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FINITE ELEMENT ERROR ANALYSIS OF SPACE AVERAGED FLOW FIELDS DEFINED BY A DIFFERENTIAL FILTER

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This paper analyzes finite element approximations of space averaged flow fields which are given by filtering, i.e. averaging in space, the solution of the steady state Stokes and Navier–Stokes equations with a differential filter. It is shown that $\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2}$, the error of the filtered velocity $\bar{\mathbf{u}}$ and the filtered finite element approximation of the velocity $\overline{\mathbf{u}^h}$, converges under certain conditions of higher order than $\|\mathbf{u} - \mathbf{u}^h\|_{L^2}$, the error of the velocity and its finite element approximation. It is also proved that this statement stays true if the L^2-error of finite element approximations of $\bar{\mathbf{u}}$ and $\overline{\mathbf{u}^h}$ is considered. Numerical tests in two and three space dimensions support the analytical results.

Keywords: Differential filter; convergence of finite element method.

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1. Introduction

Large eddy simulation (LES) is currently one of the most promising and popular approaches for modelling turbulence. In LES, one seeks to compute only the large structures of a flow field whereas the influence of the small (subgrid scale) structures on the flow field is modelled. The large flow structures are defined by a spatial average of the flow field. This process is called filtering and many different filters may be used, see Ref. 15 for an overview. Analyzing LES models requires the analysis of the underlying filter.

One of the most popular filters and certainly the best analyzed one is the Gaussian filter. Let u be the function to be filtered and

$$g_{\hat{\delta}}(\mathbf{x}) = \left(\frac{6}{\hat{\delta}^2\pi}\right)^{d/2} \exp\left(-\frac{6}{\hat{\delta}^2}\|\mathbf{x}\|_2^2\right)$$

be the Gaussian filter function where $d \in \{2,3\}$ is the space dimension and $\|\mathbf{x}\|_2$ is the Euclidean norm of $\mathbf{x} \in \mathbb{R}^d$. The filtered function $\hat{u}$ is defined by convolving u and $g_{\hat{\delta}}$: $\hat{u} = g_{\hat{\delta}} * u$. All flow structures of size $\mathcal{O}(\hat{\delta})$ are the large ones and the parameter $\hat{\delta}$ is called the filter width. The application of the convolution requires that u is defined in $\mathbb{R}^d$.

An alternative way of defining a filter $\bar{u}$ of u is by using a regularizing partial differential equation. Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded convex domain with polygonal or polyhedral boundary $\partial\Omega$, $\delta \in (0, \mathrm{diam}(\Omega))$ be a constant and $u \in H^k(\Omega)$, $k \geq 0$ be given. Consequently, $\partial\Omega$ is a Lipschitz boundary. Throughout this paper, standard notations for Lebesgue and Sobolev spaces are used, e.g., see Ref. 1. The regularizing equation is given by the elliptic boundary value problem

$$\begin{aligned} -\delta^2 \Delta \bar{u} + \bar{u} &= u \quad \text{in } \Omega\,, \\ \bar{u} &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.1}$$

The weak formulation of (1.1) reads as follows: Find $\bar{u} \in H_0^1(\Omega)$ such that

$$\delta^2(\nabla\bar{u}, \nabla v) + (\bar{u}, v) = (u, v) \quad \forall\, v \in H_0^1(\Omega)\,, \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

Definition 1.1. The solution $\bar{u}$ of (1.2) is called differential filter of u with respect to the filter width δ.

The differential filter was introduced in Refs. 5 and 6. Since the differential filter is defined as a solution of a Helmholtz equation, it is sometimes called Helmholtz filter. In Ref. 16, it is discussed that the differential filter (which is called exponential filter in Ref. 16) has some advantages over the Gaussian filter, especially a simpler representation of the so-called subgrid scales $u - \bar{u}$ in comparison to $u - \hat{u}$. The use of the differential filter for LES computations in complex 3D domains is recommended in Ref. 14.

There is a relationship between the Gaussian and the differential filter. Let $\Omega = \mathbb{R}^d$ and let the Fourier transform of the Gaussian filter be approximated by the second-order subdiagonal Padé approximation

$$\mathcal{F}(g_{\hat{\delta}}) = \exp\left(-\frac{\hat{\delta}^2}{24}\|\mathbf{y}\|_2^2\right) = \frac{1}{1 + \frac{\|\mathbf{y}\|_2^2}{24}\hat{\delta}^2} + \mathcal{O}(\hat{\delta}^4)\,.$$

Then, one obtains

$$\mathcal{F}(g_{\hat{\delta}} * u) = \mathcal{F}(g_{\hat{\delta}})\mathcal{F}(u) \approx \frac{1}{1 + \frac{\|\mathbf{y}\|_2^2}{24}\hat{\delta}^2}\mathcal{F}(u) = \mathcal{F}\left(\left(I - \frac{\hat{\delta}^2}{24}\Delta\right)^{-1} u\right),$$

such that

$$g_{\hat{\delta}} * u \approx \left(I - \frac{\hat{\delta}^2}{24}\Delta\right)^{-1} u\,. \tag{1.3}$$

This shows that the differential filter is an approximation of the Gaussian filter with $\delta = \hat{\delta}/\sqrt{24}$.

The effect of applying the Gaussian filter to the solution $(\mathbf{u}, p)$ of the steady state Stokes and Navier–Stokes equations and to the corresponding finite element solution $(\mathbf{u}^h, p^h)$ was analyzed in Refs. 4 and 10. It was proved that under certain conditions $\|\hat{\mathbf{u}} - \widehat{\mathbf{u}^h}\|_{L^2}$ converges of higher order than $\|\mathbf{u} - \mathbf{u}^h\|_{L^2}$. Thus, there is additional accuracy for the filtered error.

This paper extends the analysis of Refs. 4 and 10 to the differential filter. It will be shown that $\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2}$ converges of higher order than $\|\mathbf{u} - \mathbf{u}^h\|_{L^2}$ if certain conditions are satisfied. The need of extending the analysis of Refs. 4 and 10 to obtain this result arises from two important differences between the Gaussian and the differential filter. First, from (1.3) it was concluded that the differential filter is an approximation of the Gaussian filter if $\Omega = \mathbb{R}^d$. This statement stays valid for a bounded domain only if the differential filter is equipped with appropriate boundary conditions. However, to our knowledge, such boundary conditions are not known. Thus, the use of homogeneous Dirichlet boundary conditions will result most probably in a considerable difference of the differential filter and the Gaussian filter near the boundary $\partial\Omega$. Second, whereas $\hat{u}$ is infinitely smooth under moderate regularity assumptions on u, e.g. $u \in L^2(\Omega)$, the differential filter $\bar{u}$ will be in general in $H^2(\Omega)$ if $u \in L^2(\Omega)$. The limited regularity of $\bar{u}$ requires additional studies of its properties which will be presented in Sec. 2.

In general, it will only be possible to compute discrete approximations of the filtered functions. An important question is if the error of these discrete approximations possesses the same, possibly higher, order of convergence as $\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2}$. This question, which had not been addressed for the Gaussian filter in Refs. 4 and 10, will be answered positively by the analysis given in Sec. 5.

The plan of the paper is as follows. Analytical properties of the differential filter are studied in Sec. 2. Section 3 provides an estimate of $\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2}$ for the solution of the Stokes equations. This analysis is extended to the steady state Navier–Stokes equations in Sec. 4. The error of finite element approximations of the filtered functions is analyzed in Sec. 5. Finally, Sec. 6 presents numerical examples in two and three dimensions which support the analytical results.

2. Some Properties of the Differential Filter

In this section, some essential properties of the differential filter are proved.

It is well known that (1.2) admits a unique solution $\bar{u} \in H_0^1(\Omega)$. If Ω is convex, then $\bar{u} \in H^2(\Omega)$, see Ref. 8, Theorem 3.2.1.2. The following two lemmas give estimates for several norms of $\bar{u}$.

Lemma 2.1. *For every $u \in L^2(\Omega)$, the solution $\bar{u}$ of* (1.2) *satisfies*

$$\delta^4 \|\Delta \bar{u}\|_{L^2}^2 + \delta^2 \|\nabla \bar{u}\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2 \le 5 \|u\|_{L^2}^2 \,. \tag{2.1}$$

Moreover, there is a constant $C(\Omega)$ *such that*

$$\delta\|\bar{u}\|_{H^1} \le C(\Omega)\|u\|_{L^2}\,. \tag{2.2}$$

Proof. Choosing in (1.2) $v = \bar{u}$ and applying the Cauchy–Schwarz inequality and Young's inequality on the right-hand side give

$$\delta^2\|\nabla\bar{u}\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2 \le \frac{1}{2}\|u\|_{L^2}^2 + \frac{1}{2}\|\bar{u}\|_{L^2}^2\,.$$

Consequently

$$\delta^2\|\nabla\bar{u}\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2 \le \|u\|_{L^2}^2\,. \tag{2.3}$$

From the differential equation (1.1), the triangle inequality and (2.3), we obtain

$$\delta^2\|\Delta\bar{u}\|_{L^2} = \|\bar{u} - u\|_{L^2} \le 2\|u\|_{L^2}\,.$$

This proves (2.1).

Estimate (2.2) follows directly from Poincaré's inequality: there is a constant $C(\Omega)$ such that

$$\|v\|_{H^1} \le C(\Omega)\|\nabla v\|_{L^2} \quad \forall\, v \in H_0^1(\Omega)\,.$$

□

Lemma 2.2. *For given* $u \in L^2(\Omega)$ *there is a constant* $C(\Omega)$ *independent of* δ *such that*

$$\|\bar{u}\|_{H^2} \le C(\Omega)\delta^{-2}\|u\|_{L^2}\,. \tag{2.4}$$

Proof. From (1.1) follows that

$$-\Delta\bar{u} = \frac{1}{\delta^2}(u - \bar{u}) \quad \text{in } \Omega\,, \tag{2.5}$$

i.e. $\bar{u}$ is the solution of a Poisson problem with right-hand side $(u - \bar{u})/\delta^2$.

The Laplacian operator $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is a bounded linear operator and an isomorphism (one-to-one and onto), see Ref. 8, Theorem 3.2.1.2, or Ref. 7, Chap. I, Remark 1.2. An application of the open mapping theorem gives that its inverse

$$(-\Delta)^{-1} : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$$

is a linear and continuous operator and hence a bounded operator. Consequently there exists $C = C(\Omega)$ such that

$$\|v\|_{H^2} \le C\|\Delta v\|_{L^2}$$

for every $v \in H^2(\Omega) \cap H_0^1(\Omega)$. From (2.5) follows now

$$\|\bar{u}\|_{H^2} \le \frac{C(\Omega)}{\delta^2}\|u - \bar{u}\|_{L^2} \le \frac{C(\Omega)}{\delta^2}(\|u\|_{L^2} + \|\bar{u}\|_{L^2})\,.$$

An application of the previous lemma finishes the proof. □

The following lemma states that the filtering procedure is self-adjoint.

Lemma 2.3. *For every* $u, v \in L^2(\Omega)$ *holds*

$$(\bar{u}, v) = (u, \bar{v})\,. \tag{2.6}$$

Proof. Since $\bar{v} \in H_0^1(\Omega)$ if $v \in H_0^1(\Omega)$, one gets from (1.2) and the symmetry of the bilinear forms

$$(u, \bar{v}) = \delta^2(\nabla\bar{u}, \nabla\bar{v}) + (\bar{u}, \bar{v}) = (\bar{u}, v)\,. \qquad \square$$

As δ tends to 0 the average $\bar{u}$ approaches u in $L^2(\Omega)$. This property of the differential filter is proved in the following lemma and theorem. Note that this result does not require compatibility conditions on traces of u and $\bar{u}$ on $\partial\Omega$.

Lemma 2.4. *For every* $u \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $\delta > 0$ *holds*

$$\|\bar{u} - u\|_{L^2} \le \delta^2 \|\Delta u\|_{L^2}\,.$$

Proof. We denote by T_δ the bounded linear operator

$$T_\delta : L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)\,, \quad u \mapsto \bar{u}\,,$$

i.e. $T_\delta u = (I - \delta^2\Delta)^{-1}(u)$. Estimate (2.3) implies that the sequence of operators $\{T_\delta\}_\delta$ is uniformly bounded

$$\|T_\delta\|_{\mathcal{L}(L^2,L^2)} = \sup_{u \in L^2(\Omega), u \ne 0} \frac{\|\bar{u}\|_{L^2}}{\|u\|_{L^2}} \le 1 \tag{2.7}$$

for every $\delta > 0$.

Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Since T_δ is the inverse of $(I - \delta^2\Delta)$, we have

$$(I - \delta^2\Delta)T_\delta u = u$$

and consequently

$$\bar{u} - u = T_\delta u - u = \delta^2 \Delta T_\delta u\,.$$

From this equation follows that $\Delta T_\delta u \in H_0^1(\Omega) \cap H^2(\Omega)$ since $\bar{u} - u$ belongs to this space. In particular, $\Delta T_\delta u$ vanishes on $\partial\Omega$.

The operators T_δ and Δ commute for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$. We have $\Delta u \in L^2(\Omega)$ since $u \in H^2(\Omega)$. The unique solution of (1.2) with right-hand side Δu is $T_\delta \Delta u$. If $u \in H^2(\Omega)$, we can apply the Laplacian to (1.1). The resulting equation also has the right-hand side Δu and its unique solution with homogeneous Dirichlet boundary conditions is $\Delta T_\delta u$. Hence $T_\delta \Delta u = \Delta T_\delta u$. Using this identity and (2.7), the proof of the lemma is concluded as follows:

$$\|\bar{u} - u\|_{L^2} = \|\delta^2 T_\delta \Delta u\|_{L^2} \le \delta^2 \|T_\delta\|_{\mathcal{L}(L^2,L^2)} \|\Delta u\|_{L^2} \le \delta^2 \|\Delta u\|_{L^2}$$

for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$. $\square$

Lemma 2.4 shows that the sequence of operators $\{T_\delta\}_\delta$ converges to the identity on a dense subset of $L^2(\Omega)$. This sequence is also uniformly bounded and consequently it converges pointwise to the identity:

Theorem 2.1. *For every* $u \in L^2(\Omega)$

$$\|\bar{u} - u\|_{L^2} \to 0$$

as $\delta \to 0$.

3. The Filtered Solution of the Stokes Equations

In this section, an L^2-estimate of the filtered error between the solution of the continuous Stokes problem and its finite element approximation is proved. It turns out that the filtered error converges of higher order than the non-filtered error provided the solution of a dual Stokes problem is sufficiently regular and the filter width is chosen appropriately.

Let $V = (H_0^1(\Omega))^d$, $Q = L_0^2(\Omega)$ and $\mathbf{f} \in (L^2(\Omega))^d$. We consider the Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega\,, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega\,, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega\,, \\ \int_\Omega p\, d\mathbf{x} &= 0\,. \end{aligned} \tag{3.1}$$

Its variational formulation reads as follows: Find $(\mathbf{u}, p) \in V \times Q$ such that

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) = (\mathbf{f}, \mathbf{v}) \quad \forall\ (\mathbf{v}, q) \in V \times Q\,. \tag{3.2}$$

Let $\mathcal{T}^h$ denote a decomposition of Ω into mesh cells. We denote by h_K the diameter of a mesh cell K and we set $h = \max_{K \in \mathcal{T}^h}\{h_K\}$. Each family of triangulations is assumed to be admissible and shape regular in the usual sense, e.g., see Ref. 3. With the mesh $\mathcal{T}^h$, we can construct conforming velocity-pressure finite element spaces $V^h \times Q^h$ with $V^h \subset V$ and $Q^h \subset Q$. These spaces are assumed to satisfy the inf–sup or Babuška–Brezzi condition, i.e. there exists a constant $\beta > 0$ independent of the triangulation such that

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in V^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|\nabla \mathbf{v}^h\|_{L^2} \|q^h\|_{L^2}} \geq \beta > 0\,. \tag{3.3}$$

We assume the following regularity of the solution of (3.2)

$$\mathbf{u} \in (H^{k+1}(\Omega))^d \cap V\,, \quad p \in H^k(\Omega) \cap Q\,, \quad k \geq 1\,, \tag{3.4}$$

and the approximation properties of the finite element spaces

$$\begin{aligned} &V^h \text{ contains piecewise polynomials of degree } k\,, \\ &Q^h \text{ contains piecewise polynomials of degree } k-1\,. \end{aligned} \tag{3.5}$$

For a wide variety of velocity-pressure finite element spaces satisfying (3.3), the following optimal *a priori* error estimate has been proved under the assumptions (3.4) and (3.5), see, e.g. Ref. 7,

$$\|\nabla(\mathbf{u}-\mathbf{u}^h)\|_{L^2} + h^{-1}\|\mathbf{u}-\mathbf{u}^h\|_{L^2} + \|p-p^h\|_{L^2} \le Ch^k(\|\mathbf{u}\|_{H^{k+1}} + \|p\|_{H^k})\,. \quad (3.6)$$

Moreover, the approximation properties

$$\inf_{\mathbf{v}^h\in V^h}(\|\mathbf{u}-\mathbf{v}^h\|_{L^2} + h\|\nabla(\mathbf{u}-\mathbf{v}^h)\|_{L^2}) \le Ch^{k+1}\|\mathbf{u}\|_{H^{k+1}}\,, \quad (3.7)$$

$$\inf_{q^h\in Q^h}\|p-p^h\|_{L^2} \le Ch^k\|p\|_{H^k} \quad (3.8)$$

are valid, if the finite element spaces are defined on meshes which are obtained by regular refinement from an initial mesh. They are not valid for general quadrilateral and hexahedral meshes, see Refs. 2 and 12.

The finite element approximation of the solution of (3.2) is given by: Find $(\mathbf{u}^h, p^h) \in V^h \times Q^h$ such that

$$(\nabla\mathbf{u}^h, \nabla\mathbf{v}^h) - (p^h, \nabla\cdot\mathbf{v}^h) + (q^h, \nabla\cdot\mathbf{u}^h) = (\mathbf{f}, \mathbf{v}^h) \quad \forall\,(\mathbf{v}^h, q^h) \in V^h \times Q^h\,. \quad (3.9)$$

Let $\boldsymbol{\phi} \in (L^2(\Omega))^d$, then the filter $\bar{\boldsymbol{\phi}} \in (H^2(\Omega))^d \cap V$ since Ω is assumed to be convex. For the error analysis, we have to consider the dual problem of (3.1) with the right-hand side $\bar{\boldsymbol{\phi}}$

$$\begin{aligned} -\Delta\boldsymbol{\psi} + \nabla(-\lambda) &= \bar{\boldsymbol{\phi}} \quad && \text{in } \Omega \\ \nabla\cdot\boldsymbol{\psi} &= 0 && \text{in } \Omega \\ \boldsymbol{\psi} &= \mathbf{0} && \text{on } \partial\Omega \\ \int_\Omega \lambda\, d\mathbf{x} &= 0\,. \end{aligned}$$

Its weak formulation is: Find $(\boldsymbol{\psi}, \lambda) \in V \times Q$ such that

$$(\nabla\boldsymbol{\psi}, \nabla\mathbf{v}) + (\lambda, \nabla\cdot\mathbf{v}) - (q, \nabla\cdot\boldsymbol{\psi}) = (\bar{\boldsymbol{\phi}}, \mathbf{v}) \quad (3.10)$$

for all $(\mathbf{v}, q) \in V \times Q$. The solution $(\boldsymbol{\psi}, \lambda)$ of (3.10) exists uniquely. We assume that the stability estimate

$$\|\boldsymbol{\psi}\|_{H^{\hat{k}+1}} + \|\lambda\|_{H^{\hat{k}}} \le C\|\bar{\boldsymbol{\phi}}\|_{H^{\hat{k}-1}} \quad (3.11)$$

holds for $\hat{k} \in \{1, 2, 3\}$.

The following theorem gives an estimate of the error between the filtered continuous velocity $\bar{\mathbf{u}}$ and the filter $\overline{\mathbf{u}^h}$ of its finite element approximation $\mathbf{u}^h$.

Theorem 3.1. *Let $\delta \ge h$ be given, let $(\mathbf{u}, p)$ be the solution of the Stokes equations* (3.2) *and $(\mathbf{u}^h, p^h)$ be its finite element approximation defined in* (3.9). *Assuming that $(\mathbf{u}, p)$ possesses the regularity given in* (3.4), *the solution of the dual problem* *satisfies the stability estimate* (3.11), *the finite element spaces have the approximation properties* (3.7), (3.8) *and the a priori error estimate* (3.6) *is valid, then there exists a constant C independent of* $\mathbf{u}$, p, h *and δ such that*

$$\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2} \leq Ch^{k+1}\left(\frac{h}{\delta}\right)^{\hat{k}-1}(\|\mathbf{u}\|_{H^{k+1}} + \|p\|_{H^k}) \tag{3.12}$$

with $\hat{k} \leq k$.

Proof. The estimate of $\bar{\mathbf{u}} - \overline{\mathbf{u}^h}$ is derived with the help of the dual problem (3.10). Using the self-adjointness property (2.6) of differential filters gives

$$(\bar{\mathbf{u}} - \overline{\mathbf{u}^h}, \boldsymbol{\phi}) = (\mathbf{u} - \mathbf{u}^h, \bar{\boldsymbol{\phi}})\,. \tag{3.13}$$

The errors $\mathbf{u} - \mathbf{u}^h$ and $p - p^h$ satisfy the Galerkin orthogonality

$$(\nabla(\mathbf{u} - \mathbf{u}^h), \nabla\mathbf{v}^h) - (p - p^h, \nabla\cdot\mathbf{v}^h) + (q^h, \nabla\cdot(\mathbf{u} - \mathbf{u}^h)) = 0 \tag{3.14}$$

for every $(\mathbf{v}^h, q^h) \in V^h \times Q^h$. Choosing $\mathbf{v} = \mathbf{u} - \mathbf{u}^h$, $q = p - p^h$ in (3.10), subtract (3.14) from the obtained equation and using (3.13) give

$$\begin{aligned}
&(\bar{\mathbf{u}} - \overline{\mathbf{u}^h}, \boldsymbol{\phi}) \\
&\quad = (\nabla(\boldsymbol{\psi} - \mathbf{v}^h), \nabla(\mathbf{u} - \mathbf{u}^h)) + (\lambda - q^h, \nabla\cdot(\mathbf{u} - \mathbf{u}^h)) - (p - p^h, \nabla\cdot(\boldsymbol{\psi} - \mathbf{v}^h)) \\
&\quad \leq (\|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} + \|p - p^h\|_{L^2})\|\nabla(\boldsymbol{\psi} - \mathbf{v}^h)\|_{L^2} + \|\lambda - q^h\|_{L^2}\|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2}
\end{aligned}$$

for every $(\mathbf{v}^h, q^h) \in V^h \times Q^h$. We pick $(\mathbf{v}^h, q^h)$ to be the best approximation of $(\boldsymbol{\psi}, \lambda)$ in $V^h \times Q^h$. Using the approximation assumptions (3.7), (3.8) for $\hat{k} \leq k$, where we exploit the regularity of $(\boldsymbol{\psi}, \lambda)$, and applying the stability estimate (3.11) yield

$$(\bar{\mathbf{u}} - \overline{\mathbf{u}^h}, \boldsymbol{\phi}) \leq Ch^{\hat{k}}(\|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} + \|p - p^h\|_{L^2})\|\bar{\boldsymbol{\phi}}\|_{H^{\hat{k}-1}}$$

for $\hat{k} \in \{1, 2, 3\}$. Now the right-hand side can be estimated by the inequalities for the filter proved in Lemmas 2.1 and 2.2 which gives

$$(\bar{\mathbf{u}} - \overline{\mathbf{u}^h}, \boldsymbol{\phi}) \leq C\frac{h^{\hat{k}}}{\delta^{\hat{k}-1}}(\|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} + \|p - p^h\|_{L^2})\|\boldsymbol{\phi}\|_{L^2}\,,$$

$\hat{k} \in \{1, 2, 3\}$. Dividing by $\|\boldsymbol{\phi}\|_{L^2}$, taking the supremum over all $\boldsymbol{\phi} \in (L^2(\Omega))^d$ with $\|\boldsymbol{\phi}\|_{L^2} \neq 0$ and applying the *a priori* error estimate (3.6) finish the proof. □

Remark 3.1. Choosing $\delta = Ch^\alpha, \alpha \in [0, 1]$ yields the order of convergence $k + 1 + (1 - \alpha)(\hat{k} - 1)$ in (3.12). This is better than the order of convergence of $\|\mathbf{u} - \mathbf{u}^h\|_{L^2}$ given in (3.6) if $\alpha < 1$ and $\hat{k} > 1$. If $\alpha = 1$, then the filter width is proportional to the mesh width. Since flow structures smaller than the mesh width cannot be approximated, it is consistent to obtain in this case the same order of convergence as for $\|\mathbf{u} - \mathbf{u}^h\|_{L^2}$. Thus, the essential condition which must be fulfilled for a higher order convergence is a higher order regularity of the solution of the dual problem.

4. The Filtered Solution of the Navier–Stokes Equations

This section extends the analysis of the Stokes equations to the Navier–Stokes equations.

The steady state Navier–Stokes equations are given by

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega\,,\\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega\,,\\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega\,,\\ \int_\Omega p\,d\mathbf{x} &= 0\,, \end{aligned} \tag{4.1}$$

where ν is the kinematic viscosity and $\mathbf{f} \in (L^2(\Omega))^d$ is given. The variational formulation of (4.1) reads as follows: Find $(\mathbf{u}, p) \in V \times Q$ such that

$$(\nu\nabla\mathbf{u}, \nabla\mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + (q, \nabla\cdot\mathbf{u}) = (\mathbf{f}, \mathbf{v}) \quad \forall\,(\mathbf{v}, q) \in V \times Q \tag{4.2}$$

with the trilinear form

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w})\,.$$

We assume that there is a unique solution $(\mathbf{u}, p)$ of (4.2). This is known to be true if ν is sufficiently large or $\|\mathbf{f}\|_{H^{-1}}$ is sufficiently small, see Ref. 7, Chap. IV.

We consider the Galerkin finite element approximation of (4.2): Find $(\mathbf{u}^h, p^h) \in V^h \times Q^h$ such that

$$(\nu\nabla\mathbf{u}^h, \nabla\mathbf{v}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - (p^h, \nabla\cdot\mathbf{v}^h) + (q^h, \nabla\cdot\mathbf{u}^h) = (\mathbf{f}, \mathbf{v}^h) \tag{4.3}$$

for all $(\mathbf{v}^h, q^h) \in V^h \times Q^h$. If ν is sufficiently large, optimal error estimates of the form (3.6) can be proved for this finite element method. For the error analysis of $\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2}$, we have to consider a linearized adjoint problem of (4.1). The variational formulation of this linearized adjoint problem is: Find $(\boldsymbol{\psi}, \lambda) \in V \times Q$ such that

$$(\nu\nabla\boldsymbol{\psi}, \nabla\mathbf{v}) + b(\mathbf{u}, \mathbf{v}, \boldsymbol{\psi}) + b(\mathbf{v}, \mathbf{u}, \boldsymbol{\psi}) + (\lambda, \nabla\cdot\mathbf{v}) - (q, \nabla\cdot\boldsymbol{\psi}) = (\bar{\boldsymbol{\phi}}, \mathbf{v}) \tag{4.4}$$

for all $(\mathbf{v}, q) \in V \times Q$. We assume that the solution of (4.4) exists uniquely and fulfils a stability estimate of form (3.11).

The following theorem shows that an error estimate of form (3.12) holds also for the Navier–Stokes equations.

Theorem 4.1. *Let $\delta \geq h$ be given, let $(\mathbf{u}, p)$ be the solution of the Navier–Stokes equations* (4.2) *and $(\mathbf{u}^h, p^h)$ be its finite element approximation defined in* (4.3)*. Assuming that $(\mathbf{u}, p)$ possesses the regularity given in* (3.4)*, the solution of the linearized dual problem* (4.4) *satisfies the stability estimate* (3.11)*, the finite element spaces have the approximation properties* (3.7), (3.8) *and the a priori error estimate* (3.6) *is valid, then there exists a constant C independent of* $\mathbf{u}$, p, h *and* δ *such that*

$$\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2} \leq C h^{k+1} \left(\frac{h}{\delta}\right)^{\hat{k}-1} (\|\mathbf{u}\|_{H^{k+1}} + \|p\|_{H^k}) \tag{4.5}$$

with $\hat{k} \leq k$.

Proof. Starting as in the proof of Theorem 3.1, one obtains

$$(\bar{\mathbf{u}} - \overline{\mathbf{u}^h}, \boldsymbol{\phi}) = (\nu\nabla(\boldsymbol{\psi} - \mathbf{v}^h), \nabla(\mathbf{u} - \mathbf{u}^h)) + b(\mathbf{u}, \mathbf{u} - \mathbf{u}^h, \boldsymbol{\psi}) + b(\mathbf{u} - \mathbf{u}^h, \mathbf{u}, \boldsymbol{\psi})$$
$$- b(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) + b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + (\lambda - q^h, \nabla \cdot (\mathbf{u} - \mathbf{u}^h))$$
$$- (p - p^h, \nabla \cdot (\boldsymbol{\psi} - \mathbf{v}^h)) \tag{4.6}$$

for every $(\mathbf{v}^h, q^h) \in V^h \times Q^h$. A direct calculation gives

$$b(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) = b(\mathbf{u}, \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{u} - \mathbf{u}^h, \mathbf{u}, \mathbf{v}^h)$$
$$- b(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h)\,. \tag{4.7}$$

This is substituted into (4.6). The terms in (4.6) which also appear in the Stokes equations are estimated analogously as in the proof of Theorem 3.1.

We consider now the trilinear terms. Let $(\mathbf{v}^h, q^h)$ be the best approximation of $(\boldsymbol{\psi}, \lambda)$ in $V^h \times Q^h$. Using Hölder's inequality, the Sobolev imbeddings $H^1(\Omega) \to L^6(\Omega)$ and $H^{1/2}(\Omega) \to L^3(\Omega)$, the interpolation of $H^{1/2}(\Omega)$ between $L^2(\Omega)$ and $H^1(\Omega)$ (see Ref. 1, Theorem 4.17) and the interpolation error estimate (3.7) for $\hat{k} \le k$ give

$$\begin{aligned}
|b(\mathbf{u}, \mathbf{u} - \mathbf{u}^h, \boldsymbol{\psi} - \mathbf{v}^h)| &\le \|\mathbf{u}\|_{L^6} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} \|\boldsymbol{\psi} - \mathbf{v}^h\|_{L^3}\\
&\le C\|\nabla\mathbf{u}\|_{L^2} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} \|\boldsymbol{\psi} - \mathbf{v}^h\|_{H^{1/2}}\\
&\le C\|\nabla\mathbf{u}\|_{L^2} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} \|\boldsymbol{\psi} - \mathbf{v}^h\|_{L^2}^{1/2} \|\nabla(\boldsymbol{\psi} - \mathbf{v}^h)\|_{L^2}^{1/2}\\
&\le Ch^{\hat{k}+1/2} \|\nabla\mathbf{u}\|_{L^2} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} \|\boldsymbol{\psi}\|_{H^{\hat{k}+1}}
\end{aligned}$$

and in the same way

$$|b(\mathbf{u} - \mathbf{u}^h, \mathbf{u}, \boldsymbol{\psi} - \mathbf{v}^h)| \le Ch^{\hat{k}+1/2} \|\nabla\mathbf{u}\|_{L^2} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} \|\boldsymbol{\psi}\|_{H^{\hat{k}+1}}\,.$$

The last trilinear term in (4.7) is split into two parts

$$b(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) = b(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h - \boldsymbol{\psi}) + b(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h, \boldsymbol{\psi})\,,$$

where the first one can be estimated in the same way as above:

$$|b(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h - \boldsymbol{\psi})| \le Ch^{\hat{k}+1/2} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2}^2 \|\boldsymbol{\psi}\|_{H^{\hat{k}+1}}\,.$$

The estimate of the second term uses the Sobolev imbedding $H^{\hat{k}+1}(\Omega) \to L^\infty(\Omega)$, $\hat{k} \ge 1$,

$$\begin{aligned}
|b(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h, \boldsymbol{\psi})| &\le \|\mathbf{u} - \mathbf{u}^h\|_{L^2} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} \|\boldsymbol{\psi}\|_{L^\infty}\\
&\le C\|\mathbf{u} - \mathbf{u}^h\|_{L^2} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_{L^2} \|\boldsymbol{\psi}\|_{H^{\hat{k}+1}}\,.
\end{aligned}$$

Collecting all estimates, using the stability (3.11), the estimates for the filter given in Lemmas 2.1 and 2.2 and the *a priori* error estimate (3.6) give

$$\begin{aligned}
&\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2}\\
&\quad\le C\left(\frac{h}{\delta}\right)^{\hat{k}-1} h^{k+1}(1 + h^{1/2}\|\nabla\mathbf{u}\|_{L^2} + h^{k+1/2} + h^{2k+1})(\|\mathbf{u}\|_{H^{k+1}} + \|p\|_{H^k})\,.
\end{aligned}$$

Since $\|\nabla \mathbf{u}\|_{L^2} < \infty$ by the smoothness assumptions on $\mathbf{u}$, it can be seen that the terms coming from the trilinear forms are of higher order. □

5. The Finite Element Approximation of the Filtered Functions

In general, it will be possible to compute only finite-dimensional approximations $\tilde{\mathbf{u}}$ and $\widetilde{\mathbf{u}^h}$ of the filtered functions $\bar{\mathbf{u}}$ and $\overline{\mathbf{u}^h}$, respectively. This section provides an estimate of $\|\tilde{\mathbf{u}} - \widetilde{\mathbf{u}^h}\|_{L^2}$ and shows that the order of convergence is the same as for $\|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{L^2}$.

We pick a finite element space $V^H \subset V$ which is connected with a triangulation $\mathcal{T}^H$ and a mesh parameter H. Let $\tilde{\mathbf{u}} \in V^H$ be the Galerkin approximation of $\bar{\mathbf{u}}$ satisfying

$$\delta^2(\nabla \tilde{\mathbf{u}}, \nabla \mathbf{v}^H) + (\tilde{\mathbf{u}}, \mathbf{v}^H) = (\mathbf{u}, \mathbf{v}^H) \quad \forall\, \mathbf{v}^H \in V^H \tag{5.1}$$

and $\widetilde{\mathbf{u}^h} \in V^H$ be the Galerkin approximation of $\overline{\mathbf{u}^h}$ satisfying

$$\delta^2(\nabla \widetilde{\mathbf{u}^h}, \nabla \mathbf{v}^H) + (\widetilde{\mathbf{u}^h}, \mathbf{v}^H) = (\mathbf{u}^h, \mathbf{v}^H) \quad \forall\, \mathbf{v}^H \in V^H\,. \tag{5.2}$$

Lemma 5.1. *Let* $(\mathbf{u}, p)$ *be the solution of the Stokes equations* (3.2) *or of the Navier–Stokes equations* (4.2), *respectively. Let the a priori error estimate* (3.6) *be fulfilled, then*

$$\|(\bar{\mathbf{u}} - \overline{\mathbf{u}^h}) - (\tilde{\mathbf{u}} - \widetilde{\mathbf{u}^h})\|_{L^2} \le C \left(\frac{H}{\delta}\right)^2 h^{k+1} \|\mathbf{u}\|_{H^{k+1}}\,. \tag{5.3}$$

Proof. Subtraction of (5.1) and (5.2) gives

$$\delta^2(\nabla(\tilde{\mathbf{u}} - \widetilde{\mathbf{u}^h})), \nabla \mathbf{v}^H) + (\tilde{\mathbf{u}} - \widetilde{\mathbf{u}^h}, \mathbf{v}^H) = (\mathbf{u} - \mathbf{u}^h, \mathbf{v}^H)$$

for all $\mathbf{v}^H \in V^H$. This is the Galerkin finite element formulation of

$$\begin{aligned} -\delta^2 \Delta \Theta + \Theta &= \mathbf{u} - \mathbf{u}^h \quad &&\text{in } \Omega\,, \\ \Theta &= \mathbf{0} &&\text{on } \partial\Omega \end{aligned} \tag{5.4}$$

in V^H. The solution of this problem is $\Theta = \bar{\mathbf{u}} - \overline{\mathbf{u}^h}$. Since $\mathbf{u}, \mathbf{u}^h \in (H_0^1(\Omega))^d$, we have at least $\Theta \in (H^2(\Omega))^d$. The standard finite element estimate for (5.4) gives

$$\|\Theta - (\tilde{\mathbf{u}} - \widetilde{\mathbf{u}^h})\|_{L^2} = \|(\bar{\mathbf{u}} - \overline{\mathbf{u}^h}) - (\tilde{\mathbf{u}} - \widetilde{\mathbf{u}^h})\|_{L^2} \le C H^2 \|\bar{\mathbf{u}} - \overline{\mathbf{u}^h}\|_{H^2}\,. \tag{5.5}$$

Note that for $\delta \to 0$, (5.1) and (5.2) are just the standard L^2-projections of $\mathbf{u}$ and $\mathbf{u}^h$ into V^H. From the L^2-stability of the L^2-projection, which holds with $C = 1$, follows that the constant C in (5.5) tends to one for $\delta \to 0$. Thus, it can be chosen independently of δ.

The proof is finished by applying (2.4) to the right-hand side of (5.5) and by using the *a priori* estimate (3.6). □

The triangle inequality applied to (5.3) and the error estimates (3.12) and (4.5) give

Theorem 5.1. *Let the assumptions of Theorem* 3.1 *for the Stokes equations or of Theorem* 4.1 *for the Navier–Stokes equations be fulfilled. Then, the* L^2*-error of the finite element approximations* $\tilde{\mathbf{u}}$ *and* $\widetilde{\mathbf{u}^h}$ *can be estimated as follows*

$$\|\tilde{\mathbf{u}} - \widetilde{\mathbf{u}^h}\|_{L^2} \leq Ch^{k+1}\left(\left(\frac{h}{\delta}\right)^{\hat{k}-1} + \left(\frac{H}{\delta}\right)^2\right)(\|\mathbf{u}\|_{H^{k+1}} + \|p\|_{H^k}) \tag{5.6}$$

with $\hat{k} \leq k$.

The most convenient choice is $V^H = V^h$ because one can use the same mesh for computing the solution of the Stokes or Navier–Stokes equations and the finite element approximations of the filters. Estimate (5.6) shows that this choice is sufficient to retain the same order of convergence as given in Remark 3.1.

6. Numerical Studies

This section presents numerical examples which support the analytical results. This will be done only for the Navier–Stokes equations since we obtained similar results for the Stokes equations.

The finite element spaces $V^h = Q_k$, $Q^h = P_{k-1}^{\text{disc}}$, $k \in \{2,3\}$ were used for the computations on the quadrilateral and hexahedral grid. That means, the velocity is approximated by a continuous function of order k and the pressure by a discontinuous function of order $k-1$. These finite element spaces are conforming. On the triangular and tetrahedral grid, the computations were carried out with the Taylor–Hood finite element $V^h = P_2, Q^h = P_1$. All used pairs of finite element spaces fulfil the inf–sup condition (3.3), see Refs. 7 and 13. The finite element space for approximating the filter was chosen to be $V^H = V^h$. The computations were performed with the code MooNMD, see Refs. 9 and 11.

Example 6.1. 2d *example supporting the error estimate* (5.6). We consider the Navier–Stokes equations (4.1) in $\Omega = (0,1)^2$ with $\nu = 0.1$ and with the prescribed solution $\mathbf{u} = (u_1, u_2)$ and p given by

$$\begin{aligned} u_1 &= 2\pi \sin^3(\pi x)\sin(\pi y)\cos(\pi y)\,, \\ u_2 &= -3\pi \sin^2(\pi x)\cos(\pi x)\sin^2(\pi y)\,, \\ p &= \cos(\pi x) + \cos(\pi y)\,. \end{aligned}$$

This problem fits exactly into the framework of the analysis presented in this paper.

Tables 1–3 present results for finite element discretizations with velocity spaces of order k and pressure spaces of order $k-1$, $k \in \{2,3\}$ and different values of δ. The computations were carried out on quadrilateral and triangular grids. The initial grids are presented in Fig. 1. Because of their irregularity, higher order rates of convergence as an effect of superconvergence can be excluded. The number of degrees of freedom on the finest level is nearly $725{,}000$ for the Q_2/P_1^{disc} finite element, nearly $400{,}000$ for the Q_3/P_2^{disc} finite element and about $665{,}000$ for the

Table 1. Example 6.1, Q_2/P_1^{disc} finite element discretization, order of convergence for different choices of δ.

Level	$\|\mathbf{u}-\mathbf{u}^h\|_{L^2}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = h^{0.5}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = 0.5$	
0	1.145145e-1		2.055072e-2		2.654022e-2	
1	1.377608e-2	3.055	1.800340e-3	3.513	1.603269e-3	4.049
2	1.741729e-3	2.984	1.568772e-4	3.521	1.040759e-4	3.945
3	2.184065e-4	2.995	1.226431e-5	3.677	6.596570e-6	3.980
4	2.732056e-5	2.999	8.850454e-7	3.793	4.138766e-7	3.994
5	3.415639e-6	3.000	6.077102e-8	3.864	2.589344e-8	3.999
6	4.269705e-7	3.000	4.060803e-9	3.904	1.618760e-9	4.000
Theory		3.000		3.500		4.000

Table 2. Example 6.1, P_2/P_1 finite element discretization, order of convergence for different choices of δ.

Level	$\|\mathbf{u}-\mathbf{u}^h\|_{L^2}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = h^{0.5}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = 0.5$	
0	1.084089e-1		1.691105e-2		2.400509e-2	
1	1.927985e-2	2.491	4.175891e-3	2.018	3.771400e-3	2.670
2	2.584938e-3	2.899	4.400397e-4	3.246	3.094694e-4	3.607
3	3.281124e-4	2.978	3.492728e-5	3.655	2.019232e-5	3.938
4	4.130777e-5	2.990	2.571142e-6	3.764	1.276136e-6	3.984
5	5.181897e-6	2.995	1.820701e-7	3.820	8.011974e-8	3.993
6	6.488892e-7	2.997	1.262811e-8	3.850	5.018199e-9	3.997
Theory		3.000		3.500		4.000

Table 3. Example 6.1, Q_3/P_2^{disc} finite element discretization, order of convergence for different choices of δ.

Level	$\|\mathbf{u}-\mathbf{u}^h\|_{L^2}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = h^{0.5}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = 0.5$	
0	1.125285e-2		9.517286e-4		1.271797e-3	
1	9.467422e-4	3.571	4.810401e-5	4.306	4.022989e-5	4.982
2	6.054851e-5	3.967	1.342504e-6	5.163	6.530215e-7	5.945
3	3.809419e-6	3.990	3.806302e-8	5.140	1.030669e-8	5.985
4	2.384798e-7	3.998	1.108255e-9	5.102	1.613809e-10	5.997
5	1.491051e-8	3.999	3.305694e-11	5.067	2.524241e-12	5.998
Theory		4.000		5.000		6.000

P_2/P_1 finite element. The discrete Navier–Stokes problems were solved up to a Euclidean norm of the residual vector less than 10^{-12}.

The order of convergence depends on the regularity $\hat{k}$ of the dual problem given in (3.11). In the case of a polygonal domain, one can expect in general only $\hat{k} = 1$. However, the regularity might be higher for particular examples. The order of convergence obtained in the numerical studies suggest that $\hat{k} = 3$ in Example 6.1,

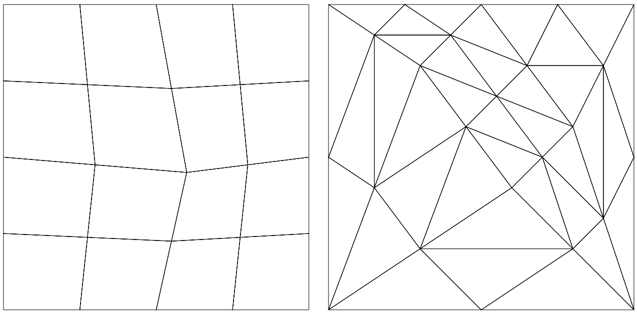

Fig. 1. Example 6.1, initial grids (level 0).

see also Ref. 4. Let $\delta = ch^\alpha, \alpha \in [0,1]$, then the order of convergence for $\|\tilde{\mathbf{u}} - \widetilde{\mathbf{u}^h}\|_{L^2}$ which can be expected from (5.6) is $h^{4-\alpha}$ if $k = 2$ and $h^{6-2\alpha}$ if $k = 3$.

The numerical studies presented in Tables 1–3 show that the expected and the computed order of convergence coincide very well in the case of constant δ. For $\delta = h^\alpha, \alpha \in (0,1)$, the computed order of convergence is in some cases higher than its expected asymptotic value. The asymptotic range might not be reached yet. If the grids are refined further, we observed that the influence of round off errors becomes visible. In all cases, the order of convergence of the filtered error is higher than the order of $\|\mathbf{u} - \mathbf{u}^h\|_{L^2}$.

Example 6.2. 3*d example.* This example shows also that in 3d the filtered L^2-error might converge faster than the L^2-error.

Let $\Omega = (0,1)^3$ and let $\mathbf{u} = (u_1, u_2, u_3)$ and p given by

$$\begin{aligned}
u_1(x,y,z) &= \sin(\pi x)\sin(\pi y)\sin(\pi z) + x^4\cos(\pi y)\\
u_2(x,y,z) &= \cos(\pi x)\cos(\pi y)\cos(\pi z) - 3y^3 z\\
u_3(x,y,z) &= \cos(\pi x)\sin(\pi y)\cos(\pi z) + \cos(\pi x)\sin(\pi y)\sin(\pi z)\\
&\quad - 4x^3 z\cos(\pi y) + 4.5y^2z^2\\
p(x,y,z) &= 3x - \sin(y + 4z) + c\,.
\end{aligned}$$

The constant c is chosen such that $p \in L_0^2(\Omega)$ and the right-hand side $\mathbf{f}$ is chosen such that $(\mathbf{u}, p)$ fulfil the Navier–Stokes equations (4.1) with $\nu = 0.1$. In contrast to Example 6.1, we have here a solution with nonhomogeneous Dirichlet boundary conditions.

The computations were carried out with the Q_2/P_1^{disc} finite element discretization on a hexahedral grid and with the P_2/P_1 finite element discretization on a tetrahedral grid. The initial hexahedral grid (level 0) consists of eight cubes of edge length 0.5. On level 5, there are nearly 7.5 million degrees of freedom. The initial tetrahedral grid consists of six tetrahedra. On level 5, the system has about 6.7 million degrees of freedom.

Table 4. Example 6.2, Q_2/P_1^{disc} finite element discretization, order of convergence for different choices of δ.

Level	$\|\mathbf{u}-\mathbf{u}^h\|_{L^2}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = h^{0.5}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = 0.5$	
0	3.742528e-2		5.150531e-3		9.744671e-3	
1	4.478076e-3	3.063	5.664402e-4	3.185	7.137835e-4	3.771
2	5.343528e-4	3.067	4.930463e-5	3.522	4.687405e-5	3.929
3	6.569613e-5	3.024	3.848715e-6	3.679	2.994656e-6	3.968
4	8.173159e-6	3.007	2.787300e-7	3.787	1.883923e-7	3.991
5	1.020367e-6	3.002	1.920575e-8	3.859	1.179532e-8	3.997
Theory		3.000		3.500		4.000

Table 5. Example 6.2, P_2/P_1 finite element discretization, order of convergence for different choices of δ.

Level	$\|\mathbf{u}-\mathbf{u}^h\|_{L^2}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = h^{0.4}$		$\|\tilde{\mathbf{u}}-\widetilde{\mathbf{u}^h}\|_{L^2}$, $\delta = 0.5$	
0	1.636325e-1		1.953558e-2		4.525972e-2	
1	2.324915e-2	2.815	3.776680e-3	2.371	5.918700e-3	2.935
2	2.744844e-3	3.082	4.247209e-4	3.153	4.806004e-4	3.622
3	3.297820e-4	3.057	3.835012e-5	3.469	3.430517e-5	3.808
4	4.055963e-5	3.023	3.023162e-6	3.665	2.270324e-6	3.917
5	5.037018e-6	3.009	2.192793e-7	3.785	1.450954e-7	3.968
Theory		3.000		3.600		4.000

The computational results presented in Tables 4 and 5 suggest that the regularity of the dual problem is $\hat{k} \geq 2$.

For both finite element discretization, the expected and the computed orders of convergence coincide very well if δ is constant. For $\delta = h^{\alpha}$, $\alpha < 1$, the computed order of convergence is somewhat higher than the predicted asymptotic value. There is always a higher order of convergence of the filtered errors than of $\|\mathbf{u}-\mathbf{u}^h\|_{L^2}$.

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References

1. R. A. Adams, *Sobolev Spaces* (Academic Press, 1975).
2. D. N. Arnold, D. Boffi and R. S. Falk, Approximation by quadrilateral finite elements, *Math. Comp.* **71** (2002) 909–922.
3. P. G. Ciarlet, Basic error estimates for elliptic problems, in *Handbook of Numerical Analysis II*, eds. P. G. Ciarlet and J. L. Lions (North-Holland, 1991), pp. 19–351.
4. A. Dunca, V. John and W. J. Layton, Approximating local averages of fluid velocities: The equilibrium Navier–Stokes equations, to appear in *Appl. Numer. Math.*

5. M. Germano, Differential filters for the large eddy numerical simulation of turbulent flows, *Phys. Fluids* **29** (1986) 1755–1757.
6. M. Germano, Differential filters of elliptic type, *Phys. Fluids* **29** (1986) 1757–1758.
7. V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes Equations* (Springer-Verlag, 1986).
8. P. Grisvard, *Elliptic Problems in Nonsmooth Domains* (Pitman, 1985).
9. V. John, *Large Eddy Simulation of Turbulent Incompressible Flows. Analytical and Numerical Results for a Class of LES Models*, Lecture Notes in Computational Science and Engineering, Vol. 34 (Springer-Verlag, 2003).
10. V. John and W. J. Layton, Approximating local averages of fluid velocities: The Stokes problem, *Computing* **66** (2001) 269–287.
11. V. John and G. Matthies, MooNMD — a program package based on mapped finite element methods, *Comput. Visual. Sci.* **6** (2004) 163–170.
12. G. Matthies, Mapped finite elements on hexahedra. Necessary and sufficient conditions for optimal interpolation errors, *Numer. Alg.* **27** (2001) 317–327.
13. G. Matthies and L. Tobiska, The inf–sup condition for the mapped $Q_k/P_{k-1}^{\mathrm{disc}}$ element in arbitrary space dimensions, *Computing* **69** (2002) 119–139.
14. J. S. Mullen and P. F. Fischer, Filtering techniques for complex geometry fluid flows, *Comm. Numer. Meth. Engrg.* **15** (1999) 9–18.
15. P. Sagaut, *Large Eddy Simulation for Incompressible Flows* (Springer-Verlag, 2001).
16. Y. Zhou, M. Hossain and G. Vahala, A critical look at the use of filters in large eddy simulation, *Phys. Lett.* **A139** (1989) 330–332.