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AN INF-SUP STABLE AND RESIDUAL-FREE BUBBLE ELEMENT FOR THE OSEEN EQUATIONS*

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Abstract. We investigate the residual-free bubble method for the linearized incompressible Navier–Stokes equations. Starting with a nonconforming inf-sup stable element pair for approximating the velocity and pressure, we enrich the velocity space by discretely divergence-free bubble functions to handle the influence of strong convection. An important feature of the method is that the stabilization does not generate an additional coupling between the mass equation and the momentum equation as is the case for the streamline upwind Petrov–Galerkin method applied to equal-order interpolation. Furthermore, the discrete solution is piecewise divergence-free, a property which is useful for the mass balance in transport equations coupled with the incompressible Navier–Stokes equations.

Key words. stabilized finite elements, Navier–Stokes equations, nonconforming finite elements

AMS subject classifications. 65N12, 65N30, 65N15

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1. Introduction.

Finite element approximations of the Oseen equations need stability for advective-dominated flows and compatibility between the velocity and pressure spaces. The latter is also necessary for the Stokes flow.

Starting with the streamline upwind Petrov–Galerkin (SUPG) stabilization of Brooks and Hughes [9] for the advective term, this idea has been extended to the Stokes equations in [21], where a stabilized method is proposed accommodating low equal-order interpolation to be stable and convergent. This formulation circumvents the need to abide by inf-sup condition for many interpolations. In an attempt to get the stability features of these works, a method is proposed in [14] that at the same time is advective stable and overcomes the inf-sup restrictions of the standard Galerkin method. The analysis of these SUPG-type stabilizations, including the case of equal-order interpolations, can be found in [31]. The drawback of these methods is that various terms need to be added to the weak formulation. Residual-based stabilization methods which use inf-sup stable pairs of elements reduce the number of terms which have to be added to the Galerkin formulation [17, 25]. However, there is still a strong coupling of the form $(\nabla p, (\mathbf{b}\cdot\nabla)\mathbf{v}_h)$ which is difficult to handle, and an optimal L^2 error estimate for the pressure is missing in [17]. Several attempts have been made to relax the strong coupling of velocity and pressure and to introduce symmetric versions of the stabilizing terms; for an overview see [5]. Local projection-type methods have

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been introduced for the Stokes problem in [2], extended to the transport equation in [3], and analyzed for low-order discretizations of the Oseen equations in [4]. They are designed for equal-order interpolation and allow a separation of the velocity and pressure in the stabilization terms. The disadvantage is that the finite element stencil is less compact than for the SUPG-type stabilization. They also suffer from the weak fulfillment of the incompressiblity constraint which is important for mass conservation in a transport equation coupled with the Navier–Stokes problem. In the edge-oriented stabilization technique, introduced in [10], we find the same problem of a much wider stencil which needs also some special data structure or an implicit defect correction.

Our method of enriching the velocity space of an inf-sup stable pair of finite elements by discretely divergence-free functions will always suppress additional coupling terms in the discrete formulation and lead to a separation of the velocity and pressure in the stabilization terms. Due to the use of inf-sup stable finite element pairs, the computed velocity field is always discretely divergence-free. As a first step in this paper, we analyze the simplest version of such an enrichment method, the Crouzeix–Raviart element of lowest order, i.e., piecewise linear nonconforming velocity and piecewise constant pressure approximations.

The plan of the paper is as follows. In section 2, the weak formulation of the Oseen equations and its Galerkin discretization is considered. Next, in section 3, we apply the residual-free bubble approach and highlight the advantages of using discretely divergence-free enrichments. The relation to the classical SUPG method is studied in section 4. Finally, an a priori error estimate for an approximate residual-free bubble method is derived in section 5. A numerical test example confirms the convergence rates.

Notations. We use the Sobolev spaces $W^{k,p}(D)$, $H^k(D) = W^{k,2}(D)$, $H_0^k(D)$, $L^2(D) = H^0(D)$, and write $\mathbf{W}^{k,p}(D)$, $\mathbf{H}^k(D)$, $\mathbf{H}_0^k(D)$, $\mathbf{L}^2(D)$ for their vector-valued versions. The norms and seminorms in the scalar and vector-valued versions of $W^{k,p}(D)$ are denoted by $\|\cdot\|_{k,p,D}$ and $|\cdot|_{k,p,D}$, respectively [12]. To simplify the notation, we drop D if $D = \Omega$ and p if $p = 2$. Moreover, we introduce the broken H^1 seminorm and norm for piecewise H^1 functions defined on a triangulation $\mathcal{T}_h$ by

$$|v|_{1,h} := \left(\sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 \right)^{1/2}, \qquad \|v\|_{1,h} := \left(|v|_{1,h}^2 + \|v\|_0^2 \right)^{1/2}.$$

2. A linearized Navier–Stokes model. We consider the steady linearized Navier–Stokes model given by

$$-\nu \Delta \mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega \subset \mathbb{R}^d, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = \mathbf{0} \qquad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma = \partial\Omega, \tag{2.3}$$

where $\mathbf{b} \in \mathbf{W}^{1,\infty}(\Omega)$ with $\nabla \cdot \mathbf{b} = 0$ in Ω, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and Ω denotes a bounded domain in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. Homogeneous Dirichlet boundary conditions are considered for simplicity of presentation. The extension to nonhomogeneous Dirichlet boundary conditions is straightforward when the boundary data are interpolated in the space of restrictions of discretely divergence-free functions. For smooth boundary data, this is always possible and requires only additional technical details which do not lead to further insight into the method. The weak formulation of (2.1)–(2.3) reads:

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Find $(\mathbf{u},p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that for all $(\mathbf{v},q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$,

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + (q, \nabla\cdot\mathbf{u}) = (\mathbf{f},\mathbf{v}), \tag{2.4}$$

where the bilinear forms a and b are defined by

$$a(\mathbf{u},\mathbf{v}) := \nu(\nabla\mathbf{u},\nabla\mathbf{v}), \quad b(\mathbf{u},\mathbf{v}) := ((\mathbf{b}\cdot\nabla)\mathbf{u},\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.5}$$

$(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or its vector-valued and tensor-valued versions, and

$$L_0^2(\Omega) = \{q \in L^2(\Omega) \,:\, (q,1) = 0\}.$$

The property

$$b(\mathbf{v},\mathbf{v}) = ((\mathbf{b}\cdot\nabla)\mathbf{v},\mathbf{v}) = \frac{1}{2}(\mathbf{b}\cdot\nabla(\mathbf{v}\cdot\mathbf{v}),1) = -\frac{1}{2}(\nabla\cdot\mathbf{b},\mathbf{v}\cdot\mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$

of the bilinear form b guarantees that the Lax–Milgram lemma can be applied in the subspace of divergence-free functions. A unique pressure in $L_0^2(\Omega)$ follows from the Babuška–Brezzi condition for the pair $(\mathbf{H}_0^1(\Omega), L_0^2(\Omega))$ [18]. Therefore, there is a unique solution $(\mathbf{u},p)$ of (2.4) for all positive ν.

For the finite element approximation, we use the nonconforming P_1^{nc}/P_0 element pair of Crouzeix–Raviart [13]. Let $\mathcal{T}_h$ be a regular decomposition of the domain $\Omega \subset \mathbb{R}^d$ into d-dimensional simplices $K \in \mathcal{T}_h$, where the mesh parameter h represents the maximum diameter of the elements $K \in \mathcal{T}_h$. We denote by $\mathcal{E}_h$ the set of all $(d-1)$-dimensional faces E of cells $K \in \mathcal{T}_h$. We choose for any face $E \in \mathcal{E}_h$ a unit normal $\mathbf{n}_E$ with an arbitrary but fixed orientation where $\mathbf{n}_E$ on boundary faces is the outer unit normal of Ω. We will write $\mathbf{n}_K$ for the outer unit normal with respect to the cell K. For a scalar piecewise continuous function ψ, the jump $[\psi]_E$ and the average $\{\psi\}_E$ on a face E are defined by

$$[\psi]_E := \begin{cases} (\psi|_K)|_E - (\psi|_{\widetilde{K}})|_E & \text{if} \quad E \not\subset \Gamma, \\ (\psi|_K)|_E & \text{if} \quad E \subset \Gamma, \end{cases}$$

$$\{\psi\}_E := \begin{cases} \frac{1}{2}\left((\psi|_K)|_E + (\psi|_{\widetilde{K}})|_E\right) & \text{if} \quad E \not\subset \Gamma, \\ \frac{1}{2}(\psi|_K)|_E & \text{if} \quad E \subset \Gamma, \end{cases}$$

where K and $\widetilde{K}$ are chosen such that $E = \partial K \cap \partial\widetilde{K}$ and $\mathbf{n}_K = \mathbf{n}_E$.

Note that the definition of the jump and the average on a boundary face corresponds to that on an inner face when extending the functions outside of Ω by zero. Furthermore, we have the relation

$$[\varphi\psi]_E = [\varphi]_E\{\psi\}_E + \{\varphi\}_E[\psi]_E$$

on both inner and boundary faces E. The jump and the average of vector-valued functions are defined in a componentwise manner.

Now our approximate spaces $\mathbf{V}_h \approx \mathbf{H}_0^1(\Omega)$ and $Q_h \approx L_0^2(\Omega)$ can be defined to be

$$\mathbf{V}_h := \left\{\mathbf{v}_h \in \mathbf{L}^2(\Omega) \,:\, \mathbf{v}_h\Big|_K \in P_1(K)^d \;\forall K \in \mathcal{T}_h,\; \int_E [\mathbf{v}_h]_E \, d\gamma = 0 \;\forall E \in \mathcal{E}_h\right\}, \tag{2.6}$$

$$Q_h := \left\{q_h \in L_0^2(\Omega) \,:\, q_h\Big|_K \in P_0(K) \;\forall K \in \mathcal{T}_h\right\}, \tag{2.7}$$

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where $P_n(K)$ is the set of all polynomials on K of degree less than or equal to n. Note that a function $\mathbf{v}_h \in \mathbf{V}_h$—in general—is discontinuous across the inner faces E and does not vanish on the boundary.

Now, we introduce the discrete bilinear forms elementwise to be

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \nu \sum_{K\in\mathcal{T}_h} (\nabla_h \mathbf{u}_h, \nabla_h \mathbf{v}_h)_K, \tag{2.8}$$

$$b_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{K\in\mathcal{T}_h} ((\mathbf{b}\cdot\nabla_h)\mathbf{u}_h, \mathbf{v}_h)_K - \sum_{E\in\mathcal{E}_h} \langle \mathbf{b}\cdot\mathbf{n}_E[\mathbf{u}_h]_E, \{\mathbf{v}_h\}_E\rangle_E\,. \tag{2.9}$$

Here, the discrete versions of the gradient and the divergence operators, ∇ and $\nabla\cdot$, respectively, are understood in the following sense:

$$(\nabla_h \mathbf{v}_h)\big|_K := \nabla\left(\mathbf{v}_h\big|_K\right) \quad \forall \mathbf{v}_h \in \mathbf{V}_h,\ \forall K\in\mathcal{T}_h,$$

$$(\nabla_h \cdot \mathbf{v}_h)\big|_K := \nabla\cdot\left(\mathbf{v}_h\big|_K\right) \quad \forall \mathbf{v}_h \in \mathbf{V}_h,\ \forall K\in\mathcal{T}_h,$$

and $\langle\cdot,\cdot\rangle_E$ denotes the inner product in $L^2(E)$ and its vector-valued versions. To simplify the notation, we briefly write ∇ instead of ∇_h in expressions like (2.8) and (2.9). Clearly, we have

$$a_h(\mathbf{u},\mathbf{v}) = a(\mathbf{u},\mathbf{v}), \quad b_h(\mathbf{u},\mathbf{v}) = b(\mathbf{u},\mathbf{v}), \quad \mathbf{u},\mathbf{v}\in\mathbf{H}^1(\Omega).$$

The additional term in the elementwise-defined bilinear form b_h (compare (2.9)) vanishes for $\mathbf{v}_h \in \mathbf{H}^1(\Omega)$. For functions $\mathbf{v}_h$ belonging to our nonconforming finite element space $\mathbf{V}_h$, it guarantees that we have

$$\begin{aligned} b_h(\mathbf{v}_h,\mathbf{v}_h) &= \frac{1}{2}\sum_{K\in\mathcal{T}_h} ((\mathbf{b}\cdot\nabla)(\mathbf{v}_h\cdot\mathbf{v}_h), 1)_K - \sum_{E\in\mathcal{E}_h} \langle \mathbf{b}\cdot\mathbf{n}_E[\mathbf{v}_h]_E, \{\mathbf{v}_h\}_E\rangle_E \\ &= \sum_{E\in\mathcal{E}_h}\left(\frac{1}{2}\langle \mathbf{b}\cdot\mathbf{n}_E[\mathbf{v}_h\cdot\mathbf{v}_h]_E, 1\rangle_E - \langle \mathbf{b}\cdot\mathbf{n}_E[\mathbf{v}_h]_E, \{\mathbf{v}_h\}_E\rangle_E\right) = 0, \end{aligned}$$

in analogy to $b(\mathbf{v},\mathbf{v}) = 0$ for all $\mathbf{v}\in\mathbf{H}^1_0(\Omega)$.

The standard Galerkin finite element method reads:

Find $(\mathbf{u}_h, p_h)\in \mathbf{V}_h\times Q_h$ such that for all $(\mathbf{v}_h, q_h)\in\mathbf{V}_h\times Q_h$,

$$a_h(\mathbf{u}_h,\mathbf{v}_h) + b_h(\mathbf{u}_h,\mathbf{v}_h) - (p_h, \nabla_h\cdot\mathbf{v}_h) + (q_h, \nabla_h\cdot\mathbf{u}_h) = (\mathbf{f},\mathbf{v}_h). \tag{2.10}$$

The finite element pair $(\mathbf{V}_h, Q_h)$ satisfies the discrete inf-sup stability condition

$$\exists\beta_0 > 0 \quad \forall q_h\in Q_h: \qquad \beta_0\|q_h\|_0 \le \sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{(q_h, \nabla_h\cdot\mathbf{v}_h)}{|\mathbf{v}_h|_{1,h}}; \tag{2.11}$$

see [6, 13]. As a result, we have the unique solvability of (2.10). Error estimates which do not take into consideration the size of ν are standard, e.g., in the energy norm we have

$$\nu^{1/2}|\mathbf{u}-\mathbf{u}_h|_{1,h} + \|p-p_h\|_0 \le C(\nu)\ h\ (|u|_2 + |p|_1) \tag{2.12}$$

with a constant $C(\nu)$ depending on ν. We are interested in the case of small ν (high Reynolds numbers) in which numerical experiments show the need for stabilization [11, 29, 30]. In the next section, we will follow the concept of residual-free bubble stabilizations, which has been already successfully applied to scalar convection-diffusion equations [1, 7, 8, 16].

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3. Residual-free bubble method. Let us enrich the velocity space $\mathbf{V}_h$ by the space of residual-free bubbles

$$\mathbf{B}_h := \bigoplus_{K\in\mathcal{T}_h} \mathbf{H}_0^1(K)$$

and denote the enriched space by $\mathbf{V}_{RFB}$. Since a piecewise linear function which vanishes at the boundary of each cell is identically zero, we conclude $\mathbf{V}_{RFB} = \mathbf{V}_h \oplus \mathbf{B}_h$. The pair $(\mathbf{V}_{RFB}, Q_h)$ satisfies the discrete inf-sup stability (2.11) as well. Note that a function from the bubble space $\mathbf{B}_h$ is discretely divergence-free since we have, for all $q_h \in Q_h,\ \mathbf{v}_B \in \mathbf{B}_h$,

$$(q_h, \nabla_h \cdot \mathbf{v}_B) = \sum_{K\in\mathcal{T}_h} q_h\big|_K (1, \nabla \cdot \mathbf{v}_B)_K = \sum_{K\in\mathcal{T}_h} q_h\big|_K \langle 1, \mathbf{v}_B \cdot \mathbf{n}_K\rangle_{\partial K} = 0.$$

In this sense the inf-sup stability will not be improved by enriching $\mathbf{V}_h$ by $\mathbf{B}_h$. Each element $\mathbf{u}_{RFB} \in \mathbf{V}_{RFB}$ can be uniquely represented in the form

$$\mathbf{u}_{RFB} = \mathbf{u}_h + \mathbf{u}_B \quad \text{with } \mathbf{u}_h \in \mathbf{V}_h,\ \mathbf{u}_B \in \mathbf{B}_h.$$

The Galerkin approximation of (2.4) with respect to the pair $(\mathbf{V}_{RFB}, Q_h)$ reads:

Find $(\mathbf{u}_h, \mathbf{u}_B, p_h) \in \mathbf{V}_h \times \mathbf{B}_h \times Q_h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{u}_B, \mathbf{v}_h) - (p_h, \nabla_h \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.1}$$

$$a_h(\mathbf{u}_B, \mathbf{v}_B) + b_h(\mathbf{u}_B, \mathbf{v}_B) + b_h(\mathbf{u}_h, \mathbf{v}_B) = (\mathbf{f}, \mathbf{v}_B) \quad \forall \mathbf{v}_B \in \mathbf{B}_h, \tag{3.2}$$

$$(q_h, \nabla_h \cdot \mathbf{u}_h) = 0 \qquad \forall q_h \in Q_h. \tag{3.3}$$

Note that in deriving (3.1)–(3.3) we have taken into consideration the orthogonality property

$$\begin{aligned} a_h(\mathbf{v}_B, \mathbf{w}_h) = a_h(\mathbf{w}_h, \mathbf{v}_B) &= \nu \sum_{K\in\mathcal{T}_h} (\nabla \mathbf{w}_h, \nabla \mathbf{v}_B)_K \\ &= \nu \sum_{K\in\mathcal{T}_h} \left(\left\langle \frac{\partial \mathbf{w}_h}{\partial \mathbf{n}_K}, \mathbf{v}_B \right\rangle_{\partial K} - (\Delta \mathbf{w}_h, \mathbf{v}_B)_K \right) = 0 \end{aligned}$$

and the property that $\mathbf{u}_B$ and $\mathbf{v}_B$ are discretely divergence-free. Equation (3.2) can be considered to define $\mathbf{u}_B$ as a functional of $\mathbf{u}_h$. In order to find a representation for $\mathbf{u}_B$, we define $M(\mathbf{u}_h)$, $F(\mathbf{f}) \in \mathbf{B}_h$ as the solutions of the problems:

Find $M(\mathbf{u}_h),\ F(\mathbf{f}) \in \mathbf{B}_h$ such that for all $\mathbf{v}_B \in \mathbf{B}_h$,

$$a_h(M(\mathbf{u}_h), \mathbf{v}_B) + b_h(M(\mathbf{u}_h), \mathbf{v}_B) = -b_h(\mathbf{u}_h, \mathbf{v}_B),$$

$$a_h(F(\mathbf{f}), \mathbf{v}_B) + b_h(F(\mathbf{f}), \mathbf{v}_B) = (\mathbf{f}, \mathbf{v}_B).$$

Then, the solution $\mathbf{u}_B$ of (3.2) can be represented in the form $\mathbf{u}_B = M(\mathbf{u}_h) + F(\mathbf{f})$. Elimination of $\mathbf{u}_B$ from (3.1) gives the residual-free bubble method for solving (2.4):

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$a_{RFB}(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla_h \cdot \mathbf{v}_h) = l_{RFB}(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.4}$$

$$(q_h, \nabla_h \cdot \mathbf{u}_h) = 0 \qquad \forall q_h \in Q_h, \tag{3.5}$$

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where

$$a_{RFB}(\mathbf{u}_h, \mathbf{v}_h) = a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(M(\mathbf{u}_h), \mathbf{v}_h), \tag{3.6}$$

$$l_{RFB}(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - b_h(F(\mathbf{f}), \mathbf{v}_h). \tag{3.7}$$

The difficulty in realizing the *exact* residual-free method (3.4)–(3.5) is that we have to evaluate the terms $b_h(M(\mathbf{u}_h), \mathbf{v}_h)$ and $b_h(F(\mathbf{f}), \mathbf{v}_h)$, which essentially means solving an infinite-dimensional problem. Therefore, in practice some sort of approximation is used. We mention in particular the following approaches:

- stabilizing subgrid methods [8],
- pseudo-residual-free bubble method [7],
- two-level and three-level approaches [15, 16, 19, 20].

In the following we will reformulate the method (3.4)–(3.5) by looking at the constant coefficient case.

4. Relation to other stabilized methods. The case of continuous P_1 pressure and velocity approximations on triangles has been considered in [28]; for a systematic study on quadrilaterals with a continuous Q_1 pressure approximation and a sufficiently large velocity space see [24]. In that paper the fully nonlinear case of the Navier–Stokes equations has also been considered.

In the following we consider a discretization within the space $(\mathbf{V}_h \times Q_h)$, i.e., nonconforming piecewise linear velocity and piecewise constant pressure approximations. Let us assume that $\mathbf{b}$ and $\mathbf{f}$ are constants. Moreover, let $\varphi_K \in H_0^1(K)$ be the solution of the scalar convection-diffusion problem

$$-\nu\Delta\varphi_K + \mathbf{b}\cdot\nabla\varphi_K = 1 \text{ in } K, \quad \varphi_K = 0 \text{ on } \partial K.$$

Then, we obtain

$$M(\mathbf{u}_h)\Big|_K = -(\mathbf{b}\cdot\nabla)\mathbf{u}_h\Big|_K \varphi_K, \quad F(\mathbf{f})\Big|_K = \mathbf{f}\Big|_K \varphi_K.$$

The terms which appear in (3.6)–(3.7), in addition to the standard Galerkin approach, become

$$\begin{aligned} b_h(M(\mathbf{u}_h), \mathbf{v}_h) &= \sum_{K\in\mathcal{T}_h} ((\mathbf{b}\cdot\nabla)M(\mathbf{u}_h), \mathbf{v}_h)_K - \sum_{E\in\mathcal{E}_h} \langle \mathbf{b}\cdot\mathbf{n}_E[M(\mathbf{u}_h)]_E, \{\mathbf{v}_h\}_E\rangle_E \\ &= -\sum_{K\in\mathcal{T}_h} ((\mathbf{b}\cdot\nabla)\mathbf{v}_h, M(\mathbf{u}_h))_K \\ &= \sum_{K\in\mathcal{T}_h} ((\mathbf{b}\cdot\nabla)\mathbf{v}_h, (\mathbf{b}\cdot\nabla)\mathbf{u}_h\ \varphi_K)_K \\ &= \sum_{K\in\mathcal{T}_h} \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v}_h, (\mathbf{b}\cdot\nabla)\mathbf{u}_h)_K, \end{aligned}$$

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$$\begin{aligned}
-b_h(F(\mathbf{f}),\mathbf{v}_h) &= -\sum_{K\in\mathcal{T}_h}((\mathbf{b}\cdot\nabla)F(\mathbf{f}),\mathbf{v}_h)_K + \sum_{E\in\mathcal{E}_h}\langle \mathbf{b}\cdot\mathbf{n}_E[F(\mathbf{f})]_E,\{\mathbf{v}_h\}_E\rangle_E \\
&= \sum_{K\in\mathcal{T}_h}((\mathbf{b}\cdot\nabla)\mathbf{v}_h, F(\mathbf{f}))_K \\
&= \sum_{K\in\mathcal{T}_h}((\mathbf{b}\cdot\nabla)\mathbf{v}_h, \mathbf{f}\,\varphi_K)_K \\
&= \sum_{K\in\mathcal{T}_h}\tau_K((\mathbf{b}\cdot\nabla)\mathbf{v}_h, \mathbf{f})_K,
\end{aligned}$$

since $M(\mathbf{u}_h),\ F(\mathbf{f})\in\mathbf{B}_h$ where

$$\tau_K = \frac{1}{|K|}\int_K \varphi_K\,dx.$$

Thus, the exact residual-free bubble method for constant $\mathbf{b}$ and $\mathbf{f}$ is equal to:

Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ such that

$$\tilde{a}_{RFB}(\mathbf{u}_h,\mathbf{v}_h) - (p_h,\nabla_h\cdot\mathbf{v}_h) = \tilde{l}_{RFB}(\mathbf{v}_h) \qquad \forall \mathbf{v}_h\in\mathbf{V}_h, \tag{4.1}$$

$$(q_h,\nabla_h\cdot\mathbf{u}_h) = 0 \qquad \forall q_h\in Q_h, \tag{4.2}$$

where

$$\tilde{a}_{RFB}(\mathbf{u}_h,\mathbf{v}_h) = a_h(\mathbf{u}_h,\mathbf{v}_h) + b_h(\mathbf{u}_h,\mathbf{v}_h) + \sum_{K\in\mathcal{T}_h}\tau_K((\mathbf{b}\cdot\nabla)\mathbf{u}_h,(\mathbf{b}\cdot\nabla)\mathbf{v}_h)_K,$$

$$\tilde{l}_{RFB}(\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h) + \sum_{K\in\mathcal{T}_h}\tau_K(\mathbf{f},(\mathbf{b}\cdot\nabla)\mathbf{v}_h)_K.$$

Since on each $K\in\mathcal{T}_h$ it holds that $-\nu\Delta\mathbf{u}_h+\nabla p_h=0$, the method corresponds to the SUPG method analyzed in [26] for the fully nonlinear case of the Navier–Stokes equations. However, the influence of small ν on the error constants has not been investigated in that paper.

5. Error estimate for the generalized Oseen equations. We now turn to estimates with Reynolds-number-independent constants. It has been shown in a series of papers [22, 23, 27] that for nonconforming finite element discretizations applied to scalar convection-diffusion equations, one has to add certain jump terms to the discretization to recover the error estimates of the SUPG method known for conforming finite elements. Therefore, we expect to meet the same situation in the more complex problem of linearized Navier–Stokes equations and add

$$j_h(\mathbf{u}_h,\mathbf{v}_h) := \sum_{E\in\mathcal{E}_h}\gamma_E\langle[\mathbf{u}_h]_E,[\mathbf{v}_h]_E\rangle_E$$

with positive constants γ_E to the discrete formulation. In the case of a scalar convection-diffusion equation it turns out that it is enough to choose $\gamma_E\sim 1$ (see [22]), but due to the coupling with the pressure we have to choose γ_E differently; see Lemma 5.2. Note that the solution $(\mathbf{u},p)\in\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$ satisfies $[\mathbf{u}]_E=0$ and consequently $j_h(\mathbf{u},\mathbf{v})=0$ for all $\mathbf{v}\in\mathbf{H}_0^1(\Omega)+\mathbf{V}_h$.

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We shall consider and analyze the case of the generalized Oseen equations,

$$-\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma = \partial\Omega$$

which appears as a result of time discretizations of the nonstationary Navier–Stokes equations with $\sigma = (1/\Delta t)$. Its weak formulation reads:

Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$,

$$a^\sigma(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + (q, \nabla\cdot\mathbf{u}) = (\mathbf{f},\mathbf{v}), \tag{5.1}$$

where the bilinear form $a(\cdot,\cdot)$ in (2.4) has been replaced by the bilinear form

$$a^\sigma(\mathbf{u},\mathbf{v}) := \nu(\nabla\mathbf{u},\nabla\mathbf{v}) + \sigma(\mathbf{u},\mathbf{v}).$$

Let us introduce the following notations:

$$\begin{aligned} A\left((\mathbf{u},p),(\mathbf{v},q)\right) &= a_h^\sigma(\mathbf{u},\mathbf{v}) + b_h(\mathbf{u},\mathbf{v}) + j_h(\mathbf{u},\mathbf{v}) + \sum_{K\in\mathcal{T}_h} \tau_K((\mathbf{b}\cdot\nabla)\mathbf{u}, (\mathbf{b}\cdot\nabla)\mathbf{v})_K \\ &\quad - (p, \nabla_h\cdot\mathbf{v}) + (q, \nabla_h\cdot\mathbf{u}), \\ L\left((\mathbf{v},q)\right) &= (\mathbf{f},\mathbf{v}) + \sum_{K\in\mathcal{T}_h} \tau_K(\mathbf{f}, (\mathbf{b}\cdot\nabla)\mathbf{v})_K \end{aligned}$$

with $a_h^\sigma(\cdot,\cdot)$ being the discrete analogue of $a^\sigma(\cdot,\cdot)$, more precisely

$$a_h^\sigma(\mathbf{u}_h,\mathbf{v}_h) := \sum_{K\in\mathcal{T}_h} \left(\nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h)_K + \sigma(\mathbf{u}_h,\mathbf{v}_h)_K\right) \quad \forall\mathbf{u}_h,\mathbf{v}_h \in \mathbf{V}_h + \mathbf{H}_0^1(\Omega).$$

The discrete problem to be studied now becomes:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$,

$$A\left((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)\right) = L\left((\mathbf{v}_h,q_h)\right). \tag{5.2}$$

The bilinear form $A(\cdot,\cdot)$ generates a norm on the product space $\mathbf{V}_h \times Q_h$

$$\begin{aligned} |||(\mathbf{v},q)||| = \Bigg(& \nu|\mathbf{v}|_{1,h}^2 + \sigma\|\mathbf{v}\|_0^2 + (\nu+\sigma)\|q\|_0^2 \\ & + j_h(\mathbf{v},\mathbf{v}) + \sum_{K\in\mathcal{T}_h} \tau_K\|(\mathbf{b}\cdot\nabla)\mathbf{v}\|_{0,K}^2 \Bigg)^{1/2}. \end{aligned}$$

First we show an inf-sup condition for the bilinear form $A(\cdot,\cdot)$ on the product space $\mathbf{V}_h \times Q_h$.

LEMMA 5.1. *Assume that* $\max(\nu,\sigma,\tau_K,\gamma_E h_E) \le C$. *Then, there is a positive constant* β *independent of* $\nu > 0$ *such that for all* $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$,

$$|||(\mathbf{v}_h,q_h)||| \le \frac{1}{\beta} \sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h} \frac{A\left((\mathbf{v}_h,q_h),(\mathbf{w}_h,r_h)\right)}{|||(\mathbf{w}_h,r_h)|||}. \tag{5.3}$$

Proof. Let us consider an arbitrary $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$. Choosing $(\mathbf{w}_h, r_h) = (\mathbf{v}_h, q_h)$, we have

$$\begin{aligned} & A\big((\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)\big) \\ & = \nu|\mathbf{v}_h|_{1,h}^2 + \sigma\|\mathbf{v}_h\|_0^2 + j_h(\mathbf{v}_h,\mathbf{v}_h) + \sum_{K\in\mathcal{T}_h} \tau_K\|(\mathbf{b}\cdot\nabla)\mathbf{v}_h\|_{0,K}^2 \end{aligned} \tag{5.4}$$

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due to the property $b_h(\mathbf{v}_h, \mathbf{v}_h) = 0$ which has been shown in section 2.

Now let us consider another choice of $(\mathbf{w}_h, r_h)$. For any $q_h \in Q_h$ the discrete Babuška–Brezzi condition (2.11) guarantees the existence of a function $\mathbf{v}_{q_h} \in \mathbf{V}_h$ such that

$$(\nabla_h \cdot \mathbf{v}_{q_h}, q_h) = -(q_h, q_h), \qquad \|\mathbf{v}_{q_h}\|_{1,h} \le C\|q_h\|_0.$$

Thus, by choosing $(\mathbf{w}_h, r_h) = (\mathbf{v}_{q_h}, 0)$ we obtain

$$\begin{aligned} A\left((\mathbf{v}_h, q_h), (\mathbf{v}_{q_h}, 0)\right) &= \|q_h\|_0^2 + a_h^\sigma(\mathbf{v}_h, \mathbf{v}_{q_h}) + b_h(\mathbf{v}_h, \mathbf{v}_{q_h}) + j_h(\mathbf{v}_h, \mathbf{v}_{q_h}) \\ &\quad + \sum_{K\in\mathcal{T}_h} \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v}_h, (\mathbf{b}\cdot\nabla)\mathbf{v}_{q_h})_K. \end{aligned} \tag{5.5}$$

Now, the second term on the right-hand side of (5.5) can be bounded as follows:

$$\begin{aligned} |a_h^\sigma(\mathbf{v}_h, \mathbf{v}_{q_h})| &\le C\ (\nu|\mathbf{v}_h|_{1,h} + \sigma\|\mathbf{v}_h\|_0)\ \|q_h\|_0 \\ &\le C\ \left(\nu^2|\mathbf{v}_h|_{1,h}^2 + \sigma^2\|\mathbf{v}_h\|_0^2\right) + \frac{1}{8}\|q_h\|_0^2. \end{aligned}$$

Elementwise integration by parts of the third term on the right-hand side of (5.5) gives

$$\begin{aligned} b_h(\mathbf{v}_h, \mathbf{v}_{q_h}) &= \sum_{E\in\mathcal{E}_h} \langle \mathbf{b}\cdot\mathbf{n}_E, [\mathbf{v}_h\cdot\mathbf{v}_{q_h}]_E\rangle_E - \sum_{K\in\mathcal{T}_h}((\mathbf{b}\cdot\nabla)\mathbf{v}_{q_h}, \mathbf{v}_h)_K \\ &\quad - \sum_{E\in\mathcal{E}_h} \langle \mathbf{b}\cdot\mathbf{n}_E[\mathbf{v}_h]_E, \{\mathbf{v}_{q_h}\}_E\rangle_E \\ &= \sum_{E\in\mathcal{E}_h} \langle \mathbf{b}\cdot\mathbf{n}_E[\mathbf{v}_{q_h}]_E, \{\mathbf{v}_h\}_E\rangle_E - \sum_{K\in\mathcal{T}_h}((\mathbf{b}\cdot\nabla)\mathbf{v}_{q_h}, \mathbf{v}_h)_K. \end{aligned}$$

Let $\omega(E)$ denote the union of the cells K sharing a common face E. For any $\mathbf{v}_h \in V_h$ we have

$$\|[\mathbf{v}_h]_E\|_{0,E} \le C\, h_E^{1/2}|\mathbf{v}_h|_{1,h,\omega(E)}, \qquad \|\{\mathbf{v}_h\}_E\|_{0,E} \le C\, h_E^{-1/2}\|\mathbf{v}_h\|_{0,\omega(E)},$$

from which

$$|b_h(\mathbf{v}_h, \mathbf{v}_{q_h})| \le C\,|\mathbf{v}_{q_h}|_{1,h}\,\|\mathbf{v}_h\|_0 \le C\,\|q_h\|_0\,\|\mathbf{v}_h\|_0 \le C\,\|\mathbf{v}_h\|_0^2 + \frac{1}{8}\|q_h\|_0^2$$

follows. Similarly, for the fourth term on the right-hand side of (5.5) we obtain

$$\begin{aligned} j_h(\mathbf{v}_h, \mathbf{v}_{q_h}) &\le C \sum_{E\in\mathcal{E}_h} \gamma_E\|[\mathbf{v}_h]_E\|_{0,E}\, h_E^{1/2}|\mathbf{v}_{q_h}|_{1,h,\omega(E)} \\ &\le C \sum_{E\in\mathcal{E}_h} \gamma_E^2 h_E\|[\mathbf{v}_h]_E\|_{0,E}^2 + \frac{1}{8}\|q_h\|_0^2. \end{aligned}$$

Finally, the fifth term on the right-hand side of (5.5) is estimated by

$$\begin{aligned} \left|\sum_{K\in\mathcal{T}_h} \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v}_h, (\mathbf{b}\cdot\nabla)\mathbf{v}_{q_h})_K\right| &\le \sum_{K\in\mathcal{T}_h} \tau_K\|(\mathbf{b}\cdot\nabla)\mathbf{v}_h\|_{0,K}\|(\mathbf{b}\cdot\nabla)\mathbf{v}_{q_h}\|_{0,K} \\ &\le C \sum_{K\in\mathcal{T}_h} \tau_K^2\|(\mathbf{b}\cdot\nabla)\mathbf{v}_h\|_{0,K}^2 + \frac{1}{8}\|q_h\|_0^2. \end{aligned}$$

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Combining the inequalities and taking into consideration that ν, τ_K, and $\gamma_E h_E$ are bounded from above, we get from (5.5)

$$
\begin{aligned}
&A\big((\mathbf{v}_h, q_h), (\mathbf{v}_{q_h}, 0)\big) \geq \frac{1}{2}\|q_h\|_0^2 \\
&\qquad - C_1 \left[\nu|\mathbf{v}_h|_{1,h}^2 + \|\mathbf{v}_h\|_0^2 + j_h(\mathbf{v}_h, \mathbf{v}_h) + \sum_{K\in\mathcal{T}_h} \tau_K \|(\mathbf{b}\cdot\nabla)\mathbf{v}_h\|_{0,K}^2\right].
\end{aligned} \tag{5.6}
$$

Multiplying this inequality by $(\nu+\sigma)$, using the estimate $\nu+\sigma \leq C$ to bound

$$
\begin{aligned}
(\nu+\sigma)\nu|\mathbf{v}_h|_{1,h}^2 &\leq C\nu|\mathbf{v}_h|_{1,h}^2, \\
(\nu+\sigma) j_h(\mathbf{v}_h, \mathbf{v}_h) &\leq C j_h(\mathbf{v}_h, \mathbf{v}_h), \\
(\nu+\sigma)\sum_{K\in\mathcal{T}_h} \tau_K \|(\mathbf{b}\cdot\nabla)\mathbf{v}_h\|_{0,K}^2 &\leq C \sum_{K\in\mathcal{T}_h} \tau_K \|(\mathbf{b}\cdot\nabla)\mathbf{v}_h\|_{0,K}^2,
\end{aligned}
$$

and hiding the $\nu\|\mathbf{v}_h\|_0^2$ term by the discrete Poincaré's inequality

$$
(\nu+\sigma)\|\mathbf{v}_h\|_0^2 = \nu\|\mathbf{v}_h\|_0^2 + \sigma\|\mathbf{v}_h\|_0^2 \leq C\nu|\mathbf{v}_h|_{1,h}^2 + \sigma\|\mathbf{v}_h\|_0^2,
$$

we end up with

$$
\begin{aligned}
&A\big((\mathbf{v}_h, q_h), ((\nu+\sigma)\mathbf{v}_{q_h}, 0)\big) \geq \frac{\nu+\sigma}{2}\|q_h\|_0^2 \\
&\qquad - C_2 \left[\nu|\mathbf{v}_h|_{1,h}^2 + \sigma\|\mathbf{v}_h\|_0^2 + j_h(\mathbf{v}_h, \mathbf{v}_h) + \sum_{K\in\mathcal{T}_h} \tau_K \|(\mathbf{b}\cdot\nabla)\mathbf{v}_h\|_{0,K}^2\right].
\end{aligned} \tag{5.7}
$$

From (5.4) and (5.7) we get for $(\mathbf{w}_h, r_h) := (1-\alpha)(\mathbf{v}_h, q_h) + \alpha((\nu+\sigma)\mathbf{v}_{q_h}, 0)$,

$$
A\big((\mathbf{v}_h, q_h), (\mathbf{w}_h, r_h)\big) \geq \frac{\alpha}{2}|||(\mathbf{v}_h, q_h)|||^2 \tag{5.8}
$$

with $\alpha = 2/(2C_2+3) \in (0,1)$. Moreover, analyzing each individual term in the triple norm, we can show that

$$
|||(\mathbf{v}_{q_h}, 0)||| \leq C\|\mathbf{v}_{q_h}\|_{1,h} \leq C\|q_h\|_0,
$$

and with $\nu+\sigma \leq C\sqrt{\nu+\sigma}$ we conclude that

$$
\begin{aligned}
|||(\mathbf{w}_h, r_h)||| &\leq (1-\alpha)|||(\mathbf{v}_h, q_h)||| + \alpha(\nu+\sigma)|||(\mathbf{v}_{q_h}, 0)||| \\
&\leq C_3|||(\mathbf{v}_h, q_h)|||
\end{aligned}
$$

follows. Thus, we obtain (5.3) with $\beta = \alpha/(2C_3)$. ∎

Remark. Note that for $\sigma > 0$ we have control over the L^2 norm of the velocity and the pressure *uniformly with respect to* ν. However, for $\sigma = 0$ we lose this uniform L^2 norm control. In this case, the pressure is only controlled by $\nu^{1/2}\|\cdot\|_0$. Taking into consideration Poincaré's inequality we see that the velocity is also controlled by $\nu^{1/2}\|\cdot\|_0$. This behavior, that the case $\sigma > 0$ leads to a uniform (with respect to ν) control of the L^2 norm of velocity and pressure, can be also observed in other stabilized methods; see, for example, [11].

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Let the weak solution of the generalized Oseen equations belong additionally to $\mathbf{H}^2(\Omega) \times H^1(\Omega)$. Our formulation admits the following consistency property, where the parameter choice satisfies the assumption of Lemma 5.1.

LEMMA 5.2. *Let* $(\mathbf{u}, p) \in (\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)) \times (L_0^2(\Omega) \cap H^1(\Omega))$ *be the weak solution of* (5.1) *and let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the discrete solution of* (5.2). *Then, the consistency error can be represented in the form*

$$
\begin{aligned}
R(\mathbf{u}, p; \mathbf{w}_h, r_h) :&= A\left((\mathbf{u} - \mathbf{u}_h, p - p_h), (\mathbf{w}_h, r_h)\right) \\
&= \sum_{E \in \mathcal{E}_h} \left\{ \left\langle \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}_E}, [\mathbf{w}_h]_E \right\rangle_E - \langle p, [\mathbf{w}_h]_E \cdot \mathbf{n}_E \rangle_E \right\} \\
&\quad + \sum_{K \in \mathcal{T}_h} \tau_K (\nu \Delta \mathbf{u} - \sigma \mathbf{u} - \nabla p, (\mathbf{b} \cdot \nabla) \mathbf{w}_h)_K.
\end{aligned}
$$

Furthermore, assume that $\tau_K \sim h_K^2$ *and* $\gamma_E \sim h_E^{-1}$. *Then, there is a positive constant* C *independent of* ν *such that*

$$
|R(\mathbf{u}, p; \mathbf{w}_h, r_h)| \le C\, h (\|\mathbf{u}\|_2 + \|p\|_1)\, |||(\mathbf{w}_h, r_h)||| \quad \forall (\mathbf{w}_h, r_h) \in \mathbf{V}_h \times Q_h.
$$

Proof. The representation follows by testing the strong form of the problem with $\mathbf{w}_h$ and $(\mathbf{b} \cdot \nabla)\mathbf{w}_h$, respectively, elementwise integration by parts, and taking into consideration the definition of $A(\cdot, \cdot)$, (5.1), and (5.2). Following [13] we have

$$
\left| \sum_{E \in \mathcal{E}_h} \left\langle \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}_E}, [\mathbf{w}_h]_E \right\rangle_E \right| \le C\, h \nu \|\mathbf{u}\|_2\, |\mathbf{w}_h|_{1,h} \le C\, h \|\mathbf{u}\|_2\, |||(\mathbf{w}_h, r_h)|||,
$$

$$
\left| \sum_{E \in \mathcal{E}_h} \langle p, [\mathbf{w}_h]_E \cdot \mathbf{n}_E \rangle_E \right| \le C h \|p\|_1\, |\mathbf{w}_h|_{1,h},
$$

which shows that the second estimate does not lead to the desired estimate with a ν independent constant. Therefore, we bound the term in a different way as follows:

$$
\begin{aligned}
\left| \sum_{E \in \mathcal{E}_h} \langle p, [\mathbf{w}_h]_E \cdot \mathbf{n}_E \rangle_E \right| &\le C \sum_{E \in \mathcal{E}_h} \gamma_E^{-1/2} h_E^{1/2} |p|_{1,h,\omega(E)}\, \gamma_E^{1/2} \|[\mathbf{w}_h]_E\|_{0,E} \\
&\le C h \|p\|_1 \sqrt{j_h(\mathbf{w}_h, \mathbf{w}_h)}.
\end{aligned}
$$

Concerning the last term of the consistency error, we get

$$
\begin{aligned}
&\left| \sum_{K \in \mathcal{T}_h} \tau_K (\nu \Delta \mathbf{u} - \sigma \mathbf{u} - \nabla p, (\mathbf{b} \cdot \nabla) \mathbf{w}_h)_K \right| \\
&\qquad \le \sum_{K \in \mathcal{T}_h} \tau_K^{1/2} \left(\|\mathbf{u}\|_{2,K} + \|p\|_{1,K} \right) \tau_K^{1/2} \|(\mathbf{b} \cdot \nabla) \mathbf{w}_h\|_{0,K} \\
&\qquad \le C h \left(\|\mathbf{u}\|_2 + \|p\|_{1,K} \right) \left(\sum_{K \in \mathcal{T}_h} \tau_K \|(\mathbf{b} \cdot \nabla) \mathbf{w}_h\|_{0,K}^2 \right)^{1/2}.
\end{aligned}
$$

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Summarizing the individual estimates we obtain the statement of the lemma. ∎

Next we shall investigate the interpolation error. First, note that a discretely divergence-free function is divergence-free on each cell K. Indeed, if χ_K denotes the characteristic function of K, $|K|$ and $|\Omega|$ denoting the measure of K and Ω, respectively, we conclude for a discretely divergence-free function $\mathbf{v}_h \in \mathbf{V}_h$ that the function $\nabla_h \cdot \mathbf{v}_h$ is piecewise constant and, thus, by setting $q_h = \chi_K - |K|/|\Omega| \in Q_h$,

$$\begin{aligned} 0 = (q_h, \nabla_h \cdot \mathbf{v}_h) &= (1, \nabla_h \cdot \mathbf{v}_h)_K - \frac{|K|}{|\Omega|}(1, \nabla_h \cdot \mathbf{v}_h)_\Omega \\ &= |K| \left(\nabla \cdot \mathbf{v}_h\big|_K\right) - \frac{|K|}{|\Omega|} \sum_{K \in \mathcal{T}_h} \langle 1, \mathbf{v}_h \cdot \mathbf{n}_K \rangle_{\partial K} \\ &= |K| \left(\nabla \cdot (\mathbf{v}_h\big|_K)\right). \end{aligned}$$

Lemma 5.3. *The canonical interpolant* $\mathbf{I}_h : \mathbf{H}_0^1(\Omega) \to \mathbf{V}_h$ *defined by*

$$\frac{1}{|E|} \int_E (\mathbf{I}_h \mathbf{v} - \mathbf{v}) \, ds = \mathbf{0} \qquad \forall E \in \mathcal{E}_h$$

satisfies

$$(q_h, \nabla_h \cdot \mathbf{I}_h \mathbf{v}) = (q_h, \nabla \cdot \mathbf{v}) \qquad \forall\, q_h \in Q_h,\ \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{5.9}$$

$$\|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{0,K} + h_K |\mathbf{v} - \mathbf{I}_h \mathbf{v}|_{1,K} \le C\, h_K^2\, |\mathbf{v}|_{2,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega). \tag{5.10}$$

Proof. For the proof see [13]. ∎

Lemma 5.4. *Let* $(\mathbf{u}, p) \in (\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)) \times (L_0^2(\Omega) \cap H^1(\Omega))$ *be the weak solution of* (5.1) *and let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the discrete solution of* (5.2). *Assume that* $\tau_K \sim h_K^2$ *and* $\gamma_E \sim h_E^{-1}$. *Then, for the canonical interpolant* $\mathbf{I}_h : \mathbf{H}_0^1(\Omega) \to \mathbf{V}_h$ *and the* L^2 *projection* $J_h : L_0^2(\Omega) \to Q_h$ *there is a constant* C *independent of* ν *such that*

$$|A\left((\mathbf{u} - \mathbf{I}_h \mathbf{u}, p - J_h p), (\mathbf{w}_h, r_h)\right)| \le C\, h (\|\mathbf{u}\|_2 + \|p\|_1)\, |||(\mathbf{w}_h, r_h)||| \tag{5.11}$$

for all $(\mathbf{w}_h, r_h) \in \mathbf{V}_h \times Q_h$.

Proof. Taking into consideration the definition of $|||\cdot|||$, we estimate each term in $A(\cdot,\cdot)$ separately. The estimate

$$|a_h^\sigma(\mathbf{u} - \mathbf{I}_h \mathbf{u}, \mathbf{w}_h)| \le C\, h\, \|\mathbf{u}\|_2\, |||(\mathbf{w}_h, r_h)|||$$

is standard. Using elementwise integration by parts, we obtain

$$b_h(\mathbf{u} - \mathbf{I}_h \mathbf{u}, \mathbf{w}_h) = \sum_{E \in \mathcal{E}_h} \langle \mathbf{b} \cdot \mathbf{n}_E [\mathbf{w}_h]_E, \{\mathbf{u} - \mathbf{I}_h \mathbf{u}\}_E \rangle_E - \sum_{K \in \mathcal{T}_h} (\mathbf{u} - \mathbf{I}_h \mathbf{u}, (\mathbf{b} \cdot \nabla) \mathbf{w}_h)_K$$

(see also the proof of Lemma 5.1). The first term on the right-hand side is estimated by

$$\begin{aligned} &\Big| \sum_{E \in \mathcal{E}_h} \langle \mathbf{b} \cdot \mathbf{n}_E [\mathbf{w}_h]_E, \{\mathbf{u} - \mathbf{I}_h \mathbf{u}\}_E \rangle_E \Big| \\ &\qquad \le C \sum_{E \in \mathcal{E}_h} \gamma_E^{-1/2} h_E^{3/2} \|\mathbf{u}\|_{2,\omega(E)}\ \gamma_E^{1/2} \|[\mathbf{w}_h]_E\|_{0,E} \\ &\qquad \le C\, h^2\, \|\mathbf{u}\|_2 \sqrt{j_h(\mathbf{w}_h, \mathbf{w}_h)} \end{aligned}$$

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and the second one by

$$\left|\sum_{K\in\mathcal{T}_h}(\mathbf{u}-\mathbf{I}_h\mathbf{u},(\mathbf{b}\cdot\nabla)\mathbf{w}_h)_K\right| \le C\sum_{K\in\mathcal{T}_h}\tau_K^{-1/2}h_K^2\|\mathbf{u}\|_{2,K}\,\tau_K^{1/2}\|(\mathbf{b}\cdot\nabla)\mathbf{w}_h\|_{0,K}$$
$$\le C\,h\,\|\mathbf{u}\|_2\left(\sum_{K\in\mathcal{T}_h}\tau_K\|(\mathbf{b}\cdot\nabla)\mathbf{w}_h\|_{0,K}^2\right)^{1/2}.$$

The next expression is

$$|j_h(\mathbf{u}-\mathbf{I}_h\mathbf{u},\mathbf{w}_h)| \le C\sum_{E\in\mathcal{E}_h}\gamma_E^{1/2}h_E^{3/2}\|\mathbf{u}\|_{2,\omega(E)}\ \gamma_E^{1/2}\|[\mathbf{w}_h]_E\|_{0,E}$$
$$\le C\,h\,\|\mathbf{u}\|_2\sqrt{j_h(\mathbf{w}_h,\mathbf{w}_h)}$$

followed by

$$\sum_{K\in\mathcal{T}_h}\tau_K\big((\mathbf{b}\cdot\nabla)(\mathbf{u}-\mathbf{I}_h\mathbf{u}),(\mathbf{b}\cdot\nabla)\mathbf{w}_h\big)_K \le C\,h^2\,\|\mathbf{u}\|_2\left(\sum_{K\in\mathcal{T}_h}\tau_K\|(\mathbf{b}\cdot\nabla)\mathbf{w}_h\|_{0,K}^2\right)^{1/2}.$$

The orthogonality of the L^2 projection J_h and the property that any discretely divergence-free function is divergence-free on each cell yield that the last two terms become zero; i.e.,

$$(p-J_hp,\nabla_h\cdot\mathbf{w}_h)=0 \qquad \forall\mathbf{w}_h\in\mathbf{V}_h,$$
$$(r_h,\nabla_h\cdot(\mathbf{u}-\mathbf{I}_h\mathbf{u}))=0 \qquad \forall r_h\in Q_h.$$

Collecting all estimates, we get the statement of the lemma. ∎

THEOREM 5.5. *Let $(\mathbf{u},p)\in(\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^2(\Omega))\times(L_0^2(\Omega)\cap H^1(\Omega))$ be the weak solution of* (5.1) *and let $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ be the discrete solution of* (5.2). *Assume that $\tau_K\sim h_K^2$ and $\gamma_E\sim h_E^{-1}$. Then, there is a positive constant C independent of ν such that*

$$|||(\mathbf{u}-\mathbf{u}_h,p-p_h)||| \le C\,h\,(\|\mathbf{u}\|_2+\|p\|_1). \tag{5.12}$$

Proof. Starting with Lemma 5.1 we have

$$|||(\mathbf{u}_h-\mathbf{I}_h\mathbf{u},p_h-J_hp)||| \le \frac{1}{\beta}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h}\frac{A((\mathbf{u}_h-\mathbf{I}_h\mathbf{u},p_h-J_hp),(\mathbf{w}_h,r_h))}{|||(\mathbf{w}_h,r_h)|||}$$
$$\le \frac{1}{\beta}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h}\frac{A((\mathbf{u}_h-\mathbf{u},p_h-p),(\mathbf{w}_h,r_h))}{|||(\mathbf{w}_h,r_h)|||}$$
$$+\frac{1}{\beta}\sup_{(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h}\frac{A((\mathbf{u}-\mathbf{I}_h\mathbf{u},p-J_hp),(\mathbf{w}_h,r_h))}{|||(\mathbf{w}_h,r_h)|||}.$$

Now, the first term can be bounded by Lemma 5.2 and the second one by Lemma 5.4. It remains to apply the triangle inequality

$$|||(\mathbf{u}-\mathbf{u}_h,p-p_h)||| \le |||(\mathbf{u}-\mathbf{I}_h\mathbf{u},p-J_hp)|||+|||(\mathbf{u}_h-\mathbf{I}_h\mathbf{u},p_h-J_hp)|||$$

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and the approximation properties of the interpolation operators $\mathbf{I}_h$ and J_h. ☐

Remark. According to the definition of the triple norm we have for $\sigma > 0$ an additional control *uniformly with respect to* ν over the L^2 norm of the velocity and the pressure. For $\sigma = 0$ we lose this control for $\nu \to 0$.

Remark. In the SUPG method the additional stabilizing term

$$\sum_{K \in \mathcal{T}_h} \gamma_K (\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_K$$

is often used [17, 31]. In our case of the Crouzeix–Raviart element, discretely divergence-free functions are piecewise divergence-free, therefore this term vanishes.

Remark. Often the SUPG parameter in the SUPG method is chosen in the advective regime as $\tau_K \sim h_K$, which is the correct choice for equal-order interpolation [9, 14, 31]. However, using inf-sup stable elements with different-order interpolation in the SUPG method, we have to take $\tau_K \sim h_K^2$ [5].

Numerical test. We consider the generalized Oseen equations (5.2) in $\Omega = (0,1)^2$ with the prescribed solution

$$\mathbf{u} = \begin{pmatrix} 2x^2(1-x)^2 y(1-y)(1-2y) \\ -2y^2(1-y)^2 x(1-x)(1-2x) \end{pmatrix}, \quad p = x^3 + y^3 - 0.5,$$

the convection field

$$\mathbf{b} = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix},$$

and with the parameters $\nu = 10^{-3}$, $\sigma = 100$. The choice of σ corresponds to a length of the time step of 0.01 in the nonstationary Navier–Stokes equations.

The coarsest grid in the computations (level 0) consists of two triangles with the common edge from $(0,0)$ to $(1,1)$. On level 7, the system has 98 816 velocity degrees of freedom (including Dirichlet nodes) and 32 768 pressure degrees of freedom.

Results for different choices of the parameter γ_E in the jump term $j_h(\mathbf{u}_h, \mathbf{v}_h)$ are presented in Tables 5.1 and 5.2. In Table 5.1, computations without this jump term ($\gamma_E = 0$) and with the appropriate choice ($\gamma_E = 1$) known from scalar convection-diffusion equations (cf. [22]) are given. It can be observed that the order of convergence with respect to the natural norms for the Oseen equations is far below the optimal one in the convection-dominated regime; even an increase of errors occurs. However, optimal orders are obtained for the choice $\gamma_E = 1/h_E$, which is in agreement with our theoretical results presented in this section; see Table 5.2. In addition, the optimal order of convergence in the $|||\cdot|||$ norm, (5.12), can be seen.

TABLE 5.1
Results obtained with $\gamma_E = 0$ and $\gamma_E = 1$.

	$\gamma_E = 0$				$\gamma_E = 1$			
Level	$\lVert\nabla(\mathbf{u}-\mathbf{u}_h)\rVert_0$	Order	$\lVert p-p_h\rVert_0$	Order	$\lVert\nabla(\mathbf{u}-\mathbf{u}_h)\rVert_0$	Order	$\lVert p-p_h\rVert_0$	Order
3	3.057e-1	—	2.790e-1	—	2.211e-1	—	2.185e-1	—
4	5.899e-1	−0.949	2.625e-1	0.088	3.377e-1	−0.611	1.601e-1	0.449
5	1.083e+0	−0.876	2.487e-1	0.078	4.549e-1	−0.430	1.077e-1	0.572
6	1.748e+0	−0.691	2.166e-1	0.200	5.336e-1	−0.230	6.433e-2	0.744
7	2.205e+0	−0.335	1.474e-1	0.555	5.486e-1	−0.040	3.410e-2	0.916

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Table 5.2
Results obtained with $\gamma_E = 1/h_E$.

Level	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0$	Order	$\|p-p_h\|_0$	Order	$\|\|\|(\mathbf{u}-\mathbf{u}_h, p-p_h)\|\|\|$	Order
3	8.610e-2	—	1.176e-1	—	1.179e+0	—
4	5.332e-2	0.691	4.389e-2	1.422	4.409e-1	1.418
5	2.775e-2	0.942	1.776e-2	1.306	1.789e-1	1.301
6	1.386e-2	1.002	8.196e-3	1.115	8.270e-2	1.113
7	6.895e-3	1.001	4.053e-3	1.021	4.090e-2	1.021

REFERENCES

[1] M. I. Asensio, A. Russo, and G. Sangalli, *The residual-free bubble numerical method with quadratic elements*, Math. Models Methods Appl. Sci., 14 (2004), pp. 641–661.

[2] R. Becker and M. Braack, *A finite element pressure gradient stabilization for the Stokes equations based on local projections*, Calcolo, 38 (2001), pp. 173–199.

[3] R. Becker and M. Braack, *A two-level stabilization scheme for the Navier-Stokes equations*, in Numerical Mathematics and Advanced Applications, M. Feistauer, V. Dolejší, P. Knobloch, and K. Najzar, eds., Springer-Verlag, Berlin, 2004, pp. 123–130.

[4] M. Braack and E. Burman, *Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method*, SIAM J. Numer. Anal., 43 (2006), pp. 2544–2566.

[5] M. Braack, E. Burman, V. John, and G. Lube, *Stabilized finite element methods for the generalized Oseen problem*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 853–866.

[6] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[7] F. Brezzi, L. D. Marini, and A. Russo, *Applications of pseudo residual-free bubbles to the stabilization of convection-diffusion problems*, Comput. Methods Appl. Mech. Engrg., 166 (1998), pp. 51–63.

[8] F. Brezzi, L. D. Marini, and A. Russo, *On the choice of stabilizing subgrid for convection-diffusion problems*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 127–148.

[9] A. N. Brooks and T. J. R. Hughes, *Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 32 (1982), pp. 199–259.

[10] E. Burman and P. Hansbo, *The edge stabilization method for finite elements in cfd*, in Numerical Mathematics and Advanced Applications, M. Feistauer, V. Dolejší, P. Knobloch, and K. Najzar, eds., Springer-Verlag, Berlin, 2004, pp. 196–203.

[11] E. Burman and P. Hansbo, *A stabilized non-conforming finite element method for incompressible flow*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 2881–2899.

[12] P. G. Ciarlet, *Basic error estimates for elliptic problems*, in Handbook of Numerical Analysis: Finite Element Methods, Vol. 2, P. G. Ciarlet and J. L. Lions, eds., North–Holland, Amsterdam, 1991, pp. 17–351.

[13] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations.* I, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 7 (1973), pp. 33–76.

[14] L. P. Franca and S. L. Frey, *Stabilized finite element methods:* II. *The incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 99 (1992), pp. 209–233.

[15] L. P. Franca and A. Nesliturk, *On a two-level finite element method for the incompressible Navier-Stokes equations*, Internat. J. Numer. Methods Engrg., 52 (2001), pp. 433–453.

[16] L. P. Franca, A. Nesliturk, and M. Stynes, *On the stability of residual-free bubbles for convection-diffusion problems and their approximation by a two-level finite element method*, Comput. Methods Appl. Mech. Engrg., 166 (1998), pp. 35–49.

[17] T. Gelhard, G. Lube, M. A. Olshanskii, and J.-H. Starcke, *Stabilized finite element schemes with LBB-stable elements for incompressible flows*, J. Comput. Appl. Math., 177 (2005), pp. 243–267.

[18] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1986.

[19] V. Gravemeier, W. A. Wall, and E. Ramm, *A three-level finite element method for the instationary incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 193 (2004), pp. 1323–1366.

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[20] V. Gravemeier, W. A. Wall, and E. Ramm, *Large eddy simulation of turbulent incompressible flows by a three-level finite element method*, Internat. J. Numer. Methods Fluids, 48 (2005), pp. 1067–1099.

[21] T. J. R. Hughes, L. P. Franca, and M. Balestra, *A new finite element formulation for computational fluid dynamics:* V. *Circumventing the Babuška–Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations*, Comput. Methods Appl. Mech. Engrg., 59 (1986), pp. 85–99.

[22] V. John, G. Matthies, F. Schieweck, and L. Tobiska, *A streamline-diffusion method for nonconforming finite element approximations applied to convection-diffusion problems*, Comput. Methods Appl. Mech. Engrg., 166 (1998), pp. 85–97.

[23] V. John, J. Maubach, and L. Tobiska, *Nonconforming streamline-diffusion-finite-element-methods for convection-diffusion problems*, Numer. Math., 78 (1997), pp. 165–188.

[24] P. Knobloch and L. Tobiska, *Stabilization methods of bubble type for the Q_1/Q_1-element applied to the incompressible Navier-Stokes equations*, M2AN Math. Model Numer. Anal., 34 (2000), pp. 85–107.

[25] G. Lube and G. Rapin, *Residual-based stabilized higher-order FEM for a generalized Oseen problem*, Math. Models Methods Appl. Sci., 16 (2006), pp. 949–966.

[26] G. Lube and L. Tobiska, *A nonconforming finite element method of streamline diffusion type for the incompressible Navier–Stokes equations*, J. Comput. Math., 8 (1990), pp. 147–158.

[27] G. Matthies and L. Tobiska, *The streamline diffusion method for conforming and nonconforming finite elements of lowest order applied to convection-diffusion problems*, Computing, 66 (2001), pp. 343–364.

[28] A. Russo, *Bubble stabilization of finite element methods for the linearized incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 132 (1996), pp. 335–343.

[29] F. Schieweck and L. Tobiska, *A nonconforming finite element method of upstream type applied to the stationary Navier-Stokes equation*, M2AN Math. Model Numer. Anal., 23 (1989), pp. 627–647.

[30] F. Schieweck and L. Tobiska, *An optimal order error estimate for an upwind discretization of the Navier-Stokes equation*, Numer. Methods Partial Differential Equations, 12 (1996), pp. 407–421.

[31] L. Tobiska and R. Verfürth, *Analysis of a streamline diffusion finite element method for the Stokes and Navier–Stokes equations*, SIAM J. Numer. Anal., 33 (1996), pp. 107–127.

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