MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2006; **29**:1709–1719
Published online 19 May 2006 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/mma.750
MOS subject classification: 76 F 65; 35 Q 30

Asymptotic behaviour of commutation errors and the divergence of the Reynolds stress tensor near the wall in the turbulent channel flow

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Communicated by W. Sprößig

SUMMARY

The derivation of the space averaged Navier–Stokes equations for the large eddy simulation (LES) of turbulent incompressible flows introduces two groups of terms which do not depend only on the space averaged flow field variables: the divergence of the Reynolds stress tensor and commutation errors. Whereas the former is studied intensively in the literature, the latter terms are usually neglected. This note studies the asymptotic behaviour of these terms for the turbulent channel flow at a wall in the case that the commutation errors arise from the application of a non-uniform box filter. To perform analytical calculations, the unknown flow field is modelled by a wall law (Reichardt law and $1/\alpha$th power law) for the mean velocity profile and highly oscillating functions model the turbulent fluctuations. The asymptotics show that near the wall, the commutation errors are at least as important as the divergence of the Reynolds stress tensor. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: incompressible turbulent channel flow; large eddy simulation (LES); commutation errors; wall laws; non-uniform filters

1. INTRODUCTION: COMMUTATION ERRORS IN LARGE EDDY SIMULATION (LES)

Turbulent incompressible flows are governed by the incompressible Navier–Stokes equations

$$\begin{aligned} \mathbf{u}_t - 2Re^{-1}\nabla\cdot\mathbb{D}(\mathbf{u}) + \nabla\cdot(\mathbf{u}\mathbf{u}^{\mathrm{T}}) + \nabla p = \mathbf{f} \quad &\text{in } (0,T]\times\Omega \\ \nabla\cdot\mathbf{u} = 0 \quad &\text{in } (0,T]\times\Omega \end{aligned} \tag{1}$$

The vector field $\mathbf{u}=(u_1,u_2,u_3)^{\mathrm{T}}$ is the velocity, p the pressure, $Re>0$ the Reynolds number, $\Omega\subset\mathbb{R}^3$ a bounded domain, T a positive time, $\mathbf{f}$ the external force, and $\mathbb{D}(\mathbf{u})$ the velocity

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 Received 31 January 2006

deformation tensor

$$\mathbb{D}(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}}{2}$$

We prefer to write the non-linear term in its conservative form with the tensor

$$(\mathbf{u}\mathbf{u}^{\mathrm{T}})_{ij} := u_i u_j, \quad i,j = 1,2,3$$

since with this form the problems arising in space-averaging are simpler to identify. Equation (1) must be also equipped with boundary conditions on $\partial\Omega$ and an initial condition.

We recall that presently a direct numerical simulation (DNS) is not possible in the high Reynolds number case, due to the extreme richness of scales in turbulent flows, e.g. see Reference [1]. This is the reason why turbulence modelling becomes necessary. A popular approach is large eddy simulation (LES), where one seeks to simulate the behaviour of large flow structures $(\overline{\mathbf{u}}, \overline{p})$ and to model the influence of the turbulent fluctuations $(\mathbf{u}', p') = (\mathbf{u}, p) - (\overline{\mathbf{u}}, \overline{p})$ on the formers. Besides numerous promising numerical results obtained with LES models, another appealing feature of these models is the chance of achieving rigorous mathematical support, e.g. see References [2–5].

Generally, large flow structures in LES are defined by convolution with a filter function. In this note, we consider the symmetric box (or top-hat) filter with non-uniform filter width. We make this choice since the box filter is one of the most popular filters in LES. The usual form of such a filter in multiple dimensions is a tensor product of one-dimensional filters, see References [6,7]. Let $\mathbf{x} = (x, y, z) \subset \mathbb{R}^3$ and $\delta(x)$, $\delta(y)$, $\delta(z)$ be the non-uniform filter widths in the respective co-ordinate directions. Then, the non-uniform box filter $\overline{u}(\mathbf{y})$ of a scalar function $u(\mathbf{x})$ is given by

$$\overline{u}(\mathbf{y}) = \frac{1}{8\delta(x)\delta(y)\delta(z)} \int_{x-\delta(x)}^{x+\delta(x)} \int_{y-\delta(y)}^{y+\delta(y)} \int_{z-\delta(z)}^{z+\delta(z)} u(\mathbf{x})\, \mathrm{d}\mathbf{x} \tag{2}$$

An appealing advantage of non-uniform filters with compact filter kernel is that the filter width can be chosen such that the domain of filtering is always inside Ω. Hence, an extension of functions outside Ω is not necessary for the filtering to be well-defined. This property is desirable for all boundary conditions which do not allow a physically motivated extension outside Ω of the functions appearing in (1). We refer to Reference [8] for the study of a situation where the functions are extended off the domain.

In order to enforce the domain of filtering being always inside Ω, the volume of the filter box with centre $\mathbf{x}$ necessarily has to tend to zero at the boundary. This property does not hold if the point $\mathbf{x}$, where the filter is applied, is not the centre of the filter box. This kind of filter is called non-symmetric or skewed. We will not consider them in this note since such filters introduce commutation errors that are considerably larger than for non-uniform symmetric filters, see References [9,10].

The usual way of deriving equations for $(\overline{\mathbf{u}}, \overline{p})$ is first to filter the Navier–Stokes equations (1) leading to the space averaged Navier–Stokes equations in the space–time domain $(0, T] \times \Omega$:

$$\begin{aligned} \overline{\mathbf{u}}_t - 2Re^{-1}\nabla \cdot \mathbb{D}(\overline{\mathbf{u}}) + \nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\mathrm{T}}) + \nabla \overline{p} &= \overline{\mathbf{f}} + \nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\mathrm{T}} - \overline{\mathbf{u}\mathbf{u}^{\mathrm{T}}}) \\ \nabla \cdot \overline{\mathbf{u}} &= 0 \end{aligned} \tag{3}$$

Then, the divergence of the (subgrid-scale) Reynolds stress tensor $\mathscr{R}(\mathbf{u},\mathbf{u}) = \overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\mathrm{T}} - \overline{\mathbf{u}\mathbf{u}^{\mathrm{T}}}$ must be modelled in terms of $(\overline{\mathbf{u}}, \overline{p})$, see References [2,7]. LES modelling considers the second step almost exclusively.

A crucial assumption made during the first step is that of commutation between differentiation and convolution. This is true in the case of filters with constant filter width, as those freely used in problems defined on the whole space, e.g. by imposing periodic boundary conditions. However, this assumption is in general not satisfied. In particular, it fails if Ω is a bounded domain, hence if the width of a filter (with the domain of filtering always inside Ω) must be allowed to vary. In this respect, we recall that considering a bounded domain is the standard case in applications, since periodic or Cauchy problems are just mathematical idealizations.

We now define the commutation error we will deal with. Let $u : \Omega \to \mathbb{R}$ denote a generic function to be filtered (a velocity component or the pressure). Then, the so-called *commutation error* with respect to the ith partial derivative, is defined by

$$\mathscr{E}_{\mathrm{c}}(\partial_i u) := \partial_i \overline{u} - \overline{\partial_i u} \tag{4}$$

In the last decade, this error attracted increasing attention from the scientific community, see References [4,8,10–15], and its role is becoming better understood.

As a natural counterpart, a boundary commutation error (BCE) is committed if a filter with constant filter width is used in a bounded domain, as analysed in References [4,8]. The well-posedness of the filtering requires in this case an extension of $(\mathbf{u}, p)$ outside Ω. It is shown in Reference [8] that no commutation error is committed if the space averaged Navier–Stokes equations are derived in the framework of distributions. However, in comparison to the form (3), which is obtained by simply interchanging differentiation and convolution, an additional term appears. From the latter it is possible to have an analytical expression for the BCE term. The analysis of this term shows that it is large near the boundary and it does not even vanish in the sense of the Lebesgue spaces L^p, $p \in [1, \infty]$, as the filter width tends to zero. Thus, the commutation error committed (and usually omitted) in this case is of considerable importance in a neighbourhood of the boundary. For the model problem of the space averaged heat equation with $\delta = \mathrm{const}$, numerical studies in Reference [16] show that modelling the BCE is crucial for reducing the error at the boundaries.

Since there is no way to eliminate these sources of error in the derivation of LES equations, in Reference [9] enhanced estimates were proved for the commutation error in the case of non-uniform filters. The main point of interest of these estimates is that, contrary to other papers concerning this topic, they do not require strong regularity assumptions on the velocity $\mathbf{u}$ or a special form of the kernel.

Straightforward calculations show that applying a non-uniform filter to the Navier–Stokes equations (1) and taking the commutation error into account give, instead of (3),

$$\begin{aligned}
\overline{\mathbf{u}}_t - 2Re^{-1}\nabla\cdot\,\mathbb{D}(\overline{\mathbf{u}}) + \nabla\cdot\,(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\mathrm{T}}) + \nabla\overline{p}
&= \overline{\mathbf{f}} + \nabla\cdot\,(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\mathrm{T}} - \overline{\mathbf{u}\mathbf{u}^{\mathrm{T}}}) - 2Re^{-1}\nabla\cdot\,\mathscr{E}_{\mathrm{c}}(\mathbb{D}(\mathbf{u})) \\
&\quad - 2Re^{-1}\mathscr{E}_{\mathrm{c}}(\nabla\cdot\,\mathbb{D}(\mathbf{u})) + \mathscr{E}_{\mathrm{c}}(\nabla\cdot\,(\mathbf{u}\mathbf{u}^{\mathrm{T}})) + \mathscr{E}_{\mathrm{c}}(\nabla p) \\
\nabla\cdot\,\overline{\mathbf{u}} &= \mathscr{E}_{\mathrm{c}}(\nabla\cdot\,\mathbf{u})
\end{aligned} \tag{5}$$

in $(0,T]\times\Omega$. For simplicity, we use the symbol $\mathscr{E}_{\rm c}$ to denote the sum of all commutation errors involving a given differential operator. For instance, $\mathscr{E}_{\rm c}(\nabla\cdot\mathbf{u})$ involves all the terms coming from the filtering of the divergence operator applied to $\mathbf{u}$.

Note that the commutation error terms coming from the viscous term are scaled by Re^{-1} which is, for turbulent flows, a small non-dimensional factor.

On the right-hand side of the equations in (5), there are two groups of terms which do not depend just on $(\overline{\mathbf{u}},\overline{p})$, but depend also on the unfiltered variables: the divergence of the Reynolds stress tensor

$$\nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^{\rm T}-\overline{\mathbf{u}\mathbf{u}^{\rm T}}) \tag{6}$$

and the commutation errors

$$\begin{gathered}-2Re^{-1}\nabla\cdot\mathscr{E}_{\rm c}(\mathbb{D}(\mathbf{u}))-2Re^{-1}\mathscr{E}_{\rm c}(\nabla\cdot\mathbb{D}(\mathbf{u}))+\mathscr{E}_{\rm c}(\nabla\cdot(\mathbf{u}\mathbf{u}^{\rm T}))+\mathscr{E}_{\rm c}(\nabla p)\\ \mathscr{E}_{\rm c}(\nabla\cdot\mathbf{u})\end{gathered} \tag{7}$$

By far, most of the LES literature neglects the commutation errors. However, there is first numerical evidence that commutation errors are of importance, see Reference [10]. Simulations of the turbulent mixing layer problem show that the commutation errors may be large, e.g. for non-uniform filters with sharp variations in the filter width. In Reference [10], exclusively commutation errors away from solid boundaries are considered and two models for them are proposed and tested.

A complete analysis of the relation between the commutation errors and the divergence of the Reynolds stress tensor requires analytical expressions for the filter widths $\delta(x)$, $\delta(y)$, $\delta(z)$, and the velocity field $\mathbf{u}(\mathbf{x})$. Whereas $\delta(x)$, $\delta(y)$, and $\delta(z)$ can be prescribed, it seems impossible to obtain such an expression for $\mathbf{u}(\mathbf{x})$, an instantaneous velocity field in a turbulent flow.

To circumvent this problem, we use in this note known mean velocity profiles, which are perturbed with highly oscillating functions. The asymptotic behaviours of the divergence of the Reynolds stress tensor (6) and the commutation errors (7) are studied for the turbulent channel flow at a solid wall. The turbulent channel flow is a classical benchmark problem for the numerical study of turbulent flows [1,17]. Two wall laws are considered as mean velocity profiles, the Reichardt law and the $1/\alpha$th power law. We do not consider the Prandtl–Taylor law, which is linear near the wall, since a direct evaluation of the commutation error for the non-uniform box filter shows that this error vanishes identically in the case of a linear function.

2. ASYMPTOTIC ANALYSIS OF THE COMMUTATION ERRORS AND THE DIVERGENCE OF THE REYNOLDS STRESS TENSOR FOR THE TURBULENT CHANNEL FLOW

The turbulent channel flow is defined in a hexahedral domain $\Omega=(0,L)\times(0,2H)\times(0,b)$, where H is the channel half-width and L and b are chosen to be sufficiently large see, e.g. References [1,17]. Periodic boundary conditions are imposed in the x–z-directions (for simplicity, assume that after rescaling b is an integer multiple of 2π), while no-slip boundary conditions are given at $y=0$ and $y=2H$. This is a classical numerical test case and it can be easily shown that, under minimal assumptions [1], the solution $\mathbf{u}=(u_1,u_2,u_3)^{\mathrm{T}}=(u,v,w)^{\mathrm{T}}$ is of the form

$$\mathbf{u}=\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} U(y)+u'(t;x,y,z) \\ v'(t;x,y,z) \\ w'(t;x,y,z) \end{pmatrix}, \quad p=p'(t,x,y,z)$$

where $U(y)$ is the mean velocity profile and the prime denotes the turbulent fluctuations. Away from boundaries, the size of the turbulent fluctuations is much smaller than the size of the mean velocity. The standard test cases for turbulent channel flows from Reference [17] are defined for $Re\in\{180,395,590\}$, where the Reynolds number is based on the friction velocity, the channel half-width and the viscosity of the fluid.

In our study of commutations errors and of the divergence of the Reynolds stress tensor, we mainly consider terms which involve the mean velocity $U(y)$. This is motivated by the fact that the mean flow gives the non-oscillating contributions to these errors. Terms not involving the mean flow give often negligible contributions in the asymptotics, see below.

We are interested in the relation of the commutation error and the divergence of the Reynolds stress tensor near the walls and, without loss of generality, we consider the wall $y=0$. Neither the mean velocity profile $U(y)$ (apart form special Reynolds numbers, where DNS data are available) nor the fluctuations are known. However, for the near wall region, empirical laws for $U(y)$ exist, e.g. see References [1,18]. Two of them will be studied below.

Regarding fluctuations, we want to study the effect of a fluctuation mainly in the y-direction, since this is the variable with respect to, the filter must necessarily be non-uniform (in the other directions we can extend the filter by periodicity and use uniform filters).

As a model for fluctuations, we have to use an oscillation vanishing at $y=0$. A straightforward choice is $A\sin(\omega y)$. In order to have the oscillations vanishing also at $y=2H$, to keep periodicity in the x–z-directions, and to satisfy the divergence-free constraint, a simple model is the following:

$$\begin{pmatrix} u'(t;x,y,z)=A\sin\left(\dfrac{\omega\pi y}{H}\right) \\[2ex] v'(t;x,y,z)=A\sin\left(\dfrac{\omega\pi y}{H}\right)^2\cos(z) \\[2ex] w'(t;x,y,z)=-\dfrac{A\pi\omega}{H}\sin\left(\dfrac{2\omega\pi y}{H}\right)\sin(z) \end{pmatrix}$$

with $A,\omega\geqslant 0$. This model is surely an idealization (as all models), however, it captures important properties of fluctuations. Besides vanishing at solid boundaries and being divergence-free, it is small in absolute value if A is small and it becomes highly oscillating if $\omega\gg 1$, having large derivatives. In addition, it satisfies the asymptotic $v'=\mathcal{O}(y^2)$ for $y\to 0$, see, e.g. Reference [1, Chapter 7].

Finally, the filter widths $\delta(x)$, $\delta(y)$, and $\delta(z)$ need to be prescribed to compute (8)–(10). By periodicity, in the x–z-directions, there is need for a variable width just in the y-direction. With this observation and recalling that the filter is the product of three one-dimensional filters, we can restrict, without loss of generality, to consider filters acting only in y-direction. This follows since in the other directions the filter width is constant, hence no commutation error is committed. Note also that the mean velocity field in the turbulent channel flow depends just on the y-variable, so this is the most relevant one. The remaining variables concern only fluctuations. Fluctuations are mainly modelled by a function which depends on y (slowly changing dependence on z is imposed to preserve the zero divergence, even if it is not essential). Concerning $\delta(y)$, we will study the family of functions $\{\delta(y)=y^q, q\geqslant 1\}$. This choice ensures that in the evaluation of the filter the domain of integration remains always inside Ω.

Straightforward computations (see Sections 2.1, 2.2) prove that the relevant term concerning the commutation error for the viscous term is the following:

$$\begin{aligned}\mathscr{E}_{\mathrm{visc}} &:= \left|\mathscr{E}_{\mathrm{c}}\left(\frac{\partial^2 u(t;x,y,z)}{\partial y^2}\right)\right| \\ &= \left|Re^{-1}\left(\frac{\partial^2}{\partial y^2}\overline{u(t;x,y,z)} - \frac{\overline{\partial^2 u(t;x,y,z)}}{\partial y^2}\right)\right| \end{aligned} \tag{8}$$

Regarding the convective term, among all terms of the type

$$\frac{\partial u_i u_j}{\partial y} \quad \text{for } i,j=1,2,3$$

the important one is that involving the product of u and $v=v'$:

$$\begin{aligned}\mathscr{E}_{\mathrm{conv}} &:= \left|\mathscr{E}_{\mathrm{c}}\left(\frac{\partial u(t;x,y,z)v'(t;x,y,z)}{\partial y}\right)\right| \\ &= \left|\frac{\partial}{\partial y}\overline{u(t;x,y,z)v'(t;x,y,z)} - \frac{\overline{\partial u(t;x,y,z)v'(t;x,y,z)}}{\partial y}\right| \end{aligned} \tag{9}$$

The asymptotic behaviours of the other terms (involving $\partial_y(v\,v)$ and $\partial_y(w\,v)$) are given in Remarks 1 and 2.

Among all terms appearing in the divergence of the Reynolds stress tensor, again we restrict to those involving derivatives with respect to y and among them, the relevant one will turn out to be

$$\begin{aligned}\mathscr{E}_{\mathrm{reyn}} := \Bigg|&\left(\frac{\partial\overline{u(t;x,y,z)}\ \overline{v'(t;x,y,z)}}{\partial y} - \frac{\partial}{\partial y}\overline{u(t;x,y,z)v'(t;x,y,z)}\right) \\ &+ \left(\frac{\partial\overline{u(t;x,y,z)}\ \overline{w'(t;x,y,z)}}{\partial z} - \frac{\partial}{\partial z}\overline{u(t;x,y,z)w'(t;x,y,z)}\right)\Bigg| \end{aligned} \tag{10}$$

Note that the first term of $\mathscr{E}_{\mathrm{conv}}$ and the second term of $\mathscr{E}_{\mathrm{reyn}}$ are the same.

Now, replacing $U(y)$ by an empirical wall law (for convenience denoted again by $U(y)$), expressions (8)–(10) can be computed. We are interested in the asymptotic behaviour towards the wall, i.e. as $y \to 0$. For this reason, series expansions of $\mathscr{E}_{\text{visc}}$, $\mathscr{E}_{\text{conv}}$, and $\mathscr{E}_{\text{reyn}}$ are computed. Sometimes, the integrals which have to be evaluated lead to special functions that are difficult to deal with. Some of the computations require very high order Taylor series expansions and the simplification of several (dozens of) terms. We do not report them here since they do not add any insight into the problem, but just add ugly formulas to the paper. In such cases, it is more convenient to apply the series expansion already for the term in the integral, taking care of having a large enough order in the series expansion, not to loose the leading terms.

2.1. The Reichardt law

First, we consider the Reichardt law

$$U(y) = 2.5 \log(1 + 0.4y) + 7.8 \left[1 - \exp\left(-\frac{y}{11}\right) - \frac{y}{11} \exp\left(-\frac{y}{3}\right)\right] \tag{11}$$

For our purposes, the behaviour for $y \to 0$ is of importance since we want to evaluate the effects near the lower boundary. Evaluating (8)–(10), one obtains

$$\mathscr{E}_{\text{visc}} = Re^{-1} \left[\frac{2q^2 - q}{363}\right] y^{2q-2} + \mathcal{O}(y^{2q-1})$$

$$\mathscr{E}_{\text{conv}} = 2q \frac{A\pi^2\omega^2|\cos(z)|}{H^2} y^{2q} + \mathcal{O}(y^{2q+1})$$

$$\mathscr{E}_{\text{reyn}} = \frac{4q}{3} \frac{A\pi^2\omega^2|\cos(z)|}{H^2} y^{2q} + \mathcal{O}(y^{2q+1})$$

The commutation errors coming from the convective term $\mathscr{E}_{\text{conv}}$ and the divergence of the Reynolds stress tensor $\mathscr{E}_{\text{reyn}}$ possess, up to a constant factor, the same asymptotic behaviour. The leading order term of $\mathscr{E}_{\text{conv}}$ is 1.5 times larger than the leading order term of $\mathscr{E}_{\text{reyn}}$.

However, asymptotically, the commutation error $\mathscr{E}_{\text{visc}}$ coming from the viscous term dominates. One can argue that this term is in practice of less importance since it is scaled by the small factor Re^{-1}. But in the case $q = 1$, $\mathscr{E}_{\text{visc}}$ behaves qualitatively different than the other two terms since $\mathscr{E}_{\text{visc}}$ does not converge to zero as $y \to 0$.

The turbulent fluctuation u' does not contribute to the leading order term of $\mathscr{E}_{\text{visc}}$ since

$$Re^{-1}\left(\frac{d^2\,\overline{u'}}{dy^2} - \overline{\frac{d^2u'}{dy^2}}\right) = Re^{-1}\frac{(2q+1)q}{3}\frac{A\pi^3\omega^3}{H^3}\,y^{2q-1} + \mathcal{O}(y^{2q})$$

Thus, the asymptotic behaviour of $\mathscr{E}_{\text{visc}}$ is neither influenced by the smallness of the turbulent fluctuation nor by its possible large frequency.

Remark 1

In the case of the Reichardt law, also the asymptotic behaviour of some other terms, not involving $U(y)$, is of interest. We found that, concerning the other viscous terms, the commutation error involving filtering of $\partial_y^2 w$ is of one order higher, hence asymptotically smaller, while the commutation error coming from $\partial_y^2 v$ is of the same order as $\mathscr{E}_{\text{visc}}$:

$$\mathscr{E}_{\text{c}}(\partial_y^2 v) = Re^{-1}\,\frac{2(2q^2-q)}{3}\,\frac{A\pi^2\omega^2|\cos(z)|}{H^2}\,y^{2q-2} + \mathcal{O}(y^{2q-1})$$

The comparison between $\mathscr{E}_{\text{c}}(\partial_y^2 v)$ and $\mathscr{E}_{\text{visc}}$ is hard since their ratio does not depend on q but on A,ω,z,H. We focus on $\mathscr{E}_{\text{visc}}$ since for the $1/\alpha$th power law considered below, $\mathscr{E}_{\text{visc}}$ will dominate $\mathscr{E}_{\text{c}}(\partial_y^2 v)$.

Concerning the remaining convective terms, the commutation error involving the product vv is $\mathcal{O}(y^{2q+1})$, while that involving the product of v and w possesses the same asymptotic behaviour as $\mathscr{E}_{\text{conv}}$:

$$\mathscr{E}_{\text{c}}(v\,w) = 4q\,\frac{A^2\pi^4\omega^4|\sin(z)\cos(z)|}{H^4}\,y^{2q} + \mathcal{O}(y^{2q+1})$$

Again, we do not focus on this term since it does not depend on the mean flow, the ratio of $\mathscr{E}_{\text{conv}}$ and $\mathscr{E}_{\text{c}}(vw)$ depends on A,ω,z,H and $\mathscr{E}_{\text{c}}(vw)$ will be a higher-order term for the $1/\alpha$th power law, see Remark 2.

Regarding the other terms coming from the divergence of the Reynolds stress tensor, we found that $\partial_y(\overline{v}\ \overline{v}-\overline{vv})+\partial_z(\overline{v}\ \overline{w}-\overline{v\,w}) = \mathcal{O}(y^{2q+1})$. On the other hand, it is

$$|\partial_y(\overline{w}\ \overline{v}-\overline{wv})+\partial_z(\overline{w}\ \overline{w}-\overline{ww})| = \frac{8q-4}{3}\,\frac{A^2\pi^4\omega^4|\cos(z)\sin(z)|}{H^4}\,y^{2q} + \mathcal{O}(y^{2q+1})$$

Hence, this term is of the same order as $\mathscr{E}_{\text{reyn}}$. For the same reasonings as above, we further focus only on $\mathscr{E}_{\text{reyn}}$.

2.2. *The $1/\alpha$th power law*

In this section, we analyse the role of commutation errors in the case of the power law

$$U(y) = \begin{cases} U_\infty\left(\dfrac{y}{\eta}\right)^{1/\alpha} & 0 \leqslant y \leqslant \eta \\ U_\infty & \eta < y \end{cases} \tag{12}$$

with $\alpha > 0$. In practice, the value $\alpha = 7$ is used, see Reference [18]. We use this law as mean flow near the wall, to evaluate the same quantities as in the previous section.

One obtains for the near-wall region $0 \leqslant y \leqslant \eta$ the following asymptotic behaviours of (8)–(10):

$$\begin{aligned} \mathscr{E}_{\text{visc}} &= Re^{-1}c_0(\alpha,q)\,\frac{U_\infty}{\eta^{1/\alpha}}\,y^{2q-4+1/\alpha} + o(y^{2q-4+1/\alpha}) \\ \mathscr{E}_{\text{conv}} &= c_1(\alpha,q)\,\frac{U_\infty}{\eta^{1/\alpha}}\,y^{2q-1+1/\alpha} + o(y^{2q-1+1/\alpha}) \\ \mathscr{E}_{\text{reyn}} &= c_2(\alpha,q)\,\frac{U_\infty}{\eta^{1/\alpha}}\,y^{2q-1+1/\alpha} + o(y^{2q-1+1/\alpha}) \end{aligned} \tag{13}$$

Remark 2

The remaining 'viscous', 'convective' and 'Reynolds' error terms are related to products of v and w alone and do not depend on the mean flow. Hence, they are the same we evaluated

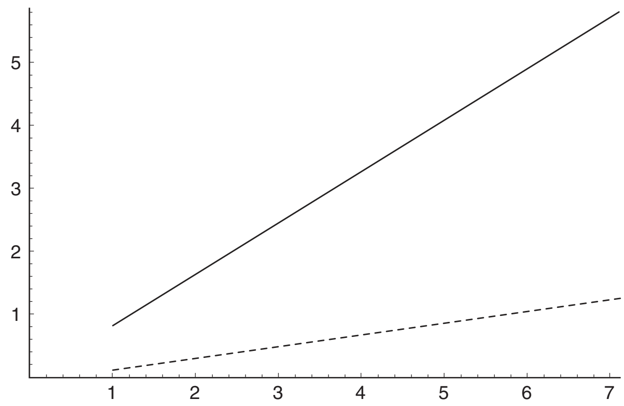

Figure 1. The constants $c_1(7,q)$ (solid line) and $c_2(7,q)$ (dashed line), (divided by $[A\pi^2\omega^2|\cos(z)|/H^2]$) in the leading order terms of $\mathscr{E}_{\text{conv}}$ and $\mathscr{E}_{\text{reyn}}$ for the 1/7th power law.

in the previous section and their order of convergence remains unchanged. Comparing (13) with the results in Remark 1 shows that for the $1/\alpha$th power law the leading terms are always those given in (13).

Again, the fluctuation $u' = A\sin(\omega\pi y/H)$ does not contribute to the asymptotic of $\mathscr{E}_{\text{visc}}$. The observations for the $1/\alpha$th power law are similar to those for the Reichardt law. The dominant term in the asymptotic $y\to 0$ is $\mathscr{E}_{\text{visc}}$. In the interval $q\in[1, 2-1/(2\alpha)]$, $\mathscr{E}_{\text{visc}}$ does not vanish as $y\to 0$ in contrast to $\mathscr{E}_{\text{conv}}$ and $\mathscr{E}_{\text{reyn}}$. However, as already mentioned, the multiplication of $\mathscr{E}_{\text{reyn}}$ with the small factor Re^{-1} will reduce the influence of this term in practice. Again, $\mathscr{E}_{\text{conv}}$ and $\mathscr{E}_{\text{reyn}}$ possess the same asymptotic behaviour.

We will report the constants $c_0(\alpha,q)$, $c_1(\alpha,q)$, $c_2(\alpha,q)$ in the most important case $\alpha=7$: for $q>1$

$$c_0(7,q) = \left|\frac{4q^2}{49} - \frac{66q}{343}\right|$$

$$c_1(7,q) = \frac{40q}{49}\,\frac{A\pi^2\,\omega^2|\cos(z)|}{H^2}$$

$$c_2(7,q) = 4\left(\frac{q}{21} - \frac{1}{49}\right)\frac{A\pi^2\omega^2|\cos(z)|}{H^2}$$

The difference of the constants $c_1(7,q)$ in $\mathscr{E}_{\text{conv}}$ and $c_2(7,q)$ in $\mathscr{E}_{\text{reyn}}$ is illustrated in Figure 1. The constant $c_2(7,q)$ of $\mathscr{E}_{\text{reyn}}$ is always smaller than $c_1(7,q)$ and the ratio of $c_1(7,q)$ and $c_2(7,q)$ is between $15/2=7.5$ for $q\to 1$ and $30/7$ for $q\to\infty$. Altogether, $\mathscr{E}_{\text{conv}}$ is again somewhat larger than $\mathscr{E}_{\text{reyn}}$ but both terms have the same order of magnitude.

2.3. Evaluation of the results, remarks

For both wall laws, the Reichardt law and the $1/\alpha$th power law, the commutation error arising from the viscous term $\mathscr{E}_{\text{visc}}$ dominates asymptotically the divergence of the Reynolds stress tensor and the commutation error from the convective term. However, $\mathscr{E}_{\text{visc}}$ is scaled by a small factor which will reduce its importance in practice. The commutation error coming from the convective term $\mathscr{E}_{\text{conv}}$ and the divergence of the Reynolds stress tensor $\mathscr{E}_{\text{reyn}}$ have for both wall laws the same asymptotic order. The constants in the leading order term are always somewhat larger for $\mathscr{E}_{\text{conv}}$. In summary, $\mathscr{E}_{\text{conv}}$ and $\mathscr{E}_{\text{reyn}}$ are of the same importance.

It was observed in Reference [10] that the commutation error coming from the convective term and the divergence of the Reynolds stress tensor have the same asymptotic order also away from solid boundaries. However, numerical simulations at the turbulent mixing layer problem show that for smoothly varying filter widths, which one can use away from boundaries, the constant in the divergence of the Reynolds stress tensor is considerably larger.

Note also that the commutation error can be asymptotically reduced by choosing, instead of the box filter, higher-order filters [19]. However, the use of higher order filters leads to an equally strong decrease of the contribution of the divergence of the Reynolds stress tensor. Thus, it does not seem to be possible to obtain a separate control over these two sources of errors simply by adopting a suitable class of filters, which is already guessed in Reference [20].

3. CONCLUSIONS

This note studied the asymptotic behaviour of commutation errors arising from non-uniform filter widths in the box filter and the asymptotic behaviour of the divergence of the Reynolds stress tensor.

In particular, we focused our attention to the behaviour near the boundary in the turbulent channel flow. The unknown flow field has been modelled near the wall by a wall law (Reichardt law, $1/\alpha$th power law) for the mean velocity field and a highly oscillating model for the turbulent fluctuations. The analytical results reveal that near the wall, the commutation errors are asymptotically not of lower importance than the divergence of the Reynolds stress tensor. This supports the point of view that the modelling of commutation errors near walls is at least as important as the modelling of the Reynolds stress tensor in LES models. Hence, a precise modelling of commutation errors must be a component of advanced and accurate LES models.

This note is a first step, performed with analytical tools, on the relationship between various sources of errors in LES. To obtain deeper insight, numerical studies of them in the turbulent channel flow will be the topic of forthcoming work.

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