Isogeometric analysis for flows around a cylinder

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Abstract

This note studies the accuracy of Isogeometric Analysis (IGA) applied in the simulation of incompressible flows around a cylinder in two and three dimensions. Quantities of interest, like the drag coefficient, the lift coefficient, and the difference of the pressure between the front and the back of the cylinder are monitored. Results computed with standard finite element methods are used for comparison.

Key words: Isogeometric Analysis (IGA); flow around a cylinder; drag coefficient; lift coefficient

1 Introduction

Isogeometric Analysis (IGA) is a rather new approach for the discretization of partial differential equations which was proposed in [9]. It can use non-uniform rational B-splines (NURBS) for the parametrization of the domain and at the

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same time as basis functions of the finite-dimensional function spaces applied in the discretization. Compared with finite element methods, the basis functions of IGA are smoother and in some situations, curved boundaries of the domain can be represented exactly. However, the implementation of IGA is somewhat more involved, the incorporation of essential boundary conditions is not as straightforward, and the desire to apply standard techniques known from finite element methods, like adaptive grid refinement, requires a non-trivial extension of the standard IGA approach. There is the question if the quality (accuracy) of the solutions obtained with IGA justifies the effort to face these difficulties. This question can be answered only with careful numerical studies. This note constitutes a contribution in this direction for the incompressible Navier–Stokes equations.

IGA for incompressible flow problems has been investigated from the analytical and numerical point of view, e.g, in [1–4,6,7,15]. In particular, it was clarified that counterparts of the popular Taylor–Hood pairs of finite element spaces satisfy a discrete inf-sup condition [1,2]. Even divergence-free versions of IGA were proposed in [4,6,7], whose implementation is however considerably more involved compared with a standard IGA.

This note aims at contributing to the assessment of IGA by studying incompressible flows around cylinders in two and three dimensions. The results for quantities of interest, like the drag and lift coefficient, are compared with the corresponding results obtained with finite element methods.

2 Flows around a cylinder

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a domain. The steady-state incompressible Navier–Stokes equations without body forces are given by

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the velocity, p is the pressure, and ν is a dimensionless viscosity. In addition, appropriate boundary conditions have to be prescribed.

The numerical studies consider flows around cylinders. Flows around bodies constitute standard situations in applications. For such flows, important quantities of interest are the drag and lift coefficients at the body. Also the difference of the pressure between upstream and downstream faces of the body is of importance. The considered examples were proposed in [14].

The finite element results presented in this note were computed with the code

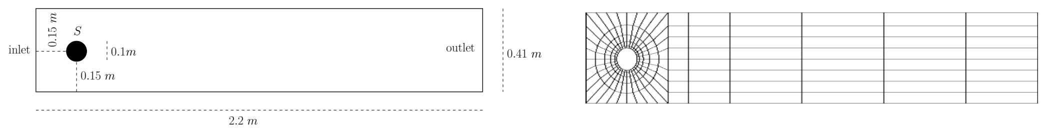

Fig. 1. Two-dimensional flow around a cylinder: Domain and coarsest finite element grid.

MooNMD [12]. An isoparametric approximation of curved boundaries was used. Drag and lift coefficients were computed, for both the IGA and the finite element simulations, with volume integrals as described in detail in [11]. The IGA was implemented in Octave and the correct implementation was checked at examples with prescribed solution [5]. The nonlinear systems of equations were solved until the Euclidean norm of the difference of two subsequent iterates (velocity and pressure) was smaller than 10^{-6}. Smaller tolerances and using also the Euclidean norm of the residual vector as stopping criterion gave quantitatively very similar results. The linear systems of equations were solved with the solver linsolve provided by Octave.

2.1 Two-dimensional flow around a cylinder

The domain for this example and the initial grid for the finite element simulations are shown in Fig. 1. This example is further given by $\nu = 10^{-3}$ in (1), the inlet boundary condition

$$\boldsymbol{u}(0, x_2) = \left(\frac{1.2}{0.41^2} x_2(0.41 - x_2), 0\right)^T,$$

no-slip boundary conditions at the upper and lower wall and at the cylinder, and the do-nothing boundary condition $(\nu\nabla\boldsymbol{u} - p\mathbb{I})\,\boldsymbol{n} = \boldsymbol{0}$ at the outlet, where $\boldsymbol{n}$ is the outward pointing normal vector and $\mathbb{I}$ the identity tensor. The Reynolds number of this flow, based on the mean inflow velocity, the diameter of the cylinder, and the viscosity of the fluid is $\mathrm{Re} = 20$.

IGA relies on computations on a reference square (or cube in three dimensions). To this end, Ω is parametrized, i.e., the domain is divided into so-called patches and these patches are pulled back to the reference domain. It was already observed in [13], where scalar convection-diffusion equations and also a domain with a hole were considered, that the parametrization possesses a considerable impact on the accuracy of the solution. To the best of our knowledge, there are no guidelines on how to parametrize more complicated domains in an optimal way. Here, results for several parametrizations of Ω will be presented, see Fig. 2.

The representation of the circular boundary requires NURBS of degree two. Thus, this degree is the smallest degree for the pressure that can be used in the

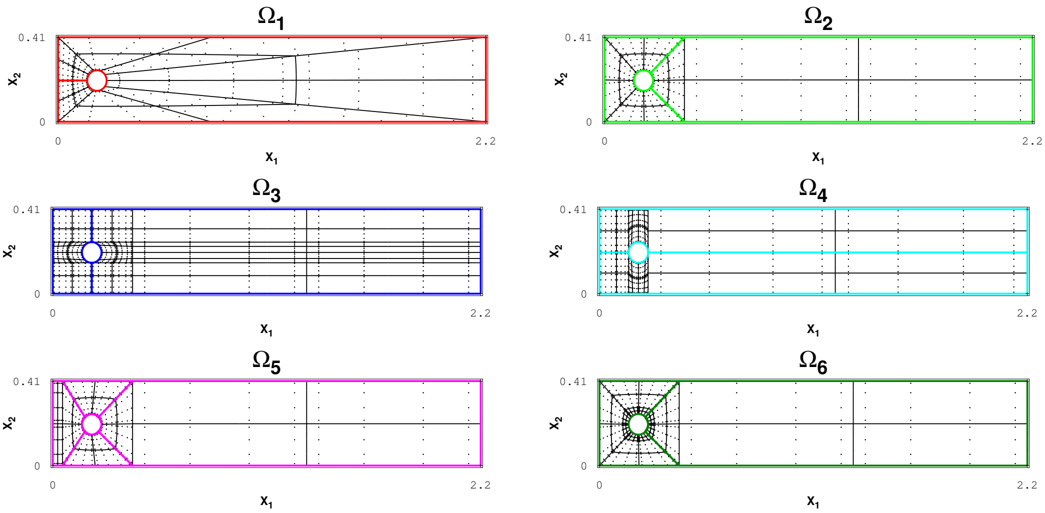

Fig. 2. Two-dimensional flow around a cylinder: Parametrizations of Ω into patches: $\Omega_1, \ldots, \Omega_6$, left to right, top to bottom. The thick colored lines give the decomposition into patches, the thin lines the corresponding coarsest grids, and the dots the quadrature points. Note that Ω_6 is more refined at the cylinder than Ω_2.

IGA. Results will be presented for velocity NURBS of order $p_{\rm v} \in \{3,4\}$ and the pressure NURBS of order $p_{\rm v}-1$. These choices are the counterparts of the Taylor–Hood spaces Q_k/Q_{k-1}, $k \in \{3,4\}$, which were used in the finite element simulations. The initial finite element mesh, Fig. 1, was refined uniformly.

For assessing the accuracy of the results, the reference values used in [8] were taken and the relative errors to these values were evaluated. E.g., let $c_{\rm drag}^h$ be a computed approximation of the drag coefficient, then the relative error is given by $|c_{\rm drag}^h - c_{\rm drag}^{\rm ref}|/|c_{\rm drag}^{\rm ref}|$, where $c_{\rm drag}^{\rm ref}$ is the reference value from [8]. The relative errors versus the number of degrees of freedom are depicted in Figs. 3 and 4. It can be observed that the parametrization of Ω indeed has a great effect on the accuracy. The best parametrizations are Ω_2, Ω_5, Ω_6, and (apart of the pressure difference) Ω_3. Note that the best parametrization in the numerical studies in [13] was the analog of Ω_3. Concerning drag and lift coefficient, the numerical results computed with IGA are not more accurate than those obtained with the finite element method. For the pressure difference, only the results obtained with parametrization Ω_6 are notably more accurate than the finite element results.

Numerical results for this example with IGA and B-splines instead of NURBS can be found in [8]. Comparing these results with the results presented in Figs. 3 and 4, one can observe that the results with NURBS are much more accurate, see also [5, Fig. 40].

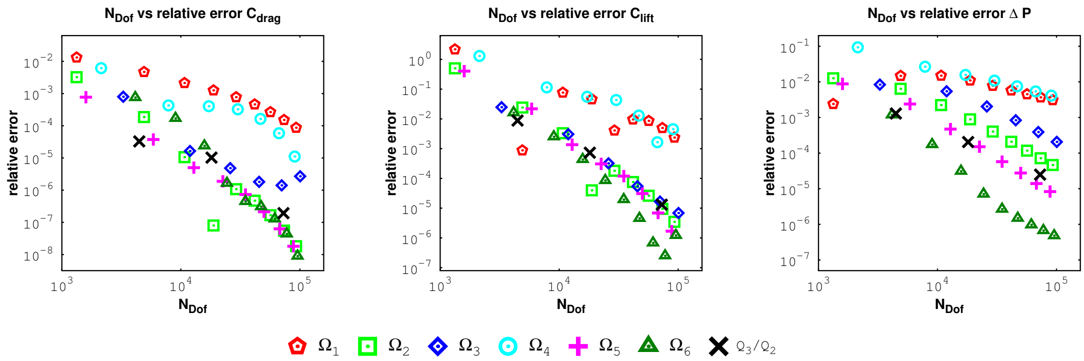

Fig. 3. Results for the two-dimensional flow around a cylinder for third order velocity and second order pressure.

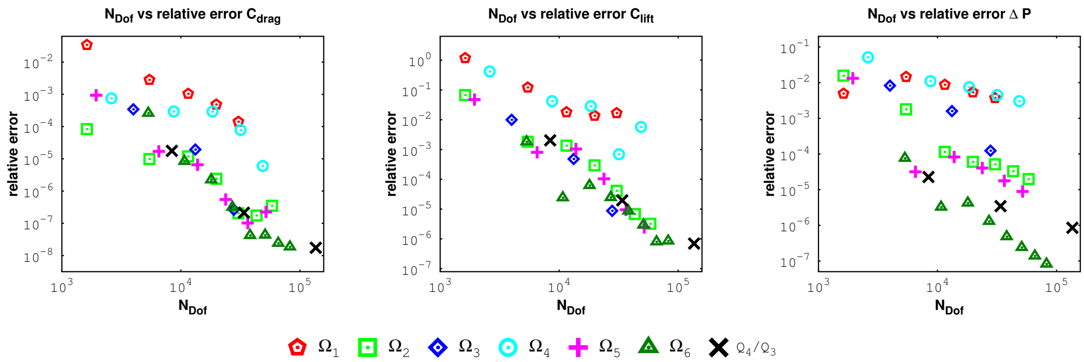

Fig. 4. Results for the two-dimensional flow around a cylinder for fourth order velocity and third order pressure.

2.2 *Three-dimensional flow around a cylinder*

The domain of this example is given by

$$\Omega = \left\{\{(0, 2.5) \times (0, 0.41)\} \setminus \overline{B_{0.05}(0.5, 0.2)}\right\} \times (0, 0.41),$$

where $B_{0.05}(0.5, 0.2)$ is a circle with center $(0.5, 0.2)$ and radius 0.05. Thus, the cross-section of this domain looks similarly like for the two-dimensional case, see Fig. 1. The viscosity is $\nu = 10^{-3}$. At the outlet, a do-nothing boundary condition was prescribed, at the inlet

$$\boldsymbol{u}(0, x_2, x_3) = \left(\frac{7.2}{0.41^4} x_2(0.41 - x_2)x_3(0.41 - x_3), 0, 0\right)^T,$$

and no-slip boundary conditions at the other walls. The Reynolds number of this flow, based on the same quantities as in the two-dimensional example, is $\mathrm{Re} = 20$.

For simulations with the IGA, parametrizations of type Ω_3, Ω_5, and Ω_6 were used, compare Fig. 5. The results obtained with the corresponding patches

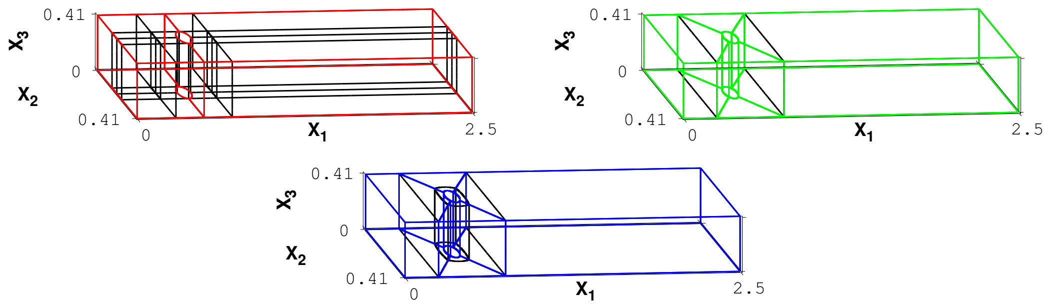

Fig. 5. Three-dimensional flow around a cylinder: Parametrizations of Ω into patches: Ω_3, Ω_5, and Ω_6.

were among the best results for the two-dimensional flow around a cylinder. For brevity, results for a parametrization of type Ω_2 will not be included in the presentation, since Ω_2 is constructed very similarly to Ω_6 and the latter provided more accurate results for the two-dimensional example. For computing the drag and lift coefficient with the volume integrals, one has to use vector-valued test functions that take the value 1 in some component at the cylinder and 0 at all other boundaries. In the simulations, the value 1 was also prescribed at the intersection of the cylinder and the wall. The obtained results are compared with results from [10] for finite element simulations with the Taylor–Hood pairs of spaces P_k/P_{k-1} and Q_k/Q_{k-1}, $k \in \{2, 3\}$.

Reference values for the drag coefficient, the lift coefficient, and the pressure difference are provided in [10]. Relative errors to these values are presented in Fig. 6. Concerning the different parametrizations used in the IGA, Ω_6 performed best. All finite element methods compute drag coefficients of a similar accuracy as the best IGA approach. Concerning the lift coefficient, only Q_2/Q_1 and Q_3/Q_2 give similarly accurate results as IGA with Ω_6. And with respect to the pressure difference, the higher order finite element methods P_3/P_2 and Q_3/Q_2 are clearly more accurate than their lower order counterparts. But only the result obtained with Q_3/Q_2 is of a similar order of accuracy as the result of the IGA with Ω_6.

Since the IGA and the finite element methods are implemented in different codes, a comparison of computing times is not meaningful. It should only be mentioned that the computing times for the IGA were reasonable.

3 Summary

This note assessed the accuracy of IGA applied to the simulation of incompressible flows around obstacles. The dependency of the results on the chosen parametrization became obvious. For a comparable number of degrees of free-

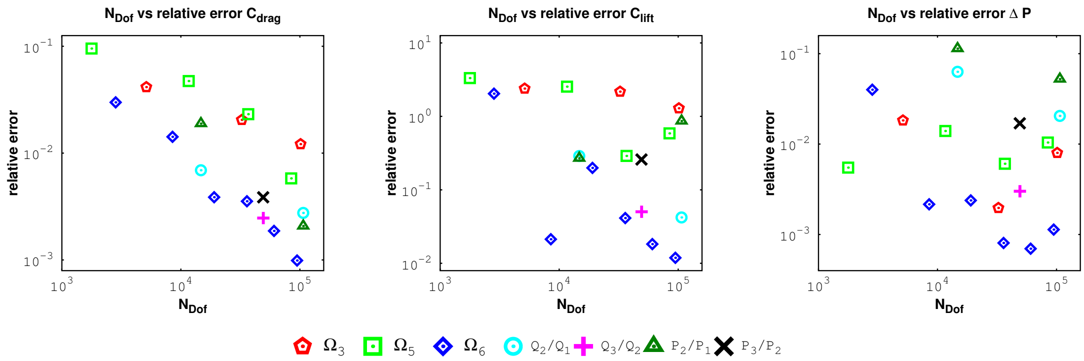

Fig. 6. Results for the three-dimensional flow around a cylinder for the IGA with third order velocity and second order pressure and for the finite element pairs Q_2/Q_1, Q_3/Q_2, P_2/P_1, and P_3/P_2.

dom, the accuracy with respect to several quantities of interest of the best studied parametrization and the best standard finite element methods is comparable.

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