite element approximations applied to convection–diffusion problems

V. John[1], G. Matthies[2], F. Schieweck[3], L. Tobiska*

Institute für Analysis und Numerik, Otto-von-Guericke-Universität Magdeburg, PF 4120, D-29016 Magdeburg, Germany

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Abstract

We consider a nonconforming streamline–diffusion finite element method for solving convection–diffusion problems. The theoretical and numerical investigation for triangular and tetrahedral meshes recently given by John, Maubach and Tobiska has shown that the usual application of the SDFEM gives not a sufficient stabilization. Additional parameter dependent jump terms have been proposed which preserve the same order of convergence as in the conforming case. The error analysis has been essentially based on the existence of a conforming finite element subspace of the nonconforming space. Thus, the analysis can be applied for example to the Crouzeix/Raviart element but not to the nonconforming quadrilateral elements proposed by Rannacher and Turek. In this paper, parameter free new jump terms are developed which allow to handle both the triangular and the quadrilateral case. Numerical experiments support the theoretical predictions.

1. Introduction

The streamline–diffusion finite element method (SDFEM) for the solution of convection–diffusion problems has been successfully applied in the case of conforming finite element spaces. This method was proposed first by Hughes and Brooks [6] and applied to several classes of problems. Starting with the fundamental work by Nävert [13], it was mainly analyzed by Johnson and his co-workers [5,9,10]. Nowadays, the convergence properties are well-understood in the conforming case [14,17,21,22,24]. However, in the nonconforming case there are some new effects which will be studied in this paper.

For applications from computational fluid dynamics, finite element methods of nonconforming type are attractive since they easily fulfil the discrete version of the Babuška–Brezzi condition. Another advantage of nonconforming finite elements is that the unknowns are associated with the element faces such that each degree of freedom belongs to at most two elements. This results in a cheap local communication when the method is parallelized on MIMD-machines (see e.g. [4,7,11,18]).

In order to stabilize finite element discretizations in the case of moderate and higher Reynolds numbers, several upwind methods (resulting in algebraic systems with M-matrices) have been developed and analyzed [2,3,16,18–20]. However, the main drawback of upwind methods is the restricted order of the discretization error. Therefore, the SDFEM which is known to combine good stability properties with high accuracy outside interior or boundary layers seems to be a suitable alternative to upwind schemes. In [12], a nonconforming

* Corresponding author. Email: tobiska@mathematik.uni-magdeburg.de

[1] Email: john@mathematik.uni-magdeburg.de

[2] Email: matthies@mathematik.uni-magdeburg.de, research partly supported by DFG grant EN 278/2-1.

[3] Email: schiewec@mathematik.uni-magdeburg.de

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SDFEM for the solution of the stationary Navier–Stokes equations is studied but no special attention has been paid to the dependency of the error constants on the Reynolds number. For the simplified model of a scalar convection–diffusion problem, a first analysis of convergence properties for a class of nonconforming SDFEM, in the following called NSDFEM, reflecting the dependency of the perturbation parameter is given in [8]. If the standard SDFEM is applied, large oscillations can occur. The theoretical and numerical investigation has shown that by adding jump terms to the discretization one can preserve the same $O(h^{k+1/2})$ order of L^2-convergence as in the conforming case for piecewise polynomials of degree k. It is important to note that the error analysis essentially uses the fact that there exists a conforming finite element subspace of the nonconforming approximation space. Thus, the analysis in [8] cannot be applied to the nonconforming Q_1^{rot} quadrilateral elements [15,18] where a suitable conforming subspace is not available. In order to overcome these difficulties, we develop new jump terms which allow to handle mixed meshes consisting of triangles and quadrilaterals. With such meshes, we have more flexibility to create locally refined meshes.

In this paper, we propose a modified nonconforming SDFEM discretization for the solution of the boundary value problem

$$-\varepsilon\,\Delta u + b\cdot\nabla u + cu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma := \partial\Omega\,. \tag{1}$$

For this modified discretization, theoretical and numerical investigations demonstrate that the order of convergence is identical to that of the corresponding conforming methods. The main advantage of the new method is that in contrast to [8] no additional parameters in the jump terms have to be chosen. Moreover, numerical experiments show for the new discretization resonable convergence rates of iterative solvers which is not the case for the theoretically supported jump term parameters in [8].

The remainder of this paper is organized as follows. In Section 2, notations and the modified nonconforming discretization of (1) are introduced. We also describe assumptions on the finite element spaces to be fulfilled. These assumptions are satisfied in particular for the nonconforming Crouzeix/Raviart element and a nonconforming quadrilateral element. In Section 3, the coerciveness of the bilinear form and the convergence properties are studied. Finally, in Section 4 several numerical experiments are presented in order to support the theoretical results.

2. Notations and preliminaries

For simplicity, we restrict our representation to the two-dimensional case but the results can also be generalized to three dimensions. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary Γ. For the data of problem (1) we assume that ε is a small positive parameter and that $b = (b_1, b_2)$, c and f are sufficiently smooth functions satisfying the assumption

$$\inf_{x\in\Omega}\left(c(x) - \frac{1}{2}\operatorname{div} b(x)\right) \geq c_0 > 0\,. \tag{2}$$

Using the space $V := H_0^1(\Omega)$, the standard weak formulation of (1) reads

Find $u \in V$ such that for all $v \in V$

$$\varepsilon(\nabla u, \nabla v) + (b\cdot\nabla u + cu, v) = (f, v)\,, \tag{3}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Note that under the assumption (2) there exists a unique solution of problem (3). Let T_h be a regular decomposition of the domain Ω into elements $K \in T_h$ which are allowed to be (open) triangles or convex quadrilaterals. By h_K we denote the diameter of the element K and by h the maximum of all h_K for $K \in T_h$. Furthermore, let $h_K^{\min}$ denote the minimum length of the edges of K and $\alpha_K^{\min}$ and $\alpha_K^{\max}$ the minimum and maximum angle between neighbouring edges of K, respectively. For the discretization of (3) we consider a family $\{T_h\}$ of decompositions with $h \to 0$ which is assumed to be *shape regular*, i.e. there are constants C_1 and $\alpha_0 \in (0, \pi)$ independent of h such that

$$h_K / h_K^{\min} \leq C_1 \quad \text{and} \quad \alpha_0 \leq \alpha_K^{\min} \leq \alpha_K^{\max} \leq \pi - \alpha_0 \quad \forall\, K \in T_h\,. \tag{4}$$

For a quadrilateral element K, we denote by γ_K the maximum angle between two opposite edges of K and formally we set $\gamma_K = 0$ for triangular elements. In order to get a reasonable estimate for the interpolation error of the nonconforming quadrilateral finite elements we assume

$$\gamma_K \leq C_2 h_K \quad \forall K \in T_h \tag{5}$$

with a constant C_2 independent of h. Assumption (5) can be realized by the mesh generation process in the following way. We start with a coarse mesh T^0 consisting of quadilaterals and triangles. Then, for a given mesh T^l we construct the next finer mesh T^{l+1} by subdividing a certain set of marked elements $K \in T^l$ into finer elements. For the refinement pattern of a quadrilateral element we assume that it is either subdivided into four 'son-elements' by connecting the midpoints of opposite edges or that it is subdivided into triangular son-elements only. For the subdivision of a triangular element only triangular son-elements are allowed. At the end of this refinement process we get for some finest level $l_{\max}$ our final mesh $T_h := T^{l_{\max}}$. With the above assumption on the refinement pattern, it can be proven that (5) is satisfied [18]. The assumption (4) admits locally adapted meshes with non-degenerating elements. However, anisotropic mesh refinement is excluded.

In order to explain the finite element space V_h we need some further notations and definitions. In the following, $\mathscr{E}(K)$ denotes the set of all edges of an element $K \in T_h$, $\mathscr{E} := \cup_{K \in T_h} \mathscr{E}(K)$ the set of all edges of T_h, $\mathscr{E}_\Gamma := \{E \in \mathscr{E} : E \subset \Gamma\}$ and $\mathscr{E}_0 := \mathscr{E} \backslash \mathscr{E}_\Gamma$ the set of the boundary and inner edges, respectively.

For a given piecewise continuous function φ, the jump $[|\varphi|]_E$ and the average $A_E\varphi$ on a face $E \in \mathscr{E}$ are defined by

$$[|\varphi|]_E(x) := \begin{cases} \lim\limits_{t\to+0} \varphi(x + tn_E) - \lim\limits_{t\to+0} \varphi(x - tn_E) & \text{if } E \not\subset \Gamma \\ -\lim\limits_{t\to+0} \varphi(x - tn_E) & \text{if } E \subset \Gamma \end{cases},$$

$$A_E\varphi(x) := \begin{cases} \frac{1}{2}\left(\lim\limits_{t\to+0} \varphi(x + tn_E) + \lim\limits_{t\to+0} \varphi(x - tn_E)\right) & \text{if } E \not\subset \Gamma \\ \frac{1}{2}\left(\lim\limits_{t\to+0} \varphi(x - tn_E)\right) & \text{if } E \subset \Gamma \end{cases},$$

where n_E is a normal unit vector on E and $x \in E$. If $E \subset \Gamma$ we choose the orientation of n_E to be outward with respect to Ω otherwise n_E has an arbitrary but fixed orientation. For an element $K \in T_h$, we denote by n_K the unit normal vector on ∂K oriented outward with respect to K. For the nonconforming finite element functions $v_h \in V_h$, the continuity condition of conforming finite elements at the edges $E \in \mathscr{E}$ is weakened to

$$J_E(v_h) := |E|^{-1} \int_E [|v_h|]_E \, d\gamma = 0 \quad \forall E \in \mathscr{E} \; \forall v_h \in V_h, \tag{6}$$

where $|E|$ denotes the length of the edge E. Condition (6) says that the mean-value $J_E(v_h)$ of the jumps of v_h on edge E is zero for all $E \in \mathscr{E}$. Now, our finite element space V_h can be defined as

$$V_h := \{v_h \in L^2(\Omega) : v_h|_K \in P(K) \; \forall K \in T_h, \; J_E(v_h) = 0 \; \forall E \in \mathscr{E}\}, \tag{7}$$

where $P(K) := P_1(K)$ if K is a triangle and $P(K) := Q_1^{\mathrm{rot}}(K)$ for a quadrilateral element K. Herein, $P_1(K)$ denotes the space of all linear functions on K and $Q_1^{\mathrm{rot}}(K)$ the space of the so-called 'rotated bilinear' functions (see [15]) defined by

$$Q_1^{\mathrm{rot}}(K) := \{\hat{q} \circ F_K^{-1} : \hat{q} \in \mathrm{span}(1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2 - \hat{x}_2^2)\},$$

where $F_K : \hat{K} \to K$ is the bilinear transformation between the reference element $\hat{K} = [-1, 1]^2$ and the element K. The degrees of freedom (nodal values) of a function $v_h \in V_h$ are the mean-values $N_E(v_h)$ at inner edges $E \in \mathscr{E}_0$ defined by

$$N_E(v_h) := |E|^{-1} \int_E v_h|_K \, d\gamma \quad \text{for } E \in \mathscr{E}(K),$$

where K is an (arbitrary) element having the edge E. Because of (6), the value $N_E(v_h)$ does not depend on the choice of K. Thus, the nodal values $N_E(v)$ are well defined for all $v \in V + V_h$. Note that from the conditions $J_E(v_h) = 0 \; \forall E \in \mathscr{E}_\Gamma$ in the definition of V_h we have the discrete boundary conditions $N_E(v_h) = 0 \; \forall E \in \mathscr{E}_\Gamma$. Let

φ_E, $E \in \mathscr{E}_0$, be the finite element basis functions of V_h with $N_E(\varphi_E) = 1$ and $N_{E'}(\varphi_E) = 0\ \forall E' \in \mathscr{E}\setminus\{E\}$. Then, we define the interpolation operator $I_h : V \rightarrow V_h$ by

$$I_h v := \sum_{E \in \mathscr{E}_0} N_E(v)\varphi_E \quad \forall v \in V . \tag{8}$$

In the following, we denote for a subdomain $\omega \subset \Omega$ by $(\cdot,\cdot)_\omega$ the inner product in $L^2(\omega)$ and by $\|\cdot\|_{m,\omega}$ and $|\cdot|_{m,\omega}$ the usual norm and semi norm in the Sobelev space $H^m(\omega)$, respectively.

Now, the new version of the NSDFEM reads:

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$a_h(u_h, v_h) = l_h(v_h) \tag{9}$$

where the bilinear form a_h and the linear form l_h are given by

$$\begin{aligned} a_h(u_h, v_h) := &\sum_{k \in T_h} \left((\varepsilon(\nabla u_h, \nabla v_h)_K + (b\cdot\nabla u_h + cu_h, v_h)_K + (-\varepsilon\,\Delta u_h + b\cdot\nabla u_h + cu_h, \delta_K b\cdot\nabla v_h)_K \right) \\ &+ \sum_{E\in\mathscr{E}} \int_E b\cdot n_E[|u_h|]_E A_E v_h \,\mathrm{d}\gamma + \sum_{E\in\mathscr{E}} \int_E |b\cdot n_E|[|u_h|]_E[|v_h|]_E \,\mathrm{d}\gamma , \\ l_h(v_h) := &(f, v_h) + \sum_{K\in T_h} \delta_K(f, b\cdot\nabla v_h)_K . \end{aligned} \tag{10}$$

The size of the positive control parameters δ_K will be discussed in the next section.

REMARK 1. In [8], different weak forms of the convection term $b\cdot\nabla u$ have been studied. The theoretical and numerical investigations have shown that the approximation error for the convective form of the convection term behaves better than for the skew symmetrized and the fully skew symmetrized form. Therefore, we restrict our study to the convective form used in (3) and (10).

REMARK 2. The jump terms introduced in [8] have the form

$$\sum_{E\in\mathscr{E}} \int_E \sigma_E[|u_h|]_E[|v_h|]_E \,\mathrm{d}\gamma ,$$

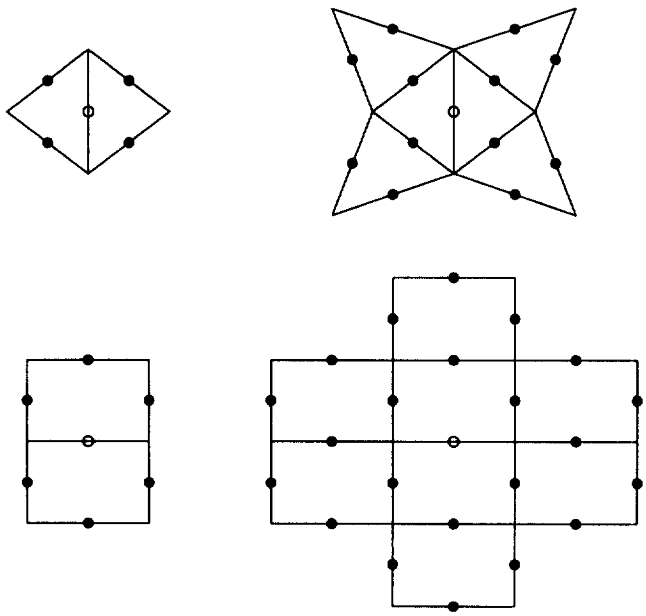

Fig. 1. Coupling of degrees of freedoms, left without, right with jump terms.

where the parameter σ_E has to be chosen appropriately. In contrast, now we have two parts, the first one to guarantee coerciveness of the bilinear form and the second one to ensure an ε-uniform estimate of the consistency error.

REMARK 3. The jump terms in the modified NSDFEM discretization lead to an increased number of couplings between the local basis functions, see Fig. 1. Therefore, the number of non-zero entries in the system matrix increases considerably by a factor of about 2.6 for triangular meshes and 3.28 for quadrilateral meshes. In addition, the implementation of this discretization on a parallel computer becomes more difficult.

3. Error analysis

In this section, we will provide a discretization error estimate which shows the convergence of the modified NSDFEM (9), (10) for $h \to 0$ assuming an appropriate choice of the parameter δ_K.

For the error analysis, we use the mesh-dependent norm on $V + V_h$

$$|||v|||^2 := \sum_{K\in T_h} \left(\varepsilon|v|_{1,K}^2 + c_0\|v\|_{0,K}^2 + \delta_K\|b\cdot\nabla v\|_{0,K}^2\right) + \sum_{E\in\mathscr{E}} \int_E |b\cdot n_E|[|v|]_E^2 \,\mathrm{d}\gamma\,. \tag{11}$$

Due to the assumption of a shape regular family of triangulations, there exists a positive constant μ independent of K and the triangulation T_h such that the following local inverse inequalities hold

$$\|\Delta v_h\|_{0,K} \le \mu h_K^{-1}|v_h|_{1,K} \quad \forall\, v_h \in V_h, \forall\, K\in T_h\,. \tag{12}$$

We set

$$c_{\max} = \sup_{x\in\Omega}|c(x)|\,.$$

For the control parameter δ_K, we assume

$$0 < \delta_K \le \frac{1}{2}\min\left(\frac{c_0}{c_{\max}^2}, \frac{h_K^2}{\varepsilon\mu^2}\right). \tag{13}$$

Let us first consider the coerciveness of the bilinear form a_h on V_h.

LEMMA 4. Let V_h be the finite element space defined by (7) and the control parameter δ_K fulfil the assumption (13). Then, the discrete bilinear form a_h is coercive on V_h, i.e.

$$a_h(v_h, v_h) \ge \frac{1}{2}|||v_h|||^2 \quad \forall\, v_h \in V_h\,. \tag{14}$$

PROOF. Using the definition of a_h, element wise integration by parts and

$$[|v_h w_h|]_E = [|v_h|]_E A_E w_h + A_E v_h [|w_h|]_E\,, \tag{15}$$

we obtain

$$\begin{aligned} a_h(v,v) := &\sum_{K\in T_h} \varepsilon|v|_{1,K}^2 + \sum_{K\in T_h}\left(c - \frac{1}{2}\nabla\cdot b, v^2\right)_K + \sum_{K\in T_h}\frac{1}{2}\int_{\partial K} b\cdot n_K v^2\,\mathrm{d}\gamma \\ &+ \sum_{K\in T_h}\delta_K\|b\cdot\nabla v\|_{0,K}^2 + \sum_{K\in T_h}\delta_K(-\varepsilon\Delta v + cv, b\cdot\nabla v)_K \\ &+ \sum_{E\in\mathscr{E}}\int_E b\cdot n_E\frac{1}{2}[|v^2|]_E\,\mathrm{d}\gamma + \sum_{E\in\mathscr{E}}\int_E |b\cdot n_E|[|v|]^2\,\mathrm{d}\gamma\,. \end{aligned}$$

Taking into consideration the definition of the jump, the third term and the first jump term cancel. Using (2), we get

$$a_h(v,v) \geq |||v|||^2 + \sum_{K\in T_h} \delta_K(-\varepsilon\Delta v + cv, b\cdot\nabla v)_K .$$

For the remaining term, we start with

$$\left|\sum_{K\in T_h} \delta_K(-\varepsilon\Delta v + cv, b\cdot\nabla v)_K\right| \leq \sum_{K\in T_h} \delta_K|(-\varepsilon\Delta v, b\cdot\nabla v)_K| + \sum_{K\in T_h} \delta_K|(cv, b\cdot\nabla v)_K| . \tag{16}$$

Using the Cauchy–Schwarz-inequality, we get for the first term of (16):

$$\begin{aligned}\sum_{K\in T_h} \delta_K|(-\varepsilon\Delta v, b\cdot\nabla v)_K| &\leq \sum_{K\in T_h} \varepsilon\delta_K^{1/2}\|\Delta v\|_{0,K}\,\delta_K^{1/2}\|b\cdot\nabla v\|_{0,K}\\ &\leq \sum_{K\in T_h} \varepsilon^2\delta_K\|\Delta v\|_{0,K}^2 + \frac{1}{4}\sum_{K\in T_h}\delta_K\|b\cdot\nabla v\|_{0,K}^2 .\end{aligned}$$

In the same way, we obtain for the second term of (16):

$$\begin{aligned}\sum_{K\in T_h} \delta_K|(cv, b\cdot\nabla v)_K| &\leq \sum_{K\in T_h} \delta_K^{1/2}c_{\max}\|v\|_{0,K}\delta_K^{1/2}\|b\cdot\nabla v\|_{0,K}\\ &\leq \sum_{K\in T_h} \delta_K c_{\max}^2\|v\|_{0,K}^2 + \frac{1}{4}\sum_{K\in T_h}\delta_K\|b\cdot\nabla v\|_{0,K}^2 .\end{aligned}$$

Applying the inverse inequality (12) and the assumption (13), we get

$$\varepsilon^2\delta_K\|\Delta v\|_{0,K}^2 \leq \varepsilon^2\delta_K\mu^2\, h_K^{-2}|v|_{1,K}^2 \leq \frac{1}{2}\,\varepsilon|v|_{1,K}^2$$

and $\delta_K c_{\max}^2 \leq c_0/2$. Summarizing all estimates, we obtain

$$\left|\sum_{K\in T_h} \delta_K(-\varepsilon\Delta v + cv, b\cdot\nabla v)_K\right| \leq \frac{1}{2}\,|||v|||^2$$

which concludes the proof of (14). □

In order to prove our main result on the convergence of the proposed NSDFEM we need the following approximation properties of the space V_h. For the finite element interpolation operator I_h defined by (8), the estimates

$$|v - I_h v|_{m,K} \leq Ch_K^{2-m}\|v\|_{2,K} \quad \forall\, v\in H^2(K) , \tag{17}$$

$$\|v - (I_h v)|_K\|_{0,E} \leq Ch_K^{3/2}\|v\|_{2,K} \quad \forall\, v\in H^2(K) ,\quad E\in \mathscr{E}(K) \tag{18}$$

are satisfied for an arbitrary element $K\in T_h$ and $m\in\{0,1,2\}$ provided that $\{T_h\}$ is shape regular and the assumption (5) is fulfilled for the quadrilateral elements (see [8,18]). In the following, for an edge $E\in\mathscr{E}$, the operator $P_E : L^2(E)\to P_0(E)$ denotes the $L^2(E)$-projection into the space $P_0(E)$ of the constant functions on E. The estimate

$$\|v - P_E v\|_{0,E} \leq Ch_K^{1/2}|v|_{1,K} \quad \forall\, v\in H^1(K),\quad E\in\mathscr{E}(K) \tag{19}$$

holds, where $K\in T_h$ is an element having the edge E. For triangular elements, (19) is proven in [1]. To prove (19) for a quadrilateral element, we divide it into two triangles and apply (19) to the triangle with the edge E.

The following theorem shows the convergence of the new NSDFEM (9), (10).

THEOREM 5. Let the assumption of Lemma 4 be fulfilled. Moreover, let the solution u of (3) belong to $H_0^1(\Omega)\cap H^2(\Omega)$ and u_h be the solution of (9), (10). Then, the error estimate

$$|||u - u_h||| \leq C\Big(\sum_{K\in T_h} (\lambda_K\, h_K^2 + h_K^3)\|u\|_{2,K}^2\Big)^{1/2} \tag{20}$$

holds with an ε-independent constant C. The parameter λ_K is defined by

$$\lambda_K := \varepsilon + h_K^2 + \delta_K + \delta_K^{-1} h_K^2 . \tag{21}$$

PROOF. The error is split into two parts by applying the triangle inequality

$$|||u - u_h||| \leq |||I_h u - u_h||| + |||u - I_h u||| . \tag{22}$$

Put $w_h := I_h u - u_h$. From the coerciveness of the bilinear form a_h, we get

$$\begin{aligned} \frac{1}{2} |||I_h u - u_h|||^2 &\leq a_h(I_h u - u, w_h) + a_h(u - u_h, w_h) \\ &= a_h(I_h u - u, w_h) + \sum_{K \in T_h} \int_{\partial K} \varepsilon \frac{\partial u}{\partial n_K} w_h \, d\gamma . \end{aligned} \tag{23}$$

We set $w := I_h u - u$. In order to estimate the first term of the right-hand side in (23), we consider each term of $a_h(I_h u - u, w_h)$. For the first term, we get

$$\begin{aligned} \left| \varepsilon \sum_{K \in T_h} (\nabla w, \nabla w_h)_K \right| &\leq \varepsilon \sum_{K \in T_h} \|\nabla w\|_{0,K} \|\nabla w_h\|_{0,K} \\ &\leq C \varepsilon^{1/2} \Big(\sum_{K \in T_h} h_K^2 \|u\|_{2,K}^2 \Big)^{1/2} |||w_h||| . \end{aligned}$$

Using an element wise integration by parts, we obtain for the second term

$$(b \cdot \nabla w + cw, w_h)_K = (c - \nabla \cdot b, w w_h)_K - (b \cdot \nabla w_h, w)_K + \int_{\partial K} b \cdot n_K w w_h \, d\gamma . \tag{24}$$

We sum over all cells K and get the term

$$T := \sum_{K \in T_h} \int_{\partial K} b \cdot n_K w w_h \, d\gamma , \tag{25}$$

which will be estimated together with the first jump term in (10). The remaining parts of (24) are handled as usual

$$\begin{aligned} \left| \sum_{K \in T_h} (c - \nabla \cdot b, w w_h)_K \right| &\leq C \sum_{K \in T_h} \|w\|_{0,K} \|w_h\|_{0,K} \\ &\leq C \Big(\sum_{K \in T_h} h_K^4 \|u\|_{2,K}^2 \Big)^{1/2} |||w_h||| , \\ \left| \sum_{K \in T_h} (b \cdot \nabla w_h, w)_K \right| &\leq \sum_{K \in T_h} \delta_K^{1/2} \|b \cdot \nabla w_h\|_{0,K} \delta_K^{-1/2} \|w\|_{0,K} \\ &\leq C \Big(\sum_{K \in T_h} \delta_K^{-1} h_K^4 \|u\|_{2,K}^2 \Big)^{1/2} |||w_h||| . \end{aligned}$$

The streamline–diffusion term can be estimated in the following way

$$\begin{aligned} \left| \sum_{K \in T_h} (-\varepsilon \Delta w + b \cdot \nabla w + cw, \delta_K b \cdot \nabla w_h)_K \right| &\leq C \sum_{K \in T_h} \delta_K^{1/2} (\varepsilon \|\Delta w\|_{0,K} + \|\nabla w\|_{0,K} + \|w\|_{0,K}) \delta_K^{1/2} \|b \cdot \nabla w_h\|_{0,K} \\ &\leq C \sum_{K \in T_h} \delta_K^{1/2} (\varepsilon + h_K + h_K^2) \|u\|_{2,K} \, \delta_K^{1/2} \|b \cdot \nabla w_h\|_{0,K} \\ &\leq C \Big(\sum_{K \in T_h} (\varepsilon^2 \delta_K + \delta_K h_K^2 + \delta_K h_K^4) \|u\|_{2,K}^2 \Big)^{1/2} |||w_h||| \\ &\leq C \Big(\sum_{K \in T_h} (\varepsilon + \delta_K) h_K^2 \|u\|_{2,K}^2 \Big)^{1/2} |||w_h||| . \end{aligned}$$

To get this, we used $\varepsilon^2 \delta_K \leq C \varepsilon h_K^2$ which is a part of assumption (13). Now, we estimate the second jump term of (10)

$$\Big|\sum_{E\in\mathscr{E}}\int_E |b\cdot n_E|[|w|]_E[|w_h|]_E\,\mathrm{d}\gamma\Big| \le C\sum_{E\in\mathscr{E}} \|[|w|]_E|_{0,E}\,\||b\cdot n_E|^{1/2}[|w_h|]_E|_{0,E}$$
$$\le C\Big(\sum_{E\in\mathscr{E}}\|[|w|]_E|_{0,E}^2\Big)^{1/2}\Big(\sum_{E\in\mathscr{E}}\||b\cdot n_E|^{1/2}[|w_h|]_E|_{0,E}^2\Big)^{1/2}.$$

Using the definition of the jump and (18), we get

$$\|[|w|]_E|_{0,E}\le Ch_E^{3/2}\|u\|_{2,N(E)}$$

where $N(E)$ is the union of all cells K with $E\subset\partial K$ and h_E is the length of the edge E. It follows:

$$\Big|\sum_{E\in\mathscr{E}}\int_E |b\cdot n_E|\,[|w|]_E[|w_h|]_E\,\mathrm{d}\gamma\Big| \le C\Big(\sum_{E\in\mathscr{E}} h_E^3\|u\|_{2,N(E)}^2\Big)^{1/2}|||w_h|||.$$

Now, we consider the term T in (25) and the first jump term in (10). Using (15), we obtain

$$\sum_{K\in T_h}\sum_{E\in\mathscr{E}(K)}\int_E b\cdot n_K w w_h\,\mathrm{d}\gamma + \sum_{E\in\mathscr{E}}\int_E b\cdot n_E[|w|]_E A_E w_h\,\mathrm{d}\gamma$$
$$= -\sum_{E\in\mathscr{E}}\int_E b\cdot n_E[|ww_h|]_E\,\mathrm{d}\gamma + \sum_{E\in\mathscr{E}}\int_E b\cdot n_E[|w|]_E A_E w_h\,\mathrm{d}\gamma$$
$$= -\sum_{E\in\mathscr{E}} b\cdot n_E A_E w[|w_h|]_E\,\mathrm{d}\gamma\,.$$

This term can be estimated in the following way

$$\Big|-\sum_{E\in\mathscr{E}}\int_E b\cdot n_E A_E w[|w_h|]_E\,\mathrm{d}\gamma\Big| \le \sum_{E\in\mathscr{E}}\int_E |b\cdot n_E|^{1/2}|A_E w||b\cdot n_E|^{1/2}|\,[|w_h|]_E\,\mathrm{d}\gamma$$
$$\le C\sum_{E\in\mathscr{E}}\|A_E w|_{0,E}\||b\cdot n_E|^{1/2}[|w_h t]_E|_{0,E}]$$
$$\le C\Big(\sum_{E\in\mathscr{E}}\|A_E w|_{0,E}^2\Big)^{1/2}\Big(\sum_{E\in\mathscr{E}}\||b\cdot n_E|^{1/2}[|w_h|]_E|_{0,E}^2\Big)^{1/2}.$$

For each term in the first sum, we use

$$\|A_E w\|_{0,E}\le Ch_E^{3/2}\|u\|_{2,N(E)}$$

which follows with the help of (18). Summarizing the estimates, we get

$$\Big|-\sum_{E\in\mathscr{E}}\int_E b\cdot n_E A_E w[|w_h|]_E\,\mathrm{d}\gamma\Big| \le C\Big(\sum_{E\in\mathscr{E}} h_E^3\|u\|_{2,N(E)}^2\Big)^{1/2}|||w_h|||.$$

In order to bound the consistency error, we use (6)

$$\sum_{K\in T_h}\varepsilon\int_{\partial K}\frac{\partial u}{\partial n_K}w_h\,\mathrm{d}\gamma = -\sum_{E\in\mathscr{E}}\int_E \varepsilon\frac{\partial u}{\partial n_E}[|w_h|]_E\,\mathrm{d}\gamma$$
$$= -\sum_{E\in\mathscr{E}}\int_E \varepsilon\Big(\frac{\partial u}{\partial n_E} - P_E\frac{\partial u}{\partial n_E}\Big)[|w_h|]_E\,\mathrm{d}\gamma\,.$$

Applying the projection estimate (19), we see that

$$\Big|\sum_{K\in T_h}\varepsilon\int_{\partial K}\frac{\partial u}{\partial n_K}w_h\,\mathrm{d}\gamma\Big| \le C\sum_{K\in T_h}\varepsilon h_K\|u\|_{2,K}|w_h|_{1,K}$$
$$\le C\sum_{K\in T_h}\varepsilon^{1/2}h_K\|u\|_{2,K}\varepsilon^{1/2}|w_h|_{1,K}$$
$$\le C\Big(\sum_{K\in T_h}\varepsilon h_K^2\|u\|_{2,K}^2\Big)^{1/2}|||w_h|||$$

holds. Thus, the first term of (22) is estimated by

$$|||I_h u - u_h||| \leq C\Big(\sum_{K \in T_h} \lambda_K h_K^2 \|u\|_{2,K}^2\Big)^{1/2} + C\Big(\sum_{E \in \mathscr{E}} h_E^3 \|u\|_{2,N(E)}^2\Big)^{1/2}, \tag{26}$$

with λ_K defined in (21).The interpolation estimates show that

$$|||w|||^2 = \sum_{K \in T_h} \big(\varepsilon |w|_{1,K}^2 + c_0 \|w\|_{0,K}^2 + \delta_K \|b \cdot \nabla w\|_{0,K}^2 \big) + \sum_{E \in \mathscr{E}} \int_E |b \cdot n_E| \, [|w|]_E^2 \, d\gamma$$

$$\leq C \sum_{K \in T_h} h_K^2 (\varepsilon + h_K^2 + \delta_K) \|u\|_{2,K}^2 + C \sum_{E \in \mathscr{E}} h_E^3 \|u\|_{2,N(E)}^2$$

and, using $\varepsilon + h_K^2 + \delta_K \leq \lambda_K$, we get for the second term of (22)

$$|||w|||^2 \leq C \sum_{K \in T_h} \lambda_K h_K^2 \|u\|_{2,K}^2 + C \sum_{E \in \mathscr{E}} h_E^3 \|u\|_{2,N(E)}^2 .$$

This gives in combination with (26) and the shape regularity of the triangulation the convergence estimate (20). □

REMARK 6. For the choice $\delta_K \sim h_K$ in (21), which gives $\lambda_K \sim h_K$ in the case $\varepsilon < h_k$, the error estimate (20) yields

$$|||u - u_h||| \leq C h^{3/2} \|u\|_{2,\Omega} .$$

Thus, the new NSDFEM has the same order of convergence in the $|||\cdot|||$-norm, and consequently in the L^2-norm, as the standard SDFEM for conforming P_1-triangular and Q_1-quadrilateral elements, respectively, [17].

4. Numerical Studies

Now, we want to investigate the numerical behaviour of the discretization schemes (9), (10) for some test examples. The goals are first to verify the order of convergence proven in Theorem 5 and that the constant in the error estimate (20) is independent of ε, second to compare the results using triangular and quadrilateral elements and third to demonstrate the effect on the stability of the discrete solution if the jump terms are omitted in the discretization schemes.

In our test examples, problem (1) is solved using the domain $\Omega = (0,1)^2$. The error will be measured in the L^2-norm $\|\cdot\|_{0,\Omega}$ and the streamline–diffusion norm $\|v\|_S$ on $V + V_h$ defined by

$$\|v\|_S = \Big(\sum_{K \in T_h} \big(\varepsilon |v|_{1,K}^2 + c_0 \|v\|_{0,K}^2 + \delta_K \|b \cdot \nabla v\|_{0,K}^2 \big) \Big)^{1/2} .$$

The order of convergence is computed using the errors on the two finest grids.

The grids are generated by uniform refinement of a given coarse grid (level 0). Here, we present results for two kinds of coarse grids, a quadrilateral and a triangular one shown in Fig. 2.

In the following tables, the new discretization (9), (10) will be referred as 'new-j', the discretization with the

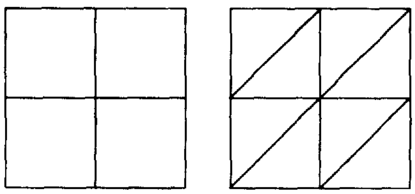

Fig. 2. Quadrilateral and triangular coarse grid (level 0).

jump terms introduced [8] using $\sigma_E = 1$ (see Remark 2) as 'old-j' and the standard NSDFEM discretization without jump terms as 'no-j'.

EXAMPLE 7: *Smooth polynomial solution.* We take $b = (3,2)$, $c = 2$ and the right-hand side f is chosen such that

$$u(x, y) = 100(1-x)^2x^2y(1-2y)(1-y)$$

is the solution of (1). The streamline–diffusion parameters are chosen to be $\delta_K = h_K \forall K \in T_h$.

All discretizations using jump terms show the expected order of convergence of 1.5 in the streamline–diffusion norm and second order convergence in the L^2-norm, see Tables 1 and 2. The same order of convergence can be observed for the standard NSDFEM without jump terms on the quadrilateral mesh whereas a reduction occurs for $\varepsilon = 10^{-8}$ on the triangular mesh. Between the nonconforming streamline–diffusion-method proposed in [8] ('old-j') and the new one defined in (9), (10) ('new-j'), there are often only minor differences. However, for the discretization (9), (10), no parameters in the jump terms have to be chosen. Considering the same level, the results on the quadrilateral mesh are better than on the triangular mesh although the quadrilateral elements need less degrees of freedom.

Tables 3 and 4 present results for a fixed mesh size h and different values of the diffusion parameter ε. For all discretizations with jump terms and the standard NSDFEM discretization on the quadrilateral meshes, the error is independent of ε. This confirms that the constant C in the error estimate (20) does not depend on ε. The order of convergence for the standard NSDFEM on triangular meshes has been studied in detail in [8].

EXAMPLE 8: *Layers at the outflow part of the boundary.* Let $b = (2,3)$ and $c = 1$. The right-hand side f and the Dirichlet boundary condition are chosen such that

$$u(x, y, \varepsilon) = xy^2 - y^2 \exp\left(\frac{2(x-1)}{\varepsilon}\right) - x \exp\left(\frac{3(y-1)}{\varepsilon}\right) + \exp\left(\frac{2(x-1)+3(y-1)}{\varepsilon}\right)$$

Table 1
Example 7, error and order of convergence in the norm $\|\cdot\|_{0,\Omega}$

		$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$		
Level	Grid	no-j	old-j	new-j	no-j	old-j	new-j
5	△	4.702 − 4	3.110 − 4	3.134 − 4	2.887 − 3	3.211 − 4	3.221 − 4
	□	2.902 − 4	2.931 − 4	2.521 − 4	3.046 − 4	3.075 − 4	2.587 − 4
6	△	8.154 − 5	7.574 − 5	7.641 − 5	9.611 − 4	8.123 − 5	8.102 − 5
	□	7.090 − 5	7.132 − 5	6.019 − 5	7.874 − 5	7.916 − 5	6.323 − 5
7	△	1.710 − 5	1.852 − 5	1.902 − 5	3.139 − 4	2.047 − 5	2.032 − 5
	□	1.639 − 5	1.634 − 5	1.481 − 5	1.991 − 5	1.996 − 5	1.559 − 5
order	△	2.254	2.032	2.006	1.614	1.989	1.996
order	□	2.122	2.126	2.022	1.984	1.987	2.020

Table 2
Example 7, error and order of convergence in the norm $\|\cdot\|_S$

		$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-8}$		
Level	Grid	no-j	old-j	new-j	no-j	old-j	new-j
5	▽	2.330 − 2	2.428 − 2	2.428 − 2	2.463 − 2	2.427 − 2	2.427 − 2
	□	2.066 − 2	2.066 − 2	2.065 − 2	2.065 − 2	2.065 − 2	2.064 − 2
6	□	8.685 − 3	8.690 − 3	8.690 − 3	8.792 − 3	8.682 − 3	8.682 − 3
	□	7.306 − 3	7.306 − 3	7.304 − 3	7.299 − 3	7.299 − 3	7.296 − 3
7	▽	3.090 − 3	3.093 − 3	3.094 − 3	3.120 − 3	3.087 − 3	3.087 − 3
	□	2.586 − 3	2.586 − 3	2.585 − 3	2.580 − 3	2.580 − 3	2.579 − 3
order	△	1.491	1.490	1.490	1.494	1.492	1.492
order	□	1.499	1.499	1.498	1.500	1.500	

Table 3
Example 7, error on level 7 for different ε in the norm $\|\cdot\|_{0,\Omega}$

	Triangular mesh		Quadrilateral mesh	
ε	no-j	new-j	no-j	new-j
10^{-4}	1.710 − 5	1.902 − 5	1.628 − 5	1.481 − 5
10^{-6}	1.500 − 4	2.028 − 5	1.986 − 5	1.557 − 5
10^{-8}	3.139 − 4	2.032 − 5	1.991 − 5	1.559 − 5
10^{-10}	3.191 − 4	2.032 − 5	1.991 − 5	1.559 − 5

Table 4
Example 7, error on level 7 for different ε in the norm $\|\cdot\|_S$

	Triangular mesh		Quadrilateral mesh	
ε	no-j	new-j	no-j	new-j
10^{-4}	3.090 − 3	3.094 − 3	2.586 − 3	2.585 − 3
10^{-6}	3.117 − 3	3.087 − 3	2.580 − 3	2.579 − 3
10^{-8}	3.120 − 3	3.087 − 3	2.580 − 3	2.579 − 3
10^{-10}	3.142 − 3	3.087 − 3	2.580 − 3	2.579 − 3

is the solution of (1), see Fig. 3. The solution has boundary layers at the lines $x = 1$ and $y = 1$. The absolute value of the Dirichlet boundary condition becomes exponentially small for $\varepsilon \to 0$.

For the computations, the parameters $\varepsilon = 10^{-8}$ and $\delta_K = 0.25 h_K \forall K \in T_h$ have been used. The thickness of the boundary layers is smaller than h for all meshes. Thus, the interpolation error in the layer region reduces the order of convergencein the global L^2-norm to 0.5. The discretization methods with jump terms reach this order, see Table 5. Streamline–diffusion methods are known to combine good stability with high accuracy outside the layers. The latter can be seen in Table 6 where the local L^2-error measured in $\tilde{\Omega} = (0, 0.75)^2$ is presented.

Using the standard NSDFEM without jump terms ('no-j') on a triangular mesh can lead to huge oscillations inside the layers, see Fig. 4. In contrast, on a quadrilateral mesh the discretization 'no-j' remains stable, see Table 5. Here, the use of a quadrilateral mesh shows advantages. However, a theoretical explanation of this phenomenon is an open problem.

REMARK 9. The linear systems have been solved using CGS with ILU as preconditioner or a multigrid method with ILU_β-smoothing (see e.g. [23]). In the numerical tests we have observed rates of convergence which were clearly below 1for the discretizations with jump terms and the standard discretization ('no-j') on quadrilateral meshes (e.g. not worse than 0.6 for the multigrid method). In contrast, using the standard NSDFEM discretization on a triangular mesh, the rate of convergence has been very close to 1 (e.g. 0.99 and worse in the

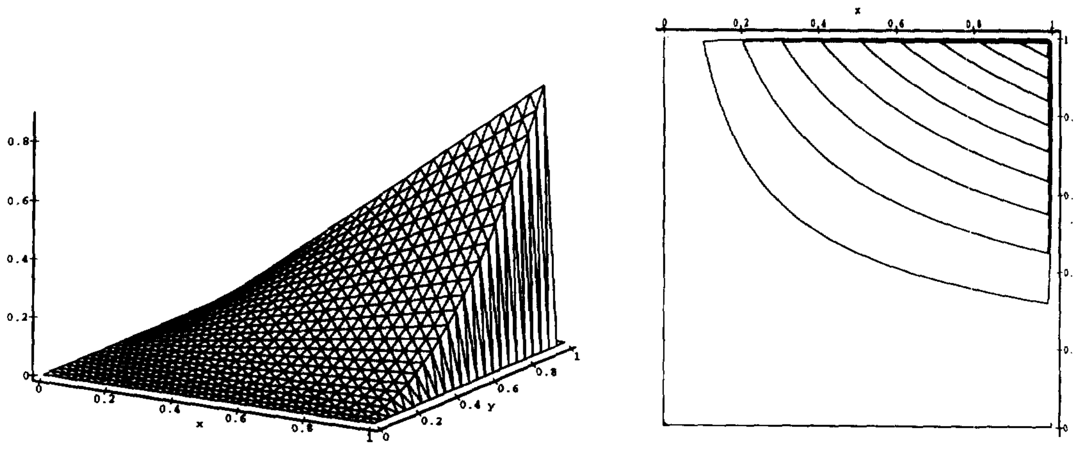

Fig. 3. Exact solution and contour-lines of Example 8.

Table 5
Example 8, Error and order of convergence in the norm $\|\cdot\|_{0,\Omega}$

	Triangular mesh		Quadrilateral mesh	
Level	no-j	new-j	no-j	new-j
4	1.729 + 0	7.129 − 2	7.728 − 2	7.758 − 2
5	2.461 + 0	5.068 − 2	5.466 − 2	5.479 − 2
6	3.487 + 0	3.593 − 2	3.865 − 2	3.872 − 2
7	4.898 + 0	2.544 − 2	2.733 − 2	2.737 − 2
order	− 0.4900	0.498	0.500	0.500

Table 6
Example 8, local L^2-error and order of convergence in the norm $\|\cdot\|_{0,\tilde{\Omega}}$

	Triangular mesh		Quadrilateral mesh	
Level	no-j	new-j	no-j	new-j
4	7.030 − 4	7.995 − 5	6.067 − 5	5.662 − 5
5	4.385 − 5	1.919 − 5	1.514 − 5	1.416 − 5
6	1.091 − 5	4.797 − 6	3.784 − 6	3.539 − 6
7	2.683 − 6	1.199 − 6	9.459 − 7	8.848 − 7
order	2.023	2.000	2.000	2.000

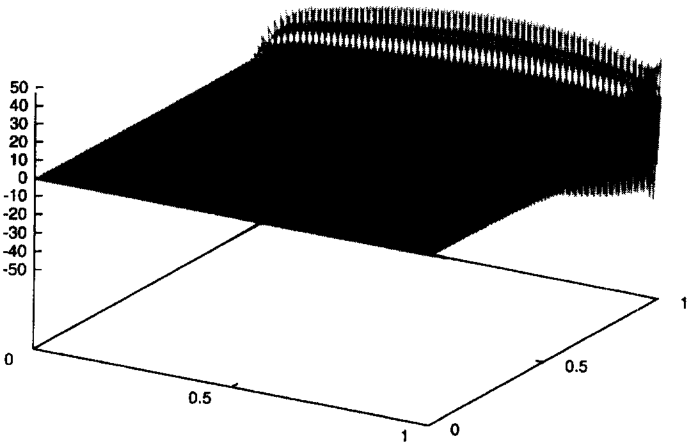

Fig. 4. Example 8, solution on a triangular mesh without jump terms, level 5.

multigrid method) and the number of iterations has increased with the level of refinement. Thus, the use of jump terms on triangular meshes improves both the accuracy of the discrete solution and the convergence properties of the solvers.

REMARK 10. Computations using non-rectangular quadrilateral grids have shown similar results as in the previous examples. The behaviour of the discretizations on mixed meshes consisting of triangles and quadrilaterals (e.g. appearing in adaptive refinements of quadrilateral meshes with triangular closure) is similar to that of pure triangular meshes. In particular, huge oscillations can appear in the standard NSDFEM which can be removed by using the jump terms introduced in (10).

References

[1] M. Crouzeix and P.A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, RAIRO Numer. Anal. 3 (1973) 33–76.

[2] O. Dorok, Eine stabilisierte finite Elemente Methode zur Lösung der Boussinesq-Approximation der Navier–Stokes-Gleichungen, Ph.D. Thesis, Otto-von-Guericke-Universität Magdeburg, 1995.
[3] O. Dorok, W. Grambow and L. Tobiska, Aspects of the finite element discretizations for solving the Boussinesq approximation of the Navier–Stokes equations, in: F.-K. Hebeker, R. Rannacher and G. Wittum, eds. Numerical Methods for the Navier–Stokes Equations. Proceedings of the International Workshop on Numerical Methods for the Navier–Stokes equations held at Heidelberg Oct 25–28, Vieweg–Verlag (1994) p. 50–61.
[4] O. Dorok, V. John, U. Risch, F. Schieweck and L. Tobiska, Parallel finite element methods for the incompressible Navier–Stokes equations, in: E.H. Hirschel, ed., Flow Simulation with High-Performance Computers II. Vieweg–Verlag, Notes on Numerical Fluid Mechanics, Vol. 52 (1996) 20–33.
[5] K. Eriksson and C. Johnson, Adaptive streamline diffusion finite element methods for stationary convection–diffusion problems, Math. Comput. 60 (1993) 167–188.
[6] T.J.R. Hughes and A.N. Brooks, A multidimensional upwind scheme with no crosswind diffusion, in: T.J.R. Hughes, ed., Finite Element Methods for Convection Dominated Flows, Vol. 34 of AMD. (ASME, New York, 1979).
[7] V. John, Parallele Lösung der inkompressiben Navier–Stokes Gleichungen auf adaptiv verfeinerten Gittern. Ph.D. Thesis, Otto-von-Guericke-Universität Magdeburg, 1997.
[8] V. John, J. Maubach and L. Tobiska, Nonconforming streamline–diffusion-finite-element-methods for convection–diffusion problems, Numer. Math. 78 (1997) 165–188.
[9] C. Johnson and J. Saranen, Streamline diffusion methods for the incompressible Euler and Navier–Stokes equations, Math. Comput. 47 (1986) 1–18.
[10] C. Johnson, A.H. Schatz and L.B. Wahlbin, Crosswind smear and pointwise errors in the streamline diffusion finite element methods, Math. Comput. 49(179) (1987) 25–38.
[11] W. Layton, J. Maubach and P. Rabier, Robustness of an elementwise parallel finite element method for convection–diffusion problems, Technical Report, ICMA-93-185 Department of Mathematics and Statistics University of Pittsburgh, 1995.
[12] G. Lube and L. Tobiska, A nonconforming finite element method of streamline-diffusion type for the incompressible Navier–Stokes equations, J. Comput. Math. 8(2) (1990) 147–158.
[13] U. Nävert, A finite element method for convection–diffusion problems, Ph.D. Thesis, Chalmers University of Technology Göteborg, 1982.
[14] K. Niijima, Pointwise error estimates for a streamline diffusion finite element scheme. Numer. Math. 56 (1990) 707–719.
[15] R. Rannacher and S. Turek, Simple nonconforming quadrilateral Stokes element, Numer. Methods Partial Diff. Eqs. 8 (1992) 97–111.
[16] U. Risch, An upwind finite element method for singularly perturbed elliptic problems and local estimates in the L^∞-norm, Math. Modél. Anal. Numér. 24(2) (1990) 235–264.
[17] H.-G. Roos, M. Stynes and L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations. Convection–Diffusion and Flow Problems (Springer-Verlag, 1996).
[18] F. Schieweck, Parallele Lösung der stationären inkompressiblen Navier–Stokes Gleichungen. Habilitationsschrift, Otto-von-Guericke-Universität Magdeburg, 1997.
[19] F. Schieweck and L. Tobiska, A nonconforming finite element method of upstream type applied to the stationary Navier–Stokes equations. RAIRO Numer. Anal. 23 (1989) 627–647.
[20] F. Schieweck and L. Tobiska, An optimal order error estimate for an upwind discretization of the Navier–Stokes equations, Numer. Methods Partial Diff. Eqs. 12 (1996) 407–421.
[21] L. Tobiska, Stabilized finite element methods for the Navier–Stokes problem, in: J.J.H. Miller, ed., Applications of Advanced Computational Methods for Boundary and Interior Layers, (Boole Press, Dublin, 1993) 173–191.
[22] L. Tobiska and R. Verfürth, Analysis of a streamline diffusion finite element method for theStokes and Navier–Stokes equations, SIAM J. Numer. Anal. 33(1) (1996) 107–127.
[23] G. Wittum, On the robustness of ILU-smoothing, SIAM J. Sci. Stat. Comput. 10 (1989) 699–717.
[24] G. Zhou, How accurate is the streamline diffusion finite element method? J. Comput. Math. 66(217) (1997) 31–44.