SIAM J. NUMER. ANAL.
Vol. 46, No. 1, pp. 151–179

CONVERGENCE OF TIME-AVERAGED STATISTICS OF FINITE ELEMENT APPROXIMATIONS OF THE NAVIER–STOKES EQUATIONS*

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Abstract. When discussing numerical solutions of the Navier–Stokes equations, especially when turbulent flows are concerned, there are at least two questions that can be raised. *What is meaningful to compute? How to determine the fidelity of the computed solution with respect to the true solution?* This paper takes a step towards the answer of these questions for turbulent flows. We consider long-time averages of weak solutions of the Navier–Stokes equations, rather than strong solutions. We present error estimates for the time-averaged energy dissipation rate, drag and lift, most of them under the assumption of small Reynolds/generalized Grashof number. For shear flows, we address the question of fidelity of the computed solution with respect to the true solution, in view of Kolmogorov's energy cascade theory.

Key words. time-averaged statistics, Navier–Stokes equations, turbulence

AMS subject classifications. 74H15, 76D05, 76F65

DOI. 10.1137/060649550

1. Introduction.

The motion of an incompressible Newtonian fluid is governed by the incompressible Navier–Stokes equations. A fundamental problem of fluid motion is turbulence and a fundamental problem in the Navier–Stokes equations is uniqueness of weak solutions in the general case of no assumed extra regularity or small data. The Leray conjecture [20] states that these two problems are connected: the lack of uniqueness of weak solutions (which he called "turbulent solutions") is not an artifact of imperfect mathematical techniques, but it reflects fundamental physical mechanisms of turbulence.

The numerical analysis of turbulent flows is caught between the gaps in the physical understanding of turbulence and those in the mathematical foundations of the Navier–Stokes equations. For example, smooth strong solutions are not expected while, if the uniqueness of the weak solution is unknown, bounding the error in a numerical simulation is currently not possible without assuming extra regularity on the solution, or without assuming both the initial data u_0 and the body force $f(x,t)$ to be very small.

On the other hand, computational simulations are carried out and statistics of computed fluid velocities and pressures often reflect rather accurately statistics of physical flows even in the absence of mathematical justification for this accuracy. Further, statistics (by which we shall mean long-time averages) are often smooth, behaving deterministically (often in accordance with the Kolmogorov theory [18]) in both numerical simulations and physical experiments. From this situation, a challenge for the numerical analysis of fluid motion arises: develop a rigorous understanding

*Received by the editors January 10, 2006; accepted for publication (in revised form) June 6, 2007; published electronically December 20, 2007.
http://www.siam.org/journals/sinum/46-1/64955.html
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of how statistics computed from numerical simulations reflect those for the unknown solution of the Navier–Stokes equations.

The incompressible Navier–Stokes equations (with homogeneous Dirichlet boundary conditions) are given by

$$\begin{aligned} u_t + u\cdot\nabla u - \nu\Delta u + \nabla p &= f && \text{in } \Omega\times(0,\infty),\\ \nabla\cdot u &= 0 && \text{in } \Omega\times[0,\infty),\\ u(x,0) &= u_0 && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma\times[0,\infty). \end{aligned} \tag{1.1}$$

Here, Ω denotes a bounded and regular flow domain in $\mathbb{R}^d$, $d=2,3$, $\Gamma=\partial\Omega$ is the boundary of Ω, $u(x,t)$, $p(x,t)$ are the fluid velocity and pressure, ν is the kinematic viscosity, $f(x,t)\in L^\infty(0,\infty;L^2(\Omega)^d)$ are the body forces, and $u_0\in L^2(\Omega)^d$ is a weakly divergence-free initial condition. The Reynolds number is defined by $Re = LU/\nu$, where the constants L and U are, respectively, a reference large-scale length and velocity. From the point of view of applications, the three-dimensional case is the important one. This case is mathematically much more challenging than the two-dimensional one, too.

We will study statistics related to the energy dissipation rate and the total kinetic energy of the flow. The energy dissipation rate per unit volume of the flow at time t is given by

$$\varepsilon(u) := \frac{\nu}{|\,\Omega\,|}\|\,\nabla u(\cdot,t)\,\|^2,$$

where $|\,\Omega\,|$ is the volume of Ω and $\|\cdot\|$ denotes the $L^2(\Omega)$-norm, and its total kinetic energy per unit volume is

$$k(u) := \frac{1}{2|\,\Omega\,|}\|\,u(\cdot,t)\,\|^2.$$

The time average $\langle q\rangle$ of a quantity q is defined by

$$\langle q\rangle = \limsup_{T\to\infty}\frac{1}{T}\int_0^T q(t)\,dt. \tag{1.2}$$

Consequently, the time average of the energy dissipation rate is

$$\langle\varepsilon(u)\rangle = \limsup_{T\to\infty}\frac{1}{T}\int_0^T \varepsilon(u)\,dt = \frac{\nu}{|\,\Omega\,|}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|\nabla u\|^2 = \frac{\nu}{|\,\Omega\,|}\langle\|\nabla u\|^2\rangle,$$

and the time average of the kinetic energy is

$$\langle k(u)\rangle = \frac{1}{2|\,\Omega\,|}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|\,u\,\|^2\,dt = \frac{1}{2|\,\Omega\,|}\langle\|\,u\,\|^2\rangle.$$

In practical simulations of turbulent flows, it is typical to compute time-averaged flow statistics, which are then matched against benchmark averages; see, e.g., [2, 17, 15, 16, 21, 5]. However, there is very little numerical analysis in support of this approach. Of course, if the error in certain norms of the velocity and the pressure is provably optimal over $0\le t<\infty$, then time averages involving these norms are convergent with optimal rates as well. But, the practical case is complementary: time averages seem to be predictable even when the dynamic flow behavior over bounded time intervals is irregular. This is the case we aim to study. However, a complete analysis seems to be still beyond the present mathematical tools.

In section 3.1, the case of large data is studied and estimates for the L^2-norm of the time-averaged error of the pressure (Theorem 3.1) and the time-averaged energy dissipation rate (Theorem 3.2) are given. Then, in section 3.2, we consider the case of arbitrary initial data u_0 and asymptotically small body forces (small generalized Grashof number) which converge to a stationary limit $f^*(x) = \lim_{t\to\infty} f(x,t)$. Let (u^h, p^h) be a finite element approximation of the velocity field and the pressure and assume that $f^*(x)$ satisfies a small data condition. Let u^* be the solution of the stationary Navier–Stokes equations with body force f^*. We show that

$$\langle \varepsilon(u - u^*) \rangle = 0, \quad \langle \varepsilon(u^h - u^{*h}) \rangle = 0,$$

and we prove (Theorem 3.3) an error estimate which shows that the problem of estimating $\langle \varepsilon(u - u^h) \rangle$ reduces to the one of estimating $\|\nabla(u^* - u^{*h})\|^2$. Then the error goes to zero optimally as the mesh width $h \to 0$. This result is plausible because the possible irregularities caused by large initial data are washed out by the time averaging.

Section 4 studies the flow through a channel around a body. Under the assumption of bounded growth of the kinetic energy (in the case of inhomogeneous inflow boundary conditions), we give estimates for the time-averaged drag and lift coefficients on an immersed body (Theorem 4.1). It is shown that the assumption holds true if the inflow boundary conditions are sufficiently small.

In section 5, we consider the complementary situation of a flow driven by a large and persistent boundary condition. We are not (yet) able to perform a complete error analysis in this case. However, following the important work of Constantin and Doering [4] in the continuous case, we show that, provided the first mesh line in the finite element mesh is within $O(1/Re)$ of the moving wall which drives the flow, the computed time-averaged energy dissipation rate for the shear flow scales as predicted for the continuous flow by the Kolmogorov theory (Theorem 5.1):

$$\langle \varepsilon(u^h) \rangle \quad \le \quad C\,\frac{U^3}{L}.$$

This restriction on the mesh size arises from mathematical analysis of constructible background flows in finite element spaces and their subsequent analysis. However, it is in accordance with an entirely different observation of the thickness of time-averaged turbulent boundary layers [24].

2. Mathematical preliminaries.

Throughout the paper, we use the standard notations $L^p(\Omega)$, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ for the Lebesgue and Sobolev spaces, respectively. The inner product in the space $L^2(\Omega)$, $L^2(\Omega)^d$, and $L^2(\Omega)^{d\times d}$ will be denoted by $(\cdot,\cdot)$ and its norm by $\|\cdot\|$. Norms in Sobolev spaces $H^k, k > 0$, are denoted by $\|\cdot\|_k$. The symbol C stands for generic constants independent of the viscosity ν and the mesh size h.

The velocity at a given time t is sought in the space

$$\mathbb{X} = H_0^1(\Omega)^d = \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{d\times d} \quad \text{and} \quad v = 0 \ \text{ on } \Gamma\},$$

equipped with the norm $\|v\|_{\mathbb{X}} = \|\nabla v\|$. The norm of the dual space $\mathbb{X}^*$ of $\mathbb{X}$ is denoted by $\|\cdot\|_{-1}$. The pressure at time t is sought in the space

$$\mathbb{Q} = L_0^2(\Omega) = \left\{ q : \, q \in L^2(\Omega)\,, \int_\Omega q\, dx = 0 \right\}.$$

In addition, the space of weakly divergence-free functions is denoted by

$$\mathbb{V} = \{ v \in \mathbb{X} : (\nabla \cdot v, q) = 0 \; \forall \, q \in \mathbb{Q} \}.$$

For Y being a Banach space and for $0 < T < \infty$, the space $L^p(0, T; Y)$ with $1 \leq p < \infty$ (and the usual modification if $p = \infty$) consists of all functions $v : [0, \infty) \to Y$ for which

$$\int_0^T \| \, v(t) \, \|_Y^p \, dt < \infty.$$

Define the trilinear forms on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$:

$$b(u, v, w) = \int_\Omega u \cdot \nabla v \cdot w \, dx \qquad \text{and} \qquad b_s(u, v, w) = \frac{1}{2} b(u, v, w) - \frac{1}{2} b(u, w, v).$$

Note that the convective form and the skew-symmetric form are equal for $u \in \mathbb{V}$: $b(u, v, w) = b_s(u, v, w)$.

LEMMA 2.1. *Let* $\Omega \subset \mathbb{R}^d, d = 2$ *or* 3. *Then there exists a constant* $C = C(\Omega) < \infty$ *such that*

$$b(u, v, w) \leq C \| \nabla u \| \, \| \nabla v \| \, \| \nabla w \| \qquad \forall \, u, v, w \in \mathbb{X}. \tag{2.1}$$

If $d = 3$, *this can be improved to*

$$b(u, v, w) \leq C \sqrt{\| \, u \, \| \, \| \nabla u \|} \, \| \nabla v \| \, \| \nabla w \| \qquad \forall \, u, v, w \in \mathbb{X}, \tag{2.2}$$

or, *respectively*, *to*

$$b(u, v, w) \leq C \| \nabla u \| \, \| \nabla v \| \sqrt{\| \, w \, \| \, \| \nabla w \|} \qquad \forall \, u, v, w \in \mathbb{X}. \tag{2.3}$$

Proof. Hölder's inequality and Sobolev imbedding theorems give

$$|b(u, v, w)| \leq \|u\|_{L^3} \|\nabla v\|_{L^2} \|w\|_{L^6} \leq C \|u\|_{1/2} \|v\|_1 \|w\|_1.$$

Estimate (2.2) is obtained now by applying the interpolation inequality between $L^2(\Omega)$ and $H^1(\Omega)$ (see Adams [1]),

$$\|u\|_{1/2} \leq C \|u\|^{1/2} \|u\|_1^{1/2},$$

and Poincaré's inequality. Similarly, (2.3) follows from

$$b(u, v, w) \leq C \|u\|_1 \|v\|_1 \|w\|_{1/2}.$$

The proof of (2.1) uses, after the application of Hölder's inequality, the imbedding $H^1(\Omega) \subset L^3(\Omega)$. $\square$

Estimates for the skew-symmetric form $b_s(\cdot,\cdot,\cdot)$ can be derived directly from (2.1)–(2.3). The norms of the trilinear forms $b_s(\cdot,\cdot,\cdot)\ :\ \mathbb{X}\to\mathbb{R}$ and of the restriction $b_s(\cdot,\cdot,\cdot)\ :\ \mathbb{V}\to\mathbb{R}$ are denoted by

(2.4)
$$M=\sup_{u,v,w\in\mathbb{X}}\frac{b_s(u,v,w)}{\|\nabla u\|\,\|\nabla v\|\,\|\nabla w\|}<\infty \text{ and } N=\sup_{u,v,w\in\mathbb{V}}\frac{b_s(u,v,w)}{\|\nabla u\|\,\|\nabla v\|\,\|\nabla w\|}<\infty.$$

We will consider in this paper finite element approximations of (1.1). Throughout the paper, we shall assume that the velocity-pressure finite element spaces $\mathbb{X}^h\subset\mathbb{X}$ and $\mathbb{Q}^h\subset\mathbb{Q}$ are conforming, have approximation properties typical of finite element spaces commonly in use, and satisfy the discrete inf-sup condition,

$$\inf_{q^h\in\mathbb{Q}^h}\sup_{v^h\in\mathbb{X}^h}\frac{(q^h,\,\nabla\cdot v^h)}{\|\nabla v^h\|\,\|q^h\|}\geq\beta^h>0, \tag{2.5}$$

where β^h is bounded away from zero uniformly in h. The space of discretely divergence-free functions is defined by

$$\mathbb{V}^h=\{v^h\in\mathbb{X}^h:(q^h,\,\nabla\cdot v^h)=0\quad\forall\, q^h\in\mathbb{Q}^h\}.$$

For examples of such spaces see, e.g., Gunzburger [13], Brezzi and Fortin [3], and Girault and Raviart [11].

Norms of the trilinear form $b_s(\cdot,\cdot,\cdot)$ restricted to $\mathbb{X}^h$ and $\mathbb{V}^h$, respectively, are defined in the same way as in (2.4), leading to corresponding constants M^h and N^h. Note that $M\geq M^h,N^h,N$ and that $N^h\to N$ as $h\to 0$ (see [11]) and by the same argument $M^h\to M$.

2.1. The continuous-in-time finite element discretization. Consider the standard finite element discretization of the Navier–Stokes equations. The semidiscrete (continuous-in-time) finite element approximations $u^h=u^h(\cdot,t)$ and $p^h=p^h(\cdot,t)$ are the maps $u^h:[\,0,\infty\,)\to\mathbb{X}^h$, $p^h:(\,0,\infty\,)\to\mathbb{Q}^h$ satisfying

$$(u^h_t,v^h)+\nu(\nabla u^h,\nabla v^h)+b_s(u^h,u^h,v^h)-(p^h,\nabla\cdot v^h)=(f,v^h)\quad\forall\, v^h\in\mathbb{X}^h, \tag{2.6}$$

$$(\nabla\cdot u^h,q^h)=0\quad\forall\, q^h\in\mathbb{Q}^h, \tag{2.7}$$

$$(u^h(\cdot,0)-u_0,v^h)=0\quad\forall\, v^h\in\mathbb{X}^h. \tag{2.8}$$

Under the inf-sup condition (2.5), this is equivalent to: find $u^h\ :\ [\,0,\infty\,)\to\mathbb{V}^h$ satisfying

$$(u^h_t,v^h)+\nu(\nabla u^h,\nabla v^h)+b_s(u^h,u^h,v^h)=(f,v^h)\quad\forall\, v^h\in\mathbb{V}^h, \tag{2.9}$$
$$(u^h(\cdot,0)-u_0,v^h)=0\quad\forall\, v^h\in\mathbb{V}^h.$$

2.2. The associated equilibrium problem. In the case that $f(x,t)\to f^*(x)$ in $H^{-1}(\Omega)$ as $t\to\infty$, we can associate with (1.1) the following equilibrium problem: find $u^*(x),\ p^*(x)$ satisfying

$$\begin{aligned}-\nu\Delta u^*+u^*\cdot\nabla u^*+\nabla p^*&=f^*\quad\text{in }\Omega,\\ \nabla\cdot u^*&=0\qquad\text{in }\Omega,\\ u^*&=0\qquad\text{on }\Gamma,\\ \int_\Omega p^*\,dx&=0.\end{aligned}\tag{2.10}$$

The variational formulation of the equilibrium problem is as follows: Find $u^* \in \mathbb{X}$ and $p^* \in \mathbb{Q}$ such that

$$\nu(\nabla u^*, \nabla v) + b_s(u^*, u^*, v) - (p^*, \nabla \cdot v) = (f^*, v) \quad \forall\, v \in \mathbb{X}, \qquad (\nabla \cdot u^*, q) = 0 \quad \forall\, q \in \mathbb{Q}, \tag{2.11}$$

or, equivalently, find $u^* \in \mathbb{V}$ such that

$$\nu(\nabla u^*, \nabla v) + b_s(u^*, u^*, v) = (f^*, v) \qquad \forall\, v \in \mathbb{V}. \tag{2.12}$$

A finite element approximation (u^{*h}, p^{*h}) is given by the solution of

$$\nu(\nabla u^{*h}, \nabla v^h) + b_s(u^{*h}, u^{*h}, v^h) - (p^{*h}, \nabla \cdot v^h) = (f^*, v^h) \quad \forall\, v^h \in \mathbb{X}^h, \qquad (\nabla \cdot u^{*h}, q^h) = 0 \quad \forall\, q^h \in \mathbb{Q}^h. \tag{2.13}$$

In $\mathbb{V}^h$ this becomes: find $u^{*h} \in \mathbb{V}^h$ such that

$$\nu(\nabla u^{*h}, \nabla v^h) + b_s(u^{*h}, u^{*h}, v^h) = (f^*, v^h) \qquad \forall\, v^h \in \mathbb{V}^h. \tag{2.14}$$

It is known that solutions of the equilibrium problem are nonsingular for small data, generically nonsingular for large data, and optimally approximated by u^{*h} when nonsingular; see [12]. Setting $v = u^*$ in (2.11) and $v^h = u^{*h}$ in (2.13), it is easy to check the a priori bounds

$$\| \nabla u^* \| \le \nu^{-1} \| f^* \|_{-1}, \qquad \| \nabla u^{*h} \| \le \nu^{-1} \| f^* \|_{-1}. \tag{2.15}$$

Both bounds can be sharpened slightly by replacing $\| f^* \|_{-1}$ with the dual norms of $\mathbb{V}$ or $\mathbb{V}^h$, respectively.

It is known that if the problem data is small enough, concretely if

$$N\nu^{-2} \| f^* \|_{-1} < 1,$$

then u^* is unique. If additionally $f(x,t) \equiv f^*(x)$, then $u(x,t) \to u^*(x)$ in $L^2(\Omega)^d$ exponentially fast as $t \to \infty$ and (u^h, p^h) approximates (u, p) optimally; see [11, 19, 13].

2.3. Weak solutions of the Navier–Stokes equations. The problem of turbulence is perhaps intimately connected with questions about weak solutions versus strong solutions of the Navier–Stokes equations. It is well known that weak solutions exist but it is not known if they are unique. (Thus, different methods of proving existence might possibly lead to different solutions.) A strong solution is generally defined as a weak solution which has enough extra regularity to ensure global uniqueness, i.e., which fulfills Serrin's condition [25]. In $\mathbb{R}^3$, it is unknown if strong solutions exist for all time; see [10, 26]. But if a strong solution exists, it is unique. Strong solutions might conceivably describe all fluid motion. However, in at least one conjecture about turbulence the case of strong solutions is associated with laminar flow.

For clearness of notation, we will give the definition of a weak and a strong solution, following [10].

DEFINITION 2.1. *Let*

1. $\mathbf{D}_T = \left\{ v \in C^\infty(\Omega \times [0,T])^d \ : \ v(t) \in C_0^\infty(\Omega)^d \text{ for each } t \right\}$,
2. $\mathcal{D} = \left\{ \psi \in C_0^\infty(\Omega)^d \ : \nabla \cdot \psi = 0 \text{ in } \Omega \right\}$,
3. $H(\Omega) = \left\{ v \in L^2(\Omega)^d : \nabla \cdot v = 0 \text{ and } v \cdot \hat{n} = 0 \text{ on } \Gamma \right\}$, *where* $\hat{n}$ *is the unit outer normal on* Γ,

4. $\mathcal{D}_T = \big\{ \phi(x,t) \in C^\infty(\Omega \times [0,T])^d : \phi(x,t) \in \mathcal{D}$ *for each* $t,\ 0 \le t \le T \big\}$.

Let $u_0 \in H(\Omega)$, $f \in L^2(0,T;L^2(\Omega)^d)$. *A measurable function* $u(x,t) : \Omega \times [0,T] \to \mathbb{R}^d$ *is a weak solution of the Navier–Stokes equations if for all* $T > 0$

1. $u \in L^2(0,T;\mathbb{V}) \cap L^\infty(0,T;H(\Omega))$,
2. u *satisfies the integral relation*

$$(u(T),\phi(T)) - \int_0^T \left[\left(u, \frac{\partial \phi}{\partial t} \right) - \nu\,(\nabla u, \nabla \phi) - b(u,u,\phi) \right] dt$$
$$= (u(0),\phi(0)) + \int_0^T (f,\phi)\, dt$$

for all $\phi \in \mathcal{D}_T$, *which is equivalent to*

$$\frac{d}{dt}(u,v) + \nu(\nabla u, \nabla v) + b(u,u,v) - (f,v) = 0$$

for all $v \in \mathbb{V}$,

3. u *is a strong solution if* u *is a weak solution and* $u \in L^\infty(0,T;\mathbb{V})$ *for any* $T > 0$.

We note that if Ω is a bounded domain with Γ satisfying a cone condition, then it is known that, given a weak solution u, there exists a pressure $p(x,t) \in L^\infty(0,T;L_0^2(\Omega))$ (see, e.g., [10, Remark 2.5]) satisfying

$$(u(T),\phi(T)) - \int_0^T \left[\left(u, \frac{\partial \phi}{\partial t} \right) - \nu\,(\nabla u, \nabla \phi) - b(u,u,\phi) + (p, \nabla \cdot \phi) \right] dt$$
$$= (u(0),\phi(0)) + \int_0^T (f,\phi)\, dt \qquad \forall\, \phi \in \mathbf{D}_T. \tag{2.16}$$

This is equivalent to

$$\frac{d}{dt}(u,v) + \nu\,(\nabla u, \nabla v) + b(u,u,v) - (p, \nabla \cdot v) = (f,v) \qquad \forall\, v \in \mathbb{X}. \tag{2.17}$$

It is well known (see [10]) that weak solutions satisfy the energy inequality: for any $t \in [0,T]$,

$$\frac{1}{2}\|\, u(T)\,\|^2 + \nu \int_0^T \|\,\nabla u(t)\,\|^2\, dt \le \frac{1}{2}\|\, u_0\,\|^2 + \int_0^T (u(t), f(t))\ dt. \tag{2.18}$$

Strong solutions satisfy even an energy equality, i.e., (2.18) with "$\le$" replaced by "$=$".

LEMMA 2.2. *Let* (u,p) *be a weak solution of the Navier–Stokes equations, let* (u^h,p^h) *be its finite element approximation defined by* (2.6)–(2.8), *and let* $e = u - u^h$. *Then, for any* C^1 *map* $v^h : [0,T] \to \mathbb{X}^h$, $q^h : (0,T] \to \mathbb{Q}^h$ *(for each* T, $0 < T < \infty$*)*,

$$(e(T), v^h(T)) - \int_0^T \Bigg[\left(e, \frac{\partial v^h}{\partial t} \right) - \nu(\nabla e, \nabla v^h) - b_s(u,u,v^h) + b_s(u^h,u^h,v^h)$$
$$+ (p - p^h, \nabla \cdot v^h) + (\nabla \cdot e, q^h) \Bigg] dt = (e(0), v^h(0)), \tag{2.19}$$

which is equivalent to

$$\frac{d}{dt}(e, v^h) - \left(e, \frac{\partial v^h}{\partial t}\right) + \nu\,(\nabla e, \nabla v^h) + b_s(u, u, v^h) \\ - b_s(u^h, u^h, v^h) - (p - p^h, \nabla \cdot v^h) = 0, \\ (\nabla \cdot e, q^h) = 0. \tag{2.20}$$

Proof. We shall prove (2.20). The connection between (2.20) and (2.19) is the same as that between (2.16) and (2.17). The second equation in (2.20) follows immediately from $u \in \mathbb{V}$ and (2.8).

First, note that both statements of the lemma follow by subtraction if the term with the time derivative of the test function is well defined, i.e., provided (2.16) can be shown to hold for $\phi \in C^1(0, T; \mathbb{X}^h)$ or (2.17) can be shown for $v \in C^1(0, T; \mathbb{X}^h)$ (since $\mathbb{X}^h \subset \mathbb{X}$). We show the latter.

Since $\mathbb{X}^h \subset \mathbb{X}$, (2.17) holds for all $v := \tilde{v}^h(x) \in \mathbb{X}^h$. Next, let $A(t)$ be a $C^1(0, T)$ function. Multiplication of (2.17) by $A(t)$ and using

$$A(t)\frac{d}{dt}(u, \tilde{v}^h) = \frac{d}{dt}(u, A(t)\tilde{v}^h) - (u, A'(t)\tilde{v}^h)$$

give that u and p satisfy

$$\frac{d}{dt}(u, \tilde{v}^h) - \left(u, \frac{\partial \tilde{v}^h}{\partial t}\right) + \nu\,(\nabla u, \nabla \tilde{v}^h) + b_s(u, u, \tilde{v}^h) - (p, \nabla \cdot \tilde{v}^h) = (f, \tilde{v}^h)$$

with $v^h = v^h(x, t) = A(t)\tilde{v}^h(x)$. The same equation can be derived from (2.6) for (u^h, p^h). Subtracting gives (2.20) for any v^h of the form $v^h = v^h(x, t) = A(t)\tilde{v}^h(x)$.

Since (2.20) is linear in v^h, it also follows for any v^h which is a finite linear combination of such functions,

$$v^h(x, t) = \sum_{i=1}^{N} A_i(t)\tilde{v}_i^h(x).$$

Picking $\tilde{v}_i^h(x)$ to be a basis for $\mathbb{X}^h$ completes the proof. $\square$

2.4. Preliminaries on the time-averaging operator. Define the temporal mean value by

$$\langle q \rangle_T = \frac{1}{T}\int_0^T q(t)\,dt.$$

By properties of integrals, $|\langle q \rangle_T| \leq \langle |q| \rangle_T$, and similarly for any function $q(t, x)$, where $\|\cdot\|$ is a spacial norm of $q(t, x)$,

$$\|\langle q \rangle_T\| \leq \langle \|q\| \rangle_T. \tag{2.21}$$

Let $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ with $p^{-1} + q^{-1} = 1$, $p, q \in [1, \infty]$. Using Hölder's inequality for Lebesgue spaces, one can show a Hölder inequality of the form

$$
\begin{aligned}
|\langle (u,v)\rangle_T| &\le \frac{1}{T}\int_0^T |(u,v)|\,dt \le \frac{1}{T}\int_0^T \|u\|_{L^p}\|v\|_{L^q}\,dt \\
&\le \left(\frac{1}{T}\int_0^T \|u\|_{L^p}^p\,dt\right)^{1/p}\left(\frac{1}{T}\int_0^T \|v\|_{L^q}^q\,dt\right)^{1/q} \\
&= \langle \|u\|_{L^p}^p\rangle_T^{1/p}\langle \|v\|_{L^q}^q\rangle_T^{1/q}.
\end{aligned}
\tag{2.22}
$$

With the same arguments, one obtains, for $u \in H_0^1(\Omega)$ and $v \in H^{-1}(\Omega)$,

$$
|\langle (u,v)\rangle_T| \;\le\; \langle \|\nabla u\|_{L^2}^2\rangle_T^{1/2}\langle \|v\|_{H^{-1}}^2\rangle_T^{1/2}. \tag{2.23}
$$

The time-averaging operator $\langle\cdot\rangle$ defined in (1.2) is just $\langle\cdot\rangle = \limsup_{T\to\infty}\langle\cdot\rangle_T$. One of the subtleties of this definition is that the limit superior is not additive. For example, let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$. Then

$$
\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n, \tag{2.24}
$$

$$
|\limsup_{n\to\infty} a_n| \le \limsup_{n\to\infty} |a_n|, \tag{2.25}
$$

$$
\limsup_{n\to\infty}(a_n\,b_n) \le \limsup_{n\to\infty} a_n \;\limsup_{n\to\infty} b_n \quad \text{if } a_n,\, b_n \ge 0\,\forall\, n, \tag{2.26}
$$

$$
\limsup_{n\to\infty}(c\,a_n) = c\,\limsup_{n\to\infty} a_n \quad \text{for } c \ge 0, \tag{2.27}
$$

$$
\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n \quad \text{if } a_n \le b_n\,\forall\, n, \tag{2.28}
$$

$$
\limsup_{n\to\infty}(a_n)^p = (\limsup_{n\to\infty} a_n)^p \quad \text{if } a_n \ge 0\,\forall\, n \text{ and } p > 0. \tag{2.29}
$$

Note that equality holds in (2.24) if one of the limits on the right-hand side exists.

From these properties, it follows that

$$
|\langle q\rangle| \le \langle |\,q\,|\rangle, \tag{2.30}
$$

$$
\|\langle q\rangle\| \;\le \langle \|\,q\,\|\rangle, \tag{2.31}
$$

$$
|\langle (u,v)\rangle| \le \langle \|u\|_{L^p}^p\rangle^{1/p}\langle \|v\|_{L^q}^q\rangle^{1/q}, \tag{2.32}
$$

$$
|\langle (u,v)\rangle| \le \langle \|\nabla u\|^2\rangle^{1/2}\langle \|v\|_{-1}^2\rangle^{1/2} \tag{2.33}
$$

if the right-hand sides of these inequalities are well defined.

3. Analysis of the time-averaged velocity, pressure, and energy dissipation. First, in section 3.1, the case of arbitrary data fulfilling the conditions of Definition 2.1 is considered. In this case, estimates for the L^2-norm of the time-averaged error of the pressure, $\limsup_{T\to\infty}\|\langle p-p^h\rangle_T\|$ (Theorem 3.1), and the time-averaged energy dissipation rate, $\langle\varepsilon(u-u^h)\rangle$ (Theorem 3.2), are proven. Section 3.2 considers the case of a small body force (but large initial data) such that the solution of the Navier–Stokes equations tends to the solution of the steady state Navier–Stokes equations. It will be shown that in this case the time average of the energy dissipation rate of the error $\langle\varepsilon(u-u^h)\rangle$ will be bounded by the energy dissipation rate of the error of the steady state problem (Theorem 3.3).

3.1. The case of large data.

LEMMA 3.1. *Let u be a weak solution to the Navier–Stokes equations (obtained by the Leray–Hopf construction). If $f \in L^\infty(0,\infty;H^{-1}(\Omega)^d)$, then $\| u \|$ is uniformly bounded,*

$$\frac{1}{2}\| u(T) \|^2 \le e^{-\nu C_{PF}^{-2} T}\| u(0) \|^2 + \frac{C_{PF}^2}{\nu^2}\| f \|^2_{L^\infty(0,\infty;H^{-1})}, \tag{3.1}$$

where C_{PF} is the Poincaré–Friedrich constant of Ω, and consequently

$$\lim_{T\to\infty} \frac{1}{T}\| u(T) \|^2 = 0.$$

Proof. Let V_N be the span of eigenfunctions of the Stokes operator. The Leray–Hopf construction of weak solutions gives a sequence $\{u_N\}$ in $\mathbb{V}_N$ satisfying

$$(u_{N,t}, v) + \nu(\nabla u_N, \nabla v) + b(u_N, u_N, v) = (f, v) \quad \forall\, v \in \mathbb{V}_N, \tag{3.2}$$

with a subsequence $\{u_{N_j}\}$ such that u_{N_j} converges to a weak solution u strongly in $L^2(0,T;H(\Omega)^d)$ and weakly in $L^2(0,T;\mathbb{V})$.

Setting $v = u_N$ in (3.2) and using the Cauchy–Schwarz and the Young inequalities, followed by the Poincaré–Friedrich inequality, we have

$$\frac{d}{dt}\| u_N(t) \|^2 + \nu C_{PF}^{-2}\| u_N(t) \|^2 \le \frac{1}{\nu}\| f(t) \|^2_{-1}.$$

Using the integrating factor $e^{\nu C_{PF}^{-2} t}$, we obtain a differential inequality which can be integrated on $(0,T)$, yielding

$$\| u_N(T) \|^2 \le e^{-\nu C_{PF}^{-2} T}\| u_N(0) \|^2 + \frac{C_{PF}^2}{\nu^2}\| f \|^2_{L^\infty(0,\infty;H^{-1})}.$$

This shows the uniform boundedness of $\| u_N(T) \|$. Taking the limit of both sides, using a weak convergence argument (which is standard for the Navier–Stokes equations and which we show in detail in the proof of Lemma 3.4) and letting $N \to \infty$, we recover (3.1). The second claim now follows from the first one. ∎

LEMMA 3.2. *Let $f \in L^\infty(0,\infty;H^{-1}(\Omega)^d)$ and let u^h satisfy* (2.9). *Then $\| u^h \|$ is uniformly bounded and consequently*

$$\lim_{T\to\infty} \frac{1}{T}\| u^h(T) \|^2 = 0 \qquad \text{and} \qquad \lim_{T\to\infty} \frac{1}{T}\| (u - u^h)(T) \|^2 = 0.$$

Proof. Take $v^h = u^h$ in (2.9) (a step not possible in the continuous case of Lemma 3.1) and proceeding as in the proof of Lemma 3.1, we prove a similar uniform bound for $\| u^h \|$. This proves the first statement and the bound on $\| u - u^h \|$ follows by the triangle inequality and Lemma 3.1. ∎

We next consider time averages.

LEMMA 3.3. *Let u be a weak solution of the Navier–Stokes equations satisfying the energy inequality* (2.18). *Then*

$$\langle \varepsilon(u) \rangle \ \le \ \limsup_{T\to\infty} \frac{1}{|\,\Omega\,|\,T}\int_0^T (\, f, u\,)\, dt = \frac{1}{|\,\Omega\,|}\langle (f,u) \rangle. \tag{3.3}$$

If u satisfies the energy equality, then the above inequality can be replaced by equality. Further, if $f \in L^\infty(0,T;H^{-1}(\Omega)^d) \cap L^2(0,T;L^2(\Omega)^d)$ for every $0 < T < \infty$, then

$$\langle \varepsilon(u) \rangle \quad \leq \quad \frac{1}{\nu|\,\Omega\,|}\langle \|\, f\,\|_{-1}^2 \rangle \quad \leq \quad \frac{1}{\nu|\,\Omega\,|}\,\|\, f\,\|^2_{L^\infty(0,\infty;H^{-1})}. \tag{3.4}$$

The semidiscrete finite element approximation u^h of u satisfies (3.3) *and* (3.4), *with u replaced by u^h, and with equality in* (3.3).

Proof. Since u satisfies the energy inequality (2.18), we have

$$\frac{1}{2T}\frac{1}{|\,\Omega\,|}\|\, u(T)\,\|^2 + \frac{1}{T}\int_0^T \frac{\nu}{|\,\Omega\,|}\|\,\nabla\, u(t)\,\|^2\, dt \leq \frac{1}{2T}\frac{1}{|\,\Omega\,|}\|\, u_0\,\|^2 + \frac{1}{T|\,\Omega\,|}\int_0^T (f,u)\, dt.$$

Since $\frac{1}{2T}\|\, u(T)\,\|^2 \to 0$ by Lemma 3.1 and $\frac{1}{2T}\|\, u_0\,\|^2 \to 0$ as $T \to \infty$, we obtain (3.3). If we use as a starting point the energy equality, the equal sign will be preserved.

For proving (3.4), we apply inequality (2.33) and Young's inequality to (3.3)

$$\begin{aligned}\langle \varepsilon(u)\rangle &\leq \frac{\nu}{2|\,\Omega\,|}\langle \|\,\nabla u\,\|^2\rangle + \frac{1}{2\nu|\,\Omega\,|}\langle \|\, f\,\|^2_{-1}\rangle \\ &\leq \frac{1}{2}\langle \varepsilon(u)\rangle + \frac{1}{2\nu|\,\Omega\,|}\|f\|^2_{L^\infty(0,\infty;H^{-1})}.\end{aligned}$$

In the semidiscrete case, take u^h as a test function in (2.6). This gives

$$\frac{1}{2}\frac{d}{dt}\|\, u^h(t)\,\|^2 + \nu\|\,\nabla u^h(t)\,\|^2 = (u^h(t), f(t)).$$

Integration in $(0,T)$ shows that u^h fulfills an energy equality. Now, the arguments to derive the estimates of the form (3.3) and (3.4) for u^h are the same as in the continuous case. □

Remark 3.1.

1. Let $f \in L^\infty(0,T;H^{-1}(\Omega)^d) \cap L^2(0,T;L^2(\Omega)^d)$ for every $0 < T < \infty$; then one obtains from the triangle inequality that $\langle \varepsilon(u-u^h)\rangle$ is bounded by the data of the problem

$$|\langle \varepsilon(u-u^h)\rangle| \leq \frac{2}{\nu|\,\Omega\,|}\,\|\, f\,\|^2_{L^\infty(0,\infty;H^{-1})}.$$

2. From the Poincaré–Friedrichs inequality it follows that

$$\langle k(v)\rangle \leq \frac{C_{PF}^2}{2\nu}\langle \varepsilon(v)\rangle$$

for all $v \in \mathbb{X}$. Thus, all estimates of Lemma 3.3 as well as the first part of this remark carry over to the time average of the kinetic energy.

Next, we consider the time-averaged errors. It is important to note that there is a difference between $\|\langle \nabla(u-u^h)\rangle\|$ and $\langle \|\nabla(u-u^h)\|\rangle$; the second term is an upper bound for the first one, by (2.31). Experience with turbulent flows suggests that $\langle \nabla u\rangle$ might be smooth (and thus approximable). Thus, ideally we would like estimates for the first term, $\|\langle \nabla(u-u^h)\rangle\|$. In the case of the error in the pressure, we are able to prove such a bound.

THEOREM 3.1. *Let $f \in L^\infty(0,\infty;H^{-1}(\Omega)^d) \cap L^2(0,T;L^2(\Omega)^d)$ and let $(\mathbb{X}^h, \mathbb{Q}^h)$ satisfy the discrete inf-sup condition* (2.5)*; then*

$$\limsup_{T\to\infty} \|\langle p - p^h\rangle_T\| \le \frac{\nu}{\beta^h}\left(1 + \frac{2M}{\nu^2}\|f\|_{L^\infty(0,\infty;H^{-1})}\right)\langle \|\nabla(u-u^h)\|^2\rangle^{1/2}$$
$$+\left(1+\frac{\sqrt{d}}{\beta^h}\right)\inf_{q^h\in\mathbb{Q}^h}\limsup_{T\to\infty}\|\langle p - q^h\rangle_T\|.$$

Proof. A straightforward calculation, using (2.7), shows that (2.19) is equivalent to

$$-\int_0^T (p^h - q^h, \nabla\cdot v^h)\,dt = (e(T), v^h(T)) - (e(0), v^h(0))$$
$$-\int_0^T \left[\left(e, \frac{\partial v^h}{\partial t}\right) - \nu(\nabla e, \nabla v^h) - b_s(u,e,v^h) - b_s(e,u^h,v^h) + (p - q^h, \nabla\cdot v^h)\right] dt$$

for all $(v^h, q^h) \in \mathbb{X}^h \times \mathbb{Q}^h$. Since the velocity finite element functions are continuous in $\overline{\Omega}$, all terms are well defined. Let $v^h = v^h(x)$. Division by T gives

$$(\langle q^h - p^h\rangle_T, \nabla\cdot v^h) = \frac{1}{T}(e(T), v^h) - \frac{1}{T}(e(0), v^h) + \nu\,(\langle\nabla e\rangle_T, \nabla v^h)$$
$$+\langle b_s(u,e,v^h)\rangle_T + \langle b_s(e,u^h,v^h)\rangle_T$$
$$-(\langle p - q^h\rangle_T, \nabla\cdot v^h). \tag{3.5}$$

For estimating (3.5), we use that v^h does not depend on time, (2.4), and $\|\nabla\cdot v^h\| \le \sqrt{d}\,\|\nabla v^h\|$ to obtain

$$\left|(\langle q^h - p^h\rangle_T, \nabla\cdot v^h)\right| \le \frac{C_{PF}}{T}\|e(T)\|\,\|\nabla v^h\| + \frac{C_{PF}}{T}\|e(0)\|\,\|\nabla v^h\|$$
$$+\nu\,\|\langle\nabla e\rangle_T\|\,\|\nabla v^h\| + M\langle\|\nabla u\|\,\|\nabla e\|\rangle_T\|\nabla v^h\|$$
$$+M\langle\|\nabla e\|\,\|\nabla u^h\|\rangle_T\|\nabla v^h\|$$
$$+\sqrt{d}\,\|\langle p - q^h\rangle_T\|\,\|\nabla v^h\|.$$

Dividing by $\|\nabla v^h\|$ and applying the discrete inf-sup condition (2.5) and (2.21) to the left-hand side of this inequality leads to

$$\beta^h\|\langle q^h - p^h\rangle_T\| \le \frac{C_{PF}}{T}\|e(T)\| + \frac{C_{PF}}{T}\|e(0)\| + \nu\,\|\langle\nabla e\rangle_T\|$$
$$+M\langle\|\nabla u\|\,\|\nabla e\|\rangle_T + M\langle\|\nabla e\|\,\|\nabla u^h\|\rangle_T$$
$$+\sqrt{d}\,\|\langle p - q^h\rangle_T\|.$$

The terms on the right-hand side are estimated by (2.21) and an estimate similar to (2.22), resulting in

$$\beta^h\|\langle q^h - p^h\rangle_T\| \le \frac{C_{PF}}{T}\|e(T)\| + \frac{C_{PF}}{T}\|e(0)\|$$
$$+(\nu + M\langle\|\nabla u\|^2\rangle_T^{1/2} + M\langle\|\nabla u^h\|^2\rangle_T^{1/2})\langle\|\nabla e\|^2\rangle_T^{1/2}$$
$$+\sqrt{d}\,\|\langle p - q^h\rangle_T\|.$$

The triangle inequality implies that

$$\|\langle p - p^h\rangle_T\| \le \frac{C_{PF}}{\beta^h T}\|\, e(T)\,\| + \frac{C_{PF}}{\beta^h T}\|\, e(0)\,\|$$

$$+\frac{1}{\beta^h}\left(\nu + M\langle\|\,\nabla\, u\,\|^2\rangle_T^{1/2} + M\langle\|\,\nabla\, u^h\,\|^2\rangle_T^{1/2}\right)\langle\|\,\nabla\, e\,\|^2\rangle_T^{1/2}$$

$$+\left(1+\frac{\sqrt{d}}{\beta^h}\right)\|\langle p - q^h\rangle_T\|.$$

Taking lim sup as $T \to \infty$ on both sides of the inequality and using Lemmas 3.1 and 3.2, together with properties of lim sup, give

$$\limsup_{T\to\infty}\|\langle p - p^h\rangle_T\|$$

$$\le \frac{1}{\beta^h}(\nu + M\langle\|\,\nabla\, u\,\|^2\rangle^{1/2} + M\langle\|\,\nabla\, u^h\,\|^2\rangle^{1/2})\langle\|\,\nabla\, e\,\|^2\rangle^{1/2}$$

$$+\left(1+\frac{\sqrt{d}}{\beta^h}\right)\limsup_{T\to\infty}\|\langle p - q^h\rangle_T\|.$$

The norms of the weak and the discrete solution can be estimated with the results of Lemma 3.3. The proof concludes by taking the infimum over $q^h \in \mathbb{Q}^h$. ◻

COROLLARY 3.1. *If the assumptions of Theorem* 3.1 *hold true*, *then*

$$\|\langle p - p^h\rangle\| \le \frac{\nu}{\beta^h}\left(1+\frac{2M}{\nu^2}\|f\,\|_{L^\infty(0,\infty;H^{-1})}\right)\langle\,\|\,\nabla(u-u^h)\|^2\,\rangle^{1/2}$$

$$+\left(1+\frac{\sqrt{d}}{\beta^h}\right)\inf_{q^h\in\mathbb{Q}^h}\limsup_{T\to\infty}\|\langle p - q^h\rangle_T\|.$$

Proof. The lower bound on the left-hand side is a consequence of (2.25). ◻

The key idea in the above proofs was to restrict $v^h \in \mathbb{V}^h$ to be time independent. Then, time averaging can be applied and brought inside upon the pressure error directly. It is interesting that the equations of motion give a different realization of the time-averaged error for the velocity and pressure ($\langle\|\,\nabla(u-u^h)\,\|\rangle$ versus $\|\langle p-p^h\rangle\|$). This appears also in the time-averaged lift and drag error estimates in Theorem 4.1. At this point, we do not know if this distinction has other deeper causes or implications.

We next turn to the error inequalities for the time-averaged error of the energy dissipation rate $\langle\varepsilon(u-u^h)\rangle$.

THEOREM 3.2. *Let* $Y = L^2(0,\infty;H^1(\Omega)^d) \cap L^\infty(0,\infty;\mathbb{V}^h)$ *and assume* $u_t \in L^1(0,T;\mathbb{X}^*)$ *for every* $0 < T < \infty$. *Then the time-averaged error of the energy* *dissipation rate satisfies the following inequalities*:

$$
\begin{aligned}
\langle \varepsilon(u-u^h)\rangle \leq \inf_{\tilde{u}\in Y} \Big[& C\big(\langle \varepsilon(u-\tilde{u})\rangle + \nu^{-1}\langle \| (u-\tilde{u})_t \|_{-1}^2\rangle\big) \\
& + \frac{2}{|\Omega|}\Big(\langle | b_s(u, u-u^h, u-\tilde{u}) |\rangle + \langle | b_s(u-u^h, u, u-\tilde{u}) |\rangle \\
& + \langle | b_s(u-u^h, u-u^h, u-\tilde{u}) |\rangle\Big)\Big] \\
& + C \inf_{q^h\in \mathbb{Q}^h} \Big[\nu^{-1}\langle \| p-q^h \|^2\rangle\Big] + \frac{2}{|\Omega|}\langle | b_s(u-u^h, u, u-u^h) |\rangle
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
\langle \varepsilon(u-u^h)\rangle \leq C \inf_{\tilde{u}\in Y} \Big[& \langle \varepsilon(u-\tilde{u})\rangle + \nu^{-1}\langle \| (u-\tilde{u})_t \|_{-1}^2\rangle \\
& + \nu^{-3}\langle \| u-u^h \|^{2/3} \| \nabla u \|^{4/3} \| \nabla (u-\tilde{u}) \|^{4/3}\rangle \\
& + \nu^{-3}\langle \| u-u^h \|^{2} \| \nabla (u-\tilde{u}) \|^{4}\rangle \\
& + \nu^{-1}\langle \|u\| \, \| \nabla u \| \, \| \nabla (u-\tilde{u}) \|^{2}\rangle\Big] \\
& + C \inf_{q^h\in \mathbb{Q}^h} \Big[\nu^{-1}\langle \| p-q^h \|^2\rangle\Big] + C\nu^{-3}\langle \| \nabla u \|^4 \| u-u^h \|^2\rangle.
\end{aligned} \tag{3.7}
$$

Remark 3.2. Estimate (3.7) is not closed in the sense that $\langle \varepsilon(u-u^h)\rangle$ is not estimated only by approximation errors $u-\tilde{u}$ but also by the last term. For this term, we cannot provide an estimate under the assumptions of Theorem 3.2. However, if $\| \nabla u \|$ is uniformly bounded in time, $\| \nabla u \| \in L^\infty(0,\infty; L^2(\Omega)^{d\times d})$, then these estimates can be closed provided $\langle \| e \|^2\rangle \leq C h^\alpha \langle \| \nabla e \|^2\rangle$ for some $\alpha > 0$ and provided h is small enough,

$$
\begin{aligned}
\nu^{-3}\langle \| \nabla u \|^4 \| u-u^h \|^2\rangle &\leq \nu^{-3}\|\nabla u\|_{L^\infty(0,\infty;L^2)}\langle \| u-u^h \|^2\rangle \leq C h^\alpha \langle \|\nabla e\|^2\rangle \\
&\leq (1-\beta)\langle \varepsilon(u-u^h)\rangle
\end{aligned}
$$

with $\beta \in (0,1)$ if h is sufficiently small. Now, the term can be absorbed by the left-hand side. However, the assumption on the regularity of u is again the case when pointwise accuracy in time is reasonable to expect rather than accuracy in time-averaged statistics. Thus, the problem of closing the circle in the velocity error equation for the time-averaged statistics seems to catch at the same point as in the standard error analysis.

We shall see, in section 3.2, that in at least one case the circle of analysis is closable. In more general cases, we believe the problem is due to the fact that we are estimating $\langle \| \nabla (u-u^h) \|^2\rangle$ rather than $\|\langle \nabla (u-u^h)\rangle\|^2$. Of course, an estimate for the latter term follows immediately from (2.31) and (3.7), but it is most probably not optimal.

Proof. The weak solution obtained by the Leray–Hopf construction satisfies

$$
(u_t, v^h) + \nu(\nabla u, \nabla v^h) + b_s(u,u,v^h) - (p, \nabla\cdot v^h) = (f, v^h) \qquad \forall\, v^h \in L^\infty(0,T;\mathbb{X}^h)
$$

as in the proof of Lemma 3.1. A similar equation holds for u^h. Subtraction and the fact that $(q^h, \nabla \cdot v^h) = 0$ for $q^h \in \mathbb{Q}^h$ give an equation for the error $e = u - u^h$:

$$(e_t, v^h) + \nu(\nabla e, \nabla v^h) + b_s(u, u, v^h) - b_s(u^h, u^h, v^h)$$

$$-(p - q^h, \nabla \cdot v^h) = 0 \quad \forall\, v^h \in L^\infty(0, T; \mathbb{V}^h),$$

where $q^h \in \mathbb{Q}^h$ is arbitrary.

Let $\tilde{u}$ be an interpolant of u with $\tilde{u} \in L^2(0, \infty; H^1(\Omega)^d) \cap L^\infty(0, \infty; \mathbb{V}^h)$ and write $e = \eta - \phi^h$, where $\eta = u - \tilde{u}$ and $\phi^h = u^h - \tilde{u}$. Adding

$$-b_s(e, e, e) + b_s(u, u^h, \phi^h) - b_s(u, u^h, \phi^h),$$

where the first term vanishes, we get with $v^h = \phi^h$

$$\frac{1}{2}\frac{d}{dt}\| \phi^h \|^2 + \nu \| \nabla \phi^h \|^2 = (\eta_t, \phi^h) + \nu(\nabla \eta, \nabla \phi^h) - (p - q^h, \nabla \cdot \phi^h)$$

$$+ b_s(e, u, \eta) - b_s(e, e, \eta) - b_s(e, u, e) + b_s(u, e, \eta).$$

Time-averaging this equation and observing that $\| \phi^h \|$ is uniformly bounded in time (since $\| u^h \|$ is bounded and $\tilde{u} \in L^\infty(0, \infty; L^2(\Omega)^d)$), we have

$$\langle \varepsilon(\phi^h) \rangle \le \frac{1}{|\Omega|}\Big[|\langle\, (\eta_t, \phi^h)\, \rangle| + |\langle \nu\, (\nabla \eta, \nabla \phi^h)\, \rangle| + |\langle\, (p - q^h, \nabla \cdot \phi^h)\, \rangle|$$

$$+ |\langle\, b_s(e, u, \eta)\, \rangle| + |\langle\, b_s(e, e, \eta)\, \rangle| + |\langle\, b_s(e, u, e)\, \rangle|$$

$$+ |\langle\, b_s(u, e, \eta)\, \rangle| \Big].$$

Inequalities (2.30), (2.32), (2.33), and Young's inequality give

$$\langle \varepsilon(\phi^h) \rangle \le C\Big[\nu^{-1} \langle \| \eta_t \|_{-1}^2 \rangle + \langle \varepsilon(\eta) \rangle + \nu^{-1} \langle \| p - q^h \|^2 \rangle \Big]$$

$$+ \frac{2}{|\Omega|}\Big[\langle |\, b_s(e, u, \eta)\, | \rangle + \langle |\, b_s(e, e, \eta)\, | \rangle + \langle |\, b_s(e, u, e)\, | \rangle$$

$$+ \langle |\, b_s(u, e, \eta)\, | \rangle \Big].$$

Now, (3.6) follows from the triangle inequality.

For proving (3.7), we use the following bounds on the trilinear forms, which can be derived from Lemma 2.1, together with Young's inequality:

$$\begin{aligned}
\langle | b_s(e,u,e) | \rangle &\le C \langle \| \nabla u \| \, \| e \|^{1/2} \, \| \nabla e \|^{3/2} \rangle \\
&\le \frac{\nu}{16} \langle \| \nabla e \|^2 \rangle + C\, \nu^{-3} \langle \| \nabla u \|^4 \, \| e \|^2 \rangle, \\
\langle | b_s(e,e,\eta) | \rangle &\le C \langle \| e \|^{1/2} \, \| \nabla e \|^{3/2} \, \| \nabla \eta \| \rangle \\
&\le \frac{\nu}{16} \langle \| \nabla e \|^2 \rangle + C\, \nu^{-3} \langle \| e \|^2 \, \| \nabla \eta \|^4 \rangle, \\
\langle | b_s(e,u,\eta) | \rangle &\le C \langle \| e \|^{1/2} \, \| \nabla e \|^{1/2} \, \| \nabla u \| \| \nabla \eta \| \rangle \\
&\le \frac{\nu}{16} \langle \| \nabla e \|^2 \rangle + C\, \nu^{-3} \langle \| e \|^{2/3} \, \| \nabla u \|^{4/3} \, \| \nabla \eta \|^{4/3} \rangle, \\
\langle | b_s(u,e,\eta) | \rangle &\le C \langle \| u \|^{1/2} \| \nabla u \|^{1/2} \| \nabla e \| \, \| \nabla \eta \| \rangle \\
&\le \frac{\nu}{16} \langle \| \nabla e \|^2 \rangle + C \langle \nu^{-1} \| u \| \, \| \nabla u \| \, \| \nabla \eta \|^2 \rangle.
\end{aligned}$$

Inserting these estimates into (3.6) proves (3.7). ∎

Remark 3.3. Alternative estimates of the trilinear terms in the proof are possible. Using (2.4), one obtains for $\langle | b_s(e,u,\eta) | \rangle$ the term $\nu^{-1} \langle \|\nabla u\|^2 \|\nabla \eta\|^2 \rangle$ which stays on the right-hand side of the estimate (instead of the term

$$\nu^{-3} \langle \| u - u^h \|^{2/3} \| \nabla u \|^{4/3} \| \nabla \eta \|^{4/3} \rangle).$$

The dependency of the alternative term on ν is more favorable, however assuming $\|\nabla \eta\| = \mathcal{O}(h^k)$ and $\| u - u^h \| = \mathcal{O}(h^{k+\alpha})$, $\alpha > 0$, the order of convergence for the term in estimate (3.7) is higher.

3.2. The case of large u_0 and small $f^*(x)$. There is at least one interesting case in which the error equations for the time-averaged velocity error, $\langle \varepsilon(u - u^h) \rangle$, can be closed: the case of large initial condition u_0 and asymptotically small body force $f(x,t)$. In this subsection, we assume

$$f(x,t) \in L^\infty(0,\infty; H^{-1}(\Omega)^d), \qquad f(x,t) \to f^*(x) \text{ as } t \to \infty,$$

and

$$\nu^{-2} M \, \| f^* \|_{-1} =: \alpha < 1.$$

In this case, time averaging will eventually wash out the irregularities caused by the large initial condition. We show that this is indeed the case. To shorten the presentation, we shall simplify the condition on f to

$$f(x,t) \equiv f^*(x) \qquad \text{with} \qquad \nu^{-2} M \, \| f^* \|_{-1} = \alpha < 1. \tag{3.8}$$

This small data (or small Reynolds number) condition can also be formulated as a small generalized Grashof number condition. By a scaling argument, we find that there are constants M_0, M_1, and M_2 depending on the geometry of Ω, but not on $L = diam(\Omega)$, such that $M = M_0 \, L^{2-d/2}$, and for $f^*(x) \in C^0(\Omega)$, $\| f^* \|_{-1} =$

$M_1 L^{1+d/2}$ and $\| f^* \| = M_2 L^{d/2}$. This implies that $\| f^* \|_{-1} = \frac{M_1}{M_2} L \| f^* \|$, so that (3.8) is equivalent to

$$C\, G_r < 1,$$

where G_r is the Grashof number, defined in [7] as

$$G_r = \frac{L^{3-d/2}}{\nu^2} \| f^* \|.$$

LEMMA 3.4. *Let u be a weak solution to the Navier–Stokes equations obtained by the Leray–Hopf construction and suppose* (3.8) *holds. Then*

$$\langle \varepsilon(u - u^*) \rangle = 0,$$

where u^* is the solution of the equilibrium Navier–Stokes equations (2.10).

Proof. Let V_N be the span of eigenfunctions of the Stokes operator. The Leray–Hopf construction gives a sequence $\{u_N\}$ in V_N satisfying

$$(u_{N,t}, v) + \nu(\nabla u_N, \nabla v) + b(u_N, u_N, v) = (f, v) \quad \forall\, v \in V_N, \tag{3.9}$$

with a subsequence $\{u_{N_j}\}$ in V_N converging to a weak solution u, as $N_j \to \infty$, strongly in $L^2(0, T; H(\Omega))$ and weakly in $L^2(0, T; \mathbb{V})$. Let $u_N^* \in V_N$ be the Galerkin projection of u^* in V_N. Then u_N^* satisfies

$$\nu(\nabla u_N^*, \nabla v) + b(u_N^*, u_N^*, v) = (f, v) \quad \forall\, v \in V_N, \tag{3.10}$$

and $u_N^* \to u^*$ in $\mathbb{X}$ and $\mathbb{V}$ as $N \to \infty$.

Set $\phi_N = u_N(x, t) - u_N^*(x)$ and subtract (3.10) from (3.9) to get

$$(\phi_{N,t}, v) + \nu(\nabla \phi_N, \nabla v) + b(u_N, u_N, v) - b(u_N^*, u_N^*, v) = 0 \qquad \forall\, v \in V_N.$$

Let $v = \phi_N$, add and subtract $b(u_N, u_N^*, v)$, and integrate from 0 to T to get

$$\frac{1}{2}\|\phi_N(T)\|^2 + \int_0^T \nu \|\nabla \phi_N\|^2\, dt = \frac{1}{2}\|\phi_N(0)\|^2 - \int_0^T b(\phi_N, u_N^*, \phi_N)\, dt.$$

Setting $v = u_N^*$ in (3.10) gives immediately the a priori bound $\|\nabla u_N^*\| \le \nu^{-1}\|f\|_{-1}$. Using this bound, the bound (2.4) on the trilinear form and the small data assumption (3.8) give

$$\frac{1}{2}\|\phi_N(T)\|^2 + (1 - \alpha)\nu \int_0^T \|\nabla \phi_N\|^2\, dt \le \frac{1}{2}\|\phi_N(0)\|^2.$$

Thus, dropping the first term leads to

$$\int_0^T \nu \| \nabla \phi_N \|^2\, dt \le \frac{1}{2(1 - \alpha)} \| \phi_N(0) \|^2. \tag{3.11}$$

Using classical properties of weak limits, we have

$$\liminf_{N \to \infty} \left(\int_0^T \| \nabla \phi_N \|^2\, dt \right) \ge \int_0^T \| \nabla \phi \|^2\, dt$$

with $\phi = u - u^*$. Therefore, taking the limit inferior on both sides of (3.11) gives

$$\int_0^T \nu \| \nabla \phi \|^2 \, dt \leq \frac{1}{2(1-\alpha)} \| \phi(0) \|^2.$$

Dividing by $T|\Omega|$ and taking the limit superior as $T \to \infty$ proves the result. □

The next lemma is needed to prove the desired error estimate on $\langle \varepsilon(u-u^h) \rangle$ given in Theorem 3.3.

LEMMA 3.5. *Assume that* (3.8) *holds. Then*

$$\langle \varepsilon(u^h - u^{*\,h}) \rangle = 0.$$

Proof. The proof works in the same way as that of Lemma 3.4. It is based on subtracting (2.9) and (2.14). □

Remark 3.4. By the Poincaré–Friedrichs inequality, the statements of Lemmas 3.4 and 3.5 hold for the kinetic energy as well.

THEOREM 3.3. *Suppose that the assumptions of Lemmas* 3.4 *and* 3.5 *hold. Then*

$$\langle \varepsilon(u - u^h) \rangle \ \leq \ 3 \ \varepsilon(u^* - u^{*\,h}). \tag{3.12}$$

Proof. The triangle inequality gives

$$\|\nabla(u - u^h)\|^2 \leq 3 \left(\|\nabla(u - u^*)\|^2 + \|\nabla(u^* - u^{*\,h})\|^2 + \|\nabla(u^{*\,h} - u^h)\|^2 \right).$$

Hence, we get

$$\langle \varepsilon(u - u^h) \rangle \leq 3 \left(\langle \varepsilon(u - u^*) \rangle + \varepsilon(u^* - u^{*\,h}) + \langle \varepsilon(u^{*\,h} - u^h) \rangle \right).$$

The first and the third terms on the right-hand side vanish by Lemmas 3.4 and Lemma 3.5, respectively. □

The statement of Theorem 3.3 says that the problem of estimating $\langle \varepsilon(u-u^h) \rangle$ reduces to the one of estimating $\|\nabla(u^* - u^{*\,h})\|^2$. Standard finite element error estimates thus immediately imply that $\langle \varepsilon(u - u^h) \rangle$ is optimal.

COROLLARY 3.2. *Suppose that the assumptions of Lemmas* 3.4 *and* 3.5 *hold and that* ($\mathbb{X}^h$, $\mathbb{Q}^h$) *satisfies the inf-sup condition* (2.5). *Then*

$$\langle \varepsilon(u - u^h) \rangle \leq C \left[\inf_{v^h \in \mathbb{X}^h} \nu \| \nabla(u^* - v^h) \|^2 + \inf_{q^h \in \mathbb{Q}^h} \nu^{-1} \| p^* - q^h \|^2 \right].$$

Proof. This follows by inserting the error estimates for $\| \nabla(u^* - u^{*\,h}) \|$ from [11] into the right-hand side of (3.12). □

Concerning the time-averaged error in the pressure, we have the following corollary. It is a direct consequence of Corollary 3.1.

COROLLARY 3.3. *Let the assumptions of Theorem* 3.1 *and Lemmas* 3.4 *and* 3.5 *be fulfilled. Then*

$$\|\langle p - p^h \rangle\| \leq \frac{3\sqrt{3}\nu}{\beta^h} \| \nabla(u^* - u^{*\,h}) \| + \left(1 + \frac{\sqrt{d}}{\beta^h} \right) \inf_{q^h \in \mathbb{Q}^h} \limsup_{T \to \infty} \|\langle p - q^h \rangle_T\|.$$

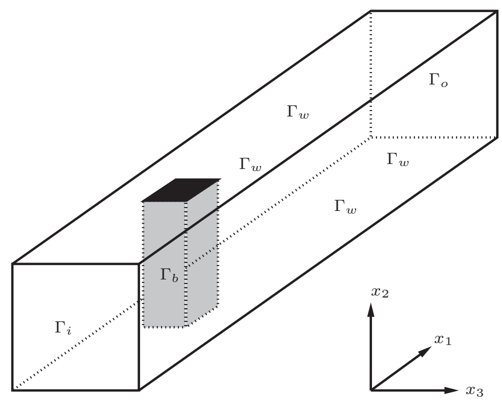

FIG. 1. *A channel with a body.*

4. Drag and lift. Consider the flow around a body in a channel with flow region Ω and boundary Γ, which consists of Γ_b (boundary of the body) and $\Gamma_c = \Gamma_i \cup \Gamma_o \cup \Gamma_w$ (where Γ_i, Γ_o correspond to the inflow and outflow and Γ_w correspond to the walls); see Figure 1 for an example.

Define

$$\sigma = -p\,\mathbb{I} + 2\,\mu\,\nabla^s u,$$

where ∇^s indicates the symmetric part of the operator ∇, and μ the dynamic viscosity. Then, we consider the Navier–Stokes equations written in the form

$$\begin{aligned} \rho(u_t + u\cdot\nabla u) &= \nabla\cdot\sigma + f && \text{in } \Omega\times(0,T], \\ \nabla\cdot u &= 0 && \text{in } \Omega\times[0,T], \\ u &= g && \text{on } \Gamma\times[0,T], \\ u(x,0) &= u_0(x) && \text{in } \Omega, \end{aligned} \tag{4.1}$$

satisfying the compatibility condition $\int_\Gamma g\cdot\hat{n}\,dS = 0$, where $\hat{n}$ is the outward pointing unit normal to Γ_b. We assume $g = 0$ on $\Gamma_w \cup \Gamma_b$.

We introduce the spaces

$$\mathbb{X}_g = \left\{ v \in H^1(\Omega)^d : v|_\Gamma = g \right\}, \quad \mathbb{X}_0 = H_0^1(\Omega)^d.$$

A weak formulation of (4.1) reads as follows: Find $u : [0,T] \to \mathbb{X}_g$ and $p : (0,T] \to \mathbb{Q}$ such that

$$\rho\,(u_t, v) + 2\,\mu(\nabla^s u, \nabla^s v) + \rho\, b(u,u,v) - (p, \nabla\cdot v) + (\nabla\cdot u, q) = (f, v) \qquad \forall (v,q) \in \mathbb{X}_0\times\mathbb{Q}. \tag{4.2}$$

We assume that the asymptotic growth of the kinetic energy of the solutions of (4.2) is bounded by

$$\limsup_{T\to\infty} \frac{1}{T} k(u(T)) = 0. \tag{4.3}$$

In particular, assumption (4.3) is valid if the solutions of (4.2) have bounded kinetic energy.[1]

Drag and lift are defined by

$$D = -\int_{\Gamma_b} \hat{e}_1 \cdot \sigma \cdot \hat{n}\, d\gamma, \qquad L = -\int_{\Gamma_b} \hat{e}_2 \cdot \sigma \cdot \hat{n}\, d\gamma,$$

respectively, where $\hat{e}_i$ is the unit vector in the ith direction. We assume that $+\hat{e}_1$ is the direction of motion and $-\hat{e}_2$ is the direction of gravity. We note that if (u, p) are sufficiently regular and $v \in L^\infty(0, T; H^1(\Omega)^d)$ with $v = 0$ on Γ_c, then we have

$$\frac{1}{T}\int_0^T \int_{\Gamma_b} v \cdot \sigma \cdot \hat{n}\, d\gamma\, dt$$
$$= \frac{1}{T}\int_0^T \Big[\rho\,(u_t, v) + 2\,\nu(\nabla^s u, \nabla^s v) + \rho\, b(u, u, v) - (p, \nabla \cdot v) - (f, v)\Big]\, dt. \tag{4.4}$$

Whenever the right-hand side of (4.4) is well defined, it can be taken as the definition of the force on the immersed body, i.e., the left-hand side of (4.4). We will use this definition under the following regularity assumptions on (u, p):

$$u \in L^1(0, T; H^1(\Omega)^d) \cap \mathbb{X}_g, \quad u_t \in L^1(0, T; H^{-1}(\Omega)^d), \quad p \in L^1(0, T; L^2(\Omega)).$$

Choosing in (4.4) a function v satisfying $v = -\hat{e}_1$ on Γ_b gives a formula for the drag and $v = -\hat{e}_2$ gives a formula for the lift.

THEOREM 4.1. *Let the force on the immersed body be well defined by* (4.4), *let the kinetic energy of the solutions of* (4.2) *be bounded by* (4.3), *and let a similar bound be valid for the kinetic energy of the finite element approximation* u^h. *Then, the time-averaged drag and lift can be estimated as*

$$|\langle D - D^h \rangle|, |\langle L - L^h \rangle|$$
$$\le C\Big[(\mu + M\rho\langle\|\nabla u\|^2 + \|\nabla u^h\|^2\rangle^{1/2})\langle\| \nabla (u - u^h) \|^2\rangle^{1/2}$$
$$+ \limsup_{T\to\infty} \|\langle p - p^h\rangle_T\|\Big]. \tag{4.5}$$

Proof. We present here a proof for the drag estimate. With the same arguments follows the estimate for the lift, according to an appropriate choice of v.

Let $v \in H^2(\Omega)^d$ be a time-independent vector field satisfying $v = -\hat{e}_1$ on Γ_b and $v = 0$ on Γ_c. From (4.4) it follows that

$$\int_0^T D\, dt = \int_0^T \Big[\rho\,(u_t, v) + 2\,\mu(\nabla^s u, \nabla^s v) + \rho\, b(u, u, v) - (p, \nabla v) - (f, v)\Big]\, dt.$$

Let $i_h v$ be a finite element interpolant to v such that $i_h v = -\hat{e}_1$ on Γ_b. In particular, we have $v, i_h v \in L^\infty(0, T; H^1(\Omega)^d)$. Then,

$$\int_0^T \Big[D - D^h\Big]\, dt = \int_0^T \Big[\rho\,(u_t, v) + 2\,\mu(\nabla^s u, \nabla^s v) + \rho\, b(u, u, v)$$
$$- (p, \nabla \cdot v) - (f, v) - \rho\,(u_t^h, i_h v) - 2\,\mu(\nabla^s u^h, \nabla^s i_h v)$$
$$-\rho\, b_s(u^h, u^h, i_h v) + (p^h, \nabla \cdot i_h v) + (f, i_h v)\Big]\, dt.$$

[1]This is known for geometries, such as simply connected $\partial\Omega$, which admit a Hopf extension of the boundary condition into Ω. It is unknown, for instance, if $\partial\Omega = \cup\Gamma_j$ is multiply connected and $\int_{\Gamma_j} g\, d\gamma = 0$, but $\int_{\Gamma_j} g\, d\gamma \neq 0$ for some connected component; see Theorem 4.1, p. 33, of Galdi [9].

Adding and subtracting terms and using $b(u,u,v) = b_s(u,u,v)$, this becomes

$$\begin{aligned}\int_0^T \left[D - D^h\right] dt = \int_0^T \Big[&\rho\,(u_t, v - i_h v) + \rho\,((u-u^h)_t, i_h v) + 2\,\mu(\nabla^s u, \nabla^s(v - i^h v)) \\ &+ 2\,\mu(\nabla^s (u-u^h), \nabla^s i_h v) + \rho\, b_s(u,u,v-i_h v) + \rho\, b_s(u,u,i_h v) \\ &- \rho\, b_s(u^h,u^h,i_h v) - (p, \nabla\cdot(v - i_h v)) \\ &- (p - p^h, \nabla\cdot i_h v) - (f, v - i_h v)\Big]\, dt. \end{aligned} \tag{4.6}$$

Observe now that the term containing $(f, v - i_h v)$ in (4.6) can be rewritten by multiplying (4.1) by $v - i_h v$ and integrating:

$$(f, v - i_h v) = \rho\,(u_t, v - i_h v) + 2\,\mu(\nabla^s u, \nabla^s (v - i_h v)) + \rho\, b_s(u,u,v-i_h v) - (p, \nabla\cdot(v - i_h v)). \tag{4.7}$$

Hence, combining (4.6) and (4.7) gives

$$\begin{aligned}\int_0^T \left[D - D^h\right] dt = \int_0^T \Big[&\rho\,((u-u^h)_t, i_h v) + 2\,\mu(\nabla^s (u-u^h), \nabla^s i^h v) \\ &+ \rho\, b_s(u,u,i_h v) - \rho\, b_s(u^h,u^h,i_h v) + (p - p^h, \nabla\cdot i_h v)\Big]\, dt. \end{aligned} \tag{4.8}$$

Let $e = u - u^h$ and divide each term in (4.8) by T. The first term on the right-hand side yields, with the time independence of $i_h v$,

$$\frac{1}{T}(e(T) - e(0), i_h v) \le \frac{C_{PF}}{T}\,(\|e(T)\| + \|e(0)\|)\,\|\nabla i_h v\|.$$

Note that the Poincaré–Friedrich inequality can be applied since $i_h v = 0$ on Γ_c. Now, by (2.22) and $\|\nabla^s v\| \le \|\nabla v\|$ for all $v \in H^1(\Omega)^d$, we obtain

$$\langle(\nabla^s e, \nabla^s i_h v)\rangle_T \quad\le\quad \langle\|\nabla e\|^2\rangle_T^{1/2}\,\|\nabla i_h v\|.$$

Next, the nonlinear terms are estimated by (2.4), (2.22), and Young's inequality:

$$\begin{aligned}\langle b_s(u,u,i_h v) - b_s(u^h,u^h,i_h v)\rangle_T &= \langle b_s(u,e,i_h v) + b_s(e,u^h,i_h v)\rangle_T \\ &\le M\langle(\|\nabla u\| + \|\nabla u^h\|)\|\nabla e\|\|\nabla i_h v\|\rangle_T \\ &\le M\|\nabla i_h v\|\langle(\|\nabla u\| + \|\nabla u^h\|)^2\rangle_T^{1/2}\langle\|\nabla e\|^2\rangle_T^{1/2} \\ &\le 2M\|\nabla i_h v\|\langle\|\nabla u\|^2 + \|\nabla u^h\|^2\rangle_T^{1/2}\langle\|\nabla e\|^2\rangle_T^{1/2}.\end{aligned}$$

For the pressure term, we obtain, with the time independence of $i_h v$ and the Cauchy–Schwarz inequality,

$$\langle(p - p^h, \nabla\cdot i_h v)\rangle_T = (\langle p - p^h\rangle_T, \nabla\cdot i_h v) \le \sqrt{d}\,\|\langle p - p^h\rangle_T\|\,\|\nabla i_h v\|.$$

Putting everything together, (4.8) becomes

$$\begin{aligned}|\langle D - D^h\rangle_T| \le \rho\frac{C_{PF}}{T}\,&(\|e(T)\| + \|e(0)\|)\,\|\nabla i_h v\| \\ &+ (2\mu + 2M\rho\langle\|\nabla u\|^2 + \|\nabla u^h\|^2\rangle_T^{1/2})\,\langle\|\nabla e\|^2\rangle_T^{1/2}\,\|\nabla i_h v\| \\ &+ \sqrt{d}\,\|\langle p - p^h\rangle_T\|\,\|\nabla i_h v\|.\end{aligned}$$

Taking the lim sup on both sides of the inequality, using (4.3) and the same property for u^h, and (2.26) give

$$\limsup_{T\to\infty} |\langle D - D^h \rangle_T|$$
$$\leq (2\mu + 2M\rho\langle\|\nabla u\|^2 + \|\nabla u^h\|^2\rangle^{1/2})\,\langle\|\nabla e\|^2\rangle^{1/2}\,\|\nabla i_h v\|$$
$$+\sqrt{d}\,\limsup_{T\to\infty}\|\langle p - p^h\rangle_T\|\,\|\nabla i_h v\|.$$

Finally, applying (2.25) to the left-hand side of the estimate and $\|\nabla i_h v\| \leq 2\|v\|_2 = C$ give the statement of theorem. ▯

The constant C in the estimate of Theorem 4.1 depends on the $H^2(\Omega)^d$-norm of the test function v and the size of Ω by the Poincaré–Friedrich constant.

In the last part of the proof of Theorem 4.1, the assumption on the boundedness of the growth of the kinetic energy is used. The boundedness of the kinetic energy has already been proven for the case $g = 0$ on Γ; see Lemmas 3.1 and 3.2. Thus, the statement of Theorem 4.1 applies for this case, and the respective bounds of Lemmas 3.1 and 3.2 can be inserted into (4.5).

The assumptions on g can be extended to sufficiently small Dirichlet data. We consider for simplicity of presentation the case when g does not depend on time. Let G be the Hopf extension of the boundary condition g to Ω, i.e., $G \in H^1(\Omega)^d$, $\nabla\cdot G = 0$, $G_t = 0$, and define $\tilde u = u - G$. Then $\tilde u \in \mathbb{V}$ and this function satisfies

$$\begin{aligned}
\rho(\tilde u_t + \tilde u\cdot\nabla\tilde u) &= \nabla\cdot\tilde\sigma + f + 2\mu\nabla\cdot\nabla^s G + \rho G\cdot\nabla G + \rho G \\
&\quad \cdot\nabla\tilde u + \rho\tilde u\cdot\nabla G \qquad \text{in } \Omega\times(0,T], \\
\nabla\cdot\tilde u &= 0 \qquad \text{in } \Omega\times[0,T], \\
\tilde u &= 0 \qquad \text{on } \Gamma\times[0,T], \\
\tilde u(x,0) &= (u_0 - G)(x) \qquad \text{in } \Omega,
\end{aligned} \tag{4.9}$$

where $\tilde\sigma = -\tilde p\,\mathbb{I} + 2\,\mu\,\nabla^s\tilde u$. Denote $\tilde f = f + 2\mu\nabla\cdot\nabla^s G + \rho G\cdot\nabla G$. Transforming (4.9) in the usual way to a variational formulation using $\tilde u$ as a test function, and applying the estimate for the dual pairing, Korn's inequality ($\|\nabla\tilde u\| \leq C_{\text{Korn}}\|\nabla^s\tilde u\|$) and Young's inequality give

$$\frac{\rho}{2}\frac{d}{dt}\|\tilde u\|^2 + \frac{3\mu}{2}\|\nabla^s\tilde u\|^2 \leq \frac{C}{\mu}\|\tilde f\|_{-1}^2 + \rho C_{\text{Korn}}^2\|\nabla^s\tilde u\|^2\|G\|_1. \tag{4.10}$$

If G is sufficiently small, for instance if

$$\rho C_{\text{Korn}}^2\|G\|_1 < \mu, \tag{4.11}$$

the last term on the right-hand side of (4.10) can be absorbed into the left-hand side and the boundedness of the kinetic energy of $\tilde u$ is obtained by integrating (4.10) on $(0,T)$. The kinetic energy of u is bounded by using the triangle inequality and the fact that $\|G\|$ is a constant. In the discrete case, by subtracting an interpolant of G^h from u^h and using a similar argument, one obtains this property for u^h. Thus for small G, the assumption of Theorem 4.1 of bounded kinetic energy holds.

Corollary 4.1. *Let the force on the immersed body be well defined by* (4.4), *let* $\tilde f = f + 2\mu\nabla\cdot\nabla^s G + \rho G\cdot\nabla G \in L^2(0,T;H^{-1}(\Omega))$, *and let the smallness condition*

(4.11) *on the inflow be fulfilled. Then, the time-averaged drag and lift can be estimated by* (4.5).

Remark 4.1. The estimate in Corollary 4.1 is not closed (the same problem as in Theorem 3.1) due to the term $\langle \| \nabla (u - u^h) \|^2 \rangle$ in (4.5). Closing this estimate requires the investigation (beyond the scope of this paper) of (4.9), which has additional terms in comparison with the Navier–Stokes equations. It is very likely that a small data assumption on $\tilde{f}$ (like in section 3.2) has to be used.

Note that the condition (4.11) can be reformulated with the Poincaré–Friedrichs inequality since G vanishes on $\Gamma_w \cup \Gamma_b$.

5. Persistent shear flows. We have seen in the previous sections that, provided that both the portion of the persistent body force driving the flow and the (nonhomogeneous) Dirichlet data are small, statistics, such as the time-averaged energy dissipation rate (or drag and lift), can be accurately predicted by a flow simulation. This accuracy holds quite generally without any further assumptions on u_0, ν, and Re typically needed to prove accuracy over bounded time intervals.

The case when the persistent forces driving the flow are not small is much more difficult. We shall prove in this section that the analogous estimate of the time-averaged energy dissipation rate is physically reasonable under a condition on the finite element mesh near the walls. Briefly, we consider the finite element approximation to the following shear flow problem: let $\Omega = [0, L]^3$, and find (u, p) satisfying

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p &= 0 && \text{in } \Omega \times (0, T], \\ \nabla \cdot u &= 0 && \text{in } \Omega \times [0, T], \\ u &= u_0 && \text{at } t = 0, \\ u(x_1, x_2, x_3, t) &= \phi(x_3) && \text{for } x_3 \in \{0, L\}, \end{aligned}$$

where

$$\phi(x_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ if } x_3 = 0 \quad \text{and} \quad \phi(x_3) = \begin{pmatrix} U \\ 0 \\ 0 \end{pmatrix} \text{ if } x_3 = L,$$

and periodic boundary conditions in the x_1, x_2 directions; see Figure 2.

In this problem, the persistent force driving the flow is clearly the motion of the top wall and the time-averaged energy dissipation rate must balance the drag exerted by the walls on the fluid. For such problems, the Richardson–Kolmogorov energy

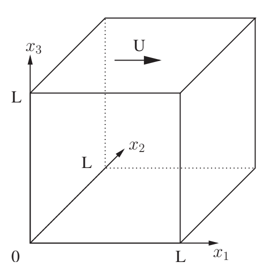

FIG. 2. *The shear flow problem.*

cascade predicts quite simply[2] that

$$\langle \varepsilon(u) \rangle \approx \frac{U^3}{L}$$

independently of ν and Re, respectively. Remarkably, the upper estimate has also been proven for weak solutions of the Navier–Stokes equations in full generality,

$$\langle \varepsilon(u) \rangle \leq C \frac{U^3}{L}$$

by Constantin and Doering [4] and Wang [27]; see also Doering and Foias [6]. Herein, we show in essence that provided the first mesh line of the finite element space is within $O(1/Re)$ of the top wall, then

$$\langle \varepsilon(u^h) \rangle \leq C \frac{U^3}{L};$$

i.e., the computed energy dissipation has the correct mathematical and physical scaling. This result depends on the precise construction of a discrete background flow. The mesh condition is necessary for the construction. Since the proof adapts the ideas of [27], we expect that a similar analysis would hold for other variational methods as well.

Let

$$\begin{aligned}
\mathbb{X} = \{v \in H^1(\Omega)^d \ :\ & v(x_1,x_2,x_3,t) = v(x_1+L,x_2,x_3,t) \text{ for } x_1 \in \Gamma,\\
& v(x_1,x_2,x_3,t) = v(x_1,x_2+L,x_3,t) \text{ for } x_2 \in \Gamma,\\
& v(x_1,x_2,x_3,t) = \phi(x_3) \text{ for } x_3 \in \Gamma\},\\
\mathbb{X}_0 = \{v \in H^1(\Omega)^d \ :\ & v(x_1,x_2,x_3,t) = v(x_1+L,x_2,x_3,t) \text{ for } x_1 \in \Gamma,\\
& v(x_1,x_2,x_3,t) = v(x_1,x_2+L,x_3,t) \text{ for } x_2 \in \Gamma,\\
& v(x_1,x_2,x_3,t) = 0 \text{ for } x_3 \in \Gamma\},\\
\mathbb{Q} = L_0^2(\Omega) &
\end{aligned}$$

and denote the corresponding conforming finite element spaces with a superscript h. We assume that the finite element space for the velocity contains linears, which is usually the case for conforming velocity finite element spaces. The finite element problem reads as follows: find $u^h : [0,T] \to \mathbb{X}^h$, $p^h : (0,T] \to \mathbb{Q}^h$ such that

$$\begin{aligned}
(u_t^h, v^h) + \nu(\nabla u^h, \nabla v^h) + b_s(u^h,u^h,v^h) - (p^h, \nabla \cdot v^h) &= 0 \quad \forall\, v^h \in \mathbb{X}_0^h,\\
(\nabla \cdot u^h, q^h) &= 0 \quad \forall\, q^h \in \mathbb{Q}^h,\\
(u^h(x,0) - u_0(x), v^h) &= 0 \quad \forall\, v^h \in \mathbb{X}_0^h.
\end{aligned} \tag{5.1}$$

[2]Briefly, the largest coherent structures are associated with the motion of the upper wall. They thus have length scale L and characteristic velocity U. Their local Reynolds number is thus $(\frac{U\,L}{\nu}(= Re))$ and the viscous dissipation is negligible on them. These break up into smaller eddies (velocity u, length l, $Re\,(l) = \frac{u\,(l)\,l}{\nu}$) until $Re\,(l)$ is small enough for viscous dissipation to drive their kinetic energy to zero exponentially fast. Since viscous dissipation is negligible through this cascade, the energy dissipation rate is related then to the power input to the largest scales at the first step in the cascade. These largest eddies have energy $\frac{1}{2}U^2$ and time scale $\tau = \frac{L}{U}$ so the rate of energy transfer is $O(\frac{U^2}{\tau}) = O(\frac{U^3}{L})$.

Consider the background flow (an extension of the boundary condition ϕ to the interior of Ω) given by

$$\tilde{\phi}(x_3) = \begin{cases} 0 & \text{if } x_3 \in [0, L - \gamma\, L], \\ \frac{U}{\gamma\, L}(x_3 - (L - \gamma\, L)) & \text{if } x_3 \in [L - \gamma\, L, L], \end{cases}$$

and define

$$\Phi = \begin{bmatrix} \tilde{\phi}(x_3) \\ 0 \\ 0 \end{bmatrix},$$

where γ is a positive number, referred to as the "boundary layer thickness." For $\gamma \in (0, 1)$ this function is piecewise linear, continuous, and satisfies the boundary conditions. Straightforward calculations give

$$\| \Phi \|_{L^\infty} = U, \tag{5.2}$$

$$\| \nabla\Phi \|_{L^\infty} = \frac{U}{\gamma\, L}, \tag{5.3}$$

$$\begin{aligned} \| \Phi \|^2 &= L^2 \int_0^L |\, \tilde{\phi}(x_3)\, |^2 dx_3 \\ &= L^2 \int_{L-\gamma\, L}^{L} \frac{U^2}{(\gamma\, L)^2} (\, x_3 - (L - \gamma\, L)\,)^2 dx_3 = \frac{U^2 \gamma L^3}{3}, \end{aligned} \tag{5.4}$$

$$\| \nabla\, \Phi \|^2 = \frac{U^2\, L}{\gamma}. \tag{5.5}$$

For completeness, we include a short proof of the scaling of the constant in the Poincaré–Friedrichs inequality that will be helpful in the proof of the next theorem.

LEMMA 5.1. *Let $\mathcal{O}_{\gamma\, L} = \{(x_1, x_2, x_3) \in \Omega : L - \gamma L \le x_3 \le L\}$ be the region close to the upper boundary (where the background flow Φ does not vanish). Then*

$$\| u^h - \Phi \|_{L^2(\mathcal{O}_{\gamma\, L})} \le \gamma\, L \,\| \nabla(u^h - \Phi) \|_{L^2(\mathcal{O}_{\gamma\, L})}. \tag{5.6}$$

Proof. First, let v be a C^1 function on $\mathcal{O}_{\gamma\, L}$ that vanishes for $x_3 = L$. Then, componentwise ($i = 1, 2, 3$), we have

$$v_i(x_1, x_2, x_3) = v_i(x_1, x_2, L) - \int_{x_3}^{L} \frac{dv_i}{d\xi_3}(x_1, x_2, \xi_3)\; d\xi_3.$$

Observing that $v_i(x_1, x_2, L) = 0$, squaring both sides, and using the Cauchy–Schwarz inequality, we get

$$v_i^2(x_1, x_2, x_3) \le \gamma L \int_{L-\gamma L}^{L} \left(\frac{dv_i}{d\xi_3}(x_1, x_2, \xi_3) \right)^2 d\xi_3.$$

Integrating both sides with respect to x_3 gives

$$\int_{L-\gamma L}^{L} v_i^2(x_1, x_2, x_3)\; dx_3 \le (\gamma L)^2 \int_{L-\gamma L}^{L} \left(\frac{dv_i}{d\xi_3}(x_1, x_2, \xi_3) \right)^2 d\xi_3.$$

Then, integrating with respect to x_1 and x_2 and summing from $i = 1$ to 3, we obtain

$$\| v \|^2_{\mathcal{O}_{\gamma L}} \leq (\gamma L)^2 \| \nabla v \|^2_{\mathcal{O}_{\gamma L}},$$

which proves the lemma in the case of a C^1 function. The case $v \in \mathbb{X}^h \subset H^1(\Omega)$ follows by density. Finally, just take $v = u^h - \Phi$. ◻

THEOREM 5.1. *Let*

$$\gamma = \frac{1}{C_\gamma \, Re}, \quad \textit{where} \quad Re = \frac{U\,L}{\nu}, \quad C_\gamma \geq 5, \tag{5.7}$$

selected such that Φ *belongs to the finite element space. Then, the time-averaged energy dissipation rate of the finite element velocity satisfies*

$$\langle \varepsilon(u^h) \rangle \quad \leq C\,\frac{U^3}{L},$$

where $C = C(C_\gamma)$.

Proof. We take $v^h = u^h - \Phi$ (this is possible due to the choice of γ) in (5.1) to get

$$(u^h_t, u^h) - (u^h_t, \Phi) + \nu \| \nabla u^h \|^2 - \nu (\nabla u^h, \nabla \Phi) - b_s(u^h, u^h, \Phi) = 0 \tag{5.8}$$

since $b_s(\cdot,\cdot,\cdot)$ is skew-symmetric and Φ is divergence free.

Observing that $(u^h_t, \Phi) = \frac{d}{d\,t}(u^h, \Phi)$, since $\frac{d\,\Phi}{d\,t} = 0$, we rewrite (5.8) as

$$\frac{1}{2}\frac{d}{d\,t}\| u^h \|^2 + \nu \| \nabla u^h \|^2 = \frac{d}{d\,t}(u^h, \Phi) + b_s(u^h, u^h, \Phi) + \nu (\nabla u^h, \nabla \Phi)$$

and integrate in time to get

$$\frac{1}{2}\| u^h(T) \|^2 - \frac{1}{2}\| u^h(0) \|^2 + \nu \int_0^T \| \nabla u^h \|^2 \, d\,t$$

$$= (u^h(T), \Phi) - (u^h(0), \Phi) + \int_0^T b_s(u^h, u^h, \Phi)\, d\,t + \nu \int_0^T (\nabla u^h, \nabla \Phi)\, d\,t. \tag{5.9}$$

We need to estimate each term on the right-hand side of (5.9). For that, we use (5.2)–(5.5) to derive upper bounds for the following terms:

$$(u^h(T), \Phi) \leq \frac{1}{2}\| u^h(T) \|^2 + \frac{1}{2}\| \Phi \|^2 = \frac{1}{2}\| u^h(T) \|^2 + \frac{1}{6}U^2 \gamma L^3,$$

$$(u^h(0), \Phi) \leq \| u^h(0) \| \, \| \Phi \| = \sqrt{\frac{\gamma}{3}}\, U L^{3/2} \| u^h(0) \|,$$

$$\nu \int_0^T (\nabla u^h, \nabla \Phi)\, d\,t \leq \frac{\nu}{2} \int_0^T \| \nabla u^h \|^2 \, d\,t + \frac{\nu}{2} \int_0^T \| \nabla \Phi \|^2 \, d\,t$$

$$= \frac{\nu}{2} \int_0^T \| \nabla u^h \|^2 \, d\,t + \frac{\nu}{2\,\gamma} L\, U^2\, T.$$

For the nonlinear term, we add and subtract terms, and use the fact that $b_s(\cdot,\cdot,\cdot)$ is skew-symmetric to write

$$\begin{aligned} b_s(u^h, u^h, \Phi) &= b_s(u^h - \Phi, u^h - \Phi, \Phi) + b_s(\Phi, u^h - \Phi, \Phi) \\ &= \frac{1}{2} b(u^h - \Phi, u^h - \Phi, \Phi) - \frac{1}{2} b(u^h - \Phi, \Phi, u^h - \Phi) \\ &\quad + \frac{1}{2} b(\Phi, u^h - \Phi, \Phi) - \frac{1}{2} b(\Phi, \Phi, u^h - \Phi). \end{aligned} \tag{5.10}$$

We use Lemma 5.1 together with (5.2)–(5.5) to analyze the terms in (5.10). In all cases, integration is restricted to $\mathcal{O}_{\gamma\,L}$ since $\mathrm{supp}(\Phi) = \overline{\mathcal{O}}_{\gamma\,L}$:

$$\begin{aligned} b(u^h - \Phi, u^h - \Phi, \Phi) &\le \| \Phi \|_{L^\infty(\mathcal{O}_{\gamma\,L})} \| \nabla(u^h - \Phi) \|_{L^2(\mathcal{O}_{\gamma\,L})} \| u^h - \Phi \|_{L^2(\mathcal{O}_{\gamma\,L})} \\ &\le U\,\gamma\,L \,\| \nabla(u^h - \Phi) \|^2_{L^2(\mathcal{O}_{\gamma\,L})} \\ &\le 2\,U\,\gamma\,L \,\| \nabla u^h \|^2 + 2\,U^3 L^2, \\ b(u^h - \Phi, \Phi, u^h - \Phi) &\le \| \nabla\Phi \|_{L^\infty(\mathcal{O}_{\gamma\,L})} \| u^h - \Phi \|^2_{L^2(\mathcal{O}_{\gamma\,L})} \\ &\le U\,\gamma\,L \,\| \nabla(u^h - \Phi) \|^2 \\ &\le 2\,U\,\gamma\,L \,\| \nabla u^h \|^2 + 2\,U^3 L^2, \\ b(\Phi, u^h - \Phi, \Phi) &\le \| \Phi \|_{L^\infty(\mathcal{O}_{\gamma\,L})} \| \Phi \|_{L^2(\mathcal{O}_{\gamma\,L})} \| \nabla(u^h - \Phi) \|_{L^2(\mathcal{O}_{\gamma\,L})} \\ &\le U \left(\frac{1}{3} U^2 \gamma L^3 \right)^{1/2} \| \nabla u^h \| + U \left(\frac{1}{3} U^2 \gamma L^3 \right)^{1/2} \left(\frac{U^2 L}{\gamma} \right)^{1/2} \\ &\le \frac{1}{2} U\,\gamma\,L \| \nabla u^h \|^2 + \frac{5}{6}\,U^3 L^2, \\ b(\Phi, \Phi, u^h - \Phi) &\le \| \Phi \|_{L^\infty(\mathcal{O}_{\gamma\,L})} \| \nabla\Phi \|_{L^2(\mathcal{O}_{\gamma\,L})} \| u^h - \Phi \|_{L^2(\mathcal{O}_{\gamma\,L})} \\ &\le U\,\gamma\,L \left(\frac{U^2\,L}{\gamma} \right)^{1/2} \| \nabla u^h \| + U\,\gamma\,L \left(\frac{U^2\,L}{\gamma} \right)^{1/2} \left(\frac{U^2\,L}{\gamma} \right)^{1/2} \\ &\le \frac{1}{2} U\,\gamma\,L \| \nabla u^h \|^2 + \frac{3}{2}\,U^3 L^2. \end{aligned}$$

Inserting these estimates into (5.10) gives

$$|b_s(u^h, u^h, \Phi)| \le \frac{5}{2} U\,\gamma\,L \,\| \nabla u^h \|^2 + \frac{19}{6} U^3 L^2.$$

Hence, estimate (5.9) becomes

$$\begin{aligned} \nu \int_0^T \| \nabla u^h \|^2 \,d\,t &\le \frac{1}{6} U^2 \gamma L^3 + \frac{1}{2} \| u^h(0) \|^2 + \sqrt{\frac{\gamma}{3}}\, U L^{3/2} \| u^h(0) \| + \frac{\nu}{2} \int_0^T \| \nabla u^h \|^2 \,d\,t \\ &\quad + \frac{\nu}{2\,\gamma} L\,U^2\,T + \frac{5}{2} \gamma\,L\,U \int_0^T \| \nabla\,u^h \|^2 \,d\,t + \frac{19}{6} U^3 L^2\,T. \end{aligned}$$

Dividing by T, taking lim sup as $T \to \infty$, and using $|\Omega| = L^3$ lead to

$$L^3 \left(\frac{1}{2} - \frac{5}{2} \frac{L\,\gamma\,U}{\nu} \right) \langle \varepsilon(u^h) \rangle \leq \frac{\nu}{2\,\gamma} L\,U^2 + \frac{19}{6} U^3 L^2.$$

Inserting (5.7) on both sides of this estimate proves the theorem. ◻

6. Summary and outlook. There are many important and interesting questions concerning the computational approximation of statistics in different flow settings. We studied the simple yet interesting case of internal flow, for instance, for small data (Reynolds number, body force) but large initial condition. In this setting, we proved convergence of the time-averaged energy dissipation rate in three dimensions without assuming that the solution is a strong, smooth, and unique solution, or the initial data is smooth. We then analyzed the convergence of time-averaged drag and lift coefficients. The crucial assumption of the general theory, a certain control on the kinetic energy, was proven for sufficiently small inflow boundary conditions. Lastly, we considered a simple case of shear flow and showed that the statistics of the computed solutions scale as predicted by the Kolmogorov theory.

One next step could be to consider discretization in time and approximate infinite time averaging by finite time averaging [8]. Our long-term goal is to develop a theory paralleling and inspired by the theory of shadowing in approximation of dynamical systems [23, 22, 14]. Are the computed statistics the exact statistics of a perturbed flow? Answering this numerical analysis question is fundamental for turbulent flow simulation.

Acknowledgment. The authors would like to thank the referees for their careful analysis of this manuscript and their important suggestions.

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