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Stabilized finite element methods for the generalized Oseen problem

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Abstract

The numerical solution of the non-stationary, incompressible Navier–Stokes model can be split into linearized auxiliary problems of Oseen type. We present in a unique way different stabilization techniques of finite element schemes on isotropic meshes. First we describe the state-of-the-art for the classical residual-based SUPG/PSPG method. Then we discuss recent symmetric stabilization techniques which avoid some drawbacks of the classical method. These methods are closely related to the concept of variational multiscale methods which seems to provide a new approach to large eddy simulation. Finally, we give a critical comparison of these methods.

Keywords: Incompressible flow; Navier–Stokes equations; Variational multiscale methods; Stabilized finite elements

1. Introduction

The motivation of the present paper stems from the finite element simulation of the incompressible Navier–Stokes problem

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \tilde{\mathbf{f}}, \tag{1}$$
$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

for the velocity $\mathbf{u}$ and the pressure p in a polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \leqslant 3$, with a given source term $\tilde{\mathbf{f}}$. A standard algorithmic treatment of (1) and (2) is to semidiscretize in time (with possible step length control) using an A-stable method and to apply a fixed point or Newton-type iteration per time step. This leads to the following auxiliary problem of Oseen type in each step of this iteration:

$$L_{\mathrm{Os}}(\mathbf{b};\mathbf{u},p) := -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{3}$$
$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega. \tag{4}$$

Also the iterative solution of the steady state Navier–Stokes equations using a fixed point iteration leads to problems of type (3) and (4) with $c = 0$.

The standard Galerkin finite element method (FEM) for (3) and (4) may suffer from two problems:

- dominating advection (and reaction) in the case of $0 < \nu \ll \|\mathbf{b}\|_{L^\infty(\Omega)}$,
- violation of the discrete inf–sup (or Babuška–Brezzi) stability condition for the velocity and pressure approximations.

The well-known *streamline upwind/Petrov–Galerkin (SUPG) method*, introduced in [5], and the *pressure-stabilization/Petrov–Galerkin* (PSPG) method, introduced in [31,26], opened the possibility to treat both problems in a unique framework using rather arbitrary FE approximations of velocity and pressure, including equal-order pairs. Additionally to the Galerkin part, the elementwise residual $L_{\mathrm{Os}}(\mathbf{b};\mathbf{u},p) - \mathbf{f}$ is tested against the (weighted) non-symmetric part $(\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q$ of $L_{\mathrm{Os}}(\mathbf{b};\mathbf{v},q)$. Moreover, it was shown in [18,23,40] that an additional element-wise stabilization of the divergence constraint (4), henceforth denoted

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as *grad–div stabilization*, is important for the robustness if $0 < \nu \ll 1$. Due to its construction, we will classify the SUPG/PSPG/grad–div approach as an (*element-wise*) *residual-based stabilization* technique.

Despite the success of this classical stabilization approach to incompressible flows over the last 20 years, one can find in recent papers a critical evaluation of this approach, see e.g. [20,12]. Drawbacks are basically due to the strong coupling between velocity and pressure in the stabilizing terms. (For a more detailed discussion, cf. Section 7.) Several attempts have been made to relax the strong coupling of velocity and pressure and to introduce symmetric versions of the stabilization terms:

- Recently, the interior penalty technique of the discontinuous Galerkin (DG) method was applied in the framework of continuous approximation spaces as proposed in [17] leading to the *edge/face oriented stabilization* introduced in [12]. It can be classified as well as a residual-based stabilization technique since it controls the inter-element jumps of the non-symmetric terms in (3) and (4).
- Another approach consists in *projection-based stabilization* techniques. The first step was done in [16] where weighted *global* orthogonal projections of the non-symmetric terms in (3) and (4) are added to the Galerkin scheme. A related *local* projection technique has been applied to the Oseen problem in [3] with low-order equal-order interpolation. Another projection-based stabilization was introduced in [32,29].

The projection-based methods are closely related to the framework of *variational multiscale methods* introduced in [25]. The latter method provides a new approach to large eddy simulation (LES) of incompressible flows which does not possess important drawbacks of the classical LES like commutation errors.

The goal of the present paper is a unique presentation of residual-based and projection-based stabilization techniques to the numerical solution of the Oseen problem (3) and (4), together with a critical comparison.

For brevity, we consider only *conforming* FEM. An extension to a non-conforming approach like DG-methods in an element- or patch-wise version can be found, e.g., in [14,21]. The latter methods are not robust with respect to the viscosity ν. An overview of appropriate stabilization mechanisms in the DG framework was given in [4].

The paper is organized as follows: In Section 2, we describe the basic Galerkin discretization of the Oseen problem. Then, we consider residual-based stabilization methods including the classical SUPG/PSPG/grad–div stabilization following [36], see Section 3, and the edge/face-stabilization method following [12,13], see Section 4. Next, we present projection-based stabilization techniques. Here, we review the local projection approach proposed in [3], see Section 5, and another projection-based stabilized scheme due to [32,29], see Section 6. A critical comparison of the schemes can be found in Section 7.

2. The standard Galerkin FEM for the Oseen problem

Throughout this paper, we will use standard notations for Lebesgue and Sobolev spaces. The L^2-inner product in a domain ω is denoted by $(\cdot,\cdot)_\omega$. Without index, the L^2-inner product in Ω is meant.

This section describes the standard Galerkin FEM for the Oseen-type problem (3) and (4), for simplicity of presentation with homogeneous Dirichlet data:

$$L_{\mathrm{Os}}(\mathbf{b};\mathbf{u},p) := -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{5}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{6}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \tag{7}$$

with $\mathbf{b} \in [H^1(\Omega)\cap L^\infty(\Omega)]^d$, $\nu,\ c \in L^\infty(\Omega)$, $\mathbf{f} \in [L^2(\Omega)]^d$ and

$$\nu > 0, \quad (\nabla\cdot\mathbf{b})(x) = 0, \quad c(x) \geqslant c_{\min} \geqslant 0, \quad \text{a.e. in } \Omega. \tag{8}$$

Let $H_0^1(\Omega) := \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}$ and $L_0^2(\Omega) := \{q \in L^2(\Omega) | \int_\Omega q\,\mathrm{d}x = 0\}$. The variational formulation reads: find $U = \{\mathbf{u},p\} \in \mathbf{V}\times\mathbf{Q} := [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ s.t.

$$\mathscr{A}(\mathbf{b};U,V) = \mathscr{L}(V) \quad \forall V = \{\mathbf{v},q\} \in \mathbf{V}\times\mathbf{Q} \tag{9}$$

with

$$\mathscr{A}(\mathbf{b};U,V) = (\nu\nabla\mathbf{u},\nabla\mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - b(\mathbf{u},q), \tag{10}$$

$$\mathscr{L}(V) = (\mathbf{f},\mathbf{v}), \tag{11}$$

$$b(\mathbf{v},p) = -(p,\nabla\cdot\mathbf{v}). \tag{12}$$

Suppose an admissible triangulation $\mathscr{T}_h$ of the polyhedral domain Ω. We assume that $\mathscr{T}_h$ is shape-regular, i.e., there exists a constant C_{sh}, independent of the meshsize h with $h_T = h|_T$, such that $C_{sh}h_T^d \leqslant \mathrm{meas}(T)$ for all $T \in \bigcup_h \mathscr{T}_h$. In particular, we exclude anisotropic elements throughout the paper.

Moreover, we assume that each element $T \in \mathscr{T}_h$ is a smooth bijective image of a given reference element $\hat{T}$, i.e., $T = F_T(\hat{T})$ for all $T \in \mathscr{T}_h$. Here, $\hat{T}$ is the (open) unit simplex or the (open) unit hypercube in $\mathbb{R}^d$. For $p \in \mathbb{N}$, we denote by $P_p(\hat{T})$ the set $\{\hat{x}^\alpha : 0 \leqslant \alpha_i, 0 \leqslant \sum_{i=1}^d \alpha_i \leqslant p\}$ on a simplex $\hat{T}$ or $\{\hat{x}^\alpha : 0 \leqslant \alpha_i \leqslant k, 1 \leqslant i \leqslant p\}$ on the unit hypercube $\hat{T}$ and define

$$X_h^p = \{v \in C(\bar{\Omega}) \mid v|_T \circ F_T \in P_p(\hat{T})\ \forall T \in \mathscr{T}_h\}. \tag{13}$$

We introduce *conforming* FE spaces on $\mathscr{T}_h$ for velocity and pressure, respectively, by

$$\mathbf{V}_h^r := [H_0^1(\Omega)\cap X_h^r]^d, \quad \mathbf{Q}_h^s := L_0^2(\Omega)\cap X_h^s \tag{14}$$

with $r,\ s \in \mathbb{N}$ and we set $\mathbf{W}_h^{r,s} := \mathbf{V}_h^r \times \mathbf{Q}_h^s$. Clearly, other conforming discrete spaces for the velocity and the pressure can be chosen (e.g., enriched with bubble functions). Moreover, for brevity, we will not present possible extensions to non-conforming methods.

A key point in the analysis of some methods is local inverse inequalities on $T \in \mathscr{T}_h$ with a constant μ_{inv} depending only on the shape-regularity

$$\|\nabla\cdot\mathbf{w}\|_{L^2(T)} \leqslant \sqrt{d}\|\nabla\mathbf{w}\|_{[L^2(T)]^{d\times d}} \leqslant \mu_{\text{inv}} r^2 h_T^{-1}\|\mathbf{w}\|_{[L^2(T)]^d} \quad \forall \mathbf{w}\in\mathbf{V}_h^r. \tag{15}$$

For simplicity, we assume that the solution $U\in\mathbf{W}:=\mathbf{V}\times\mathbf{Q}$ of (9) is smooth enough such that $\{I_{h,r}^u\mathbf{u}, I_{h,s}^p p\}\in\mathbf{W}_h^{r,s}$ can be chosen as the global Lagrange interpolants of $\{\mathbf{u},p\}$. More precisely, we want to apply the local interpolation result

$$\|v - I_{h,r}^T v\|_{H^m(T)} \leqslant C_I \frac{h_T^{l-m}}{r^{k-m}}\|v\|_{H^k(T)}, \quad 0\leqslant m\leqslant l=\min(r+1,k) \tag{16}$$

for the Lagrange interpolation $I_{h,r}^T v$ of $v\in H^k(T)$ with $k>\frac{d}{2}$, [24], Section 4. Here C_I is a constant independent of h_T, r, v, T but dependent on m, k, C_{sh}.

The standard Galerkin FEM of (9) reads as follows: find $U_h=\{\mathbf{u}_h,p_h\}\in\mathbf{W}_h^{r,s}$, s.t.

$$\mathscr{A}(\mathbf{b};U_h,V_h)=\mathscr{L}(V_h) \quad \forall V_h=\{\mathbf{v}_h,q_h\}\in\mathbf{W}_h^{r,s}. \tag{17}$$

Well-known sources of instabilities stem from the case of dominating advection, $\|\mathbf{b}\|_{L^\infty(\Omega)}\gg\nu$, and from the violation of the discrete inf–sup condition for $\mathbf{V}_h^r\times\mathbf{Q}_h^s$

$$\exists\beta_0>0: \quad \inf_{q_h\in\mathbf{Q}_h^s}\sup_{\mathbf{v}_h\in\mathbf{V}_h^r}\frac{(q_h,\nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_{[L^2(\Omega)]^{d\times d}}\|q_h\|_{L^2(\Omega)}}\geqslant\beta_0, \tag{18}$$

where β_0 can be chosen independent of h. This is the case, e.g., for equal-order velocity–pressure finite element spaces. Note that the discrete inf–sup constant β_0 depends in general on r and s.

3. Classical residual-based stabilization methods

The classical stabilization of the Galerkin scheme is a combination of pressure stabilization (PSPG) and streamline-upwind stabilization for advection (SUPG) together with a stabilization of the divergence constraint: find $U_h=\{\mathbf{u}_h,p_h\}\in\mathbf{W}_h^{r,s}$, s.t.

$$\mathscr{A}_{rbs}(\mathbf{b};U_h,V_h)=\mathscr{L}_{rbs}(V_h) \quad \forall V_h=\{\mathbf{v}_h,q_h\}\in\mathbf{W}_h^{r,s} \tag{19}$$

with

$$\mathscr{A}_{rbs}(\mathbf{b};U,V):=\mathscr{A}(\mathbf{b};U,V)+\underbrace{\sum\nolimits_{T\in\mathscr{T}_h}\gamma_T(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_T}_{\text{grad–div stabilization}} + \underbrace{\sum\nolimits_{T\in\mathscr{T}_h}(L_{\text{Os}}(\mathbf{b};\mathbf{u},p),\delta_T^u(\mathbf{b}\cdot\nabla)\mathbf{v}+\delta_T^p\nabla q)_T}_{\text{SUPG/PSPG stabilization}}, \tag{20}$$

$$\mathscr{L}_{rbs}(V):=\mathscr{L}(V)+\overbrace{\sum\nolimits_{T\in\mathscr{T}_h}(\mathbf{f},\delta_T^u(\mathbf{b}\cdot\nabla)\mathbf{v}+\delta_T^p\nabla q)_T} \tag{21}$$

containing the three parameter sets $\{\delta_T^u\}$, $\{\delta_T^p\}$ and $\{\gamma_T\}$ depending on the choice of the FE spaces, see below. The method simultaneously stabilizes spurious oscillations coming from dominating advection and the violation of the discrete inf–sup condition (18). For details and full proofs of the following presentation, we refer to [36].

3.1. Stability of the method

Stability of the stabilized method (19)–(21) with $\delta_T=\delta_T^u=\delta_T^p$ is proved w.r.t.

$$|||V|||_{rbs}:=\left([|V|]_{rbs}^2+\sigma\|q\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}, \tag{22}$$

$$[|V|]_{rbs}^2:=\|\nu^{\frac{1}{2}}\nabla\mathbf{v}\|_{L^2(\Omega)}^2+\|c^{\frac{1}{2}}\mathbf{v}\|_{L^2(\Omega)}^2+J_{rbs}(V,V), \tag{23}$$

$$J_{rbs}(V,V):=\sum_T\delta_T\|(\mathbf{b}\cdot\nabla)\mathbf{v}+\nabla q\|_{L^2(T)}^2+\sum_T\gamma_T\|\nabla\cdot\mathbf{v}\|_{L^2(T)}^2 \tag{24}$$

with parameters δ_T, γ_T, $\sigma>0$ to be determined. A simplified analysis is possible since $[\![\cdot]\!]_{rbs}$ is a mesh-dependent norm on $\mathbf{W}_h^{r,s}$ if $\delta_T>0$. Assume that

$$0<\delta_T\leqslant\frac{1}{2}\min\left\{\frac{h_T^2}{\mu_{\text{inv}}^2 r^4\nu};\frac{1}{\|c\|_{L^\infty(T)}}\right\}, \quad 0\leqslant\gamma_T. \tag{25}$$

The inverse inequalities (15) and (25) imply that the bilinear form $\mathscr{A}_{rbs}(\mathbf{b};\cdot,\cdot)$ defined in (20) satisfies

$$\mathscr{A}_{rbs}(\mathbf{b};V_h,V_h)\geqslant\frac{1}{2}[|V_h|]_{rbs}^2, \quad \forall V_h\in\mathbf{W}_h^{r,s}. \tag{26}$$

The coercivity estimate (26) yields uniqueness and existence of the discrete solution, however it provides no control of the L^2-norm of the pressure. Assume now additionally

$$0<\mu_0\frac{h_T^2}{r^2}\leqslant\delta_T\leqslant\frac{1}{2}\min\left\{\frac{h_T^2}{\mu_{\text{inv}}^2 r^4\nu};\frac{1}{\|c\|_{L^\infty(T)}}\right\}, \quad 0\leqslant\delta_T\|\mathbf{b}\|_{L^\infty(T)}^2\leqslant\gamma_T \tag{27}$$

with some positive constant μ_0. Taking advantage of Verfürth's trick, cf. [19,41], we can show that there exists a constant $\beta>0$, independent of ν, h and the spectral orders r and s, such that the bilinear form $\mathscr{A}_{rbs}(\mathbf{b};\cdot,\cdot)$ in (20) satisfies

$$\inf_{U_h\in\mathbf{W}_h^{r,s}}\sup_{V_h\in\mathbf{W}_h^{r,s}}\frac{\mathscr{A}_{rbs}(\mathbf{b};U_h,V_h)}{|||U_h|||_{rbs}|||V_h|||_{rbs}}\geqslant\beta \tag{28}$$

with the weight

$$\sqrt{\sigma}\sim\left(\sqrt{\gamma}+\frac{1}{\mu_0}+\sqrt{\nu}+\|c\|_{L^\infty(\Omega)}^{\frac{1}{2}}C_{\text{F}}+\frac{C_{\text{F}}\|\mathbf{b}\|_{L^\infty(\Omega)}}{\sqrt{\nu+c_{\min}C_{\text{F}}^2}}+\max_T\frac{h_T\|\mathbf{b}\|_{L^\infty(T)}}{\sqrt{\nu}}\right)^{-1} \tag{29}$$

of the L^2-norm of the pressure in (22). Moreover, it denotes $\gamma=\max_{T\in\mathscr{T}_h}\gamma_T$ and C_{F} the Friedrichs constant. Note that σ is only used for the analysis.

Remark. The lower bound of δ_T in (27) can be removed in case of div-stable velocity–pressure interpolations. But then one has to replace the constant β in (28) by the inf–sup constant $\beta_0=\beta_0(r,s)$ from (18).

3.2. A priori error estimates

The following continuity result is derived using standard inequalities. It reflects the effect of stabilization with assumption (25): For each $U \in \mathbf{W}$ with $\Delta\mathbf{u}|_T \in [L^2(T)]^d \ \forall T \in \mathcal{T}_h$ and $V_h \in \mathbf{W}_h^{r,s}$ there holds

$$\mathscr{A}_{rbs}(\mathbf{b}; U, V_h) \preceq \mathscr{Q}_{rbs}(U)|||V_h|||_{rbs} \tag{30}$$

with

$$\mathscr{Q}_{rbs}(U) := |[U]|_{rbs} + \left(\sum_{T\in\mathcal{T}_h} \frac{1}{\delta_T}\|\mathbf{u}\|^2_{L^2(T)}\right)^{\frac{1}{2}} + \left(\sum_{T\in\mathcal{T}_h} \frac{1}{\max(\nu,\gamma_T)}\|p\|^2_{L^2(T)}\right)^{\frac{1}{2}} + \left(\sum_{T\in\mathcal{T}_h} \delta_T\| - \nu\Delta\mathbf{u} + c\mathbf{u}\|^2_{L^2(T)}\right)^{\frac{1}{2}}. \tag{31}$$

The L^2-terms in (31) explode for $\nu, c \to 0$ if $\delta_T = \gamma_T = 0$.

Consider solutions $U \in \mathbf{W}$ and $U_h \in \mathbf{W}_h^{r,s}$ of the continuous and of the discrete problem, respectively. Let $\{I^u_{h,r}\mathbf{u}, I^p_{h,s}p\} \in \mathbf{W}_h^{r,s}$ be an appropriate interpolant of U, e.g., the Lagrange interpolant. Then we obtain the quasi-optimal a priori estimate of scheme (19)–(21):

$$|||U - U_h|||_{rbs} \preceq \mathscr{Q}_{rbs}\{\mathbf{u} - I^u_{h,r}\mathbf{u}, p - I^p_{h,s}p\}. \tag{32}$$

Now we have to fix the stabilization parameters δ_T, γ_T using (32) and the local interpolation inequalities (16). Let the assumptions (8) and (27) be valid. Then, we obtain

$$|||U-U_h|||^2_{rbs} \preceq \sum_{T\in\mathcal{T}_h}\left(M^u_T \frac{h_T^{2(l_u-1)}}{r^{2(k_u-1)}}\|\mathbf{u}\|^2_{H^{k_u}(T)} + M^p_T \frac{h_T^{2(l_p-1)}}{s^{2(k_p-1)}}\|p\|^2_{H^{k_p}(T)}\right) \tag{33}$$

with $l_p := \min\{s+1, k_p\}$, $l_u := \min\{r+1, k_u\}$ and

$$M^u_T = \frac{h_T^2}{r^2\delta_T} + \delta_T\left(\frac{\|c\|^2_{L^\infty(T)}h_T^2}{r^2} + \|\mathbf{b}\|^2_{L^\infty(T)} + \frac{r^2\nu^2}{h_T^2}\right) + \gamma_T + \nu + \frac{\|c\|_{L^\infty(T)}h_T^2}{r^2},$$

$$M^p_T = \delta_T + \frac{h_T^2}{s^2\max(\nu,\gamma_T)}.$$

First, we consider the case of equal-order interpolation of velocity and pressure, i.e., $r = s \in \mathbb{N}$. Such pairs do not fulfill the discrete inf–sup condition (18). The equilibration of the δ_T- and γ_T-dependent terms in M^u_T and M^p_T together with the stability conditions (25) and (27) yields

$$\delta_T \sim \left(\frac{r^4\nu}{h_T^2} + \frac{r\|\mathbf{b}\|_{L^\infty(T)}}{h_T} + \|c\|_{L^\infty(T)}\right)^{-1}, \qquad \gamma_T \sim \frac{h_T^2}{r^2\delta_T}. \tag{34}$$

Then, a sufficiently smooth solution U of (9) with $U|_T \in [H^k(T)]^d \times H^k(T)$ for each $T \in \mathcal{T}_h$, obeys the error estimate (with $l = \min(r+1, k)$)

$$|||U - U_h|||^2_{rbs} \preceq \sum_{T\in\mathcal{T}_h} \frac{h_T^{2(l-1)}}{r^{2(k-1)}} M_T\left(\|\mathbf{u}\|^2_{H^k(T)} + \|p\|^2_{H^k(T)}\right),$$

$$M_T = \nu r^2 + \frac{\|\mathbf{b}\|_{L^\infty(T)}h_T}{r} + \frac{\|c\|_{L^\infty(T)}h_T^2}{r^2}. \tag{35}$$

Remark. The estimate (35) is optimal with respect to h_T. Unfortunately, it is suboptimal in the spectral order r in a transition region between the diffusion-dominated and the advection-dominated limits. This is caused by the term $\frac{r^4\nu}{h_T^2}$ in (34) in order to fulfill the stability conditions (27). It is possible to refine the coefficient in front of the L^2-term of $\mathbf{u}$ on the right-hand side of (31), thus giving an optimal estimate w.r.t. r at least in the diffusion-dominated limit, see [36].

Next, we consider interpolation pairs $\mathbf{V}_h^r \times \mathbf{Q}_h^s$ with $r = s+1$. (An extension to $r \geqslant s+1$ is straightforward.) This includes the div-stable Taylor–Hood pairs with $s = r-1 \in \mathbb{N}$ on a shape-regular mesh $\mathcal{T}_h$. A balance of the γ_T- and δ_T-dependent terms in M^u_T and M^p_T yields

$$\delta_T \sim \frac{h_T^2}{r^2(\nu+\gamma)}, \quad \gamma_T \sim \frac{h_T^2}{r^2\delta_T} \sim \nu + \gamma \tag{36}$$

with $\gamma \sim 1$. In this case, a sufficiently smooth solution U of the Oseen problem (9) with $U|_T \in [H^{k+1}(T)]^d \times H^k(T)$ for each $T \in \mathcal{T}_h$ obeys the error estimate

$$|||U - U_h|||^2_{rbs} \preceq \sum_{T\in\mathcal{T}_h} \frac{h_T^{2l}}{r^{2k}}\left((\nu+\gamma)\|\mathbf{u}\|^2_{H^{k+1}(T)} + \frac{1}{\nu+\gamma}\|p\|^2_{H^k(T)}\right) \tag{37}$$

with $l = \min(r+1, k)$, provided that h_T is sufficiently small. The estimate (37) is optimal w.r.t. both h_T and r. The choice (36) reflects the importance of the grad–div stabilization term and a decreasing influence of the SUPG/PSPG term with increasing spectral order r.

3.3. Variants of the method

Other variants containing the SUPG-/PSPG-stabilization with $\delta_T = \delta^u_T = \delta^p_T$ are the Galerkin/least-squares (GLS) method [18] and the Douglas/Wang- or algebraic subgrid-scale (ASGS) method [16] adding

$$\sum_{T\in\mathcal{T}_h} (L_{\mathrm{Os}}(\mathbf{b}; U) - \mathbf{f}, \delta_T L_{\mathrm{Os}}(\mathbf{b}; V))_T$$

and

$$-\sum_{T\in\mathcal{T}_h} (L_{\mathrm{Os}}(\mathbf{b}; U) - \mathbf{f}, \delta_T L^*_{\mathrm{Os}}(\mathbf{b}; V))_T,$$

respectively, to the Galerkin formulation (17). L^*_{Os} denotes the adjoint operator of L_{Os}. The analysis of these methods is similar to the SUPG/PSPG/grad–div scheme using the stabilizing effect of the term $J_s(\cdot,\cdot)$ defined in (24).

For div-stable interpolation pairs, a *reduced stabilized* scheme by omitting the PSPG terms $\sum_{T\in\mathcal{T}_h}(L_{\mathrm{Os}}(\mathbf{b}; \mathbf{u}, p) - \mathbf{f}, \delta_T\nabla q)_T$ from scheme (19)–(21) is analyzed in [20]. Practical calculations surprisingly show that the schemes with and without PSPG give almost identical results. The grad–div stabilization is always necessary for $0 < \nu \ll 1$, whereas the SUPG stabilization is useful for problems with layers. Moreover, an order reduction of $\frac{1}{2}$ was observed by using instead of the parameter choice (36) the design (34) for equal-order interpolations.

3.4. Implementation issues

The system matrices of the Galerkin FEM and the SUPG/PSPG stabilized scheme have the form

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_s & B_1 \\ B_2 & C \end{pmatrix},$$

respectively, with $B_1 \neq B_2$, $C \neq 0$. The blocks A and A_s as well as B, B_1 and B_2 have a similar sparsity pattern. Thus, the SUPG/PSPG method can be easily incorporated into an existing code for solving the Galerkin FEM. One has to store one additional off-diagonal block and the additional sparse matrix C for the pressure couplings arising from the term $\sum_{T\in\mathscr{T}_h}(\nabla p, \delta_T^p \nabla q)_T$ in the stabilization. Note that for the reduced stabilized scheme from [20] it holds $B_2 = B$ and $C = 0$.

One drawback of the SUPG/PSPG scheme consists in needing to evaluate second order derivatives of the velocity if $r \geqslant 2$. However, these derivatives are multiplied with the small factor ν such that their omission is an option in practical computations.

3.5. Coupled vs. decoupled stabilization

Several drawbacks of the classical stabilization methods presented so far stem from the strong velocity–pressure coupling in the stabilization terms, see the discussion in Section 7. In Sections 4–6, we will consider techniques with decoupled stabilization terms.

Let us take as a starting point the stabilization terms of Eq. (20)

$$\sum_{T\in\mathscr{T}_h} (L_{\mathrm{Os}}(\mathbf{b};\mathbf{u},p), \delta_T^u(\mathbf{b}\cdot\nabla)\mathbf{v} + \delta_T^p\nabla q)_T.$$

The subgrid viscosity concept in the sense of Guermond [22] leads to the idea that the stabilization of the residual does not have to act on the whole residual but only on its projection into some appropriate subspace. We introduce an abstract projection operator $(I-P)$ and the modified stabilization term

$$\sum_{T\in\mathscr{T}_h} ((I-P)L_{\mathrm{Os}}(\mathbf{b};\mathbf{u},p), (I-P)(\delta_T^u(\mathbf{b}\cdot\nabla)\mathbf{v} + \delta_T^p\nabla q))_T \tag{38}$$

(with a similar modification of the right-hand side of the equation). Taking now $\mathbf{v}=\mathbf{u}$, $q=p$ and (for simplicity) $\delta_T = \delta_T^u = \delta_T^p$, we deduce that (38) becomes

$$\sum_{T\in\mathscr{T}_h} \Big\{((I-P)(-\nu\Delta\mathbf{u} + c\mathbf{u}), \delta_T(I-P)((\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p))_T + \|(\delta_T)^{\frac{1}{2}}(I-P)((\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p)\|^2_{L^2(T)}\Big\}.$$

Clearly, the first part of this sum is necessary for consistency and the last part gives the positivity. If the projection operator $(I-P)$ is chosen in such a way that the first part vanishes sufficiently fast as $h\to 0$, then the consistency part could be dropped without spoiling the rate of convergence. In this case, we may also drop the Petrov–Galerkin type modification of the right-hand side.

Moreover, the positive part may be split into two in order to decouple velocities and pressure. Introducing separate stabilization terms for pressure and velocity does not change the consistency properties of the scheme since the weak consistency is given by the approximation properties of the projection and not by the residual. Then, (38) is transformed to the decoupled and symmetric form

$$\sum_{T\in\mathscr{T}_h} (\delta_T^u(I-P)((\mathbf{b}\cdot\nabla)\mathbf{u}), (I-P)((\mathbf{b}\cdot\nabla)\mathbf{v}))_T + (\delta_T^p(I-P)\nabla p, (I-P)\nabla q)_T.$$

A similar argument can be applied to the grad–div stabilization term.

Choosing the subspaces and the projection operators in a specific way, we obtain the stabilization techniques proposed in Sections 4–6 where we will use for the stabilized bilinear form the unified notation

$$\mathscr{A}(\mathbf{b};U,V) + \mathscr{S}_*(\mathbf{b};U,V).$$

4. Face oriented stabilization method

The face oriented stabilization method (or edge oriented for $d=2$) takes its origin in the paper [17] on interior penalty procedures for elliptic and parabolic problems. The idea was to increase the robustness of the Galerkin approximation of elliptic problems (using continuous approximation spaces) by introducing additional least squares control of the gradient jump over element boundaries. This method was revived more than 20 years later in [8]. For the advection–diffusion problem, it was shown that the added penalty term yields a method that is stable independent of the local Peclet number $Pe_T := \frac{\|\mathbf{b}\|_{L^\infty(T)}h_T}{\nu}$ and allows optimal a priori error bounds uniform in Pe_T. The method was then extended to the generalized Stokes problem in [9] and to the Oseen equation with arbitrary polynomial degree in [12]. Other work on the face oriented stabilization includes the papers [6] where a discrete maximum principle is rigorously proved for a face oriented shock capturing scheme and [11] where the method is extended to high order polynomial approximations in a hp-framework.

Since the stabilization is based on the faces of the elements, we introduce the set of all interior faces of the mesh $\mathscr{E}$, we denote the jump of the quantity x over some face e by $[x]_e$ (the orientation of the jump is arbitrary, but fixed). The jump is extended to vector valued functions componentwise. For each face we set a fix (but arbitrary) orientation of the normal vector $\mathbf{n}_e$. We let h_e denote the diameter of the face e and $h_T = \max_{e\subset T} h_e$ the meshsize of the element T. Moreover, we assume that the mesh is locally quasi uniform in the sense that for any two elements $T, T' \in \mathcal{T}_h$ having at least one common node there holds $h_T \leqslant \rho h_{T'}$ where $\rho \geqslant 1$ is a parameter depending on the mesh regularity. The formulation takes the form, find $U_h \in \mathbf{W}_h^{r,r}$, such that

$$\mathcal{A}_{fos}(\mathbf{b}; U_h, V_h) = \mathcal{L}(V_h) \quad \forall V_h = \{\mathbf{v}_h, q_h\} \in \mathbf{W}_h^{r,r}, \tag{39}$$

where

$$\mathcal{A}_{fos}(\mathbf{b}; U_h, V_h) := \mathcal{A}(\mathbf{b}; U_h, V_h) + \mathcal{S}_{fos}(\mathbf{b}; U_h, V_h),$$

$$\mathcal{S}_{fos}(\mathbf{b}; U_h, V_h) := \sum_{e\in\mathcal{E}} \left\{ \int_e \gamma_e^u(\mathbf{b}, h_e)[\nabla \mathbf{u}_h \mathbf{n}_e]_e \cdot [\nabla \mathbf{v}_h \mathbf{n}_e]_e \mathrm{d}s \right. \tag{40}$$

$$+ \int_e \gamma_e(\mathbf{b}, h_e)[\nabla\cdot \mathbf{u}_h]_e[\nabla\cdot \mathbf{v}_h]_e \mathrm{d}s \tag{41}$$

$$\left. + \int_e \gamma_e^p(\mathbf{b}, \nu, h_e)[\nabla p_h \cdot \mathbf{n}_e]_e[\nabla q_h \cdot \mathbf{n}_e]_e \mathrm{d}s \right\}. \tag{42}$$

The numerical analysis shows that the three parameter sets in the stabilization term should be chosen as

$$\gamma_e^u(\mathbf{b}, h) := \|\mathbf{b}\cdot\mathbf{n}_e\|_{L^\infty(e)} \frac{h_e^2}{r^\alpha},$$
$$\gamma_e(\mathbf{b}, h) := \|\mathbf{b}\|_{L^\infty(e)} \frac{h_e^2}{r^\alpha},$$
$$\gamma_e^p(\mathbf{b}, \nu, h) := \min(1, \mathrm{Re}_e) \frac{h_e^2}{\|\mathbf{b}\|_{L^\infty(e)} r^\alpha},$$

with $Re_e := \frac{\|\mathbf{b}\|_{L^\infty(e)} h_e}{\nu r^{\frac{1}{2}}}$ and $\alpha := \frac{7}{2}$.

4.1. Stability for face oriented stabilization

The stability of the method is obtained by the key observation that the operator controlling the jump of the gradient actually controls the part of the gradient of the discrete approximation that is orthogonal to the finite element space. Thanks to this observation, one may obtain the crucial control of the streamline derivative and the pressure gradient independently. To prove stability we need the following interpolation result from discontinuous to continuous spaces.

There exists an interpolation operator $\pi_h^* : [W_{r,h}^{\mathrm{disc}}]^d \to [W_{r,h}^c]^d$, where $W_{r,h}^{\mathrm{disc}}$ denotes the space of discontinuous functions being piecewise polynomials of order r on each element and $W_{r,h}^c = \{v \in W_{r,h}^{\mathrm{disc}} : v \in C^0\}$, and constants γ_0, γ_1 depending on the local mesh geometry and the polynomial degree, but not on the local mesh size, such that

$$\gamma_0 \mathcal{J}(\mathbf{v}_h, \mathbf{v}_h) \preceq \|h^{\frac{1}{2}}(\nabla \mathbf{v}_h - \pi_h^* \nabla \mathbf{v}_h)\|_{L^2(\Omega)}^2 \preceq \gamma_1 \mathcal{J}(\mathbf{v}_h, \mathbf{v}_h)$$

for all $\mathbf{v}_h \in X_h^r$, where

$$\mathcal{J}(\mathbf{v}_h, \mathbf{v}_h) = \sum_{e\in\mathcal{E}} \int_e h_e^2 |[\nabla \mathbf{v}_h]_e|^2 \,\mathrm{d}s.$$

Then one uses the fact that $|[\nabla p_h]_e| = |[\nabla p_h \cdot \mathbf{n}_e]_e|$ for continuous finite element spaces. Well-posedness of the discrete problem is assured thanks to the following discrete inf–sup condition. Independently of ν and h there holds

$$|||U_h|||_{fos} \preceq \sup_{0\neq V_h \in \mathbf{W}_h^r} \frac{\mathcal{A}_{fos}(\mathbf{b}; U_h, V_h)}{|||V_h|||_{fos}},$$

for all $U_h \in \mathbf{W}_h^r$ where the triple norm is defined by

$$|||V_h|||_{fos}^2 := |[V_h]|_{fos}^2 + \sigma\|p_h\|_{L^2(\Omega)}^2 \tag{43}$$

with

$$|[V_h]|_{fos}^2 := \|\nu^{\frac{1}{2}}\nabla \mathbf{v}_h\|_{L^2(\Omega)}^2 + \|c^{\frac{1}{2}}\mathbf{v}_h\|_{L^2(\Omega)}^2 + \mathcal{S}_{fos}(\mathbf{b}; V_h, V_h) \tag{44}$$

and σ similar to (29). Note that (43) defines a norm for both velocity and pressure whereas (44) only is a seminorm on the product space.

4.2. A priori error estimates for face oriented stabilization

The use of coercivity in the seminorm (44), Galerkin orthogonality, continuity and finally approximation leads to the following a priori error estimate

$$|[U - U_h]|_{fos}^2 \preceq M_u \left(\frac{h}{r}\right)^{2(l_u - 1)} \|\mathbf{u}\|_{H^{k_u}(\Omega)}^2 + M_p \left(\frac{h}{r}\right)^{2(l_p - 1)} \|p\|_{H^{k_p}(\Omega)}^2, \tag{45}$$

where $l_p := \min\{r+1, k_p\}$, $l_u := \min\{r+1, k_u\}$ and the constants are given by

$$M_u = c\frac{h^2}{r^2} + \max_{T\in\mathcal{T}_h} \min\left\{ \frac{h_T}{r^{\frac{1}{2}}}\|\mathbf{b}\|_{L^\infty(T)} + \frac{h_T^2 r^2 \|\mathbf{b}\|_{W^{1,\infty}(T)}^2}{c}, \nu\frac{Re_T^2}{r^{\frac{1}{2}}} \right\} + \nu,$$

and

$$M_p = \max_{T\in\mathcal{T}_h} \left\{ \frac{h_T}{r^{\frac{1}{2}}} \min\left\{ \|\mathbf{b}\|_{L^\infty(T)}^{-1}, \frac{h_T}{\nu r^{\frac{3}{2}}} \right\} \right\}.$$

Here, we denote by $Re_T := \frac{\|\mathbf{b}\|_{L^\infty(T)} h_T}{\nu r^{\frac{1}{2}}}$ and assume additionally that $\mathbf{b} \in [W^{1,\infty}(\Omega)]^d$. The convergence of the pressure in the L^2-norm may then be estimated leading to

$$\|p - p_h\|_{L^2(\Omega)}^2 \preceq \left(\frac{h}{r}\right)^{2l_p} \|p\|_{H^{k_p}(\Omega)}^2 + \sigma^{-1} r |[U - U_h]|_{fos}^2. \tag{46}$$

The above dependencies on the polynomial order may so far be proven rigorously only on quasi-uniform tensor product meshes. Note the slight suboptimality of the *hp*-estimates in the high Reynolds number regime. In the case $h < r^{-\frac{5}{2}}$, the estimate in the triple norm is suboptimal by a power of $r^{\frac{1}{4}}$ and for the estimate of the pressure in the L^2-norm we get an additional factor of $r^{\frac{1}{2}}$ due to the use of H^1-stability of the L^2-projection. In the low Reynolds number regime, the triple norm estimate is optimal, but the suboptimality of the pressure remains. For details on the *hp*-analysis for face oriented stabilization, see [11].

4.3. Variants of the face oriented stabilization method

In (40), one may take the jump in the streamline derivative only, instead of building the streamline diffusion character into the stabilization parameter γ_e^u in the form of the factor $\|\mathbf{b}\cdot\mathbf{n}_e\|_{L^\infty(e)}$. Following [7], note that $|[\mathbf{b}\cdot\nabla\mathbf{u}_h]_e|^2=(\mathbf{b}\cdot\mathbf{n}_e)^2\ |[\nabla\mathbf{u}_h\mathbf{n}_e]_e|^2$. Another possibility is not to emphasize the streamline direction in the stabilization. In fact, if γ_e is used as stabilizing parameter in (40), then the divergence stabilization (41) may be omitted, at the expense of a possibly larger constant in the a priori error estimate. It is noteworthy that such a modification, introducing crosswind diffusion, will have no influence on the order.

Note also that here only the pressure stabilization is depending on the viscosity ν. This can be understood as using the local Reynolds number as non-dimensional weight for a higher order viscosity term in the velocity stabilization. An equivalent term, controlling both instabilities due the convective terms and the divergence free constraint (hence replacing (40) and (41)), is

$$\sum_{e\in\mathscr{E}}\int_e \mathrm{Re}_e\frac{h_e}{r^3}[\nu\nabla\mathbf{u}_h\mathbf{n}_e]_e\cdot[\nabla\mathbf{v}_h\mathbf{n}_e]_e\mathrm{d}s. \tag{47}$$

Remark. To simplify the analysis the boundary conditions are imposed weakly. Weak imposition of the boundary conditions was considered in [12] using an approach due to [37]. Details have been omitted above for brevity.

Remark. For velocity/pressure pairs of Taylor–Hood type satisfying the discrete inf–sup condition, the choice $\gamma_e^p=0$ is allowed. However, in this case it is recommended to replace the jump stabilization (41) of the divergence by a term $\gamma_T(\nabla\cdot\mathbf{u}_h,\nabla\cdot\mathbf{v}_h)$ with $\gamma_T=1$, compare Section 3.2. It is unclear whether the use of the term (47) is sufficient to stabilize both advection and incompressibility in such a case also. For details on finite element methods with velocity–pressure pairs satisfying the discrete inf–sup condition, see [9,13].

4.4. Implementation issues

The implementation of face oriented stabilization techniques requires an additional nearest neighbor data structure, or a table giving the two elements associated to each face. Such structures are necessary also for a posteriori error estimation and hence for adaptive finite element codes. The task of implementing the gradient jumps only requires the addition of the stabilizing matrix. An efficient implementation will use the symmetry of the matrix and moreover consider each face only once. On the other hand in case the streamline diffusion character is abandoned both velocities and pressures are stabilized using isotropic gradient jumps. In this case the stabilization matrix for both velocities and pressures may be set up (on a fixed mesh) as a preprocessing step. At each time step the stabilization matrix for velocities or pressures are constructed from this precomputed matrix simply by multiplying the indices with the appropriate weights accounting for varying $\mathbf{b}$. This may allow to diminish the computational cost compared to the residual based stabilization where for consistency reasons the whole matrix has to be recomputed at each time step.

5. Local projection-based stabilization method

The local projection-based stabilization (LPS) is designed for equal-order interpolation of pressure and velocities, i.e. $r=s$, and the stabilization of convective terms. For the formulation of the local projection, we restrict ourselves to a certain class of meshes. We assume that the mesh $\mathscr{T}_h$ results from a coarser mesh $\mathscr{T}_{2h}$ by one global refinement. Hence, the mesh $\mathscr{T}_h$ consists of patches of elements; for instance in two dimensions, three triangles can be grouped together in order to form one triangle of $\mathscr{T}_{2h}$. This restriction can be omitted for a certain variant of the local projection discussed in Section 5.3. As further notation, we introduce the space of patch-wise discontinuous finite elements of degree $r-1$:

$$\overline{X}_{2h}^{r-1}:=\{v\in L^2(\bar\Omega)\mid v|_T\circ F_T\in P_{r-1}(\hat T)\ \forall T\in\mathscr{T}_{2h}\}. \tag{48}$$

We introduce the L^2-projection $\overline{\pi}_{2h,r-1}:X_h^r\to\overline{X}_{2h}^{r-1}$, characterized by

$$(\phi-\overline{\pi}_{2h,r-1}\phi,\psi)=0\quad\forall\phi\in X_h^r\quad\forall\psi\in\overline{X}_{2h}^{r-1}$$

and the fluctuation operator with respect to $\overline{\pi}_{2h,r-1}$ by

$$\bar\varkappa_h:=I-\overline{\pi}_{2h,r-1},$$

where I stands for the identity mapping. For the Stokes system, it was proposed in [1] to account for the violation of the inf–sup condition (18) by adding the stabilization term

$$(\bar\varkappa_h\nabla p,\delta^p\bar\varkappa_h\nabla q)$$

to the Galerkin formulation. Similar to the PSPG method, the parameter δ^p depends on the local mesh size: $\delta^p\sim h_T^2$ for the Stokes problem. For the Oseen system, the same term is added but the parameter δ^p should be chosen differently. This will be specified later.

Remark. Due to the orthogonality property

$$(\bar\varkappa_h\nabla p,\overline{\pi}_{2h,r-1}\nabla q)=0\quad\forall p,q\in X_h^r,$$

the local projection can be applied only onto the test function or onto the ansatz function. Hence, it holds

$$(\bar\varkappa_h\nabla p,\delta^p\bar\varkappa_h\nabla q)=(\bar\varkappa_h\nabla p,\delta^p\nabla q)=(\nabla p,\delta^p\bar\varkappa_h\nabla q). \tag{49}$$

Then, the convective term is stabilized by introducing

$$(\bar\varkappa_h((\mathbf{b}\cdot\nabla)\mathbf{u}),\delta^u(\mathbf{b}\cdot\nabla)\mathbf{v}).$$

In order to avoid further notations, we extend the definition of $\bar\varkappa_h$ onto vector-valued functions. Additional control over the divergence is obtained by the term

$$(\bar\varkappa_h(\nabla\cdot\mathbf{u}),\gamma\nabla\cdot\mathbf{v}).$$

Note that these terms are also symmetric by the same argument as before (49). Summarizing all these terms leads to the stabilization

$$\mathscr{S}_{lps}(\mathbf{b};U,V) := (\bar{\varkappa}_h\nabla p, \delta^p\nabla q) + (\bar{\varkappa}_h\nabla\cdot\mathbf{u}, \gamma\nabla\cdot\mathbf{v}) + (\bar{\varkappa}_h((\mathbf{b}\cdot\nabla)\mathbf{u}), \delta^u(\mathbf{b}\cdot\nabla)\mathbf{v}), \tag{50}$$

and the discrete bilinear form of the Oseen problem:

$$\mathscr{A}_{lps}(\mathbf{b};U,V) = \mathscr{A}(\mathbf{b};U,V) + \mathscr{S}_{lps}(\mathbf{b};U,V).$$

In contrast to the residual-based stabilization techniques, the right-hand side keeps unchanged, such that the discrete system reads: find $U_h = \{\mathbf{u}_h, p_h\} \in \mathbf{W}_h^{r,r}$, s.t.

$$\mathscr{A}_{lps}(\mathbf{b};U_h,V) = \mathscr{L}(V) \quad \forall V = \{\mathbf{v},q\} \in \mathbf{W}_h^{r,r}. \tag{51}$$

By the same argument as in the remark above, the stabilization term can be written as

$$\mathscr{S}_{lps}(\mathbf{b};U,V) = (\bar{\varkappa}_h\nabla p, \delta^p\bar{\varkappa}_h\nabla q) + (\bar{\varkappa}_h(\nabla\cdot\mathbf{u}), \gamma\bar{\varkappa}_h(\nabla\cdot\mathbf{v})) + (\bar{\varkappa}_h((\mathbf{b}\cdot\nabla)\mathbf{u}), \delta^u\bar{\varkappa}_h((\mathbf{b}\cdot\nabla)\mathbf{v})).$$

Similar to PSPG/SUPG, this stabilization contains three sets of parameters $\{\delta^u\}$, $\{\delta^p\}$ and $\{\gamma\}$.

5.1. Stability for local projection stabilization

For the stabilization (50), we define the mesh-dependent semi-norm $|||V|||_{lps}$ by

$$|||V|||_{lps}^2 := \|\nu^{1/2}\nabla\mathbf{v}\|_{L^2(\Omega)}^2 + \|c^{1/2}\mathbf{v}\|_{L^2(\Omega)}^2 + \|\delta^{p1/2}\bar{\varkappa}_h\nabla q\|_{L^2(\Omega)}^2 + \|\gamma^{1/2}\bar{\varkappa}_h\nabla\cdot\mathbf{v}\|_{L^2(\Omega)}^2 + \|\delta^{u1/2}\bar{\varkappa}_h((\mathbf{b}\cdot\nabla)\mathbf{v})\|_{L^2(\Omega)}^2,$$

which contains the fluctuations with respect to $\bar{\varkappa}_h$. The main parts in this semi-norm including the energy-norm and the L^2-norm of $\mathbf{v}$ are the same as for the residual-based methods, see (22). The difference is in the h-dependent parts because the pressure and velocity in $|||\cdot|||_{lps}$ are separated, but these parts include only the fluctuations.

Stability is achieved directly by diagonal testing:

$$\mathscr{A}_{lps}(\mathbf{b};V,V) = |||V|||_{lps}^2 \quad \forall V \in \mathbf{W}.$$

Control over the L^2-norm of the pressure is obtained by the upper bound for the discrete solution U_h in the semi-norm $|||\cdot|||_{lps}$ and the data $\mathbf{f}$, cf. [3]:

$$\|p_h\|_{L^2(\Omega)}^2 \preceq |||U_h|||_{lps}^2 + \|\mathbf{f}\|_{L^2(\Omega)}^2.$$

This result induces uniqueness of the pressure.

5.2. A priori estimate for local projection stabilization

The a priori estimate for this stabilization becomes

$$|||U-U_h|||_{lps}^2 \preceq \sum_{T\in\mathscr{T}_h}\left(M_T^u\frac{h_T^{2(l_u-1)}}{r^{2(k_u-1)}}\|\mathbf{u}\|_{H^{k_u}(T)}^2 + M_T^p\frac{h_T^{2(l_p-1)}}{r^{2(k_p-1)}}\|p\|_{H^{k_p}(T)}^2\right) \tag{52}$$

with

$$M_T^u = \left(\frac{1}{\delta_T^u}+\frac{1}{\delta_T^p}+\|c\|_{L^\infty(T)}\right)\frac{h_T^2}{r^2}+\nu+r^2\mu_{\rm inv}^2\gamma_T+\delta_T^u r^2\mu_{\rm inv}^2\|\mathbf{b}\|_{W^{1,\infty}(T)}^2,$$

$$M_T^p = r^2\mu_{\rm inv}^2\delta_T^p + \frac{h_T^2}{r^2\max(\nu,\gamma_T)}.$$

This estimate has some similarities to the one for the residual based stabilization (33). However, the considered (semi)-norm is different.

The choice of the stabilization parameters $\delta_T^p, \delta_T^u, \gamma_T$ can be done in dependence of the regularity of the pressure. We consider the two most important cases:

At first, we study the case of same regularity of pressure and velocity, i.e. $k{:=}k_p = k_u$, and $l := \min\{r+1, k\}$. Equilibration of the terms involving the stabilization constants leads to the optimal choice

$$\delta_T^u := \left(\frac{r^2\nu}{h_T^2} + \frac{r^2\mu_{\rm inv}\|\mathbf{b}\|_{W^{1,\infty}(T)}}{h_T} + \|c\|_{L^\infty(T)}\right)^{-1}, \tag{53}$$

$$\delta_T^p := \delta_T^u,$$

$$\gamma_T := \frac{h_T^2}{r^4\delta_T^u\mu_{\rm inv}^2}. \tag{54}$$

Provided $0 < \mu_0 h_T^2/r^2 \leqslant \delta_T^p$, this leads to $M_T^u \sim h_T^2/(\delta_T^u r^2)$ and $M_T^p \sim r^2\delta_T^p\mu_{\rm inv}^2$, such that the estimate (52) becomes

$$|||U-U_h|||_{lps}^2 \preceq \sum_{T\in\mathscr{T}_h}\frac{h^{2(l-1)}}{r^{2(k-1)}}\left(M_T^u\|\mathbf{u}\|_{H^k(T)}^2 + M_T^p\|p\|_{H^k(T)}^2\right), \tag{55}$$

with

$$M_T^u = \nu + h_T\mu_{\rm inv}\|\mathbf{b}\|_{W^{1,\infty}(T)} + \frac{h_T^2}{r^2}\|c\|_{L^\infty(T)}, \tag{56}$$

and

$$M_T^p = r^2\delta_T^p\mu_{\rm inv}^2. \tag{57}$$

Let us shortly compare this result with (35) for the residual-based stabilization. Besides the fact that the considered (semi-)norms on the left-hand sides differ, the right-hand sides are qualitatively the same at least for small r. In the case that the flow is advection-dominated, the time-step is large and higher order approximation is considered ($r \gg 1$) the estimate (55)–(57) is readily suboptimal due to the term $h_T\mu_{\rm inv}\|\mathbf{b}\|_{L^\infty(T)}$ which is not divided by r. However, for moderate ν as well as for moderate time steps, the estimate (55)–(57) is optimal.

At second, in the case of less regular pressure, i.e. $k_p = k_u - 1$, the choice of the stabilization parameters above would give a poor a priori estimate because $M_T^p \sim M_T^u\|\mathbf{b}\|_{L^\infty(T)}^{-1}$. In order to have $M_T^p \sim h_T M_T^u$, the parameter δ_T^p should scale as h_T^2. We take δ_T^u as before in (53) and

$$\delta_T^p := \frac{h_T^2}{r^4\nu\mu_{\rm inv}^2}, \quad \gamma_T := \nu.$$

This leads to $M_T^p \sim h^2/(r^2\nu)$,

$$M_T^u = (1+r^2\mu_{\rm inv}^2)\nu + h_T\mu_{\rm inv}^2\|\mathbf{b}\|_{W^{1,\infty}(T)} + \frac{h_T^2}{r^2}\|c\|_{L^\infty(T)},$$

and to the a priori estimate

$$|||U - U_h|||_{lps}^2 \preceq \sum_{T\in\mathscr{T}_h} \frac{h_T^{2l_p}}{r^{2k_p}} \left(M_T^u \|\mathbf{u}\|_{H^{k_u}(T)}^2 + \frac{1}{\nu}\|p\|_{H^{k_p}(T)}^2 \right).$$

5.3. Variants of local projection stabilization

In this section, we present some variants of the local projection stabilization. First of all, the stabilization term for the convective term can be replaced by the full derivative. In this case, the stabilization becomes

$$\mathscr{S}_{lps}(U,V) = (\bar{\varkappa}_h \nabla p, \delta^p \nabla q) + (\bar{\varkappa}_h \nabla \mathbf{u}, \delta^u \nabla \mathbf{v}). \tag{58}$$

Furthermore, instead of the fluctuation filter $\bar{\varkappa}_h$, the filter with respect to the global Lagrange interpolant onto the coarser mesh can be used:

$$\varkappa_h := I - I_{2h,r}.$$

When such a filter is used, the stabilization consists of building the gradients of the fluctuations with respect to the filter. The Oseen system can now be stabilized by adding the terms:

$$\begin{aligned}\mathscr{S}_{lps}(\mathbf{b}; U,V) := (\nabla \varkappa_h p, \delta^p \nabla \varkappa_h q) + (\nabla\cdot(\varkappa_h \mathbf{u}), \gamma \nabla\cdot(\varkappa_h \mathbf{v}))\\ + ((\mathbf{b}\cdot\nabla)\varkappa_h \mathbf{u}, \delta^u (\mathbf{b}\cdot\nabla)\varkappa_h \mathbf{v}).\end{aligned} \tag{59}$$

With this notation, the discrete equation remains as before, see (51). The semi-norm $|||\cdot|||_{lps}$ contains now the stabilization terms of (59). Stability can be shown by the same technique as for the variant presented in Section 5.1, see [1]. The a priori estimate (52) remains valid, but in order to apply the Lagrange interpolant onto the exact solution, we have to assume that $p \in H^t(\Omega)$ and $v \in H^t(\Omega)^d$, with $t > \frac{d}{2}$. This variant can be considered as a generalization of the concept of Guermond [22] proposed for advection–diffusion equations.

As last variant we will shortly discuss is the most attractive one from the practical point of view. Instead of using two different meshes $\mathscr{T}_h$, and $\mathscr{T}_{2h}$, only the principal mesh $\mathscr{T}_h$ is used. The discrete space $\mathbf{W}_h^{r,r}$ should be at least of order $r \geqslant 2$. The additional space $\mathbf{W}_h^{r-1,r-1}$ is used to formulate the local projection:

$$\tilde{\varkappa}_h := I - I_{h,r-1}.$$

The stabilized form reads as (59) when $\varkappa_h$ is replaced by $\tilde{\varkappa}_h$. This stabilization term keeps optimal for the Stokes system. However, for the Oseen system it becomes suboptimal because the advection stabilization ensures only the convergence order of the lower order space $\mathbf{W}_h^{r-1,r-1}$.

5.4. Implementation issues

The price for this symmetric minimal stabilization technique is the larger stencil of the corresponding stiffness matrix due to the projection $\bar{\varkappa}_h$ acting on patches. If the complete stencil is included in the sparsity structure of the matrix, the memory requirement is about a factor of two larger in comparison to the Galerkin part (in 2-D and in 3-D as well). However, with a cheap preconditioner, as for instance

$$\begin{aligned}\mathscr{S}_{\text{prec}}(\mathbf{b}; U,V) := (\nabla p, \delta^p \nabla q) + (\nabla\cdot\mathbf{u}, \gamma\nabla\cdot\mathbf{v})\\ + ((\mathbf{b}\cdot\nabla)\mathbf{u}, \delta^{\mathbf{u}}(\mathbf{b}\cdot\nabla)\mathbf{v}),\end{aligned}$$

the larger stiffness matrix can be avoided. We refer to [1] for theoretical and practical results of such a preconditioner in the case of the Stokes problem.

Another necessity for the use of such local projection is the availability of patches (cells of the mesh $\mathscr{T}_{2h}$).

6. A coarse space projection based method

The local projection based method presented in Section 5 can be cast into a more general framework. Let $\mathbf{G}_{H,U}$ be a finite dimensional space of $d\times(d+1)$-tensor-valued functions and δ be a non-negative function. The index H should indicate that $\mathbf{G}_{H,U}$ is a coarse or large scale space, either defined on a coarser grid or by low order finite elements on the finest grid. The abstract coarse space projection formulation seeks $\{U_h, \mathbb{G}_{H,U}\} \in \mathbf{W}_h^{r,s} \times \mathbf{G}_{H,U}$ such that

$$\begin{aligned}\mathscr{A}(\mathbf{b}; U_h, V_h) + (\nabla U_h - \mathbb{G}_{H,U}, \delta \nabla V_h) &= \mathscr{L}(V_h) & \forall V_h \in \mathbf{W}_h^{r,s},\\ (\nabla U_h - \mathbb{G}_{H,U}, \mathbb{L}_{H,U}) &= 0 & \forall \mathbb{L}_{H,U} \in \mathbf{G}_{H,U}.\end{aligned} \tag{60}$$

The second equation in (60) is the $L^2(\Omega)$ projection of the pressure gradient and of the velocity gradient into $\mathbf{G}_{H,U}$. Note that the (trivial) choice $\mathbf{G}_{H,U} := (\nabla \mathbf{V}_h^r)\times(\nabla \mathbf{Q}_h^s)$, where, e.g., $\nabla \mathbf{V}_h^r$ stands for the space consisting of all derivatives of functions in the space $\mathbf{V}_h^r$ defined in (14), avoids any projection and the Galerkin formulation is obtained. We will shortly discuss several non-trivial choices of $\mathbf{G}_{H,U}$ in order to recover several types of stabilization techniques.

1. One of the first projection methods for equal-order interpolation of the Stokes system, which was proposed in [15], projects the pressure gradient only. This method can be cast into this framework by taking $\mathbf{G}_{H,U} := (\nabla \mathbf{V}_h^r) \times (X_h^r)^d$. The gradient of the velocity is not projected, but the pressure gradient becomes projected onto a discrete space equal to the discrete velocity space without Dirichlet conditions. In this case, the projection acts globally due to the continuity of the functions of $\mathbf{G}_{H,U}$.
2. Taking a discontinuous space $\mathbf{G}_{H,U}$ leads to a local projection and has the benefit that the additional degrees of freedom, $\mathbb{G}_{H,U}$, can be locally condensed. In particular, with the notation of (48), the choice $\mathbf{G}_{H,U} := (\overline{X}_{2h}^{r-1})^{d\times d} \times (\overline{X}_{2h}^{r-1})^d$ leads to $\mathbb{G}_{H,U} = \overline{\pi}_{2h,r-1}\nabla U_h$. Due to the orthogonality property (49), the local projection terms in (58) are recovered:
$$\mathscr{S}_{lps}(U,V) = (\bar{\varkappa}_h \nabla p, \delta^p \bar{\varkappa}_h \nabla q) + (\bar{\varkappa}_h \nabla \mathbf{u}, \delta^u \bar{\varkappa}_h \nabla \mathbf{v}).$$
3. Finally, the case of inf–sup stable pairs of finite element spaces was studied in [28]. Since the pressure stabilization is not necessary in this case, $\mathbf{G}_{H,U} = \mathbf{G}_H \times (\nabla \mathbf{Q}_\mathbf{h}^\mathbf{s})$.

Possible choices of the velocity part $\mathbf{G}_H$ of $\mathbf{G}_{H,U}$ and of δ will be discussed in more detail in this section.

Let $\mathbf{W}_h^{r,s} = \mathbf{V}_h^r \times \mathbf{Q}_h^s$ be a pair of finite element spaces which fulfill the discrete inf–sup condition (18). Let $\mathbf{G}_H$ be a finite dimensional space of $d\times d$-tensor-valued functions. Since the stabilization parameter δ can be interpreted in the coarse space projection based method as an additional viscosity, it is denoted by $\nu_{\text{add}}(V,x)$. Then, the coarse space projection based method is defined as follows: find $\{U_h, \mathbb{G}_H\} \in \mathbf{W}_h^{r,s} \times \mathbf{G}_H$ such that

$$\begin{aligned} \mathscr{A}(\mathbf{b}; U_h, V_h) + (\nu_{\text{add}}(U_h, h)(\nabla \mathbf{u}_h - \mathbb{G}_H), \nabla \mathbf{v}_h) &= \mathscr{L}(V_h) \quad \forall V_h \in \mathbf{W}_h^{r,s} \\ (\nabla \mathbf{u}_h - \mathbb{G}_H, \mathbb{L}_H) &= 0 \qquad\quad \forall \mathbb{L}_H \in \mathbf{G}_H. \end{aligned} \tag{61}$$

Methods of this kind have been studied in, e.g. [28–30,32–34]. Their complete description requires to choose two parameters: the space $\mathbf{G}_H$ and the additional viscosity $\nu_{\text{add}}(U_h,h)$.

The first parameter in (61) is the space of tensor-valued functions $\mathbf{G}_H$. The second equation in (61) states that the tensor-valued function $\mathbb{G}_H$ is just the $L^2(\Omega)$-projection of $\nabla \mathbf{u}_h$ into $\mathbf{G}_H$: $\mathbb{G}_H = P_{\mathbf{G}_H} \nabla \mathbf{u}_h$. With this notation, one can reformulate (61) as follows: find $U_h \in \mathbf{W}_h^{r,s}$ such that

$$\begin{aligned} \mathscr{A}_{csp}(U_h; \mathbf{u}_h, \mathbf{v}_h) &= \mathscr{A}(\mathbf{b}; U_h, V_h) + \mathscr{S}_{csp}(U_h; \mathbf{u}_h, \mathbf{v}_h) \\ &= \mathscr{L}(V_h) \ \forall V_h \in \mathbf{W}_h^{r,s} \end{aligned} \tag{62}$$

with

$$\mathscr{S}_{csp}(U_h; \mathbf{u}_h, \mathbf{v}_h) := (\nu_{\text{add}}(U_h, h)(I - P_{\mathbf{G}_H}) \nabla \mathbf{u}_h, \nabla \mathbf{v}_h).$$

In (62), $\mathbf{G}_H$ plays the role of a large scale space such that $(I - P_{\mathbf{G}_H})\nabla \mathbf{u}_h$ represents (resolved) small scales or fluctuations of $\nabla \mathbf{u}_h$. To avoid negative additional viscosity, it is required that $\mathbf{G}_H \subseteq \{\nabla \mathbf{v}_h | \mathbf{v}_h \in \mathbf{W}_h^{r,s}\}$. In the extreme case that equality holds, the second term on the left-hand side of (62) vanishes and the Galerkin finite element discretization of the Oseen equations (5)–(7) is recovered. If $\mathbf{G}_H = \{\mathbb{O}\}$, one obtains an artificial viscosity stabilization of the Oseen equations with a possible non-linear artificial viscosity. If $\nu_{\text{add}}(U_h,h)$ is the Smagorinsky eddy viscosity model (63), the Smagorinsky LES model is recovered (in the case of the Navier–Stokes equations). Since $\mathbf{G}_H$ represents large scales, it must be in some sense a coarse finite element space. There are essentially two possibilities. If $\mathbf{W}_h^{r,s}$ is a higher order finite element space, $\mathbf{G}_H$ can be defined as low order finite element space on the same grid as $\mathbf{W}_h^{r,s}$. This approach is studied in [29]. The second possibility, in particular if $\mathbf{W}_h^{r,s}$ is a low order finite element space, consists in defining $\mathbf{G}_H$ on a coarser grid, see [30] for a study of this approach in the case of advection-dominated advection–diffusion equations.

Concerning the second parameter of (61), $\nu_{\text{add}}(U_h,h)$, almost all studies (for the Navier–Stokes equations) used an eddy viscosity of Smagorinsky type [39]

$$\nu_{\text{add}}(U_h, h) = c_{\text{Sma}} h_T^2 \|\nabla \mathbf{u}_h\|_2, \tag{63}$$

where c_{Sma} is a user-chosen constant, typically $c_{\text{Sma}} \in [0.001,0.05]$, and $\|\cdot\|_2$ denotes the Frobenius norm of a tensor. In [30], $\nu_{\text{add}}(U_h,h) = ch$ has been used in the projection based stabilization for stabilizing advection-dominated advection–diffusion equations on equi-distant meshes with mesh size h.

6.1. Stability of the method

The method (62) introduces additional viscosity by adding $\nu_{\text{add}}(U_h,h)$ to the resolved small scales. For the subsequent analysis, we consider for simplicity the case $\nu_{\text{add}}(U_h,h)$ being independent of U_h. Then, the additional viscosity $\nu_{\text{add}}(h)$ can be written in front of the second term on the left-hand side of (62) and the second equation of (61) can be used to add a helpful zero to get the following problem: find $U_h \in \mathbf{W}_h^{r,s}$ such that

$$\mathscr{A}(\mathbf{b}; U_h, V_h) + \mathscr{S}_{csp}(\mathbf{u}_h, \mathbf{v}_h) = \mathscr{L}(V_h) \ \forall V_h \in \mathbf{W}_h^{r,s} \tag{64}$$

with

$$\mathscr{S}_{csp}(\mathbf{u}_h, \mathbf{v}_h) := (\nu_{\text{add}}(h)(I - P_{\mathbf{G}_H})\nabla \mathbf{u}_h, (I - P_{\mathbf{G}_H})\nabla \mathbf{v}_h).$$

By properties of the $L^2(\Omega)$-projection, one obtains for $\|\nabla \mathbf{u}_h\|_{L^2(\Omega)} > 0$

$$\begin{aligned} \mathscr{S}_{csp}(\mathbf{u}_h, \mathbf{u}_h) &= \nu_{\text{add}}(h)\left(\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 - \|P_{\mathbf{G}_H} \nabla \mathbf{u}_h\|_{L^2(\Omega)}^2\right) \\ &= \nu_{\text{add}}(h)\left(1 - \frac{\|P_{\mathbf{G}_H} \nabla \mathbf{u}_h\|_{L^2(\Omega)}^2}{\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2}\right)\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 \\ &=: \nu^+(h, \mathbf{G}_H, \mathbf{u}_h)\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 \end{aligned} \tag{65}$$

with $0 \leqslant \nu^+(h,\mathbf{G}_H,\mathbf{u}_h) \leqslant \nu_{\text{add}}(h)$. If $\|\nabla \mathbf{u}_h\|_{L^2(\Omega)} = 0$, we set $\nu^+(h,\mathbf{G}_H, \mathbf{u}_h) = 0$. The viscosity $\nu^+(h,\mathbf{G}_H,\mathbf{u}_h)$ is small only if the $L^2(\Omega)$-projection of $\nabla \mathbf{u}_h$ into the large scale space $\mathbf{G}_H$ is close to $\nabla \mathbf{u}_h$ itself. This is the case if there are (almost) no small scales in the flow. We are not interested in this situation since a stabilization is not necessary in this case. The effective viscosity is now given by

$$\nu_{\text{eff}}(h, \mathbf{G}_H, \mathbf{u}_h) := \nu + \nu^+(h, \mathbf{G}_H, \mathbf{u}_h).$$

The stability estimate is obtained in the usual way by using U_h as test function. One obtains in the first step

$$\nu_{\text{eff}}(h, \mathbf{G}_H, \mathbf{u}_h)\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 + c_{\min}\|\mathbf{u}_h\|_{L^2(\Omega)}^2 \leqslant |(\mathbf{f}, \mathbf{u}_h)|. \tag{66}$$

We consider only the case that $c_{\min} > 0$. The modifications for $c_{\min} = 0$ are obvious. The right-hand side of (66) can be estimated by the Cauchy–Schwarz inequality or by the dual estimate. Either estimate is followed by Young's inequality. One obtains finally

$$\begin{aligned} &\nu_{\text{eff}}(h, \mathbf{G}_H, \mathbf{u}_h)\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 + c_{\min}\|\mathbf{u}_h\|_{L^2(\Omega)}^2 \\ &\quad \leqslant \min\left\{\frac{\|\mathbf{f}\|_{L^2(\Omega)}^2}{c_{\min}}, \frac{\|\mathbf{f}\|_{H^{-1}(\Omega)}^2}{\nu_{\text{eff}}(h, \mathbf{G}_H, \mathbf{u}_h)}\right\}. \end{aligned}$$

6.2. A priori error analysis

The a priori error analysis starts in the usual way by subtraction (64) from (9) for test functions from $\mathbf{W}_h^{r,s}$, splitting the error into

$$\begin{aligned}\mathbf{e} &= \{\mathbf{u}-\mathbf{u}_h, p-p_h\} = \{\mathbf{u}-\tilde{\mathbf{u}}_h, p-\tilde{p}_h\} - \{\mathbf{u}_h-\tilde{\mathbf{u}}_h, p_h-\tilde{p}_h\}\\ &= \{\boldsymbol{\eta}^u, \eta^p\} - \{\boldsymbol{\phi}_h^u, \phi_h^p\}\end{aligned}$$

with $\{\tilde{\mathbf{u}}_h, \tilde{p}_h\} \in \mathbf{W}_h^{r,s}$ and using the test function $V_h = \{\boldsymbol{\phi}_h^u, \phi_h^p\}$. It is discussed in [28] that $\{\tilde{\mathbf{u}}_h, \tilde{p}_h\}$ can be defined by the Stokes projection of $\{\mathbf{u}, p\}$ to ensure optimal interpolation estimates for $\{\boldsymbol{\eta}^u, \eta^p\}$. After reordering terms, one obtains

$$\begin{aligned}&\nu_{\mathrm{eff}}(h, \mathbf{G}_H, \boldsymbol{\phi}_h^u)\|\nabla\boldsymbol{\phi}_h^u\|_{L^2(\Omega)}^2 + c_{\min}\|\boldsymbol{\phi}_h^u\|_{L^2(\Omega)}^2\\ &\quad\leqslant |(\nu\nabla\boldsymbol{\eta}^u, \nabla\boldsymbol{\phi}_h^u) + ((\mathbf{b}\cdot\nabla)\boldsymbol{\eta}^u, \boldsymbol{\phi}_h^u) + (c\boldsymbol{\eta}^u, \boldsymbol{\phi}_h^u)\\ &\qquad - (p-\tilde{p}_h, \nabla\cdot\boldsymbol{\phi}_h^u) + \mathscr{S}_{csp}(\boldsymbol{\eta}^u, \boldsymbol{\phi}_h^u) - \mathscr{S}_{csp}(\mathbf{u}, \boldsymbol{\phi}_h^u)|\end{aligned}$$

for arbitrary $\tilde{p}_h \in \mathbf{Q}_h^s$.

The first terms on the right-hand side are estimated using techniques like the Cauchy–Schwarz inequality, Hölder's inequality, Poincaré–Friedrichs' inequality and Young's inequality. One obtains for the last terms on the right-hand side, using the definition of ν^+ from (65),

$$\begin{aligned}\mathscr{S}_{csp}(\boldsymbol{\eta}^u, \boldsymbol{\phi}_h^u) &\leqslant 2\nu^+(h, \mathbf{G}_H, \boldsymbol{\eta}^u)\|\boldsymbol{\eta}^u\|_{L^2(\Omega)}^2\\ &\quad + \frac{\nu^+(h, \mathbf{G}_H, \boldsymbol{\phi}_h^u)}{8}\|\nabla\boldsymbol{\phi}_h^u\|_{L^2(\Omega)}^2\\ &\leqslant 2\nu^+(h, \mathbf{G}_H, \boldsymbol{\eta}^u)\|\boldsymbol{\eta}^u\|_{L^2(\Omega)}^2\\ &\quad + \frac{\nu_{\mathrm{eff}}(h, \mathbf{G}_H, \boldsymbol{\phi}_h^u)}{8}\|\nabla\boldsymbol{\phi}_h^u\|_{L^2(\Omega)}^2,\\ \mathscr{S}_{csp}(\mathbf{u}, \boldsymbol{\phi}_h^u) &\leqslant 2\nu_{\mathrm{add}}(h)\|(I-P_{\mathbf{G}_H})\nabla\mathbf{u}\|_{L^2(\Omega)}^2\\ &\quad + \frac{\nu_{\mathrm{eff}}(h, \mathbf{G}_H, \boldsymbol{\phi}_h^u)}{8}\|\nabla\boldsymbol{\phi}_h^u\|_{L^2(\Omega)}^2.\end{aligned}$$

Collecting terms, using (65) and applying the triangle inequality give the final estimate

$$\begin{aligned}&\nu_{\mathrm{eff}}(h, \mathbf{G}_H, (\mathbf{u}-\mathbf{u}_h))\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\Omega)}^2 + c_{\min}\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}^2\\ &\quad\leqslant C\inf_{(\tilde{\mathbf{u}}_h, \tilde{p}_h)\in\mathbf{W}_h^{r,s}}\Bigg[\nu_{\mathrm{eff}}(h, \mathbf{G}_H, \boldsymbol{\eta}^u)\|\nabla\boldsymbol{\eta}^u\|_{L^2(\Omega)}^2\\ &\quad + \left(c_{\min} + \frac{\|c\|_{L^\infty(\Omega)}^2}{c_{\min}}\right)\|\boldsymbol{\eta}^u\|_{L^2(\Omega)}^2 + \frac{\|p-\tilde{p}_h\|_{L^2(\Omega)}^2}{\nu_{\mathrm{eff}}(h, \mathbf{G}_H, \boldsymbol{\phi}_h^u)}\\ &\quad + \|\mathbf{b}\|_{L^\infty(\Omega)}^2\min\left\{\frac{\|\boldsymbol{\eta}^u\|_{L^2(\Omega)}^2}{c_{\min}}, \frac{\|\nabla\boldsymbol{\eta}^u\|_{L^2(\Omega)}^2}{\nu_{\mathrm{eff}}(h, \mathbf{G}_H, \boldsymbol{\phi}_h^u)}\right\}\\ &\quad + \nu_{\mathrm{add}}(h)\|(I-P_{\mathbf{G}_H})\nabla\mathbf{u}\|_{L^2(\Omega)}^2\Bigg].\end{aligned}$$

Except the last term on the right-hand side of this estimate, all terms behave asymptotically as the interpolation error. The last term tends to zero as the mesh width $h \to 0$ if $\nu_{\mathrm{add}}(h) \to 0$ or if $\mathbf{G}_H$ tends to the space $\{\nabla\mathbf{v}|\mathbf{v}\in\mathbf{V}\}$. In both cases, the Galerkin finite element discretization of the Oseen equations is recovered asymptotically. To obtain an optimal order of convergence, $\nu_{\mathrm{add}}(h)$ and $\mathbf{G}_H$ have to be chosen in such a way that the last term behaves at least as the interpolation error.

L^2-error estimates for the pressure can be derived in the standard way by using the discrete inf–sup condition (18).

6.3. Implementation issues

The algebraic representation of (61) consists in a large coupled system of equations. The solution of this system in coupled form has been studied in [30] and it has been found to be a very inefficient approach. A straightforward idea consists in condensing the coupled system by eliminating the equations describing the $L^2(\Omega)$-projection into $\mathbf{G}_H$ to obtain an algebraic analog of (62). In comparison to the Galerkin finite element discretization of the Oseen equations, one gets an additional matrix on the left-hand side. In [30], a semi-implicit-in-time approach to the non-stationary Navier–Stokes problem (1) and (2) was found to be quite efficient which solves in each time step an equation of the form: Find $U_h \in \mathbf{W}_h^{r,s}$ such that

$$\begin{aligned}&\mathscr{A}(\mathbf{b}; U_h, V_h) + (\nu_{\mathrm{add}}(U_h, h)\nabla\mathbf{u}_h, \nabla\mathbf{v}_h)\\ &\quad = \mathscr{L}(V_h) + (\nu_{\mathrm{add}}(U_h, h)P_{\mathbf{G}_H}\nabla\mathbf{u}_h^{\mathrm{old}}, \nabla\mathbf{v}_h) \qquad (67)\end{aligned}$$

for all $V_h \in \mathbf{W}_h^{r,s}$, where $\mathbf{u}_h^{\mathrm{old}}$ is the solution from the previous discrete time. Note, the left-hand side of (67) is in general a stable discretization, e.g., using (63) gives the same matrices as in the linearization of the Smagorinsky LES model. The fully implicit approach after the condensation of the $L^2(\Omega)$-projection was studied in [29].

In [29,30], the efficient implementation of (61) and (67) into an existing finite element code was investigated. No matter if $\mathbf{G}_H$ is defined on the same grid as $\mathbf{W}_h^{r,s}$ or on a coarser grid, one finds for the reason of efficiency two requirements on $\mathbf{G}_H$:

- $\mathbf{G}_H$ should be a discontinuous finite element space,
- the basis of $\mathbf{G}_H$ should be $L^2(\Omega)$-orthogonal.

If higher order finite elements are used for velocity and pressure, the definition of $\mathbf{G}_H$ on the same grid with low order finite elements is appealing. In this case, the fulfillment of the two requirements on $\mathbf{G}_H$ prevent unnecessary fill-in in the matrices which describe the $L^2(\Omega)$-projection into $\mathbf{G}_H$. In particular, the discontinuity of $\mathbf{G}_H$ allows the computation of the $L^2(\Omega)$-projection by using only local information. Altogether, the conditions on $\mathbf{G}_H$ ensure that the sparsity pattern of the additional matrix is the same as of the matrix representing $\mathscr{A}(\mathbf{b}; U_h, V_h)$. Thus, adding both matrices to obtain the left-hand side of (62) causes no difficulties.

To summarize, the costs of the coarse space projection space method consist essentially in storing and assembling additional matrices which represent the second term in the first equation of (61) and the second equation of (61).

These matrices can be used either to modify the right-hand side as in the semi-implicit approach (67) or to modify the system matrix like in the fully implicit approach.

7. Critical comparison and outlook

Let us first come back to the discussion at the end of Section 3 where a rough motivation of the symmetric stabilization methods of Sections 4–6 was given. In particular, choosing the subspaces and the (abstract) projection operator P in the way proposed in Sections 5 and 6 leads to the local projection and coarse-space projection based methods. Choosing the projection operator as π_h^*, cf. Section 4.1, leads to an face oriented stabilization method of Section 4. This reasoning has given a more rigorous treatment in [3] where it was shown that SUPG stabilization on the subgrid alone is sufficient to yield optimal a priori error estimates.

The following conclusions can be drawn:

- We have traded the full element residual of the SUPG/PSPG method for a projected residual, thus loosing the Galerkin orthogonality.
- The approximation properties of the projection give a weak consistency of the right order that allows for the decoupling of the velocity and the pressure and hence leads to a decoupled stabilization.
- However, at the price of a larger stencil and/or in the case P projects onto a space of lower polynomial order or a coarser space, the approximation properties of the scheme will be given by this coarse space. This results from the weak convergence of the stabilization operator. Typically the convergence order depends on the weak consistency of the stabilization operator. If the small scale space is chosen too small then the projection error vanishes at a rate significantly lower than the approximation error of the fine scale mesh (cf. Section 5.3 and 6). The slow convergence of the projection error will then make convergence rates deteriorate.

We now compare the methods from Sections 3–6 with respect to their different properties concerning some relevant issues such as velocity–pressure approximation, design of stabilization parameters, cost of stabilization, a priori error estimates.

7.1. Velocity–pressure approximation

Although not presented for all variants, all methods share the property that rather arbitrary pairs of velocity–pressure approximation are allowed. In particular, equal-order pairs are still attractive from the implementation point of view.

The stabilization of div-stable pairs of velocity–pressure spaces is necessary to treat the advection-dominated case. In this case, the non-symmetry of the SUPG/PSPG scheme is even more bothering with the non-symmetric velocity/pressure coupling. On the other hand, the stabilization of the face oriented method or the local projection method remains symmetric since the stabilization of velocity and pressure are decoupled.

7.2. Design of stabilization parameters

In the case of equal-order pairs, the stabilization parameters of the SUPG/PSPG/grad–div scheme depend in a sensitive way on the data at the element level. This is a consequence of the non-symmetric structure of the SUPG/PSPG terms. In particular, in the proof of the stability estimate (26) is the inverse inequality needed to control the terms coupled with the Laplacian $\Delta\mathbf{u}_h$. This imposes certain upper bounds on the stabilization parameters, whereas in the case of local projection stabilization and face oriented stabilization (at least when considering higher order polynomial approximation) the method is very robust with respect to overstabilization. Choosing the stabilization parameter too large gives rise to a less well-conditioned matrix, but has remarkably little effect on the approximation error (see i.e. [10,38] for numerical examples). The decoupled velocity and pressure stabilizations also allow for stabilization parameters for the velocity that are independent of the viscosity. Such a choice might not always correspond to the least possible perturbation, but the order of the numerical scheme will not be altered. For the case of the pressure however the stabilization must be changed in order to keep optimal order estimates in the regime of low local Reynolds number. Reducing the viscosity dependence of the stabilization parameters is of interest in strongly nonlinear situations such as those arising in combustion or viscoelastic flows.

For div-stable pairs, one obtains a much simpler parameter design for the SUPG/PSPG/grad–div scheme. Moreover, it seems that the PSPG terms can be omitted in this case, see [20]. They can be omitted also for the symmetric stabilizations in Sections 4 and 5. In this case the stabilization parameter may be chosen independent of the viscosity. Some grad–div stabilization is still necessary in this case to obtain a priori error estimates that are robust when $\nu \to 0$. Simply dropping the pressure stabilization without modifying the stabilization terms of the velocity leads to a numerical scheme of order $h^{l_p-\frac{1}{2}}$ for the error in the triple norm (22). Since the pressure space is of lower polynomial order than the velocity space, this estimate looks suboptimal. Increasing the least squares control of the divergence on the other hand leads to an estimate that is optimal in the H_{div} norm, see [13,9].

7.3. A priori analysis

A striking advantage of the schemes with a symmetric stabilization is the separate control of velocity and pressure terms in the analysis. In this respect, the physical meaning of the stabilization term $\sum_T \delta_T \|(\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p\|^2_{L^2(T)}$ of the SUPG/PSPG scheme is unclear. In the case of symmetric stabilization on the other hand, the a priori error estimate is given in the physically relevant triple norm augmented with the stabilization terms that now give a measure on how much artificial dissipation has been added to the equation. Hence the artificial dissipation of energy induced by the stabilization may be monitored efficiently.

The accuracy of the methods presented in Sections 3–6 is comparable, see, e.g., the discussion in Section 5.2. The analysis of the SUPG/PSPG/grad–div scheme is the most complete so far, including local estimates for scalar advection problems, and a priori error estimates in a hp-framework. Considering the other methods, it seems that some additional work remains to obtain a more complete analysis.

In the case of the projection stabilizations, it is the approximation property of the coarse space that gives the precision of the a priori error estimates. It would therefore seem appealing to choose the coarse space as big as possible from the point of view of precision, for stability on the other hand it should be chosen small enough.

7.4. Expense of stabilization: computational costs

Only the SUPG/PSPG/grad–div scheme has a non-symmetric structure of the stabilization terms. The bulk of additional terms may lead to a time-consuming assembling of the linear system. But these terms can be easily implemented into existing codes. On the other hand, the scheme is (much) more compact than that of the edge/face-oriented stabilization method.

The latter method can be easily implemented into codes with a data structure that allows the application of standard a posteriori estimators. On the one hand, integrations have to performed on the faces of the elements as in a DG method and there are many more faces than elements. But on the other hand, these integrals are of lower dimension and can hence be evaluated at lower cost than the integrals over the elements. The computational cost of the projection-based schemes depends in an essential way on the efficient implementation of the projectors, and on the choice of the coarse spaces (see Section 6 for more details).

The construction of efficient algebraic solvers and preconditioners is simpler for the schemes with symmetric stabilization. The strong velocity–pressure couplings in the matrix of the SUPG/PSPG/grad–div scheme makes this task more complicated, see, e.g. [35]. For the face/edge oriented stabilization, the system matrix does not have the same structure as the standard Galerkin method. The same is true for projection-based schemes where the coarse space does not posses specific properties as discussed in Section 6.3, see also [16]. An appealing alternative in this case seems to be the use of quasi Newton algorithms for the solution of the non-linear problem, using only the part of the matrix that fits in the standard Galerkin stencil cf. Section 5.4.

7.5. Unsymmetric vs. symmetric stabilization

Numerical flow simulation are very often used for optimization issues where beside of the Navier–Stokes equations (primal problem) an associated adjoint (or dual) problem arises. This adjoint problem has probably also to be stabilized. Usually, there are two possibilities to handle adjoint problems numerically:

(i) Building the adjoint out of the discretized stabilized primal problem.
(ii) Building the adjoint out of the continuous primal problem and then stabilize it.

These two possibilities are not equal in general. Possibility (ii) has the drawback, that the adjoint problem is in general not consistent with the optimization problem, because the gradient is perturbed. In contrast, possibility (i) is consistent, but not necessarily discretized properly. This is exactly the situation for the classical residual based stabilization techniques in finite elements. A symmetric stabilization cures this problem, because the possibilities (i) and (ii) become equal. Due to the symmetry of face/edge stabilization, local projection and coarse space projection these schemes are advantageous for optimization problems. The local projection method has already been used in optimization, see [2].

7.6. Outlook

The accurate numerical solution of the advection-dominated Oseen equations by finite element methods requires the addition of viscosity in a sophisticated way. Moreover, if equal order finite element spaces for the velocity and the pressure are applied, the use of pressure stabilization terms becomes necessary.

We presented an overview on stabilization schemes (element and face/edge residual based, projection methods) which differ above all in their basic ideas for stabilizing dominating advection. Compared to the Galerkin discretization, the numerical overhead increases for all approaches, in particular with respect to the memory requirements. Either the sparsity pattern of matrix blocks becomes more dense (edge-stabilization and local projection method (velocity–velocity block)) or additional matrices have to be assembled (SUPG, coarse space projection scheme). Nevertheless, we discussed several aspects showing that the not fully consistent schemes of Sections 4–6 with symmetric stabilization have some potential advantages over the classical SUPG/PSPG scheme.

An a priori error analysis is available for all presented schemes, leading to information about the principal choice of the parameters involved in these schemes on shape-regular triangulations. More research is necessary to extend the analysis to the case of anisotropic meshes.

We did not include a numerical comparison of the schemes, although all statements of the paper are supported by our numerical experience. In this respect, we hope to invite other groups to contribute to such a comparison.

As discussed in Section 1, the methods under consideration have to applied as a kernel within a full code for the finite element simulation based on the incompressible Navier–Stokes model. For brevity, we did not describe the link of the stabilized schemes to the variational multiscale method proposed in [25,22,27] which provide a new approach to large eddy simulations of turbulent flows. The schemes with a symmetric stabilization, as considered in Sections 4–6 are strongly motivated from this point of view.

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