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Approximating Local Averages of Fluid Velocities: The Stokes Problem

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Abstract

As a first step to developing mathematical support for finite element approximation to the large eddies in fluid motion we consider herein the Stokes problem. We show that the local average of the usual approximate flow field $\mathbf{u}^h$ over radius δ provides a very accurate approximation to the flow structures of $O(\delta)$ or greater. The extra accuracy appears for quadratic or higher velocity elements and degrades to the usual finite element accuracy as the averaging radius $\delta \to h$ (the local meshwidth). We give both a priori and a posteriori error estimates incorporating this effect.

AMS Subject Classifications: 65N15, 65N30, 76D07.

Key Words: Space filtering, finite element method, computational fluid dynamics.

1. Introduction

The fundamental question in large eddy simulation (LES) is to approximate local averages of a fluids velocity at much less cost and at much greater accuracy than the velocity itself can be approximated. The usual approach is to average the equations of motion, address closure, find approximate boundary conditions and then solve numerically the resulting continuum approximation, e.g. see [2, 8, 10]. This approach is highly developed in engineering practice; it leads to nontrivial problems of understanding the modelling error and the model's consistency near boundaries.

We consider the question of approximating local, spacial averages of fluid's velocity. This question is already interesting for the Stokes problem. The approach we consider is that of direct approximation followed by postprocessing. This approach exploits the often oscillatory nature of finite element errors. The idea is that local averaging can possibly eliminate leading order errors.

Consider the solution of the Stokes problem, given by: finding $\mathbf{u} : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d (d = 2, 3)$ and $p : \Omega \to \mathbb{R}$ satisfying

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$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega \\
\mathbf{u} &= 0 \quad \text{on} \quad \partial\Omega \\
\int_\Omega p\, dx &= 0.
\end{aligned} \tag{1}
$$

The usual finite element approximation to the Stokes problem is a pair $(\mathbf{u}^h, p^h) \in (X^h, M^h)$, where (X^h, M^h) are velocity-pressure finite element spaces, $X^h \subset X := H_0^1(\Omega)^d$, $M^h \subset M := L_0^2(\Omega)$, satisfying:

$$
\begin{aligned}
(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h) - (p^h, \nabla \cdot \mathbf{v}^h) &= (\mathbf{f}, \mathbf{v}^h) \quad \forall\, \mathbf{v}^h \in X^h, \\
(q^h, \nabla \cdot \mathbf{u}^h) &= 0, \quad \forall\, q^h \in M^h,
\end{aligned} \tag{2}
$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product. The question we consider is the extent to which the "large eddies" or large structures of $\mathbf{u}^h$ approximate those of $\mathbf{u}$. To formulate this mathematically, let $g(\cdot)$ be a mollifer or approximate identity. Specifically, $g : \mathbb{R}^d \to \mathbb{R}$ is a non-negative, C_0^∞ function, with

$$
0 \le g \le 1,\ g(0) = 1, \quad g(x) = g(-x), \quad \text{and} \quad \int_{\mathbb{R}^d} g(x)dx = 1.
$$

Define $g_\delta(x) = \delta^{-d} g(x/\delta)$ and let $g_\delta * f$ denote the usual convolution of g_δ with f,

$$
(g_\delta * f)(x) = \int_{\mathbb{R}^d} g_\delta(x-y) f(y) dy.
$$

Since $(g_\delta * \mathbf{u})(x)$ represents an average of $\mathbf{u}$ about the point x of radius $O(\delta)$, $\mathbf{u}_\delta := g_\delta * \mathbf{u}$ can be thought of as a representation of the solution scales or eddies of size $O(\delta)$ or larger. One plausible procedure (we do not examine) to approximate these larger eddies is to calculate the approximate solution $\mathbf{u}^h$ in, e.g., a hierarchical or wavelet basis and then project it into the appropriate subspace representing scales of size $O(\delta)$ or larger. We consider herein the other plausible alternative procedure of calculating $\mathbf{u}^h$ and then postprocessing by local averaging with g_δ: approximate the "larger eddies" by $g_\delta * \mathbf{u}^h$. This is precisely the motivation behind the field of LES for high Reynolds number flow. One distinction is that in LES phenomenological models are constructed for $g_\delta * \mathbf{u}$ which are then approximated exactly by a numerical method. Herein, we approximate the exact flow equations and then approximate $g_\delta * \mathbf{u}$ by simply calculating $g_\delta * \mathbf{u}^h$.

Our ultimate goal is to provide mathematical support for this method for the time dependent (nonlinear) Navier–Stokes equations. This report is a small first step in this direction. We consider only the linear Stokes problem in a region with smooth boundary, and assume the finite element space to be conforming. A complete analysis, even for the Stokes problem, would thus require further incorporation of the effects of using isoparametric elements into the analysis.

2. Convergence of Large Eddies

In large eddy simulation, δ is interpreted as a local length scale and $\bar{\mathbf{u}} = g_\delta * \mathbf{u}$ as the eddies in the flow of size greater or equal $\mathcal{O}(\delta)$. The fluctuations, or small eddies, are defined in the usual way as $\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}}$. The main result of this section is that the large eddies in the approximate solution converge to those of the true solution at a much higher rate than the small eddies. The proof is by duality, exploiting (1) the fact the that convolution is a bounded linear operator, (2) the smoothing properties of g_δ and (3) the shift theorem for the Stokes problem in a smooth region. The approach of using duality has earlier been used in a posteriori error estimation of solution functionals, e.g. see Johnson and his co-workers [9], and Becker and Rannacher [5, 6].

First, we summarize standard properties of convolution operators, e.g. see [13]. Throughout this paper it is assumed that functions from $L^2(\Omega)^d$ are extended by zero outside Ω if they are convolved. As $\delta \to 0$, $g_\delta * \mathbf{f} \to \mathbf{f}$ for $\mathbf{f} \in L^2(\Omega)^d$, $g_\delta * \mathbf{f} \in C_0^\infty(\mathbb{R}^d)^d$ and there holds (Young's inequality)

$$\|g_\delta * \mathbf{f}\|_k \le C\|g_\delta\|_{W^{k,1}}\|\mathbf{f}\|, \tag{3}$$

where $\|\cdot\|$ is the L^2-norm and $\|\cdot\|_k$ the Sobolev $W^{k,2}(\mathbb{R}^d)$ norm. The norm $\|g_\delta\|_{W^{k,1}}$ of g_δ is easily calculated by a scaling argument to be bounded by:

$$\|g_\delta\|_{W^{k,1}} \le \delta^{-k}\|g\|_{W^{k,1}} \le C\delta^{-k}. \tag{4}$$

Since $g(x) = g(-x)$, we have

$$(g_\delta * f, v) = (f, g_\delta * v) \quad \forall f, v \in L^2(\Omega). \tag{5}$$

Let $f \in H_0^1(\mathbb{R}^d)$ and denote an arbitrary first order weak derivative by $\partial_\alpha f$, $|\alpha| = 1$. Then $(g_\delta * \partial_\alpha f, v) = -(g_\delta * f, \partial_\alpha v)\, \forall v \in H_0^1(\mathbb{R}^d)$, from which follows

$$\partial_\alpha(g_\delta * f) = g_\delta * \partial_\alpha f. \tag{6}$$

The velocity pressure finite element spaces $(X^h, M^h) \subset (X, M)$ are assumed to satisfy the usual inf-sup condition for the stability of the discrete pressure:

$$\inf_{q^h \in M^h} \sup_{\mathbf{v}^h \in X^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|q^h\|\|\nabla \mathbf{v}^h\|} \ge \beta > 0,$$

as well as an approximation assumption, typical of piecewise polynomial finite element spaces of degree $(k, k-1)$: for any $\mathbf{u} \in X \cap (H^{k+1}(\Omega))^d$ and $p \in M \cap H^k(\Omega)$:

$$\inf_{\mathbf{v}^h \in X^h} \left\{\|\mathbf{u} - \mathbf{v}^h\| + h\|\nabla(\mathbf{u} - \mathbf{v}^h)\|\right\} \le Ch^{k+1}\|\mathbf{u}\|_{k+1},$$

$$\inf_{q^h \in M^h} \|p - q^h\| \le Ch^k\|p\|_k.$$

See for example, Gunzburger [12] for examples of such finite element spaces (X^h, M^h).

Proposition 1. *Suppose $\partial\Omega$ is smooth enough and let $0 < h \leq \delta$ be given. Then, there is a constant C, independent of $(\mathbf{u}, p)$, h and δ such that*

$$\begin{aligned} \|g_\delta * \mathbf{u} - g_\delta * \mathbf{u}^h\| &\leq C\left(\frac{h}{\delta}\right)^{k-1} h(\|\nabla(\mathbf{u} - \mathbf{u}^h)\| + \|p - p^h\|) \\ &\leq C\left(\frac{h}{\delta}\right)^{k-1} h^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k). \end{aligned}$$

Proof: Let $\psi(x) \in L^2(\Omega)^d$ be given and consider the problem of finding $(\phi, (-\lambda)) \in (X, M)$ satisfying the following Stokes problem:

$$\begin{cases} -\Delta\phi + \nabla(-\lambda) &= g_\delta * \psi \quad \text{in } \Omega \\ \nabla \cdot \phi &= 0 \quad \text{in } \Omega \\ \phi &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{7}$$

It is known that (ϕ, λ) exists uniquely in (X, M) and satisfies (since $\partial\Omega$ is smooth)

$$\begin{aligned} \|\phi\|_{k+1} + \|\lambda\|_k &\leq C\|g_\delta * \psi\|_{k-1} \leq \text{ by (3)} \\ &\leq C(k)\|g_\delta\|_{W^{k-1,1}}\|\psi\| \leq C\delta^{-k+1}\|\psi\|. \end{aligned} \tag{8}$$

The variational representation of (7) is to find $(\phi, \lambda) \in (X, M)$ satisfying:

$$(\nabla\phi, \nabla\mathbf{v}) + (\lambda, \nabla \cdot \mathbf{v}) - (q, \nabla \cdot \phi) = (g_\delta * \psi, \mathbf{v}), \forall(\mathbf{v}, q) \in (X, M). \tag{9}$$

Setting $\mathbf{v} = \mathbf{e} = \mathbf{u} - \mathbf{u}^h$ and $q = p - p^h$ gives:

$$(g_\delta * \mathbf{e}, \psi) = (g_\delta * \psi, \mathbf{e}) = (\nabla\mathbf{e}, \nabla\phi) - (p - p^h, \nabla \cdot \phi) + (\lambda, \nabla \cdot \mathbf{e}).$$

The error $\mathbf{e} = \mathbf{u} - \mathbf{u}^h$ satisfies the usual Galerkin orthogonality condition: for all $(\mathbf{v}^h, q^h) \in (X^h, M^h)$

$$(\nabla\mathbf{e}, \nabla\mathbf{v}^h) - (p - p^h, \nabla \cdot \mathbf{v}^h) + (q^h, \nabla \cdot \mathbf{e}) = 0. \tag{10}$$

Thus, for all $(\mathbf{v}^h, q^h) \in (X^h, M^h)$:

$$\begin{aligned} (g_\delta * \mathbf{e}, \psi) &= (\nabla\mathbf{e}, \nabla(\phi - \mathbf{v}^h)) - (p - p^h, \nabla \cdot (\phi - \mathbf{v}^h)) + (\lambda - q^h, \nabla \cdot \mathbf{e}) \\ &\leq (\|\nabla\mathbf{e}\| + \|p - p^h\|)\|\nabla(\phi - \mathbf{v}^h)\| + \|\lambda - q^h\|\|\nabla\mathbf{e}\|. \end{aligned}$$

Picking $(\mathbf{v}^h, q^h)$ to be the best approximation of (ϕ, λ) in (X^h, M^h) gives:

$$(g_\delta * \mathbf{e}, \psi) \leq Ch^k(\|\phi\|_{k+1} + \|\lambda\|_k)\|\nabla\mathbf{e}\| + Ch^k\|\phi\|_{k+1}\|p - p^h\|.$$

Using the regularity estimate (8) gives

$$\frac{(g_\delta * \mathbf{e}, \psi)}{\|\psi\|} \leq Ch\left(\frac{h}{\delta}\right)^{k-1} (\|\nabla \mathbf{e}\| + \|p - p^h\|),$$

and the result follows by taking the supremum over $\psi \in L^2(\Omega)^d$. □

Remark 1. The smoothness of the boundary needed is that $\partial\Omega$ is $C^{k,\alpha}$ for some $\alpha, 0 < \alpha < 1$. If the boundary is less smooth then the global convergence rate is correspondingly diminished.

3. An A Posteriori Error Estimator for Larger Eddies Using the Discrete Solution

A posteriori error estimators are a standard tool in modern codes for the solution of partial differential equations. They have the tasks of estimating the global error in a given norm to serve as a stopping criterion for the iterative solution of the equation and to estimate local errors to provide information for the adaptive mesh refinement. With the pioneering work of Babuška and Rheinboldt [3], rigorous analysis of a posteriori error estimators started. During the last decades, fundamental approaches for analyzing a posteriori error estimators were developed. e.g. for residual based error estimators by Verfürth [17] and Johnson et al. [9] or for error estimates which are based on the solution of local problems by Bank and Weiser [4] and Ainsworth and Oden [1]. These techniques can be applied to derive a posteriori error estimators in different norms for the Stokes equations.

In this section we show that it is possible to develop a residual based a posteriori error estimator for $\|g_\delta * \mathbf{u} - g_\delta * \mathbf{u}^h\|$. Because the estimator in Proposition 2 reflects the increased accuracy in the larger scales exhibited in the a priori estimate in Proposition 1, it shows that it is possible and advantageous to seek the larger eddies adaptively. Indeed, the estimate (12) suggests (due to the weighting of the individual terms in (12) that the larger the solution scale one seeks (the larger δ is taken), the sooner a prearranged error tolerance is achieved. Naturally, a different mesh distribution would also result from using (12) instead of a typical energy norm or L^2-norm estimator since the individual terms in the local error estimator are scaled differently. A numerical study of some a posteriori error estimators for convection–diffusion equations, [15], showed that a L^2-norm estimator produced often the most appropriate refined meshes. In contrast to estimators for stronger norms, it did not stick only to the strongest singularity of the solution (e.g. exponential layers) and thus did not fail to refine at other singularities (e.g. parabolic layers).

To develop the error estimator, we must introduce a bit more notation. Let $\Pi^h(\Omega)$ denote a division of Ω into mesh cells $\{T : T \in \Pi^h(\Omega)\}$. If $T \in \Pi^h(\Omega)$, h_T will denote the diameter of T, $(\cdot,\cdot)_T$ the $L^2(T)$ inner product with norm $\|\cdot\|_T$ and $[\cdot]_{\partial T}$ the jump of the indicated quantity across ∂T (being the difference of the inner and outer traces.) To be precise, the jump $[v_h]_E$ of a function v_h across a face E is defined by

$$[v_h]_E := \begin{cases} \lim_{t\to+0}\{v_h(x+t\hat{\mathbf{n}}_E) - v_h(x-t\hat{\mathbf{n}}_E)\} & E \not\subset \partial\Omega \\ \lim_{t\to+0}\{-v_h(x-t\hat{\mathbf{n}}_E)\} & E \subset \partial\Omega \end{cases}$$

where $\hat{\mathbf{n}}_E$ is a normal unit vector on E and $x \in E$. If $E \subset \partial\Omega$, we choose the outer normal, otherwise $\hat{\mathbf{n}}_E$ has an arbitrary but fixed orientation. With that, every face E which separates two neighbouring cells T_1 and T_2 is associated with a uniquely oriented normal $\hat{\mathbf{n}}_E$ (for definiteness from T_1 to T_2). Further, $\tilde{\omega}(T)$ will denote the union of the elements whose boundary touches ∂T. A type of angle condition is implicitly assumed in the condition that for all $w \in H^s(\Omega)$

$$\sum_{T\in\Pi^h(\Omega)} \|w\|_{s,\tilde{\omega}(T)} \le C\|w\|_{s,\Omega},$$

where $\|\cdot\|_{s,\tilde{\omega}(T)}$ is the $H^s(\tilde{\omega}(T))$ norm. Finally, it is assumed that (X^h, M^h) possesses an interpolation operator R of Clément type, [7], such that interpolation error estimates of the type

$$\begin{cases} \|\mathbf{v} - R_{X_h}(\mathbf{v})\|_{0,T} & \le & C_i h_T^{m+1}|\mathbf{v}|_{m+1,\tilde{\omega}(T)} & \forall\, \mathbf{v} \in X, \\ |\mathbf{v} - R_{X_h}(\mathbf{v})|_{1,T} & \le & C_i h_T|\mathbf{v}|_{2,\tilde{\omega}(T)} & \forall\, \mathbf{v} \in X, \\ \|\mathbf{v} - R_{X_h}(\mathbf{v})\|_{0,E} & \le & C_i h_E^{m+1/2}|\mathbf{v}|_{m+1,\tilde{\omega}(E)} & \forall\, \mathbf{v} \in X, \\ \|q - R_{M_h}(q)\|_{0,T} & \le & C_i h_T|q|_{1,\tilde{\omega}(T)} & \forall\, q \in M, \end{cases} \tag{11}$$

with $m \in \{0, 1\}$ are valid. Here, $\tilde{\omega}(E)$ denotes the union of all mesh cells whose boundary has a common point with E. See Verfürth [17] for more details concerning the mathematical framework of residual based a posteriori error estimation.

Proposition 2. *Let $(\mathbf{u}^h, p^h)$ be the usual Galerkin approximation to the solution $(\mathbf{u}, p)$ of the Stokes problem in $\Omega \subset \mathbb{R}^d (d = 2, 3)$. Suppose $\partial\Omega$ is smooth enough and $\delta > 0$. Then, there is a constant C such that*

$$\begin{aligned} \|g_\delta * \mathbf{u} - g_\delta * \mathbf{u}^h\| &\le C\left\{\sum_{T\in\Pi^h(\Omega)} \eta_T^2\right\}^{1/2} \\ &= C\left\{\sum_{T\in\Pi^h(\Omega)} \left(\frac{h_T}{\delta}\right)^{2k-2} h_T^4\|\mathbf{r}^h\|_T^2 \right. \\ &\quad + \left(\frac{h_T}{\delta}\right)^{2k-2} h_T^2\|\nabla\cdot\mathbf{u}^h\|_T^2 \\ &\quad \left. + \left(\frac{h_T}{\delta}\right)^{2k-2} h_T^3\|[\nabla\mathbf{u}^h\cdot\hat{\mathbf{n}} - p^h\hat{\mathbf{n}}]_{\partial T}\|_{\partial T}^2\right\}^{1/2}, \end{aligned} \tag{12}$$

where, for each mesh cell T in $\Pi^h(\Omega)$,

$$\mathbf{r}^h|_T := \mathbf{f} - (-\Delta \mathbf{u}^h + \nabla p^h)|_T$$

is the strong local residual on T.

Proof: The proof is by a duality argument similar in spirit to the proof of Proposition 1. Let $\psi \in (L^2(\Omega))^d$ be given. Noting that

$$\|g_\delta * \mathbf{e}\| = \sup_{\psi \in L^2(\Omega)^d} \frac{(g_\delta * \mathbf{e}, \psi)}{\|\psi\|} = \sup_{\psi \in L^2(\Omega)^d} \frac{(\mathbf{e}, g_\delta * \psi)}{\|\psi\|},$$

consider the dual problem (7). Setting $\mathbf{v} = \mathbf{e} = \mathbf{u} - \mathbf{u}^h$ and $q = p - p^h$ in the variational formulation (9) of (7) gives:

$$(g_\delta * \psi, \mathbf{e}) = (\nabla \mathbf{e}, \nabla \phi) - (p - p^h, \nabla \cdot \phi) + (\lambda, \nabla \cdot \mathbf{e}).$$

Using Galerkin orthogonality (10) in this last equation gives:

$$(g_\delta * \psi, \mathbf{e}) = (\nabla \mathbf{e}, \nabla(\phi - \mathbf{v}^h)) - (p - p^h, \nabla \cdot (\phi - \mathbf{v}^h)) + (\lambda - q^h, \nabla \cdot \mathbf{e}),$$

for any $\mathbf{v}^h \in X^h$ and $q^h \in M^h$. Consider the right hand side of the above. Integration by parts, mesh cell by mesh cell, and collecting the boundary integral terms into jump integrals, gives:

$$\begin{aligned}(g_\delta * \psi, \mathbf{e}) = \sum_{T \in \Pi^h(\Omega)} \Big[& (-\Delta \mathbf{e}, \phi - \mathbf{v}^h)_T + (\nabla(p - p^h), \phi - \mathbf{v}^h)_T \\ & + (\lambda - q^h, \nabla \cdot (\mathbf{u} - \mathbf{u}^h))_T - \frac{1}{2} \int_{\partial T} [p^h]_{\partial T} (\phi - \mathbf{v}^h) \cdot \hat{\mathbf{n}} ds \\ & + \frac{1}{2} \int_{\partial T} [\nabla \mathbf{u}^h \cdot \hat{\mathbf{n}}]_{\partial T} \cdot (\phi - \mathbf{v}^h) ds \Big],\end{aligned}$$

where, on ∂T, $\hat{\mathbf{n}}$ denotes the outward unit normal to T. This can be rewritten more compactly using the local residual $\mathbf{r}^h$ as

$$\begin{aligned}(g_\delta * \psi, \mathbf{e}) = \sum_{T \in \Pi^h(\Omega)} \Big[& (\mathbf{r}^h, \phi - \mathbf{v}^h)_T - (\nabla \cdot \mathbf{u}^h, \lambda - q^h)_T \\ & - \frac{1}{2} \int_{\partial T} [p^h]_{\partial T} (\phi - \mathbf{v}^h) \cdot \hat{\mathbf{n}} ds + \frac{1}{2} \int_{\partial T} [\nabla \mathbf{u}^h \cdot \hat{\mathbf{n}}]_{\partial T} \cdot (\phi - \mathbf{v}^h) ds \Big].\end{aligned}$$

Thus, picking $(\mathbf{v}^h, q^h)$ to be Clément interpolants of (ϕ, λ), gives:

$$\begin{aligned}(g_\delta * \psi, \mathbf{e}) \le C \sum_{T \in \Pi^h(\Omega)} \Big[& h_T^{k+1} \|\mathbf{r}^h\|_T \|\phi\|_{k+1, \tilde{\omega}(T)} + h_T^k \|\nabla \cdot \mathbf{u}^h\|_T \|\lambda\|_{k, \tilde{\omega}(T)} \\ & + h_E^{k+1/2} \|[-p^h \hat{\mathbf{n}} + \nabla \mathbf{u} \cdot \hat{\mathbf{n}}]_{\partial T}\|_{\partial T} \|\phi\|_{k+1, \tilde{\omega}(T)} \Big]. \end{aligned} \tag{13}$$

Motivated by the regularity result (8), apply a weighted Cauchy inequality to (13) gives:

$$(g_\delta * \psi, \mathbf{e}) \leq C\Bigg\{ \sum_{T\in\Pi^h(\Omega)} h_T^{2k+2}\delta^{-2(k-1)}\|\mathbf{r}^h\|_T^2 + h_T^{2k}\delta^{-2(k-1)}\|\nabla\cdot\mathbf{u}^h\|^2$$
$$h_E^{2k+1}\delta^{-2(k-1)}\|[\nabla\mathbf{u}^h\cdot\hat{\mathbf{n}} - p^h\hat{\mathbf{n}}]_{\partial T}\|_{\partial T}^2\Bigg\}^{1/2}\|\psi\|.$$

Dividing by $\|\psi\|$ and taking the supremum over $\psi \in L^2(\Omega)^d$ yields the claimed result. □

In general, $\mathbf{f}$ is not a polynomial and accordingly $\mathbf{r}^h$ likewise. Replacing $\mathbf{f}$ by a polynomial, e.g. which represents an appropriate quadrature rule, causes an additional higher order term in the a posteriori error estimate (12), cf. Proposition 4.

Numerical Illustration. We consider the driven cavity problem for $d = 2$, i.e. $\Omega = (0,1)^2$, $\mathbf{f} = \mathbf{0}$, $\mathbf{u} = (1.0)^T$ for $y = 1$ and no slip boundary conditions on the other parts of the boundary. Since the boundary values for the driven cavity are not an $H^{1/2}(\partial\Omega)$-function, the solution is not an $H^1(\Omega)$-function. Although the solution of the driven cavity problem is not smooth, the behavior of the above estimator can be illustrated well with this example. The choice of different values of δ will result in different final adaptive meshes for a preset error tolerance. This tolerance is achieved the sooner the larger δ is. In addition, the application of the a posteriori error estimator (12) yields a different sequence of adaptively refined meshes than the use of the standard L^2-error estimator, which has the same form like η_T with $k = 1$. This is because the local estimates in η_T are weighted by an additional factor of h_T in comparison to the L^2-error estimator. Thus, it can be expected that large mesh cells are refined earlier using η_T since their local error estimates are weighted by a large factor. Also note that because this estimator has the same form as other residual based estimators, it is easy to implement in an existing code.

In the numerical test, we have used the Taylor–Hood finite element, i.e. $k = 2$, an initial grid consisting of eight triangles, and the stopping criterion

$$\Bigg\{\sum_{T\in\Pi^h(\Omega)} \eta_T^2\Bigg\}^{1/2} < 0.01.$$

The results of the numerical tests correspond to the expectations, see Fig. 1. The final mesh for $\delta = 0.60(0.45, 0.30, 0.15)$ possesses $7443(10450, 14263, 20373)$ degrees of freedom whereas the mesh obtained with the standard L^2-error estimator has 86752 degrees of freedom. Note that, away from the singularities at the upper corners, the mesh for the large eddies is coarser than that for the standard L^2-error estimator with the same tolerance, compare Figs. 1 and 2.

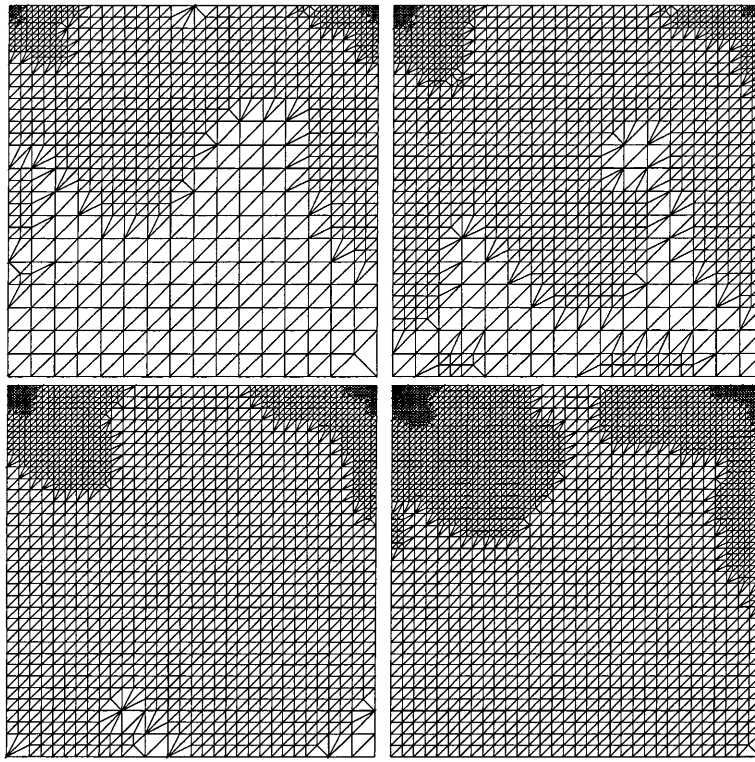

Figure 1. Final adaptive meshes for $\delta = 0.60, 0.45, 0.30, 0.15$

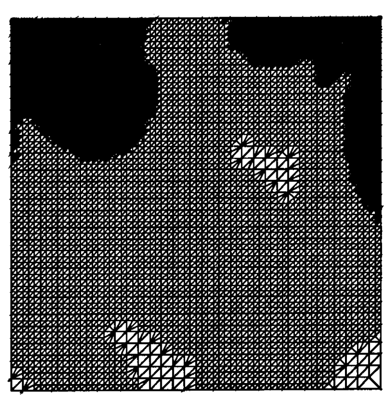

Figure 2. Final adaptive mesh for standard L^2-error estimator

We have simply picked the above termination condition to test the concept of the estimator. However, in practical applications, an order of magnitude estimate for the constant C in (12) must be also calculated to give meaning to this condition. This is typically done with the help of approximate solutions of the dual problem, [5, 6]. This difficulty is common to the type of estimators we are studying.

4. An A Posteriori Error Estimator for Larger Eddies Using Averages of the Discrete Solution

This section considers an a posteriori error estimator for $\|g_\delta * \mathbf{u} - g_\delta * \mathbf{u}^h\|$ which uses averages of the discrete solution. This estimator requires the use of interpolation operators $I^h_{\mathbf{u}}(\cdot), I^h_p(\cdot), I^h_{\mathbf{f}}(\cdot)$ which map a continuous function to some piecewise polynomial approximation (on the given mesh $\Pi^h(\Omega)$) of fixed degree, e.g. representing a quadrature rule. We assume in addition that the image of $I^h_{\mathbf{u}}(\cdot)$, is a continuous function and the image of $I^h_p(\cdot)$ is in $L^2_0(\Omega)$. These operators are somewhat arbitrary although a term in the estimates appears which represents the accuracy of their operations. Naturally, we would wish such terms to be of higher order and choose them accordingly.

Given a d-simplex $T \in \Pi^h(\Omega)$ and $(\mathbf{u}^h, p^h) \in (X^h, M^h)$. We define the local estimator

$$
\begin{aligned}
\bar{\eta}_T^2 := {} & h_T^4 \|I^h_{\mathbf{f}}(g_\delta * \mathbf{f}^h) + \Delta I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) - \nabla I^h_p(g_\delta * p^h)\|^2_{0,T} \\
& + h_T^2 \|\nabla \cdot I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)\|^2_{0,T} \\
& + \sum_{E \subset \partial T, E \neq \partial\Omega} h_E^3 \|[\nabla I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) \cdot \hat{\mathbf{n}}_E - I^h_p(g_\delta * p^h)\hat{\mathbf{n}}_E]_E\|^2_{0,E},
\end{aligned}
\tag{14}
$$

where $\mathbf{f}^h$ is a fixed degree piecewise polynomial approximation to $\mathbf{f}$. We shall show in Proposition 3 that $\bar{\eta}_T$ provides a local lower bound to the local error in the larger eddies, modulo a consistency error term which depends on the order of approximation of $I^h_{\mathbf{f}}(\cdot)$. This lower bound is proved using a scaling argument and depends on choosing the approximation to $g_\delta * \mathbf{u}$ to be piecewise polynomial (hence $I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)$) but *not* on Galerkin orthogonality. Proposition 4 proves, using duality, that $(\sum_T \bar{\eta}_T^2)^{1/2}$ provides an upper bound to the global error $\|g_\delta * \mathbf{u} - I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)\|$ modulo an extra term arising at the boundary $\partial\Omega$.

4.1. The Local Lower Bound

In addition to the assumptions of the previous section, we suppose inverse estimates hold of the form

$$
\left\{
\begin{array}{lll}
|v^h|_{m,T} \le C h_T^{-1} |v^h|_{m-1,T}, & \forall\, T \in \Pi^h(\Omega), \forall\, v^h|_T \in P_k(T), \\
\sum_{T \in \omega(E)} |v^h|_{m,T} \le C h_E^{-1} \sum_{T \subset \omega(E)} |v^h|_{m-1,T}, & \forall\, T \subset \omega(E), \forall\, v^h|_T \in P_k(T).
\end{array}
\right.
\tag{15}
$$

Here, $P_k(T)$ is the space of polynomials on mesh cell T with degree not greater than k and $\omega(E)$ is the union of the two mesh cells sharing the common face E. In addition, let $\omega(T)$ be the union of all mesh cells having a common face with T.

Proposition 3. *Let* $(\mathbf{u}, p)$ *be the solution of the Stokes problem* (1). *Given a shape regular family of triangulations* $\Pi^h(\Omega)$ *and fixed polynomial degrees for* $\mathbf{f}^h$ *and the* *interpolation operators* $I_{\mathbf{u}}^h(\cdot)$, $I_{\mathbf{f}}^h(\cdot)$ *and* $I_p^h(\cdot)$. *Then for all pairs of functions* $(\mathbf{v}^h, q^h) \in (X^h, M^h)$, *the local a posteriori error bound*

$$\bar{\eta}_T \leq C[\|g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)\|_{0,\omega(T)} + h_T\|g_\delta * p - I_p^h(g_\delta * q^h)\|_{0,\omega(T)} \\ + h_T^2\|g_\delta * \mathbf{f} - I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h)\|_{0,\omega(T)}] \tag{16}$$

holds.

Proof: Each term of $\bar{\eta}_T$ will be estimated separately from above by the local error with the use of suitable cut-off functions.

Cut-off functions are a standard tool for proving local a posteriori error estimates, see e.g. [17, 18, 14]. Let $T \in \Pi^h(\Omega)$ be an arbitrary cell and $E \subset \partial T$ a $(d-1)$-dimensional face. We denote by λ_E the linear function which is zero on E and 1 in the corner of T opposite to E. We define a cell bubble function for T by

$$B_T = \begin{cases} \left[(d+1)^{d+1}\prod_{i=1}^{d+1}\lambda_{E_i}\right]^2, & \text{in } T, \\ 0, & \text{otherwise,} \end{cases}$$

where E_i, $i = 1, \ldots, d+1$ are the faces of T.

The edge/face bubble function of an edge (in $\mathbb{R}^2$) or a face (in $\mathbb{R}^3$) is constructed as follows. Let E be the common face of the cells T_1 and T_2. The other faces of T_1 and T_2 are $E_1, \ldots, E_d$ and $E_{d+1}, \ldots, E_{2d}$, respectively. The simplex T_1 is reflected on E with the images $E_1^*, \ldots, E_d^*$ of $E_1, \ldots, E_d$. In the same way, we obtain $E_{d+1}^*, \ldots, E_{2d}^*$ by reflecting T_2 on E, see Fig. 3 for a $2d$ sketch. We define

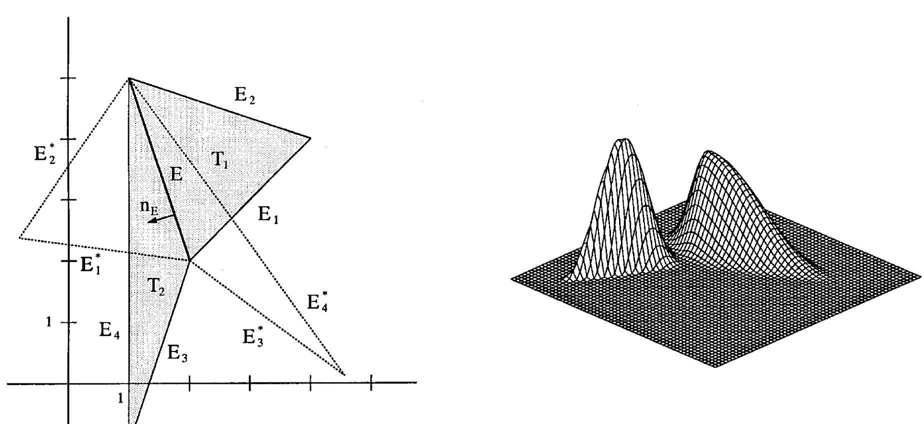

Figure 3. Cut-off function B_E

$$B_E = \begin{cases} \left(d^{2d}\prod_{i=1}^{d}\lambda_{E_i}\lambda_{E_i^*}\right)^2 & \text{on } T_1, \\ \left(d^{2d}\prod_{i=1}^{d}\lambda_{E_{i+d}}\lambda_{E_{i+d}^*}\right)^2 & \text{on } T_2, \\ 0 & \text{otherwise.} \end{cases}$$

The cut-off functions B_T and B_E have the following properties, see [17, 18, 14]:

$$\begin{cases} B_T \in C^1(\bar{\Omega}), B_E \in C^1(\bar{\Omega}), 0 \le B_T \le 1, 0 \le B_E \le C \\ B_T = 0 \text{ on } \partial T, \nabla B_T = 0 \text{ on } \partial T, \\ B_E = 0 \text{ on } \partial\omega(E), \nabla B_E = 0 \text{ on } \partial\omega(E), \\ B_T, B_E \text{ are polynomials in each mesh cell } T \in \Pi^h(\Omega). \end{cases} \tag{17}$$

The upper bound of B_E depends on the smallest aspect ratio of the mesh cells of $\omega(E)$. Thus, it is bounded independently of $\omega(E)$ for a shape regular family of triangulations.

In the following, we will need uniform norm equivalences of weighted L^2-norms in finite dimensional spaces. The proofs of the following estimates use the equivalence of norms in finite dimensional spaces, the boundedness of the cut-off functions and scaling arguments

$$\|v\|_{0,T}^2 \le C(v, vB_T)_T, \quad \forall\, v \in P_k(T), \tag{18}$$

$$\|v\|_{0,E}^2 \le C(v, vB_E)_E, \quad \forall\, v \in P_k(E), \tag{19}$$

$$\|vB_T\|_{0,T} \le \|v\|_{0,T}, \quad \forall\, v \in P_k(T), \tag{20}$$

$$\|vB_E\|_{0,\omega(E)} \le C\|v\|_{0,\omega(E)}, \quad \forall\, v \text{ with } v|_{T_i} \in P_k(T_i), T_i \subset \omega(E), \tag{21}$$

see Lemma 3.3 in [17]. The constants in (18) and (19) depend on k but not on h_T or h_E.

i) estimate of the first term in (14): Denote by $\mathbf{r}_T := I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h) + \Delta I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h) - \nabla I_p^h(g_\delta * q^h)$. Using (18), we obtain

$$\begin{aligned} \|\mathbf{r}_T\|_{0,T}^2 \le C[&(g_\delta * \mathbf{f} + \Delta I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h) - \nabla I_p^h(g_\delta * q^h), \mathbf{r}_T B_T)_T \\ &+ (I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h) - g_\delta * \mathbf{f}, \mathbf{r}_T B_T)_T]. \end{aligned}$$

Substituting $\mathbf{f} = -\Delta\mathbf{u} + \nabla p$, using (6), integrating two times by parts, using (17), the Cauchy–Schwarz inequality, the inverse inequality (15), and (20) give

$$\begin{aligned} \|\mathbf{r}_T\|_{0,T} \le C[&h_T^{-2}\|g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)\|_{0,T} + h_T^{-1}\|g_\delta * p - I_p^h(g_\delta * q^h)\|_{0,T} \\ &+ \|g_\delta * \mathbf{f} - I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h)\|_{0,T}]. \end{aligned}$$

ii) estimate of the second term in (14): Using (18), $\nabla \cdot \mathbf{u} = 0$, (6), integration by parts, and (17), we obtain

$$\|\nabla \cdot I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)\|_{0,T}^2 \leq C[(\nabla \cdot (I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h) - g_\delta * \mathbf{u}), \nabla \cdot I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)B_T)_T]$$

$$= C[(g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h), \nabla(\nabla \cdot I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)B_T))_T].$$

The Cauchy–Schwarz inequality, the inverse estimate (15) and (20) give

$$\|\nabla \cdot I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)\|_{0,T} \leq Ch_T^{-1}\|g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)\|_{0,T}.$$

iii) estimate of the third term in (14): Define $\mathbf{r}_E := [\nabla I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h) \cdot \hat{\mathbf{n}}_E - I_p^h(g_\delta * q^h)\hat{\mathbf{n}}_E]_E$ and let $\mathbf{r}_E^c B_E$ be a polynomial continuation of $r_E B_E$ to $\omega(E)$ with

$$\|\mathbf{r}_E^c B_E\|_{0,\omega(E)} \leq Ch_E^{1/2}\|\mathbf{r}_E B_E\|_{0,E}. \tag{22}$$

For details of constructing such a continuation see [18]. Since $\mathbf{r}_E^c B_E$ is defined to be a polynomial, it is continuously differentiable. Using the face bubble function B_E, (19), the continuity of $\nabla(g_\delta * \mathbf{u}), g_\delta * p, \mathbf{r}_E^c B_E$, (17), and integration by parts give

$$\|\mathbf{r}_E\|_{0,E}^2 \leq C \sum_{T \subset \omega(E)} [(\nabla(g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)), \nabla(\mathbf{r}_E^c B_E))_T - (g_\delta * p - I_p^h(g_\delta * q^h),$$

$$\nabla \cdot (\mathbf{r}_E^c B_E))_T + (\Delta(g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)) - \nabla((g_\delta * p) + I_p^h(g_\delta * q^h)), \mathbf{r}_E^c B_E)_T].$$

Each term will be estimated separately. The first term is integrated by parts once more. The boundary integral on the inner face E vanishes since all terms are continuous functions. We obtain

$$\sum_{T \subset \omega(E)} (\nabla(g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)), \nabla(\mathbf{r}_E^c B_E))_T$$

$$= \sum_{T \subset \omega(E)} -(g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h), \Delta(\mathbf{r}_E^c B_E))_T.$$

The Cauchy–Schwarz inequality, the inverse inequality (15), shape regularity of the triangulation, (21), and (22) yield

$$\sum_{T \subset \omega(E)} (\nabla(g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)), \nabla(\mathbf{r}_E^c B_E))_T\|$$

$$\leq Ch_E^{-3/2}\|g_\delta * \mathbf{u} - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)\|_{0,\omega(E)}\|\mathbf{r}_E\|_{0,E}.$$

Similarly, we obtain for the second term

$$\sum_{T\subset\omega(E)} -(g_\delta * p - I_p^h(g_\delta * q^h), \nabla\cdot(\mathbf{r}_E^c B_E))_T$$

$$\leq C h_E^{-1/2} \|g_\delta * p - I_p^h(g_\delta * q^h)\|_{0,\omega(E)} \|\mathbf{r}_E\|_{0,E}.$$

We can write the third term, using (6) and $\mathbf{f} = -\Delta\mathbf{u} + \nabla p$, in the form

$$\sum_{T\subset\omega(E)} (-I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h) - \Delta I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h) + \nabla I_p^h(g_\delta * q^h), \mathbf{r}_E^c B_E)_T$$

$$+ (I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h) - g_\delta * \mathbf{f}, \mathbf{r}_E^c B_E)_T.$$

This can be estimated using the Cauchy–Schwarz inequality, (22), (21), estimate i) for $\|I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h) + \Delta I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h) - \nabla I_p^h(g_\delta * q^h)\|_{0,T}$, and the shape regularity of the triangulation. Combining all estimates proves the proposition. □

Remark 2. (1) Since $(\mathbf{v}^h, q^h)$ may be any element of (X^h, M^h), the given error estimate is also true for approximate solutions of the discrete system.

(2) The extra term may be estimated

$$\|g_\delta * \mathbf{f} - I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h)\|_{0,\omega(T)} \leq \|g_\delta * (\mathbf{f} - \mathbf{f}^h)\|_{0,\omega(T)} + \|g_\delta * \mathbf{f}^h - I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h)\|_{0,\omega(T)}.$$

Thus, if the approximation $\mathbf{f}^h$ to $\mathbf{f}$ and the polynomial approximation $I_{\mathbf{f}}^h(g_\delta * \mathbf{f}^h)$ to $g_\delta * \mathbf{f}^h$ are accurate enough, this extra term is of higher order. The second goal can be achieved by using an appropriate interpolation formula. A careful inspection of the proof shows that $\mathbf{f}^h$ needs to be only a polynomial of fixed degree. For this reason the first goal can be achieved by choosing a sufficiently high polynomial degree for $\mathbf{f}^h$ in computing the local estimator (14).

(3) Estimates for $\|g_\delta * \mathbf{u} - g_\delta * \mathbf{v}^h\|_{0,\omega(T)}$ and $\|g_\delta * p - g_\delta * q^h\|_{0,\omega(T)}$ can be obtained by using the triangle inequality in (16). Then, we obtain the extra terms

$$\|g_\delta * \mathbf{v}^h - I_{\mathbf{u}}^h(g_\delta * \mathbf{v}^h)\|_{0,\omega(T)} \quad \text{and} \quad h_T \|g_\delta * q^h - I_p^h(g_\delta * q^h)\|_{0,\omega(T)}$$

which are of higher order if $I_{\mathbf{u}}^h(\cdot)$ and $I_p^h(\cdot)$ are accurate enough.

4.2. The Global Upper Error Bound

The global upper error bound will be proved via duality, first used in a posteriori error estimation by Johnson and his co-workers, see e.g. [9].

To construct an appropriate dual problem, we define a second piecewise polynomial interpolation of $g_\delta * \mathbf{u}^h$. Let $x_i, i = 1, \ldots, N$ be the nodal points of the finite element space X^h, then

$$I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)(x_i) := \begin{cases} I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)(x_i) & x_i \in \Omega \\ g_\delta * \mathbf{u}(x_i) & x_i \in \partial\Omega. \end{cases}$$

That means, $I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)$ and $I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)$ are different only in nodal points on the boundary, where $I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)$ takes the (unknown) values $g_\delta * \mathbf{u}(x_i)$ which are in general different from $I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)(x_i)$.

Consider the continuous dual problem of (1) with the error $g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)$ as right hand side

$$\begin{aligned} -\Delta\varphi - \nabla\lambda &= g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h) \quad \text{in} \ \ \Omega, \\ \nabla \cdot \varphi &= 0 \quad \text{in}\ \Omega, \\ \varphi &= 0 \quad \text{on} \ \ \partial\Omega. \end{aligned} \tag{23}$$

We assume the solution (φ, λ) of (23) to be H^2-regular

$$(\varphi, \lambda) \in (H^2(\Omega) \cap H^1_0(\Omega))^d \times (H^1(\Omega) \cap L^2_0(\Omega)) \tag{24}$$

and stable

$$\|\varphi\|_2 + \|\lambda\|_1 \le C\|g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\|. \tag{25}$$

Define a function $\mathbf{f}^h_\delta$ via the Riesz representation theorem by

$$\begin{aligned} &(\nabla(I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)), \nabla\mathbf{v}^h) - (I^h_p(g_\delta * p^h), \nabla \cdot \mathbf{v}^h) + (q^h, \nabla \cdot I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)) \\ &\quad =: (\mathbf{f}^h_\delta, \mathbf{v}^h), \quad \forall\ (\mathbf{v}^h, q^h) \in (X^h, M^h). \end{aligned} \tag{26}$$

Note, in general, that $(I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h),\ I^h_p(g_\delta * p^h)) \notin (X^h, M^h)$ since $I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)$ might not vanish on $\partial\Omega$. However, we assume that the approximation of $g_\delta * \mathbf{u}^h$ is continuous and consists of piecewise polynomials of some fixed degree k, and the approximation of $g_\delta * p^h$ consists of piecewise polynomials of degree $k-1$ and is normalized such that $I^h_p(g_\delta * p^h) \in L^2_0(\Omega)$.

The following propositions gives an estimate of the global L^2-error of the averages of $\mathbf{u}$ and $\mathbf{u}^h$.

Proposition 4. *Let Ω a bounded domain with Lipschitz boundary, $(\mathbf{u}, p)$ be the solution of the Stokes problem* (1) *with*

$$\mathbf{u} \in H^2(\Omega)^d \cap H^1_0(\Omega)^d,\ p \in H^1(\Omega) \cap L^2_0(\Omega)$$

and $(\mathbf{u}^h, p^h)$ be the finite element solution of (2). *Assume the solution (φ, λ) of the dual problem* (23) *is H^2-regular and depends stable on the right hand side, i.e.,* (24) *and* (25) *hold. Then, for a shape regular family of triangulations, the following global error estimate is valid*

$$\|g_\delta * \mathbf{u} - I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)\| \le C\Bigg[\Bigg(\sum_{T\in\Pi^h(\Omega)} \bar\eta_T^2\Bigg)^{1/2} + \|g_\delta * \mathbf{f} - \mathbf{f}^h_\delta\| + \|I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\|\Bigg] + |\theta|\, \|g_\delta * \mathbf{u} - I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)\|^{-1}$$

with

$$\theta = \int_{\partial\Omega} (\nabla\varphi \cdot \hat{\mathbf{n}}_{\partial\Omega} + \lambda\hat{\mathbf{n}}_{\partial\Omega})(g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h))ds.$$

Proof: The dual problem (23) is tested with $(g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h),$ $-(g_\delta * p - I^h_p(g_\delta * p^h))$ in Ω. Integration by parts gives

$$\begin{aligned}\|g_\delta * \mathbf{u} - I^h_{g_\delta*\mathbf{u}}(g_\delta * \mathbf{u}^h)\|^2 \\ &= (\nabla\varphi, \nabla g_\delta * \mathbf{u} - \nabla I^h_{g_\delta*\mathbf{u}}(g_\delta * \mathbf{u}^h)) + (\lambda, \nabla\cdot(g_\delta * \mathbf{u}) - \nabla\cdot(I^h_{g_\delta*\mathbf{u}}(g_\delta * \mathbf{u}^h))) \\ &\quad - (\nabla\cdot\varphi, g_\delta * p - I^h_p(g_\delta * p^h)) - \theta. \end{aligned} \tag{27}$$

The next step in the proof is the construction of an approximate Galerkin orthogonality. To this end, $(\mathbf{u}, p)$ are continued to $\mathbb{R}^d$ such that

$$\mathbf{u} \in H^2(\mathbb{R}^d)^d, p \in H^1(\mathbb{R}^d).$$

This continuation is possible ([11, Theorem 1.2.10]) and ensures that convolution and second order differentiation for $\mathbf{u}$ as well as first order differentiation for p commute. Now, $\mathbf{f}$ is continued such that

$$g_\delta * (-\Delta\mathbf{u} + \nabla p) = g_\delta * \mathbf{f} \quad \text{in } \mathbb{R}^d.$$

This equation and the corresponding mass balance is tested with $(\mathbf{v}^h, q^h) \in (X^h, M^h)$ in $\mathbb{R}^d$ where these functions are continued by zero outside Ω. One obtains

$$(g_\delta * \mathbf{f}, \mathbf{v}^h)_{\mathbb{R}^d} = (-\nabla\cdot(g_\delta * \nabla\mathbf{u}), \mathbf{v}^h)_{\mathbb{R}^d} + (\nabla(g_\delta * p), \mathbf{v}^h)_{\mathbb{R}^d} + (g_\delta * (\nabla\cdot\mathbf{u}), q^h)_{\mathbb{R}^d}.$$

From the definition of $\mathbf{v}^h$, q^h follows that the bilinear forms can be restricted to Ω. Integration by parts gives

$$(g_\delta * \mathbf{f}, \mathbf{v}^h)_\Omega = (g_\delta * \nabla\mathbf{u}, \nabla\mathbf{v}^h)_\Omega - (g_\delta * p, \nabla\cdot\mathbf{v}^h)_\Omega + (g_\delta * (\nabla\cdot\mathbf{u}), q^h)_\Omega. \tag{28}$$

Integrals on $\partial\Omega$ vanish since $\mathbf{v}^h$ vanishes on $\partial\Omega$. Subtracting (28) and (26) gives the approximate Galerkin orthogonality: $\forall(\mathbf{v}^h, q^h) \in (X^h, M^h)$

$$
\begin{aligned}
0 = {} & (g_\delta * \nabla \mathbf{u} - \nabla I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h), \nabla \mathbf{v}^h) - (g_\delta * p - I^h_p(g_\delta * p^h), \nabla \cdot \mathbf{v}^h) \\
& + (g_\delta * (\nabla \cdot \mathbf{u}) - \nabla \cdot I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h), q^h) - (g_\delta * \mathbf{f} - \mathbf{f}^h_\delta, \mathbf{v}^h).
\end{aligned}
$$

Note, in order to construct this approximate Galerkin orthogonality from (28) and (26), the discrete velocity in (26) must fulfil the same boundary conditions as $g_\delta * \mathbf{u}$, at least up to quadrature errors. That is the reason why $\mathbf{f}^h_\delta$ is defined by $I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)$ and not by $I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)$.

Choosing $\mathbf{v}^h = R_{X^h}\varphi =: \varphi^h$ and $q^h = R_{M^h}\lambda =: \lambda^h$ to be the Clément interpolants of (φ, λ) and subtracting the approximate Galerkin orthogonality from (27), we obtain

$$
\begin{aligned}
\|g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\|^2 = {} & (\nabla(g_\delta * \mathbf{u}) - \nabla I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h), \nabla(\varphi - \varphi^h)) \\
& + (\lambda - \lambda^h, \nabla \cdot (g_\delta * \mathbf{u}) - \nabla \cdot (I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h))) \\
& - (g_\delta * p - I^h_p(g_\delta * p^h), \nabla \cdot (\varphi - \varphi^h)) \\
& + (\nabla I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) - \nabla I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h), \nabla(\varphi - \varphi^h)) \\
& + (\nabla \cdot (I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)) - \nabla \cdot (I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)), \lambda - \lambda^h) \\
& + (g_\delta * \mathbf{f} - \mathbf{f}^h_\delta, \varphi^h) - \theta.
\end{aligned}
$$

The fourth and the fifth term are nonzero only on mesh cells which have a nodal point on the boundary $\partial\Omega$. The estimate of these terms proceeds by applying the Cauchy–Schwarz inequality and the interpolation estimates (11)

$$
\begin{aligned}
& (\nabla I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) - \nabla I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h), \nabla(\varphi - \varphi^h)) \\
& \quad + (\nabla \cdot (I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)) - \nabla \cdot (I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)), \lambda - \lambda^h) \\
& \quad \le C \sum_{T, \bar{T} \cap \partial\Omega \neq \emptyset} \Big(h_T \|\nabla I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) - \nabla I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\|_{0,T} |\varphi|_{2,T} \\
& \quad + h_T \|\nabla \cdot (I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)) - \nabla \cdot (I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h))\|_{0,T} |\lambda|_{1,T} \Big).
\end{aligned}
$$

Since both interpolants are discrete functions, we can apply an inverse inequality. Using once more the Cauchy–Schwarz inequality and the stability of the dual problem (25) concludes the estimate of these terms.

The sixth term is estimated with the Cauchy–Schwarz inequality and the stability of the dual problem (25).

The estimate of the first three terms proceeds with integration by parts. The boundary integrals on $\partial\Omega$ vanish since $\varphi - \varphi^h \equiv 0$ on $\partial\Omega$. We obtain:

$$
\begin{aligned}
&\|g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\|^2 \\
&\quad \le \sum_{T\in\Pi^h(\Omega)} \Bigg[(g_\delta * (-\Delta \mathbf{u} + \nabla p) - (-\Delta I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) + \nabla I^h_p(g_\delta * p^h)), \varphi - \varphi^h)_T \\
&\quad + (g_\delta * (\nabla \cdot \mathbf{u}) - \nabla \cdot I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h), \lambda - \lambda^h)_T \\
&\quad + \sum_{E\subset\partial T, E\not\subset\partial\Omega} (-\nabla(I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)) \cdot \hat{\mathbf{n}}_E + I^h_p(g_\delta * p^h)\hat{\mathbf{n}}_E, \varphi - \varphi^h)_E \Bigg] \\
&\quad + C\|I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\| \, \|g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\| \\
&\quad + C\|g_\delta * \mathbf{f} - \mathbf{f}^h_\delta\| \, \|g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\| + |\theta|.
\end{aligned}
$$

Using now $-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \nabla \cdot \mathbf{u} = 0$ in each mesh cell, the Cauchy–Schwarz inequality, the interpolation estimates (11), and the stability of the dual problem (25), give an estimate for $\|g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\|$. The estimate for $\|g_\delta * \mathbf{u} - I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h)\|$ follows by the triangle inequality. □

Remark 3. (1) The function $\mathbf{f}$ is continued in some unknown way outside Ω in the proof of Proposition 4. By the damping property of the Gaussian filter, this continuation practically does not influence $g_\delta * \mathbf{f}$ outside a δ-neighbourhood of $\partial\Omega$. Thus, in a subdomain $\Omega_0 \subset \Omega$ with $\mathrm{dist}(\partial\Omega, \partial\Omega_0) > \delta$, the function $g_\delta * \mathbf{f}$ is the right hand side of the Stokes equation with averaged continuous solution $(\mathbf{u}, p)$. Similarly, the function $\mathbf{f}^h_\delta$ is the right hand side of the Stokes equation with a polynomial approximation of the averaged discrete solution $(\mathbf{u}^h, p^h)$. If the difference of the continuous and the discrete solution is small, then the difference of their averages is small, too, e.g. see Proposition 1. Thus, the extra term $\|g_\delta * \mathbf{f} - \mathbf{f}^h_\delta\|_{0,\Omega_0}$ can be expected to be of higher order. On the other hand, $\mathbf{f}^h_\delta$ is computable and $g_\delta * \mathbf{f}$ can be approximated by a sufficiently accurate polynomial in Ω_0 such that $\|g_\delta * \mathbf{f} - \mathbf{f}^h_\delta\|_{0,\Omega_0}$ can be computed up to terms of higher order.

(2) The question of estimating the term $\|I^h_{\mathbf{u}}(g_\delta * \mathbf{u}^h) - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)\|$ is still open. The appearance of this term is closely connected to the commutation error in the derivation of the space averaged Stokes equations in bounded domains and the unresolved problem of appropriate boundary conditions for these equations, e.g. see the discussion in [10]. As a initial step, for certain common flow geometries, approximate boundary conditions can be assigned [10, 16].

As pointed out, the first two extra terms have their greatest impact near the boundary $\partial\Omega$. It is not clear if these terms are only caused by the technique to prove the estimate and vanish with the application of different (new) techniques or if the error estimator

$$
\left(\sum_{T\in\Pi^h(\Omega)} \bar{\eta}_T^2 \right)^{1/2}
$$

only controls the error away from the boundary.

(3) The boundary integral θ on $\partial\Omega$ measures the error of approximating the inhomogeneous Dirichlet boundary conditions of $g_\delta * \mathbf{u}$ by the polynomial $I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u}^h)$. The difference $g_\delta * \mathbf{u} - I^h_{g_\delta * \mathbf{u}}(g_\delta * \mathbf{u})$ vanishes in all nodal points of X^h on the boundary. Thus $|\theta|$ can be assumed to be small for sufficiently high finite element degree k of X^h.

References

[1] Ainsworth, M., Oden, J. T.: A unified approach to a posteriori error estimation using element residual methods. Numer. Math. *65*, 23–50 (1993).

[2] Aldama, A. A.: Filtering techniques for turbulent flow simulation Springer Lecture Notes in Engineering, Vol. 56. Berlin, Heidelberg, New York, Tokyo: Springer 1990.

[3] Babuška, I., Rheinboldt, W. C.: Error estimates for adaptive finite element computation. SIAM J. Numer. Anal. *15*, 283–301 (1978).

[4] Bank, R. E., Weiser, A.: Some a posteriori error estimators for elliptic partial differential equations. Math. Comp. *44*, 283–301 (1985).

[5] Becker, R., Rannacher, R.: A feed-back approach to error control in finite element methods: basic analysis and examples. East–West J. Numer. Math. *4*, 237–264 (1996).

[6] Becker, R., Rannacher, R.: Weighted a posteriori error control in fe methods. In: ENUMATH 97, Proceedings of the 2nd European Conference on Numerical Mathematics and Advanced Applications (Bock, H. G., ed.), pp. 621–637. Singapore: World Scientific 1998.

[7] Clément, Ph.: Approximation by finite element functions using local regularization. RAIRO Anal. Numer. *2*, 77–84 (1975).

[8] Deardoff, J.W.: A numerical study of 3-d turbulent channel flow at large Reynolds numbers. J. Fluid Mech. *41*, 453–480 (1970).

[9] Eriksson, K., Estep, D., Hansbo, P., Johnson, C.: Introduction to adaptive methods for differential equations. Acta Numer. 105–158 (1995).

[10] Galdi, P., Layton, W.: Approximation of the larger eddies in fluid motion ii: A model for space filtered flow. Math. Models Meth. Appl. Sci. *10*, 343–350 (2000).

[11] Grisvard, P.: Singularities in boundary value problems, Research Notes in Applied Mathematics, Vol. 22. Berlin, Heidelberg, New York, Tokyo: Masson & Springer, 1992.

[12] Gunzburger, M.: Finite element methods for viscous incompressible flows. Boston: Academic Press, 1989.

[13] Hörmander, L.: The analysis of partial differential operators I. 2nd ed Berlin, Heidelberg, New York, Tokyo: Springer, 1990.

[14] John, V.: A posteriori L^2-error estimates for the nonconforming P_1/P_0-finite element discretization of the Stokes equations. J. Comp. Appl. Math. *96*, 99–116 (1998).

[15] John, V.: A numerical study of a posteriori error estimators for convection–diffusion equations. Comput. Methods Appl. Mech. Engrg. *190*, 757–781 (2000).

[16] Sahin, N.: New perspectives on boundary conditions for large eddy simulation. Technical report, University of Pittsburgh, 2000.

[17] Verfürth, R.: A review of a posteriori error estimation and adaptive mesh-refinement techniques. New York: Wiley and Teubner, 1996.

[18] Verfürth, R.: A posteriori error estimates for nonlinear problems. L^r-estimates for finite element discretizations of elliptic equations. Math. Comp. *67*, 1335–1360 (1998).

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