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Finite element error analysis of a variational multiscale method for the Navier-Stokes equations

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The paper presents finite element error estimates of a variational multiscale method (VMS) for the incompressible Navier–Stokes equations. The constants in these estimates do not depend on the Reynolds number but on a reduced Reynolds number or on the mesh size of a coarse mesh.

Keywords: variational multiscale method, finite element method, error analysis, incompressible Navier–Stokes equations

Mathematics subject classifications (2000): 65M15, 65M60, 76F65

1. Introduction

The flow of an incompressible fluid is governed by the incompressible Navier–Stokes equations

$$
\begin{aligned}
\mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega,\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } [0,T]\times\Omega,\\
\mathbf{u} &= 0 \quad \text{in } [0,T]\times\partial\Omega,\\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0 \quad \text{in } \Omega,\\
\int_\Omega p\, d\mathbf{x} &= 0, \quad \text{in } (0,T].
\end{aligned} \tag{1}
$$

Here, $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$, is a bounded domain with boundary $\partial\Omega$, $[0,T]$ a finite time interval, $\mathbf{u}(t,\mathbf{x})$ is the velocity of the fluid and $p(t,\mathbf{x})$ the pressure. The viscosity $\nu > 0$, which is inverse proportional to the Reynolds number $Re = \mathcal{O}(\nu^{-1})$, the body forces

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$\mathbf{f}(t, \mathbf{x})$ and the initial velocity field $\mathbf{u}_0(\mathbf{x})$ are given. The velocity deformation tensor is the symmetric part of the velocity gradient $\mathbb{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$.

We are interested in the simulation of turbulent flows. Such flows are characterized by a small viscosity (or a large Reynolds number). Their characteristic feature is the richness of scales (flow structures). There are very large flow structures but also rather small ones, e.g., imagine a tornado. A direct discretization of equation (1), e.g., by a Galerkin finite element method, would try to simulate the behavior of all flow structures. This requires, at least, that all scales can be resolved by the given mesh. However, this is in general, in particular in three dimensions, far beyond the capacity of present day computers. A practical approach, which is the main goal of Large Eddy Simulation (LES), is to simulate only the behavior of large scales appropriately. Following Collis [5], the flow can be decomposed into three scales

- the (resolved) large scales,
- the (resolved) small scales,
- the unresolved scales.

An important aspect of turbulent flows is the mutual influence of all scales. Thus, simply neglecting the unresolved scales in numerical simulations would predict in general a completely wrong behavior of the resolved scales. Instead, one needs to model the influence of the unresolved scales onto the resolved ones by using a turbulence model.

It still has to be clarified how the resolved scales are defined. In the classical LES approach [14,20], they are defined as an average in space given by convolution with an appropriate filter function. A crucial assumption in the derivation of equations for the resolved scales is the commutation of convolution and differentiation. However, this property only holds in special cases. In particular, it does not hold in the case that Ω is a bounded domain. A so-called commutation error is committed if convolution and differentiation are simply interchanged nevertheless. Recent analytical results of different commutation errors, [3,4,6], and numerical studies, [23], show that this error is in general large near the boundary of the domain and it cannot be neglected. This error is one reason why numerical simulations with LES models need in general a special treatment at the boundary (wall models, van Driest damping).

In [13], Hughes proposed to define the large scales in a different way, namely by a projection into appropriate subspaces. This approach is called Variational Multiscale Method (VMS). VMS might be a remedy of the problems which LES encounters near the boundary. Based on ideas developed in [9,13], several VMS have been proposed in the literature [8,12,15]. The VMS from [15], which has as parameters a large scale space L^H consisting of symmetric tensors and a turbulent viscosity ν_T, is analyzed in the present paper. The analysis will be carried out for the case that ν_T is a positive constant. Error estimates are proven for the difference of the velocity $\mathbf{u}$ of the Navier–Stokes equations (1) and the finite element velocity $\mathbf{u}^h$ of the VMS. From the numerical point of view, the VMS introduces additional viscosity which acts directly, however, only on the (resolved) small scales of the discrete velocity. Because of this artificial viscosity, one can expect to obtain error estimates whose constants do not depend on the Reynolds number, like for the Galerkin finite element discretization of the Navier–Stokes equations, but on some reduced Reynolds number. These will be exactly the results proved in this paper. We will present two results of this kind.

One possibility of realizing the VMS considered in this paper is to define the large scale space L^H on a coarser grid. So-called two-grid methods for non-linear partial differential equations have been analyzed, e.g., in Xu [24,25] and Layton et al. [18,19]. We like to point out that the task of the coarse grid in these methods and in the VMS is completely different. In [18,19,24,25], the non-linear equation is solved solely on the coarse grid and this solution is used for solving on the fine grid only a linear equation whose solution converges still asymptotically optimal. Thus, an asymptotically optimal solution can be computed with less computational effort. The coarse grid (space) in a VMS serves for distinguishing the scales of the solution and not for reducing the computational costs. In the VMS considered in [15] and here, the non-linear equation will be solved on the fine grid. Since the turbulent viscosity might depend on the velocity, there is even an additional non-linearity in comparison to the Navier–Stokes equations.

The paper is organized as follows. In Section 2, the VMS is introduced. Section 3 provides some auxiliary results which are needed in the proofs and gives an outline of the proofs. The error estimates with constants depending on a reduced Reynolds number are presented in Sections 4 and 5.

2. The variational multiscale method

Standard notations for Lebesgue and Sobolev spaces are used throughout this paper, e.g., see Adams [1]. The inner product in $(L^2(\Omega))^d$, $d \in \mathbb{N}$, is denoted by $(\cdot,\cdot)$. Generic constants which do not depend on the Reynolds number Re and the mesh width h are denoted by C.

Let $V = (H_0^1(\Omega))^d$ equipped with the norm $\|\mathbf{v}\|_V = \|\nabla \mathbf{v}\|_{L^2}$ and $Q = L_0^2(\Omega)$. A variational formulation of the Navier–Stokes equations (1) is as follows: Find $\mathbf{u} : [0,T] \to V,\ p : (0,T] \to Q$ satisfying

$$
\begin{aligned}
(\mathbf{u}_t,\mathbf{v}) + (2\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v})) + b_s(\mathbf{u},\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) &= (\mathbf{f},\mathbf{v}) \quad \forall\, \mathbf{v}\in V,\\
(q,\nabla\cdot\mathbf{u}) &= 0 \qquad \forall\, q\in Q
\end{aligned} \tag{2}
$$

and $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \in V$. Here,

$$
b_s(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\left(((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})\right)
$$

is the skew-symmetric form of the convective term.

A Galerkin finite element discretization of (2) is unstable in the case of small viscosity (or high Reynolds number). The use of a turbulence model becomes necessary which should model the action of the unresolved scales onto the resolved scales and which serves as stabilization in a numerical simulation. We consider an approach which was presented in [15], see also [16] for this approach in the context of scalar convection–diffusion equations.

We will study the continuous-in-time finite element approach, i.e., only a semi-discretization in space is considered. Let $V^h \subset V$ and $Q^h \subset Q$ be conforming finite element spaces which fulfill the inf-sup condition

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in V^h} \frac{(\nabla \cdot \mathbf{v}^h, q^h)}{\|\nabla \mathbf{v}^h\|_{L^2} \|q^h\|_{L^2}} \geq C > 0, \tag{3}$$

where C is independent of h. For the VMS, a large scale space $L^H \subset L = \{\mathbb{L} \in (L^2(\Omega))^{d \times d}, \mathbb{L} = \mathbb{L}^T\}$ and a so-called turbulent viscosity $\nu_T = \nu_T(t, \mathbf{x}, \mathbf{u}^h, p^h) \geq 0$ are introduced. The semi-discrete problem reads as follows: Find $\mathbf{u}^h : [0,T] \to V^h$, $p^h : (0,T] \to Q^h$ and $\mathbb{G}^H : [0,T] \to L^H$ satisfying

$$\begin{aligned}
(\mathbf{u}_t^h, \mathbf{v}^h) + (2\nu \mathbb{D}(\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) + b_s(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) & & & \\
- (p^h, \nabla \cdot \mathbf{v}^h) + (\nu_T(\mathbb{D}(\mathbf{u}^h) - \mathbb{G}^H), \mathbb{D}(\mathbf{v}^h)) &= (\mathbf{f}, \mathbf{v}^h) & \forall\, \mathbf{v}^h \in V^h, \\
(q^h, \nabla \cdot \mathbf{u}^h) &= 0 & \forall\, q^h \in Q^h, \\
(\mathbb{G}^H - \mathbb{D}(\mathbf{u}^h), \mathbb{L}^H) &= 0 & \forall\, \mathbb{L}^H \in L^H
\end{aligned} \tag{4}$$

and $\mathbf{u}^h(0, \mathbf{x}) = \mathbf{u}_0^h \in V^h$ is a discretely divergence-free approximation of $\mathbf{u}_0$.

The parameters in (4) are the large scale space L^H and the turbulent viscosity ν_T. In the last term on the left hand side in the first equation, the turbulent viscosity is added to the difference of the deformation tensor of all resolved scales and the deformation tensor of the large scales. The large scales are defined in the last equation of (4) by projection of the resolved scales onto L^H. Altogether, ν_T acts directly only on the resolved small scales. This is the basic idea of a VMS, see [15] for a discussion of the connection of (4) to the VMS proposed in [12]. The implementation of the VMS (4) into a code for solving the Navier–Stokes equations and numerical results are presented in [15].

Consider now the limit cases for L^H:

- $L^H = \mathbb{D}(V^h)$, that means, $\mathbb{D}(\mathbf{v}^h) \in L^H$ for all $\mathbf{v}^h \in V^h$ and if $\mathbb{L}^H \in L^H$ then exists a $\mathbf{v}^h \in V^h$ such that $\mathbb{L}^H = \mathbb{D}(\mathbf{v}^h)$. In this case, $\mathbb{D}(\mathbf{u}^h) \in L^H$, $\mathbb{G}^H = \mathbb{D}(\mathbf{u}^h)$ and the turbulence model is subtracted for all scales. System (4) becomes a Galerkin finite element discretization of the Navier–Stokes equations.
- $L^H = \{\mathbb{O}\}$. Then, $\mathbb{G}^H = \mathbb{O}$ and the model ν_T acts on all resolved scales.

We require $L^H \subseteq \mathbb{D}(V^h)$. If this relation is violated, the turbulence model ν_T would be subtracted from scales where it was not added previously.

Let $P_{L^H} \; : \; L \to L^H, \; \mathbb{D}(\mathbf{v}) \to P_{L^H}\mathbb{D}(\mathbf{v})$ with

$$(P_{L^H}\mathbb{D}(\mathbf{v}) - \mathbb{D}(\mathbf{v}), \mathbb{L}^H) = 0 \quad \forall\, \mathbb{L}^H \in L^H \tag{5}$$

denote the L^2-projection from L onto L^H. Then $\mathbb{G}^H = P_{L^H}\mathbb{D}(\mathbf{u}^h)$ in (4). Since P_{L^H} is an L^2-projection, it follows for $\mathbf{v} \in V$ and $\|\mathbb{D}(\mathbf{v})\|_{L^2} > 0$

$$\begin{aligned}
\nu_T\|(I - P_{L^H})\mathbb{D}(\mathbf{v})\|_{L^2}^2 &= \nu_T\Big(\|\mathbb{D}(\mathbf{v})\|_{L^2}^2 - \|P_{L^H}\mathbb{D}(\mathbf{v})\|_{L^2}^2\Big) \\
&= \nu_T\left(1 - \frac{\|P_{L^H}\mathbb{D}(\mathbf{v})\|_{L^2}^2}{\|\mathbb{D}(\mathbf{v})\|_{L^2}^2}\right)\|\mathbb{D}(\mathbf{v})\|_{L^2}^2 \\
&=: \nu_{\text{add}}(\mathbf{v})\|\mathbb{D}(\mathbf{v})\|_{L^2}^2.
\end{aligned} \tag{6}$$

In addition, from $0 \le \|P_{L^H}\mathbb{D}(\mathbf{v})\|_{L^2} \le \|\mathbb{D}(\mathbf{v})\|_{L^2}$ follows

$$0 \le \nu_{\text{add}}(\mathbf{v}) \le \nu_T. \tag{7}$$

Note, if $\mathbf{v}$ depends on t then $\nu_{\text{add}}(\mathbf{v})$, too. From (7) follows $\nu_{\text{add}}(\mathbf{v}(t,\cdot)) \in L^\infty(0,T)$ if ν_T is bounded almost everywhere in the time interval. If $\|\mathbb{D}(\mathbf{v})\|_{L^2} = 0$ then $\mathbf{v} = \mathbf{0}$ since $\mathbf{v} \in V$. In this case, we set $\nu_{\text{add}}(\mathbf{v}) = 0$.

In this paper, we consider the case that ν_T is a non-negative constant. A straightforward calculation shows that

$$\begin{aligned}
&(\nu_T\mathbb{D}(\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) - (\nu_T P_{L^H}\mathbb{D}(\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) \\
&\quad = (\nu_T(I - P_{L^H})\mathbb{D}(\mathbf{u}^h), (I - P_{L^H})\mathbb{D}(\mathbf{v}^h)).
\end{aligned} \tag{8}$$

Thus, System (4) can be reformulated as follows: Find $\mathbf{u}^h \; : \; [0,T] \to V^h, \; p^h \; : \; (0,T] \to Q^h$ satisfying

$$\begin{aligned}
(\mathbf{u}_t^h, \mathbf{v}^h) + (2\nu\mathbb{D}(\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) \qquad & \\
+\, b_s(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - (p^h, \nabla\cdot\mathbf{v}^h) \qquad & \\
+\, (\nu_T(I - P_{L^H})\mathbb{D}(\mathbf{u}^h), (I - P_{L^H})\mathbb{D}(\mathbf{v}^h)) &= (\mathbf{f}, \mathbf{v}^h) \quad \forall\, \mathbf{v}^h \in V^h, \\
(q^h, \nabla\cdot\mathbf{u}^h) &= 0 \qquad\quad \forall\, q^h \in Q^h.
\end{aligned} \tag{9}$$

Let $V_{\text{div}}^h = \{\mathbf{v}^h \in V^h \; : \; (\nabla\cdot\mathbf{v}^h, q^h) = 0 \;\forall\, q^h \in Q^h\}$ the space of discretely divergence free functions. From the inf-sup condition (3) follows that this space is not empty. Then, (9) is equivalent to: Find $\mathbf{u}^h : \; [0,T] \to V_{\text{div}}^h$ such that

$$\begin{aligned}
&(\mathbf{u}_t^h, \mathbf{v}^h) + (2\nu\mathbb{D}(\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) + b_s(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) \\
&\qquad + (\nu_T(I - P_{L^H})\mathbb{D}(\mathbf{u}^h), (I - P_{L^H})\mathbb{D}(\mathbf{v}^h)) = (\mathbf{f}^h, \mathbf{v}^h)
\end{aligned} \tag{10}$$

for all $\mathbf{v}^h \in V_{\text{div}}^h$.

The finite element spaces (V^h, Q^h) contain all resolved scales. Let $V^H \in (H^1(\Omega))^d$ be a finite element space such that $L^H = \mathbb{D}(V^H)$. The space V^H should be coarser than V^h. The functions of V^H may be defined on the same grid as the functions of V^h with a lower piecewise polynomial degree or on a coarser grid. But no boundary conditions, like no-slip conditions, are incorporated in the definition of V^H, i.e., in general $V^H \not\subset V^h$. The large eddies of a turbulent flow generally do not fulfill no-slip boundary conditions, e.g., the large eddies of a tornado move along the surface of the earth instead of sticking on the surface. The pair of spaces for the resolved large scales is given by (V^H, Q^H). Here, Q^H is chosen such that an inf-sup condition of type (3) is fulfilled for (V^H, Q^H). The large scales $P_H\mathbf{u}$ of the velocity are defined by an elliptic projection into V^H and the large scales $P_H p$ of the pressure by the L^2-projection into Q^H; $P_H : (V, Q) \to (V^H, Q^H)$

$$\begin{aligned} (\mathbb{D}(\mathbf{u} - P_H\mathbf{u}), \mathbb{D}(\mathbf{v}^H)) &= 0 \quad \forall\ \mathbf{v}^H \in V^H, \\ (\mathbf{u} - P_H\mathbf{u}, 1) &= 0, \\ (p - P_H p, q^H) &= 0 \quad \forall\ q^H \in Q^H. \end{aligned}$$

It was proven in [15] that for $L^H = \mathbb{D}(V^H)$ holds

$$P_{L^H}\mathbb{D}(\mathbf{v}) = \mathbb{D}(P_H\mathbf{v}) \quad \forall\ \mathbf{v} \in V. \tag{11}$$

That means, the definition of the large scales by projection and differentiation commute. This property does not hold in general for the classical LES.

3. Preliminaries and the outline of the proofs

The following sections present a finite element error analysis for the space discrete velocity solution of (10). For simplicity, we assume that the characteristic length scale and velocity scale in the Navier–Stokes equations are chosen such that $Re = \nu^{-1}$. Considering the limit cases of the choice of L^H in Section 2, one has:

- $L^H = \mathbb{D}(V^h)$: a finite element error estimate which constants depending on $2Re$ or $(2\nu)^{-1}$, e.g., see Heywood and Rannacher [10,11];
- $L^H = \{\mathbb{O}\}$: a finite element error estimate where the most constants, in particular the constant in the dominating exponential factor, depend on $Re_{\nu_T} := (2\nu + \nu_T)^{-1}$, e.g., see the analysis for the Smagorinsky turbulence model in [14,17]. Since in our analysis, the finite element solution of the VMS is compared to the solution of the continuous Navier–Stokes equations (2), some constants may depend on $2Re$ instead on Re_{ν_T}.

The limit cases lead to the expectation that if L^H is chosen in between them allows finite element error estimates with constants depending on a reduced Reynolds number Re_{red} with

$$Re_{\nu_T} < Re_{\mathrm{red}} < 2Re. \tag{12}$$

In the following sections, such error estimates will be derived for the case that ν_T is constant.

For the finite element error analysis, we need some assumptions on the regularity of the parameters of the Navier–Stokes equations and the solution. We assume that

$$\mathbf{f} \in (L^2(0,T;L^2))^d, \quad \mathbf{u}_0 \in V, \tag{13}$$

and that (2) possesses a solution $(\mathbf{u}, p)$ with

$$\nabla\mathbf{u} \in (L^4(0,T;L^2))^{d\times d},\ \mathbf{u}_t \in (L^2(0,T;H^{-1}))^d,\ p \in L^4(0,T;L^2). \tag{14}$$

Note, these assumptions imply that Serrin's condition is fulfilled from what follows that the solution of (2) is unique, e.g., see Temam [22], Galdi [7] or Sohr [21]. For simplicity let $\mathbf{f} = \mathbf{f}^h$. In addition, we assume that Ω has a polygonal (in 2d) or polyhedral (in 3d) boundary such that no boundary approximation in the application of the finite element method becomes necessary.

Inequalities which will be used frequently are Young's inequality

$$ab \leq \frac{t}{p}a^p + \frac{t^{-q/p}}{q}b^q, \quad a,b,p,q,t \in \mathbb{R}, \quad \frac{1}{p}+\frac{1}{q} = 1, \quad p,q \in (1,\infty), \quad t > 0, \tag{15}$$

Poincaré's inequality in V

$$\|\mathbf{v}\|_{L^2} \leq C\|\nabla\mathbf{v}\|_{L^2} \quad \forall\ \mathbf{v} \in V \tag{16}$$

and Korn's inequality in V

$$\|\nabla\mathbf{v}\|_{L^2} \leq C\|\mathbb{D}(\mathbf{v})\|_{L^2} \quad \forall\ \mathbf{v} \in V. \tag{17}$$

The proof of the finite element error estimate uses an approach by Rannacher and Heywood [10,11]. We will first give an outline:

1. Prove stability of $\mathbf{u}$ and $\mathbf{u}^h$, i.e., certain norms of $\mathbf{u}$ and $\mathbf{u}^h$ are bounded a priori by the data of the problem: $\mathbf{f}, \mathbf{u}_0, \nu$.

2. Derive an error equation by subtracting (10) from (2) for test functions from V^h_{div}. Split the error into an approximation term $\boldsymbol{\eta}$ and a (finite element) remainder $\boldsymbol{\phi}^h$

$$\mathbf{e} = \mathbf{u} - \mathbf{u}^h = (\mathbf{u} - \tilde{\mathbf{u}}^h) - (\mathbf{u}^h - \tilde{\mathbf{u}}^h) =: \boldsymbol{\eta} - \boldsymbol{\phi}^h, \tag{18}$$

where $\tilde{\mathbf{u}}^h \in V^h_{\mathrm{div}}$ is a projection of $\mathbf{u}$ which fulfills certain interpolation estimates. An example for such a projection is the Stokes projection. Then, take $\boldsymbol{\phi}^h$ as test function in the error equation.

3. Estimate the right hand side of the error equation such that one obtains an inequality of the form

$$\frac{d}{dt}\|\boldsymbol{\phi}^h\|^2_{L^2} + g_1(t,\boldsymbol{\phi}^h) \le g_2(t,\boldsymbol{\eta},\mathbf{u}) + g_3(t,\mathbf{u})\|\boldsymbol{\phi}^h\|^2_{L^2}, \tag{19}$$

where all functions are non-negative for almost all $t \in [0,T]$.

4. Show that Gronwall's lemma can be applied to (19), i.e., show that all functions in (19) belong to $L^1(0,T)$. Apply Gronwall's lemma to get an estimate for $\boldsymbol{\phi}^h$.

5. Prove the error estimate for $\mathbf{e}$ by applying the triangle inequality to (18).

Along these lines, two estimates with constants depending on a reduced Reynolds number will be proved.

4. First error estimate with constants depending on a reduced Reynolds number

This error estimate uses the parameter ν_{add} defined in (6). We will first prove the stability of $\mathbf{u}$ and $\mathbf{u}^h$.

Lemma 4.1. The solution $\mathbf{u}^h$ of the finite element problem (4) fulfills $\mathbf{u}^h \in (L^\infty(0,T;L^2))^d$ and $\mathbb{D}(\mathbf{u}^h) \in (L^2(0,T;L^2))^{d\times d}$. The velocity solution of the continuous problem (2) fulfills $\mathbf{u} \in (L^\infty(0,T;L^2))^d$ and $\mathbb{D}(\mathbf{u}) \in (L^2(0,T;L^2))^{d\times d}$.

Proof. The proof for $\mathbf{u}^h$ and $\mathbf{u}$ is very similar. We will show the result for $\mathbf{u}^h$. Set $\mathbf{v}^h = \mathbf{u}^h$ in (10), use the skew symmetry of $b_s(\cdot,\cdot,\cdot)$, (6), the standard estimate of the dual pairing, Korn's inequality (17) and integrate over $(0,t)$ with $t \le T$:

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}^h(t)\|^2_{L^2} + \int_0^t (2\nu + \nu_{\mathrm{add}}(\mathbf{u}^h(\tau)))\|\mathbb{D}(\mathbf{u}^h)(\tau)\|^2_{L^2} d\tau \\
&\quad \le \frac{1}{2}\|\mathbf{u}^h_0\|^2_{L^2} + \int_0^t \|\mathbf{f}(\tau)\|_{H^{-1}}\|\nabla\mathbf{u}^h(\tau)\|_{L^2} d\tau \\
&\quad \le \frac{1}{2}\|\mathbf{u}^h_0\|^2_{L^2} + \frac{C}{\nu}\|\mathbf{f}\|^2_{L^2(0,t;\,H^{-1})} + \int_0^t \frac{2\nu + \nu_{\mathrm{add}}(\mathbf{u}^h(\tau))}{2}\|\mathbb{D}(\mathbf{u}^h)(\tau)\|^2_{L^2} d\tau.
\end{aligned}$$

Subtraction of the last term and the regularity (13) of the coefficients give $\mathbb{D}(\mathbf{u}^h) \in (L^2(0,T;L^2))^{d\times d}$. Taking then the supremum of $t \in (0,T)$ gives the statement $\mathbf{u}^h \in (L^\infty(0,T;L^2))^d$. □

The splitting of the error (18) is performed with the help of a projection $\tilde{\mathbf{u}}^h \in V^h_{\mathrm{div}}$ of $\mathbf{u}$. Let $t \in [0,T]$ be arbitrary. We require that this projection fulfills

$$\|\boldsymbol{\eta}\|_{L^2} + h\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2} \le Ch^k(\|\mathbf{u}(t,\cdot)\|_{H^k} + \nu^{-1}\|p(t,\cdot)\|_{H^{k-1}}), \tag{20}$$

$$\|\boldsymbol{\eta}_t\|_{L^2} + h\|(\mathbb{D}(\boldsymbol{\eta}))_t\|_{L^2} \le Ch^k(\|\mathbf{u}(t,\cdot)\|_{H^k} + \nu^{-1}\|p(t,\cdot)\|_{H^{k-1}}), \tag{21}$$

where the constants depend only on Ω. Korn's inequality (17), (20) with $k = 1$ and the regularity assumptions (14) imply

$$\nabla\boldsymbol{\eta} \in (L^4(0,T;L^2))^{d\times d}. \tag{22}$$

An example for an appropriate projection is the Stokes projection which is the solution of: Find $\tilde{\mathbf{u}}^h \in V^h_{\mathrm{div}}$ such that

$$(2\nu\mathbb{D}(\mathbf{u}(t,\cdot) - \tilde{\mathbf{u}}^h), \mathbb{D}(\mathbf{v}^h)) = (p(t,\cdot), \nabla\cdot\mathbf{v}^h) \quad \forall\ \mathbf{v}^h \in V^h_{\mathrm{div}}.$$

Let $\mathbf{u}(t,\cdot) \in (H^k(\Omega))^d, p(t,\cdot) \in H^{k-1}(\Omega)$, $k \ge 1$ and V^h possess a $(k-1)$-th order approximation property, e.g., V^h is the finite element space P^{k-1} on simplicial meshes or Q^{k-1} on quadrilateral/hexahedral meshes, then a simple scaling argument of Lemma 5.3. in [11] gives (20), (21). For $t = 0$, the pressure can be well defined, e.g., see [10,22].

Now, Step 2 of the proof is carried out by a straightforward calculation. One obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\phi}^h\|^2_{L^2} + (2\nu + \nu_{\mathrm{add}}(\boldsymbol{\phi}^h))\|\mathbb{D}(\boldsymbol{\phi}^h)\|^2_{L^2}\\
&\quad = (\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) + (2\nu\mathbb{D}(\boldsymbol{\eta}), \mathbb{D}(\boldsymbol{\phi}^h)) + (\nu_T(I - P_{L^H})\mathbb{D}(\boldsymbol{\eta}), (I - P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h))\\
&\qquad + b_s(\mathbf{u},\mathbf{u},\boldsymbol{\phi}^h) - b_s(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\phi}^h) - (\nu_T(I - P_{L^H})\mathbb{D}(\mathbf{u}), (I - P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h))\\
&\qquad - (p - \lambda^h, \nabla\cdot\boldsymbol{\phi}^h)
\end{aligned} \tag{23}$$

with arbitrary $\lambda^h \in Q^h$.

To get an inequality of form (19), the terms on the right hand side of (23) have to be estimated. All bilinear terms are estimated essentially in the same way: using the Cauchy-Schwarz inequality (or the estimate for the dual pairing), Korn's inequality (17) and Young's inequality (15). In addition, (6) is used. One obtains

$$
\begin{aligned}
(\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) &\le \|\boldsymbol{\eta}_t\|_{H^{-1}} \|\nabla \boldsymbol{\phi}^h\|_{L^2} \le C \|\boldsymbol{\eta}_t\|_{H^{-1}} \|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} \\
&\le \frac{2\nu + \nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}{8} \|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + \frac{C}{2\nu + \nu_{\mathrm{add}}(\boldsymbol{\phi}^h)} \|\boldsymbol{\eta}_t\|_{H^{-1}}^2, \\
(2\nu \mathbb{D}(\boldsymbol{\eta}), \mathbb{D}(\boldsymbol{\phi}^h)) &\le 2\nu \|\mathbb{D}(\boldsymbol{\eta})\|_{L^2} \|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} \\
&\le \frac{\nu}{8} \|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + 8\nu \|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2, \\
(p - \lambda^h, \nabla \cdot \boldsymbol{\phi}^h) &\le \|p - \lambda^h\|_{L^2} \|\nabla \cdot \boldsymbol{\phi}^h\|_{L^2} \le C \|p - \lambda^h\|_{L^2} \|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} \\
&\le \frac{2\nu + \nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}{8} \|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + \frac{C}{2\nu + \nu_{\mathrm{add}}(\boldsymbol{\phi}^h)} \|p - \lambda^h\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
&(\nu_T (I - P_{L^H}) \mathbb{D}(\boldsymbol{\eta}), (I - P_{L^H}) \mathbb{D}(\boldsymbol{\phi}^h)) \\
&\quad \le \frac{\nu_T}{16} \|(I - P_{L^H}) \mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + 4\nu_T \|(I - P_{L^H}) \mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2 \\
&\quad = \frac{\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}{16} \|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + 4\nu_{\mathrm{add}}(\boldsymbol{\eta}) \|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2, \\
&(\nu_T (I - P_{L^H}) \mathbb{D}(\mathbf{u}), (I - P_{L^H}) \mathbb{D}(\boldsymbol{\phi}^h)) \\
&\quad \le \nu_T \|(I - P_{L^H}) \mathbb{D}(\mathbf{u})\|_{L^2} \|(I - P_{L^H}) \mathbb{D}(\boldsymbol{\phi})\|_{L^2} \\
&\quad = \frac{\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}{16} \|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + 4\nu_T \|(I - P_{L^H}) \mathbb{D}(\mathbf{u})\|_{L^2}^2.
\end{aligned}
$$

The trilinear term is first decomposed into three terms. A direct calculation gives

$$
b_s(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}^h) - b_s(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\phi}^h) = b_s(\boldsymbol{\eta}, \mathbf{u}, \boldsymbol{\phi}^h) - b_s(\boldsymbol{\phi}^h, \mathbf{u}, \boldsymbol{\phi}^h) + b_s(\mathbf{u}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h).
$$

The terms on the right hand side are estimated separately using the inequality

$$
b_s(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le C \|\mathbf{u}\|_{L^2}^{1/2} \|\mathbb{D}(\mathbf{u})\|_{L^2}^{1/2} \|\mathbb{D}(\mathbf{v})\|_{L^2} \|\mathbb{D}(\mathbf{w})\|_{L^2}. \tag{24}
$$

This estimate is well known. It can be derived by applying Hölder's inequality, Sobolev imbeddings, interpolation theorems in Sobolev spaces, Poincaré's and Korn's inequality, e.g., see Layton and Tobiska [19]. One obtains by applying (24) and Young's inequality (15)

$$\begin{aligned}
&b_s(\boldsymbol{\eta}, \mathbf{u}, \boldsymbol{\phi}^h)\\
&\le C\|\boldsymbol{\eta}\|_{L^2}^{1/2}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^{1/2}\|\mathbb{D}(\mathbf{u})\|_{L^2}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}\\
&\le \frac{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}{8}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + \frac{C}{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}\|\boldsymbol{\eta}\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}\|\mathbb{D}(\mathbf{u})\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
&b_s(\boldsymbol{\phi}^h, \mathbf{u}, \boldsymbol{\phi}^h)\\
&\le C\|\boldsymbol{\phi}^h\|_{L^2}^{1/2}\|\mathbb{D}(\mathbf{u})\|_{L^2}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^{3/2}\\
&\le \frac{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}{8}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + \frac{C}{(2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h))^3}\|\boldsymbol{\phi}^h\|_{L^2}^2\|\mathbb{D}(\mathbf{u})\|_{L^2}^4,
\end{aligned}$$

$$\begin{aligned}
&b_s(\mathbf{u}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h)\\
&\le C\|\mathbf{u}^h\|_{L^2}^{1/2}\|\mathbb{D}(\mathbf{u}^h)\|_{L^2}^{1/2}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}\\
&\le \frac{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}{8}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + \frac{C}{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}\|\mathbf{u}^h\|_{L^2}\|\mathbb{D}(\mathbf{u}^h)\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2.
\end{aligned}$$

Collecting terms gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\phi}^h\|_{L^2}^2 + \frac{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}{4}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2\\
&\quad\le \Bigg[\frac{C}{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}\|\boldsymbol{\eta}_t\|_{H^{-1}}^2 + (8\nu + 4\nu_{\mathrm{add}}(\boldsymbol{\eta}))\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2\\
&\qquad + \frac{C}{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}\|p-\lambda^h\|_{L^2}^2 + 4\nu_T\|(I-P_{L^H})\mathbb{D}(\mathbf{u})\|_{L^2}^2\\
&\qquad + \frac{C}{2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h)}\Big(\|\boldsymbol{\eta}\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}\|\mathbb{D}(\mathbf{u})\|_{L^2}^2\\
&\qquad + \|\mathbf{u}^h\|_{L^2}\|\mathbb{D}(\mathbf{u}^h)\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2\Big)\Bigg]\\
&\qquad + \left[\frac{C}{(2\nu+\nu_{\mathrm{add}}(\boldsymbol{\phi}^h))^3}\|\mathbb{D}(\mathbf{u})\|_{L^2}^4\right]\|\boldsymbol{\phi}^h\|_{L^2}^2.
\end{aligned}$$

We define

$$Re_{\text{red}} := \Big(2\nu + \inf_{t\in(0,T]} \nu_{\text{add}}(\boldsymbol{\phi}^h(t))\Big)^{-1}. \tag{25}$$

It follows that Re_{red} is smaller or equal than $2Re$. Using $\nu_{\text{add}}(\boldsymbol{\eta}) \le \nu_T$ finishes Step 3 of the proof:

$$\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{\phi}^h\|_{L^2}^2 + \frac{Re_{\text{red}}^{-1}}{2}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 \\
&\le C\Big[Re_{\text{red}}\|\boldsymbol{\eta}_t\|_{H^{-1}}^2 + (Re^{-1}+\nu_T)\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2 + Re_{\text{red}}\|p-\lambda^h\|_{L^2}^2 \\
&\quad + \nu_T\|(I-P_{L^H})\mathbb{D}(\mathbf{u})\|_{L^2}^2 \\
&\quad + Re_{\text{red}}\Big(\|\boldsymbol{\eta}\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}\|\mathbb{D}(\mathbf{u})\|_{L^2}^2 + \|\mathbf{u}^h\|_{L^2}\|\mathbb{D}(\mathbf{u}^h)\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2\Big)\Big] \\
&\quad + CRe_{\text{red}}\|\mathbb{D}(\mathbf{u})\|_{L^2}^4\|\boldsymbol{\phi}^h\|_{L^2}^2.
\end{aligned} \tag{26}$$

To perform Step 4 of the proof, the $L^1(0,T)$-regularity of the terms appearing in (26) has to be studied. Let $t \in (0,T]$ be arbitrary. We have by Poincaré's inequality (16), Korn's inequality (17), the Cauchy–Schwarz inequality, (14) and (22)

$$\begin{aligned}
&\int_0^t \|\boldsymbol{\eta}(\tau)\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})(\tau)\|_{L^2}\|\mathbb{D}(\mathbf{u})(\tau)\|_{L^2}^2 d\tau \\
&\le C\int_0^t \|\mathbb{D}(\boldsymbol{\eta})(\tau)\|_{L^2}^2\|\mathbb{D}(\mathbf{u})(\tau)\|_{L^2}^2 d\tau \\
&\le C\|\mathbb{D}(\boldsymbol{\eta})\|_{L^4(0,t;L^2)}^2\|\mathbb{D}(\mathbf{u})\|_{L^4(0,t;L^2)}^2 < \infty.
\end{aligned}$$

Similarly follows by Hölders inequality, Lemma 4.1 and (22)

$$\begin{aligned}
&\int_0^t \|\mathbf{u}^h(\tau)\|_{L^2}\|\mathbb{D}(\mathbf{u}^h)(\tau)\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})(\tau)\|_{L^2}^2 d\tau \\
&\le \|\mathbf{u}^h\|_{L^\infty(0,t;L^2)}\int_0^t \|\mathbb{D}(\mathbf{u}^h)(\tau)\|_{L^2}\|\mathbb{D}(\boldsymbol{\eta})(\tau)\|_{L^2}^2 d\tau \\
&\le \|\mathbf{u}^h\|_{L^\infty(0,t;L^2)}\|\mathbb{D}(\mathbf{u}^h)\|_{L^2(0,t;L^2)}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^4(0,t;L^2)}^2 \\
&\le C\Big(Re^{1/2}\|\mathbf{u}_0^h\|_{L^2}^2 + Re^{3/2}\|\mathbf{f}\|_{L^2(0,t;H^{-1})}^2\Big)\|\mathbb{D}(\boldsymbol{\eta})\|_{L^4(0,t;L^2)}^2 < \infty.
\end{aligned}$$

The $L^1(0,T)$-regularity of the other terms is a direct consequence of (14), (20), (21) and (22).

The application of Gronwall's inequality and the last step of the proof are straightforward.

Theorem 4.2. Let $(\mathbf{u},p) \in V \times Q$ be the solution of (2) and let $\mathbf{u}^h \in V^h_{\text{div}}$ be the solution of (10) where $\nu_T \geq 0$ is a constant. Let the regularity assumptions (14) be fulfilled, $\tilde{\mathbf{u}}^h$ be a projection of $\mathbf{u}$ into V^h_{div} such that $\boldsymbol{\eta} = \mathbf{u} - \tilde{\mathbf{u}}^h$ fulfills (20) and (21). Let the reduced Reynolds number Re_{red} be defined in (25). Then, the error $\mathbf{u} - \mathbf{u}^h$ satisfies for $T \geq 0$

$$
\begin{aligned}
&\|(\mathbf{u}-\mathbf{u}^h)(T)\|_{L^2}^2 + \frac{Re_{\text{red}}^{-1}}{2}\|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_{L^2(0,T;L^2)}^2 \\
&\leq C \inf_{\substack{\tilde{\mathbf{u}}^h \in L^4(0,T;V^h_{div}) \\ \lambda^h \in L^2(0,T;Q^h)}} \Bigg\{ \|(\mathbf{u}-\tilde{\mathbf{u}}^h)(T)\|_{L^2}^2 + Re_{\text{red}}^{-1}\|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}^h)\|_{L^2(0,T;L^2)}^2 \\
&\quad + \exp\Big(C Re_{\text{red}}^3 \|\mathbb{D}(\mathbf{u})\|_{L^4(0,T;L^2)}^4\Big) \Bigg[\|\mathbf{u}_0 - \mathbf{u}_0^h\|_{L^2}^2 + \|\mathbf{u}_0 - \tilde{\mathbf{u}}^h(0)\|_{L^2}^2 \\
&\quad + Re_{\text{red}} \Big[\|(\mathbf{u}-\tilde{\mathbf{u}}^h)_t\|_{L^2(0,T;H^{-1})}^2 + \|p-\lambda^h\|_{L^2(0,T;L^2)}^2 \\
&\quad + \|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}^h)\|_{L^4(0,t;L^2)}^2 \|\mathbb{D}(\mathbf{u})\|_{L^4(0,t;L^2)}^2 \\
&\quad + \Big(Re^{1/2}\|\mathbf{u}_0^h\|_{L^2}^2 + Re^{3/2}\|\mathbf{f}\|_{L^2(0,t;H^{-1})}^2 \Big) \|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}^h)\|_{L^4(0,t;L^2)}^2 \Big] \\
&\quad + (Re^{-1} + \nu_T)\|\mathbb{D}(\mathbf{u}-\tilde{\mathbf{u}}^h)\|_{L^2(0,T;L^2)}^2 \\
&\quad + \nu_T \|(I - P_{L^H})\mathbb{D}(\mathbf{u})\|_{L^2(0,T;L^2)}^2 \Bigg] \Bigg\}. \qquad (27)
\end{aligned}
$$

Remark 4.1. Let us consider the convergence of the right hand side of (27). The crucial term is the last one on this side since it does not possess a factor where the interpolation error $\mathbf{u} - \tilde{\mathbf{u}}^h$ appears. This term tends to zero as the mesh width $h \to 0$ if $\nu_T \to 0$ or if L^H tends to $\mathbb{D}(V)$. In both cases, the Galerkin finite element discretization of the Navier-Stokes equations is recovered asymptotically. Otherwise, in particular if ν_T and L^H are fixed and $h \to 0$, one cannot expect that the solution of the discrete system converges to the solution of the continuous problem.

Let $(\mathbf{u},p) \in (H^{k+1}(\Omega))^d \times H^k(\Omega)$ for all times, $k \geq 1$, let the velocity finite element space be of piecewise order k and let the pressure finite element space be of piecewise order $k-1$. Then, the optimal order of convergence of the left hand side of (27) is h^k where h is the mesh parameter connected with the space V^h. Again, the crucial term on the right hand side is the last one. Its optimal order of convergence is H^k. Here, H is the mesh parameter connected with L^H. Depending on the ratio of h and H, the artificial viscosity ν_T can be chosen in such a way that the last term on the right hand side of (27) does not spoil the order of convergence of the estimate.

For fixed h and $\nu_T \to 0$, the estimate in Theorem 4.2 tends to the estimate for the Galerkin finite element discretization of the Navier–Stokes equations.

Remark 4.2. There is no improvement in the constant in the exponential, i.e. $Re_{\text{red}} = 2Re$, if there is a time t at which $\nu_{\text{add}}(\boldsymbol{\phi}^h(t)) = 0$. This is equivalent to $\|P_{L^H}\mathbb{D}(\boldsymbol{\phi}^h(t))\|_{L^2}^2 = \|\mathbb{D}(\boldsymbol{\phi}^h(t))\|_{L^2}^2$ or

$$(I - P_{L^H})\mathbb{D}(\mathbf{u}^h) = (I - P_{L^H})\mathbb{D}(\tilde{\mathbf{u}}^h). \tag{28}$$

That means, the resolved small scales of $\mathbf{u}^h$ and $\tilde{\mathbf{u}}^h$ are the same. However, this situation is unlikely for turbulent flows since these scales of $\mathbf{u}^h$ are considerably influenced by the model which is used for the unresolvable small scales whereas the interpolation $\tilde{\mathbf{u}}^h$ does not posses any information about this model, e.g., if $\tilde{\mathbf{u}}^h$ is defined by the Stokes projection which is asymsptotically optimal. In this case, (28) is only likely if there are solely large scales in the flow, which is not the case in turbulent flows. From the mathematical point of view, the difficulty consists in the fact that the equations for laminar flows and turbulent flows are the same, namely the Navier–Stokes (1). Since the analysis is carried out for (1), it is not possible to distinguish between the two kinds of flows and the results must also hold for the case of laminar flows. For such flows, $\nu_{\text{add}}(\boldsymbol{\phi}^h(t))$ may vanish and the error estimates of [10,11] are recovered in which the constants depend on Re.

Remark 4.3. A finite element error estimate for the $L^2(\Omega)$-error in the pressure can also be derived following Heywood and Rannacher [11], Section 7. Using the inf-sup condition (3) and the estimates for the Stokes projection (20) and (21), the pressure error can be estimated by approximation errors and the velocity error $\|(\mathbf{u} - \mathbf{u}^h)(t)\|_{L^2}$. Then, the result of Theorem 4.2 finishes the error estimate for the pressure. Since the analysis is lengthy and follows closely [11], we will not present it here.

5. Second error estimate with constants depending on a reduced Reynolds number

This section will present an error estimate with a mesh-dependent reduced Reynolds number. This reduced Reynolds number will be considerably smaller than Re if $\nu_T \gg \nu$ and if the mesh width H connected with the space L^H is also much larger than ν. This is the typical situation in turbulent flow simulations.

The starting point for this error estimate is the error equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\phi}^h\|_{L^2}^2 + 2\nu\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + \nu_T\|(I - P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 \\
&\quad = (\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) + (2\nu\mathbb{D}(\boldsymbol{\eta}), \mathbb{D}(\boldsymbol{\phi}^h)) + (\nu_T(I - P_{L^H})\mathbb{D}(\boldsymbol{\eta}), (I - P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)) \\
&\qquad + b_s(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}^h) - b_s(\mathbf{u}^h, \mathbf{u}^h, \boldsymbol{\phi}^h) - (\nu_T(I - P_{L^H})\mathbb{D}(\mathbf{u}), (I - P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)) \\
&\qquad - (p - \lambda^h, \nabla\cdot\boldsymbol{\phi}^h)
\end{aligned} \tag{29}$$

with arbitrary $\lambda^h \in Q^h$ which follows directly by subtracting (10) from (2) and taking $\boldsymbol{\phi}^h \in V^h_{\mathrm{div}}$ as test function.

We assume that the condition $L^H = \mathbb{D}(V^H)$ holds and that the finite element spaces V^h, V^H rely on quasiuniform triangulations of Ω. The first assumption was needed to prove the commutation of differentiation and the definition of the large scales by projection. From the latter assumption follows that inverse estimates for finite element functions hold.

Starting with (11) and applying the inverse estimate for V^H gives

$$\|P_{L^H}\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} = \|\mathbb{D}(P_H\boldsymbol{\phi}^h)\|_{L^2} \le CH^{-1}\|P_H\boldsymbol{\phi}^h\|_{L^2}, \tag{30}$$

where H is the mesh parameter connected with V^H. We assume now that the elliptic projection is L^2-stable for functions from V^h_{div}, i.e., there is a constant C such that

$$\|P_H\boldsymbol{\phi}^h\|_{L^2} \le C\|\boldsymbol{\phi}^h\|_{L^2} \quad \forall\ \boldsymbol{\phi}^h \in V^h_{\mathrm{div}}. \tag{31}$$

Together with (30), this gives

$$\|P_{L^H}\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} \le CH^{-1}\|\boldsymbol{\phi}^h\|_{L^2} \quad \forall\ \boldsymbol{\phi}^h \in V^h_{\mathrm{div}}. \tag{32}$$

Assumption (31) is true, e.g., for quasiuniform meshes. From Babuška and Osborn [2 equation (6.6)] follows $\|P_H\boldsymbol{\phi}^h\|_{L^2,h} \le C\|\boldsymbol{\phi}^h\|_{L^2,h}$ for a mesh-dependent norm $\|\cdot\|_{L^2,h}$. For finite element functions, this mesh-dependent norm is equivalent to $\|\cdot\|_{L^2}$, from which the L^2-stability of the elliptic projection is obtained immediately.

The first step in the estimate of the terms on the right hand side of (29) is the same as in the derivation of the error estimate leading to Theorem 4.2. But then, the term $\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}$ is estimated further using the triangle inequality and (32)

$$\begin{aligned}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} &\le \|(I-P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} + \|P_{L^H}\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}\\ &\le \|(I-P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} + CH^{-1}\|\boldsymbol{\phi}^h\|_{L^2}.\end{aligned}$$

This gives, e.g.,

$$\begin{aligned}(\boldsymbol{\eta}_t,\boldsymbol{\phi}^h) &\le \|\boldsymbol{\eta}_t\|_{H^{-1}}\|(I-P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2} + CH^{-1}\|\boldsymbol{\eta}_t\|_{H^{-1}}\|\boldsymbol{\phi}^h\|_{L^2}\\ &\le C\Big(\max\{\nu,\nu_{\mathrm{T}}\}^{-1} + H^{-2}\Big)\|\boldsymbol{\eta}_t\|^2_{H^{-1}} + \|\boldsymbol{\phi}^h\|^2_{L^2}\\ &\quad + \frac{\max\{\nu,\nu_{\mathrm{T}}\}}{64}\|(I-P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|^2_{L^2}.\end{aligned} \tag{33}$$

The last term on the right hand side of this estimate has to be hidden on the left hand side of (29). If $\nu_T \ge \nu$, then it is absorbed by the third term on the left hand side. In the case $\nu_T < \nu$, we obtain by using the triangle inequality and the $H^1(\Omega)$-stability of the elliptic projection with constant 1

$$\begin{aligned}\frac{\max\{\nu,\nu_{\mathrm{T}}\}}{64}\|(I-P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 &\le \frac{3\nu}{64}\left(\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2+\|P_{L^H}\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2\right)\\ &\le \frac{3\nu}{64}\left(\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2+\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2\right)\\ &= \frac{3\nu}{32}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2.\end{aligned}$$

In this case, this term is absorbed by the second term on the left hand side of (29).

The trilinear term which determines the most important coefficient in the exponential factor of the final estimate can be estimated in the following way

$$\begin{aligned}&b_s(\boldsymbol{\phi}^h,\mathbf{u},\boldsymbol{\phi}^h)\\ &\le C\|\boldsymbol{\phi}^h\|_{L^2}^{1/2}\|\mathbb{D}(\mathbf{u})\|_{L^2}\left(\|(I-P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^{3/2}+\|P_{L^H}\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^{3/2}\right)\\ &\le C\left(\nu_T^{-3}\|\mathbb{D}(\mathbf{u})\|_{L^2}^4+H^{-3/2}\|\mathbb{D}(\mathbf{u})\|_{L^2}\right)\|\boldsymbol{\phi}^h\|_{L^2}^2\\ &\quad+\frac{\nu_T}{64}\|(I-P_{L^H})\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2\end{aligned}$$

or in the standard way

$$b_s(\boldsymbol{\phi}^h,\mathbf{u},\boldsymbol{\phi}^h)\le\frac{\nu}{64}\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2+C\nu^{-3}\|\mathbb{D}(\mathbf{u})\|_{L^2}^4\|\boldsymbol{\phi}^h\|_{L^2}^2.$$

For the final estimate of the VMS, we will take the estimate which gives the smaller constant.

All other terms in (29) are estimated in the same fashion as (33). The conditions which allow the application of Gronwall's lemma are checked in the same way as for the error estimate given in Theorem 4.2. The second error estimate is formulated in the following theorem.

Theorem 5.1. Let $(\mathbf{u},p)\in V\times Q$ be the solution of (2) and let $\mathbf{u}^h\in V_{\mathrm{div}}^h$ be the solution of (10) where $\nu_T\ge 0$ is a constant. Let the regularity assumptions (14) be fulfilled, $\tilde{\mathbf{u}}^h$ be a projection of $\mathbf{u}$ into V_{div}^h such that $\boldsymbol{\eta}=\mathbf{u}-\tilde{\mathbf{u}}^h$ fulfills (20) and (21).

Let $L^H = \mathbb{D}(V^H)$, let V^h and V^H be defined on quasiuniform triangulations of Ω and let (31) be fulfilled. Then, the error $\mathbf{u} - \mathbf{u}^h$ satisfies for $T \geq 0$

$$
\begin{aligned}
&\|(\mathbf{u}-\mathbf{u}^h)(T)\|_{L^2}^2 + \frac{Re^{-1}}{16}\|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_{L^2(0,T;L^2)}^2 \\
&\quad + \frac{\nu_T}{16}\|(I-P_{L^H})\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_{L^2}^2 \\
&\leq C \inf_{\substack{\tilde{\mathbf{u}}^h \in L^4(0,T;V^h_{\mathrm{div}}) \\ \lambda^h \in L^2(0,T;Q^h)}} \Bigg\{ \|(\mathbf{u}-\mathbf{u}^h)(T)\|_{L^2}^2 + Re_{\mathrm{red}}^{-1}\|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_{L^2(0,T;L^2)}^2 \\
&\qquad + \exp\Bigg(4T + C\min\Big\{\nu^{-3}\|\mathbb{D}(\mathbf{u})\|_{L^4(0,T;L^2)}^4, \\
&\qquad\qquad \nu_T^{-3}\|\mathbb{D}(\mathbf{u})\|_{L^4(0,T;L^2)}^4 + H^{-3/2}\|\mathbb{D}(\mathbf{u})\|_{L^1(0,T;L^2)}\Big\}\Bigg) \\
&\qquad \times \Bigg[\|\mathbf{u}_0-\mathbf{u}_0^h\|_{L^2}^2 + \|\mathbf{u}_0-\tilde{\mathbf{u}}^h(0)\|_{L^2}^2 + Re_{\mathrm{red}}\Big[\|(\mathbf{u}-\mathbf{u}^h)_t\|_{L^2(0,T;H^{-1})}^2 \\
&\qquad + \|p-\lambda^h\|_{L^2(0,T;L^2)}^2 + \|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_{L^4(0,t;L^2)}^2\|\mathbb{D}(\mathbf{u})\|_{L^4(0,t;L^2)}^2 \\
&\qquad + \Big(Re^{1/2}\|\mathbf{u}_0^h\|_{L^2}^2 + Re^{3/2}\|\mathbf{f}\|_{L^2(0,t;H^{-1})}^2\Big)\|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_{L^4(0,t;L^2)}^2\Big] \\
&\qquad + (Re^{-1}+\nu_T)\|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_{L^2(0,T;L^2)}^2 + \nu_T\|(I-P_{L^H})\mathbb{D}(\mathbf{u})\|_{L^2(0,T;L^2)}^2\Bigg]\Bigg\}
\end{aligned}
\tag{34}
$$

where

$$
Re_{\mathrm{red}} = \max\{\nu, \nu_T\}^{-1} + H^{-2}. \tag{35}
$$

Remark 5.1. The viscosity ν is very small for turbulent flows. To have a stabilizing effect by using an artificial viscosity, one has to choose $\nu_T > \nu$, in general even $\nu_T \gg \nu$. Hence $\max\{\nu, \nu_T\}^{-1} = \nu_T^{-1}$ in (35). Then, the reduced Reynolds number (35) is not dominated by the mesh size if

$$
\nu_T \leq H^2. \tag{36}
$$

Making the same assumptions and considerations on the convergence of the individual terms in (34) as in Remark 4.1, one finds that the second term on the left hand side of (34) behaves like h^k, the third term on the left hand side like $\nu_T h^k$ and the last term on the right hand side like $\nu_T H^k$. Given h and either H or ν_T, one can choose by equilibrating these orders of convergence an appropriate value for the remaining parameter, taking into account also restriction (36).

Let ν be small, $\nu \ll h \leq H$, let h be fixed and $\nu_T \to 0$. Then, Re_{red} is very close to $\nu^{-1} = Re$ and the constants in estimate (34) have the same dependency on Re as for the Galerkin finite element discretization of the Navier–Stokes equations.

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