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A Survey on Interpolation and Projection Operators

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Contents

Introduction

The finite element method (FEM) is a very popular approach in order to discretize partial differential equations, which is based on their variational forms. It plays a dominating role in modern computer science and engineering (cf. [17]) and is the source of many theoretical explorations, especially in the theory of solving partial differential equations, which this thesis will focus on.
The idea is to transform the partial differential equation into a variational form. This is done by multiplying the strong form of the equation with a test function, integrating the equation over the domain Ω and to use integration by parts. The variational form of the partial differential equation weakens the regularity assumptions for the solution and uses functions in Sobolev[1] spaces. The solution of the resulting equations then needs to be approximated using the Ritz[2] method (cf. remark B.5 in [15]). This process of approximating the solution is called interpolation and plays a key role in the finite element method. Instead of approximating the unknown solution of the variational problem over an often complicated domain, one divides the domain into smaller subdomains, called finite elements, and approximates the solution of the variational problem over each element using "simpler" functions. Interpolation is therefore a central component of the finite element method, as it enables the construction of accurate and efficient numerical solutions to partial differential equations over complicated domains.
When it comes to the interpolation one needs to make a choice on the interpolation method that is used. A very common choice is the polynomial interpolation also often referred to as Lagrange[3] interpolation. Its popularity is due to its simplicity since it is very easy to apply and to implement. However, the Lagrange interpolation operator cannot be used in the case of non-smooth functions. That is why Clément[4] (1975) came up with another interpolation operator, which is based on local regularization and thus combines the ideas of the Lagrange interpolation and the L^2 projection (cf. [1]). So the Clément interpolation cures some difficulties of the Lagrange interpolation operator but comes with some other disadvantages. One of them being the issue that the interpolation operator constructed by Clément does not preserve homogenous boundary conditions.
Hence, another proposal was made for an interpolation operator that can be used to approximate non-smooth functions and preserves homogenous boundary conditions. It was constructed by Scott[5]-Zhang[6] (1990) in [3] and performs local projections on either an element or a facet (cf. p.483 in [3]).

[1] Sergei Sobolev (1908-1989) was a Soviet mathematician working in mathematical analysis and partial differential equations (cf. [21]).

[2] Walther Ritz (1878-1909) was a Swiss theoretical physicist (cf. [41]).

[3] Joseph-Luis Lagrange (1736-1813) was an Italian mathematician and astronomer (cf. [34]).

[4] Philippe P.J.E. Clément is a French mathematician who is working as an emeritus professor at the Delft University of Technology (cf. [23]).

[5] L. Ridgway Scott (*1948) is an American mathematician who is working as an emeritus professor at the University of Chicago (cf. [22]).

[6] Shangyou Zhang is a mathematician who is working as a professor at the University of Delaware (cf. [24]).

Instead of interpolation, also projection operators can be used. Examples are the L^2 orthogonal projection or the elliptic projection operator, which means the L^2 projection of the gradient. Both of them have the advantage of being quite easy to implement.
This thesis presents the basic ideas of the named interpolation and projection operators and explains their constructions if necessary. Furthermore, this thesis explores their approximation and stability properties in Sobolev spaces and gives some local as well as global interpolation error estimates.
This master thesis is divided into different chapters. In the first chapter the basic notions on domains and meshes, the finite element method and the theory behind interpolation operators are given. In addition, the most important definitions and results from the theory of Lebesgue[7] and Sobolev spaces are getting recalled. In the end of this chapter some general local as well as global error bounds for the interpolation in Sobolev spaces will be given.
In the second chapter the Lagrange interpolation operator will be introduced. In order to understand the construction of this operator and the ideas of proving interpolation and stability estimates the one-dimensional case will be discussed first. Then the concept of this interpolation operator will be generalized to the usage of higher degree polynomials. Also the L^p-stability of the Lagrange interpolant will be discussed. After extending the idea of the Lagrange interpolant to higher dimensions using barycentric coordinates, different error bounds for the interpolation in Sobolev spaces based on some results from the first chapter will be presented.
The third chapter deals with the construction of the Clément interpolation operator, which will be discussed in a very general setting. Moreover, stability and approximation estimates will be given.
The next chapter is then concerned with an interpolation operator based on the ideas of the Clément interpolation, namely the interpolation operator by Scott-Zhang. Here the construction as well as the stability and approximation properties will be analyzed.
The fifth chapter introduces the notion of the L^2-orthogonal projection. At first, the ordinary L^2-projection will be explained and some stability and approximation results will be proven. Then also a weighted version of the L^2 projection will be considered. Again, stability and interpolation estimates will be presented.
The sixth chapter deals with the elliptic projection. The idea of this operator will be described and its stability and approximation properties will be stated.
Finally, a conclusion will be drawn in the last chapter. There the main results will be recalled again and it will be tried to compare the different approximation and stability properties of the interpolation and projection operators. In the end there will be an outlook on further possible investigations.
This thesis is mainly based on the content in [10] and in [19]. Many definitions have also been taken from [15]. Section 5.2 is based on the presentation in [4]. The sources of the definitions and statements are mentioned next to them. If not stated otherwise the proof ideas were taken from the stated source of the statement. In the footnotes references are given for several definitions that

[7] Henri Léon Lebesgue (1875-1941) was a French mathematician (cf. [32]).

were assumed to be clear for the interested reader. Information about mathematicians were taken from Wikipedia or their individual webpages.
I would like to thank my supervisor Prof. Dr. Volker John for his advice in finding a suitable topic as well as for his very helpful support and flexibility throughout the process of writing this thesis. I'm also grateful to Prof. Shangyou Zhang from the University of Delaware for taking the time to answer some of my questions on the material covered in this thesis.

1 Preliminaries

As explained in the introduction, the key idea in the theory of finite elements is that we will look at domains that get decomposed or triangulated by a mesh, so that one can later interpolate functions locally on each of the mesh cells. We will now define those basic notions.

1.1 Domains and Meshes

The definitions and examples presented here are taken from sections 1.2 and 1.3 in [10], section 11.1 in [19] as well as from the appendix in [15].

Definition 1.1.1 (Domain, domain with Lipschitz boundary)**.** [10, Def.1.46]. In dimension 1, a *domain* is an open, bounded interval. In dimension $d \geq 2$ a *domain with Lipschitz boundary* is an open[8], bounded[9], connected[10] set in $\mathbb{R}^d$ such that its boundary $\partial\Omega = \overline{\Omega} \setminus \overset{\circ}{\Omega}$, where $\overline{\Omega}$ denotes the closure[11] and $\overset{\circ}{\Omega}$ means the interior[12] of Ω, satisfies the following property:

There are $\alpha, \beta > 0$, a finite number R of local coordinate systems $x^r = (x^{r\prime}, x^r_d), 1 \leq r \leq R$, where $x^{r\prime} \in \mathbb{R}^{d-1}$ and $x^r_d \in \mathbb{R}$, and R local maps ϕ^r that are Lipschitz[13] continuous[14] on their definition domain $\{x^{r\prime} \in \mathbb{R}^{d-1} \mid |x^{r\prime}| < \alpha\}$ and such that

$$\begin{aligned}
&\partial\Omega = \bigcup_{r=1}^{R} \{(x^{r\prime}, x^r_d) \mid x^r_d = \phi^r(x^{r\prime}) \text{ and } |x^{r\prime}| < \alpha\},\\
&\forall r : \{(x^{r\prime}, x^r_d) \mid \phi^r(x^{r\prime}) < x^r_d < \phi^r(x^{r\prime}) + \beta \text{ and } |x^{r\prime}| < \alpha\} \subset \Omega \text{ as well as}\\
&\forall r : \{(x^{r\prime}, x^r_d) \mid \phi^r(x^{r\prime}) - \beta < x^r_d < \phi^r(x^{r\prime}) \text{ and } |x^{r\prime}| < \alpha\} \subset \mathbb{R}^d \setminus \overline{\Omega},
\end{aligned}$$

where $|x^{r\prime}| \leq \alpha$ means that $|x^{r\prime}_i| \leq \alpha$ for all $1 \leq i \leq d-1$.

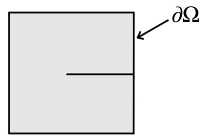

Figure 1: An example for a domain with a slit.

[8] See definition on p.8 in [16].
[9] See definition on p.96 in [42].
[10] See definition on p.793 in [11].
[11] See definition on p.15 in [16].
[12] See definition on p.15 in [16].
[13] Rudolf Lipschitz (1832-1903) was a German mathematician who made contributions to mathematical analysis and differential geometry, as well as number theory, algebras with involution and classical mechanics (cf. [38]).
[14] See definition on p.165 in [16].

Remark 1.1.2 (On the definition of a domain). (i) If not stated otherwise we will assume throughout this thesis that the considered domains are domains with Lipschitz boundary.
(ii) An implication of this definition is that a domain is located on exactly one side of its boundary $\partial\Omega$. That means in particular that sets with slits or cuts, as shown in figure 1, are not included.

Definition 1.1.3 (Polygon, polyhedron). [10, Def.1.47]. In dimension 1, a *polyhedron* is a compact interval. In dimension 2, a *polyhedron*, also called a *polygon*, is a domain, whose boundary is a finite union of segments. In dimension 3, a *polyhedron* is a domain whose boundary is a finite union of polygons.

Remark 1.1.4 (On the terminology). In the literature one often finds the terms polyhedron and polygon used interchangeably.

Remark 1.1.5 (Extension to arbitrary dimensions). One can extend the definition of a polyhedron to any dimension d using induction. One makes the observation that a polyhedron in $\mathbb{R}^d$ is a domain which has a boundary that is just a finite union of polyhedra in $\mathbb{R}^{d-1}$ (cf. remark 1.48 (iii) in [10]) .

Definition 1.1.6 (Mesh). [10, Def.1.49]. Let Ω be a domain in $\mathbb{R}^d$. A *mesh* is a union of a finite number N of compact[15], connected, Lipschitz sets[16] K_m with non-empty interior such that $\{K_m\}_{1\leq m\leq N}$ forms a partition of Ω, i.e.,

$$\overline{\Omega} = \bigcup_{m=1}^{N} K_m \ \text{ and } \ \overset{\circ}{K}_m \cap \overset{\circ}{K}_n = \emptyset \ \text{ for } \ m \neq n.$$

A mesh $\{K_m\}_{1\leq m\leq N}$ will be denoted by $\mathcal{T}_h$, where the subscript refers to the refinement level of the mesh. We set

$$\forall K \in \mathcal{T}_h : h_K = \text{diam}(K) = \max_{x_1,x_2\in K} ||x_2 - x_1||_d,$$

where $\text{diam}(K)$ means the diameter of K and $||\cdot||_d$ denotes the Euclidean[17] norm[18] in $\mathbb{R}^d$. Furthermore we define

$$h := \max_{K\in\mathcal{T}_h} h_K.$$

A sequence of successively refined meshes will be written as $\{\mathcal{T}_h\}_{h>0}$.

Definition 1.1.7 (Mesh cells, faces, edges, vertices). [15, Rmk. B.16]. A *mesh cell* K is a compact polyhedron in $\mathbb{R}^d$, $d \in \{2,3\}$, whose interior is not empty. The boundary ∂K of K consists of m-dimensional linear manifolds[19] (points, pieces of straight lines, pieces of planes), $0 \leq m \leq d-1$, which are called *m-faces*. The 0-faces are the *vertices* of the mesh cell, the 1-faces are the *edges*, and the $(d-1)$-faces are called *facets*.

[15] See definition on p.37 in [16].
[16] See definition 3.2 on p.27 in [19].
[17] Euclid (fl. 300 BC) was an ancient Greek mathematician active as a geometer and logician (cf. [29]).
[18] See definition on p.5 in [16].
[19] See definition 5.1 on p.142 in [12].

Remark 1.1.8 (Mesh cells in one dimension)**.** It is clear that the mesh cells in one dimension are just the subintervals, which decompose the domain. We will use this in section 2.1 to construct a suitable mesh.

Example 1.1.9 (A mesh on the unit square)**.** An example of a mesh can be seen in figure 2. Here the unit square was equipped with a mesh consisting of triangles and quadrangles. Of course this is just one example for a possible mesh on the unit square.

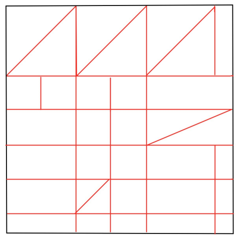

Figure 2: A possible mesh of the unit square in $\mathbb{R}^2$ (inspired by figure 1.11 in [10]).

Remark 1.1.10 (Reference cells and geometric transformations)**.** Usually a mesh will be generated by a *reference cell*, which we denote by $\widehat{K}$, through geometric transformations $T_m : \widehat{K} \longrightarrow K_m$, $K_m \in \mathcal{T}_h$, that map the reference cell to the other mesh cells K_m. We make the assumption that the geometric transformations, which we will be using throughout this thesis, are $\mathcal{C}^1$-diffeomorphisms[20].

Remark 1.1.11 (Transformations of higher degrees)**.** In this thesis we will only consider the case of geometric transformations of degree 1. The case of domains with curved boundaries, where one needs geometric transformations of higher degrees, will not be treated here. The interested reader may consult p.35 in [10].

Since we will consider simplicial meshes most of the time, we will now define the notion of a simplex. Examples of simplices will be given.

Definition 1.1.12 (Simplices)**.** [10, p.21]. Let $\{a_0, ..., a_d\}$ be a family of points in $\mathbb{R}^d, d \geq 1$. Furthermore let the vectors $\{a_1 - a_0, ..., a_d - a_0\}$ be linearly independent. Then, the convex hull[21] of $\{a_0, ..., a_d\}$ is called a *simplex*, and the points $\{a_0, ..., a_d\}$ are called *vertices* of the simplex.

Remark 1.1.13 (Choice of the reference point a_0)**.** Note that the choice of the reference point a_0 is not of importance.

Example 1.1.14 (The unit simplex)**.** [10, p.21]. The *unit simplex* of $\mathbb{R}^d$ is the set

$$\Delta^d := \left\{ x \in \mathbb{R}^d \mid x_i \geq 0, 1 \leq i \leq d \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}.$$

[20] For a definition see p.3 in [12].
[21] See p.231 in [11].

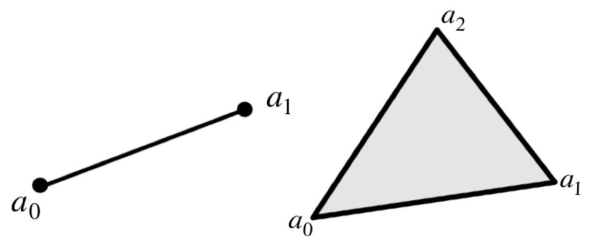

Figure 3: One-dimensional and two-dimensional simplices.

Definition 1.1.15 (Affine meshes)**.** [10, Def.1.53]. If the transformations T_m that transform the reference mesh cell to the m-th mesh cell, i.e., $T_m(\widehat{K}) = K_m$ with $m \in \{1, ..., N\}$, are affine[22], then the mesh is called *affine*. In two dimensions, if the reference cell $\widehat{K}$ is a simplex, an *affine mesh* is also often referred to as a *triangulation*.

Remark 1.1.16 (On the term triangulation)**.** One should add that there is the convention in the literature that the term *triangulation* is often used in any dimension for an affine and simplicial mesh.

Definition 1.1.17 (Affine mapping from reference simplex in $\mathbb{R}^d$)**.** The affine mapping in $\mathbb{R}^d$ that maps the reference simplex $\widehat{K} = \{\widehat{a}_i\}_{0 \le i \le d}$ to another simplex $K = \{a_i\}_{0 \le i \le d}$ is given by $x = a_0 + J_{T_K}\widehat{x}$ with

$$J_{T_K} := \left[\frac{\partial T_K^i}{\partial x_j}\right]_{ij}$$

given by column vectors $(a_j - a_0)$.

Remark 1.1.18 (Assumptions)**.** Throughout this thesis we assume the reference cell $\widehat{K}$, from which the considered mesh is generated, to be a polyhedron. Commonly used reference cells are given by the unit interval in dimension 1 and by the unit simplex (cf. example 1.1.14) or the unit square in dimension 2. In dimension 3 one may choose the three-dimensional unit simplex or the unit cube.

Definition 1.1.19 (Shape regularity)**.** [19, Def.11.2]. A sequence of affine meshes $\{\mathcal{T}_h\}_{h>0}$ is called *shape-regular* if there exists a constant C_1 such that for all mesh cells $K \in \mathcal{T}_h$ and for all h one has

$$\sigma_K := \frac{h_K}{\rho_K} \le C_1$$

with ρ_K being the diameter of the largest ball contained in K.

Definition 1.1.20 (Quasi-uniform meshes)**.** [19, Def.22.20]. A sequence of shape-regular meshes $\{\mathcal{T}_h\}_{h>0}$ is called *quasi-uniform* if there exists a constant C_2 such that for all h and for all mesh cells $K \in \mathcal{T}_h$ it holds that

$$C_2 h \le h_K.$$

[22] See p.254 in [11].

1.1.1 Geometrically Conformal Meshes

Definition 1.1.21 (Geometrically conformal meshes)**.** [10, Def.1.55]. Let Ω be a domain in $\mathbb{R}^d$ and let $\mathcal{T}_h = \{K_m\}_{1 \leq m \leq N}$ be a mesh of Ω. The mesh $\mathcal{T}_h$ is called *geometrically conformal* if the following matching criterion is satisfied:
For all mesh cells K_m and K_n having a non-empty $(d-1)$-dimensional intersection $F = K_m \cap K_n$, there is a face $\widehat{F}$ of $\widehat{K}$ and renumberings of the geometric nodes of K_m and K_n such that $F = T_m(\widehat{F}) = T_n(\widehat{F})$ and

$$T_{m|\widehat{F}} = T_{n|\widehat{F}}$$

for transformations T_m and T_n.

Remark 1.1.22 (Implications of the definition for connected domains)**.** [10, Rmk. 1.56]. For a connected domain Ω the definition of a geometrically conformal mesh tells us that for any two distinct cells $\{K_i, K_j\}$ with $i \neq j$, the intersection of the two mesh cells $K_i \cap K_j$ is
(i) either empty or a common vertex in dimension 1;
(ii) either empty or a common vertex or a common edge in dimension 2;
(iii) either empty or a common vertex or a common edge or a common face in dimension 3.

Remark 1.1.23 (Assumption on the domains)**.** In this thesis we will only consider connected domains.

Example 1.1.24 (Counterexample)**.** We want to visualize the meaning of the defintion of a geometrically conformal mesh. Looking at figure 4 we notice that due to the red node this mesh is not a geometrically conformal mesh since this node does not belong to the nodes of K_1. Remark 1.1.22 (ii) is violated.

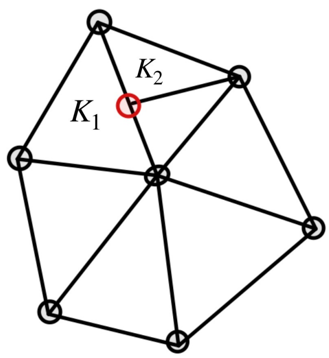

Figure 4: A mesh in $\mathbb{R}^2$ that is not geometrically conformal.

1.2 Finite Elements

In the introduction it was described that we will interpret interpolation as a part of the finite element method. Therefore we will shortly introduce the most important terminology of this theory. For a more extensive introduction to the theory of finite elements we refer to [13] or [19]. The material presented here was taken from section 5.2 in [19] and the appendix in [15].

Definition 1.2.1 (Finite elements)**.** [19, Def.5.2]. Let $d \geq 1$, an integer $n_{\text{sh}} \geq 1$, and the set $N := \{1, ..., n_{\text{sh}}\}$. A *finite element* consists of a triple (K, P, Σ) where
(i) K is a polyhedron in $\mathbb{R}^d$,
(ii) P is a finite-dimensional vector space[23] of functions $p : K \longrightarrow \mathbb{R}^q$ for some integer $q \geq 1$ and
(iii) Σ is a set of n_{sh} linear forms from P to $\mathbb{R}$, i.e., $\Sigma = \{\sigma_i\}_{i \in N}$, such that the map $\Phi_\Sigma : P \longrightarrow \mathbb{R}^{n_{\text{sh}}}$ defined by $\Phi_\Sigma(p) = (\sigma_i(p))_{i \in N}$ is an isomorphism[24]. The linear forms σ_i are called *degrees of freedom* and the bijectivity[25] of the map Φ_Σ is named *unisolvence*.

Remark 1.2.2 (On this definition)**.** (i) This definition is due to Ciarlet. The interested reader may look into [5].
(ii) A slightly more general version of this definition is definition 5.2 in [19].

Remark 1.2.3 (Basis of $\mathcal{L}(P;\mathbb{R})$)**.** From part (iii) of the prior definition it follows that Σ is a basis of the space of linear forms over P, which we denote by $\mathcal{L}(P;\mathbb{R})$. One finds for the dimension of the space of linear forms over P that $\dim(\mathcal{L}(P;\mathbb{R})) = \dim(P) = n_{sh}$. Note that the dimension of P is equal to the cardinality[26] of the set Σ (cf. remark 5.4 in [19] and p.20 in [10]).

Proposition 1.2.4 (Shape functions)**.** *[19, Prop. 5.5]. There is a basis $\{\vartheta_i\}_{i \in N}$ of the vector space P such that*

$$\forall i, j \in N : \sigma_i(\vartheta_j) := \delta_{ij}$$

with δ_{ij} denoting the Kronecker[27] symbol. The functions ϑ_i are called shape functions.

Proof. This follows directly from the bijectivity of the map from part (iii) in the definition of finite elements from above (cf. def. 1.2.1 (iii)). □

Definition 1.2.5 (Parametric finite elements)**.** [15, Def. B.27]. Let $\widehat{K}$ be a reference mesh cell with the local polynomial space $\widehat{P}(\widehat{K})$, the local linear forms $\widehat{\sigma}_1, ..., \widehat{\sigma}_{\widehat{N}}$ and a class of bijective mappings $\{T_K : \widehat{K} \longrightarrow K\}$. A finite element space is called a *parametric finite element space* if:
(i) The images $\{K\}$ of $\{T_K\}$ form the set of mesh cells.
(ii) The local spaces are given by

$$P(K) = \{p \mid p = \widehat{p} \circ T_K^{-1}, \widehat{p} \in \widehat{P}(\widehat{K})\}.$$

[23] Compare p.190 in [11].
[24] See definition on p.72 in [11].
[25] See definition on p.44 in [11].
[26] See definition on p.40 in [11].
[27] Leopold Kronecker (1823-1891) was a German mathematician who worked on number theory, algebra and logic (cf. [35]).

(iii) The local linear forms are defined by

$$\sigma_{K,i}(v(x)) = \widehat{\sigma}_i(\widehat{v}(\widehat{x})) = \widehat{\sigma}_i(v(T_K(\widehat{x}))),$$

where $\widehat{x} = (\widehat{x}_1, ..., \widehat{x}_d)^T$ are the coordinates of the reference mesh cell and it holds $x = T_K(\widehat{x})$ as well as $\widehat{v} = v \circ T_K$.

1.3 Interpolation Operators in General

In this section we will develop the basic terminology in order to talk about interpolation operators. The content we are going to present is taken from sections 5.3 and 5.5 in [19].

Definition 1.3.1 (Interpolation operator)**.** [19, Def. 5.7]. Let (K, P, Σ) be a finite element (cf. def. 1.2.1). Assume that there exists a Banach[28] space[29] $V(K) \subset L^1(K; \mathbb{R}^q)$ such that
(i) $P \subset V(K)$;
(ii) the linear forms $\{\sigma_i\}_{i\in N}$ can be extended to $\mathcal{L}(V(K); \mathbb{R})$, i.e., there exist $\{\widetilde{\sigma}_i\}_{i\in N}$ and c_Σ such that $\widetilde{\sigma}_i(p) = \sigma_i(p)$ for all $p \in P$, and $|\widetilde{\sigma}_i(v)| \leq c_\Sigma ||v||_{V(K)}$ for all $v \in V(K)$ and all $i \in N$. We will abuse the notation and use σ_i instead of $\widetilde{\sigma}_i$.
The *interpolation operator* $\mathcal{I}_K : V(K) \longrightarrow P$ will be defined by

$$\mathcal{I}_K(v)(x) := \sum_{i\in N} \sigma_i(v)\vartheta_i(x) \;\; \forall x \in K,$$

for all $v \in V(K)$.

Proposition 1.3.2 (Boundedness)**.** *[19, Prop. 5.8]. The interpolation operator $\mathcal{I}_K$, which we just defined, lies in the vector space of the bounded linear operators from $V(K)$ to P, i.e., $\mathcal{I}_K \in \mathcal{L}(V(K); P)$.*

Proof. We consider a norm $||\cdot||_P$ in P. Using the representation of the interpolation operator from above, the triangle inequality[30] and part (ii) of the prior definition we see that

$$||\mathcal{I}_K(v)||_P \overset{\text{def.}}{=} ||\sum_{i\in N} \sigma_i(v)\vartheta_i||_P \overset{\triangle\text{-inequality}}{\leq} \sum_{i\in N} ||\sigma_i(v)\vartheta_i||_P \overset{1.3.1(\text{ii})}{\leq} (c_\Sigma \sum_{i\in N} ||\vartheta_i||_P)||v||_{V(K)}$$

for all $v \in V(K)$. $\square$

Proposition 1.3.3 (P-invariance)**.** *[19, Prop. 5.9]. The space P is pointwise invariant under $\mathcal{I}_K$, i.e., $\mathcal{I}_K(p) = p$ for all $p \in P$.*

[28]Stefan Banach (1892-1945) was a Polish mathematician (cf. [39]).
[29]See definition on p. 803 in [11].
[30]cf. p.111 in [11].

Proof. We want to make use of proposition 1.2.4. Therefore let $p = \sum_{j \in N} \alpha_j \vartheta_j$. Then

$$\mathcal{I}_K(p) = \sum_{i,j \in N} \alpha_j \sigma_i(\vartheta_j) \vartheta_i \overset{\text{prop.1.2.4}}{=} \sum_{i,j \in N} \alpha_j \delta_{ij} \vartheta_i = p$$

by the property of the shape functions and the definition of the Kronecker delta δ_{ij}. Hence, the statement is verified. One says that the interpolation operator fulfills the projection property. □

Remark 1.3.4 (Choices for $V(K)$)**.** [19, Example 5.10]. If we use the values of the given function v at some points on the mesh cell K for defining the interpolation operator $\mathcal{I}_K(v)$ then a suitable choice for $V(K)$ is the space of continuous functions from K to $\mathbb{R}^q$ which we write as $\mathcal{C}^0(K; \mathbb{R}^q)$. It is also common and for our discussions of highest interest to make the choice $V(K) := W^{k,p}(K; \mathbb{R}^q)$ for some real numbers $k \geq 0$ and $p \in [1, \infty]$ such that $kp > d$. We will define the notion of this space in the next section (cf. def. 1.4.8).

We have seen that the interpolation operator $\mathcal{I}_K$ is an element in the space $\mathcal{L}(V(K); P)$ (cf. Prop. 1.3.2), i.e., $\mathcal{I}_K$ is a bounded linear operator from $V(K)$ to P. Since P is a subspace of $V(K)$ we know that P can be equipped with the norm of $V(K)$. Thus, one can view $\mathcal{I}_K$ as an element in $\mathcal{L}(V(K))$.

Definition 1.3.5 (Lebesgue constant)**.** [19, p.55]. The quantity

$$||\mathcal{I}_K||_{\mathcal{L}(V(K))} := \sup_{v \in V(K) \setminus \{0\}} \frac{||\mathcal{I}_K(v)||_{V(K)}}{||v||_{V(K)}}$$

is called *Lebesgue constant* for $\mathcal{I}_K$.

Lemma 1.3.6 (Lower bound of the Lebesgue constant)**.** *[19, Lemma 5.13]. The Lebesgue constant is bounded from below. In other words, it holds that*

$$||\mathcal{I}_K||_{\mathcal{L}(V(K))} \geq 1.$$

Proof. Firstly, we note that P is nontrivial. It was proven in proposition 1.3.3 that the interpolation operator $\mathcal{I}_K$ is P-invariant, i.e., $\mathcal{I}_K(p) = p$ for all $p \in P$. Hence, we can conclude that

$$\sup_{v \in V(K) \setminus \{0\}} \frac{||\mathcal{I}_K(v)||_{V(K)}}{||v||_{V(K)}} \geq \sup_{p \in P} \frac{||\mathcal{I}_K(p)||_{V(K)}}{||p||_{V(K)}} = \sup_{p \in P} \frac{||p||_{V(K)}}{||p||_{V(K)}} = 1.$$

□

Theorem 1.3.7 (Interpolation error involving the Lebesgue constant)**.** *[19, Thm.5.14]. For all functions $v \in V(K)$ it holds that*

$$||v - \mathcal{I}_K(v)||_{V(K)} \leq (1 + ||\mathcal{I}_K||_{\mathcal{L}(V(K))}) \inf_{p \in P} ||v - p||_{V(K)}$$

and if $V(K)$ is a Hilbert space it further holds that

$$||v - \mathcal{I}_K(v)||_{V(K)} \leq ||\mathcal{I}_K||_{\mathcal{L}(V(K))} \inf_{p \in P} ||v - p||_{V(K)}.$$

Proof. We know that by the projection property of the interpolation operator it holds for all $p \in P$ that $\mathcal{I}_K(p) = p$ (see prop. 1.3.3). Hence, we are allowed to write

$$v - \mathcal{I}_K(v) = (\mathrm{Id}_{V(K)} - \mathcal{I}_K)(v) = (\mathrm{Id}_{V(K)} - \mathcal{I}_K)(v - p)$$

with $\mathrm{Id}_{V(K)}$ being the identity operator in $V(K)$.
Using the boundedness of $(\mathrm{Id}_{V(K)} - \mathcal{I}_K)$ and then invoking the triangle inequality we obtain

$$||v - \mathcal{I}_K(v)||_{V(K)} \leq ||(\mathrm{Id}_{V(K)} - \mathcal{I}_K)(v - p)||_{V(K)} \leq (1 + ||\mathcal{I}_K||_{\mathcal{L}(V(K))})||v - p||_{V(K)}.$$

Now taking the infimum over $p \in P$ gives us the first inequality.
For the other estimate we assume that $V(K)$ is a Hilbert space. The proof relies on the fact that in a Hilbert space H an operator $T \in \mathcal{L}(H)$ with $0 \neq T \circ T = T \neq \mathrm{Id}$ fulfills the equality

$$||T||_{\mathcal{L}(H)} = ||\mathrm{Id} - T||_{\mathcal{L}(H)}.$$

For a proof of this statement see Lemma 5 in [9].
We are going to use this result with $H := V(K)$ and $T := \mathcal{I}_K$. It is true that $\mathcal{I}_K \neq 0$ because P is nontrivial. Furthermore it holds that $\mathcal{I}_K \neq \mathrm{Id}_{V(K)}$ due to the fact that P is a proper subset of $V(K)$. Also observe that $\mathcal{I}_K \circ \mathcal{I}_K = \mathcal{I}_K$ by again using the P-invariance of the interpolation operator. Now applying the mentioned result we find

$$||v - \mathcal{I}_k(v)||_{V(K)} \leq ||\mathrm{Id}_{V(K)} - \mathcal{I}_K||_{\mathcal{L}(V(K))}||v - p||_{V(K)} = ||\mathcal{I}_K||_{\mathcal{L}(V(K))}||v - p||_{V(K)}.$$

Finally, we obtain the desired estimate by again taking the infimum over all $p \in P$. □

Remark 1.3.8 (On the interpolation error)**.** The just proven estimate indicates that a large Lebesgue constant corresponds to a bad approximation behavior of the interpolation operator $\mathcal{I}_K$.

1.4 Lebesgue and Sobolev Spaces

Since we want to investigate the properties of the interpolation operators in Sobolev spaces, we introduce them together with Lebesgue spaces in a concise way. This section is based on the appendix B in [10] and the appendix A in [15].

1.4.1 Lebesgue Spaces

Definition 1.4.1 (Lebesgue spaces)**.** [15, Def.A.27]. The space of functions that are Lebesgue integrable on Ω to the power of $p \in [1, \infty)$ is denoted by

$$L^p(\Omega) = \left\{ f \mid \int_\Omega |f(x)|^p \mathrm{d}x < \infty \right\},$$

which is equipped with the norm

$$||f||_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \mathrm{d}x\right)^{1/p}.$$

For $p = \infty$, this space is given by

$$L^\infty(\Omega) = \{f \mid |f(x)| < \infty \text{ almost everywhere in } \Omega\}$$

with the norm

$$||f||_{L^\infty(\Omega)} = \operatorname{ess}\ \sup_{x\in\Omega} |f(x)| = \inf_{N\subset\Omega,\mu(N)=0} \sup_{x\in\Omega\setminus N} |f(x)|,$$

where ess sup is called the essential supremum and μ denotes the Lebesgue measure[31] on $\mathbb{R}^d$.

Example 1.4.2 ($L^1(\Omega)$)**.** For $p = 1$ one has $L^1(\Omega)$, which is the space consisting of the scalar-valued functions that are Lebesgue-integrable over the domain Ω.

Definition 1.4.3 (The space of locally integrable functions)**.** [10, Def. B.2] Let $\mathbb{M}(\Omega)$ denote the space of equivalence classes of functions, where two functions belong to the same equivalence class if they coincide almost everywhere, i.e. everywhere but on a set of zero Lebesgue measure. Then the space of locally integrable functions $L^1_{\text{loc}}(\Omega)$ is defined as

$$L^1_{\text{loc}}(\Omega) = \{f \in \mathbb{M}(\Omega) \mid \forall K \subset \Omega \text{ compact}, f \in L^1(K)\}.$$

Theorem 1.4.4 (Hölder's[32] inequality)**.** *[10, Thm. B.6]. Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \le p \le +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg| dx \le ||f||_{L^p(\Omega)} ||g||_{L^q(\Omega)}.$$

Proof. See p.11 in [18]. □

Corollary 1.4.5 (Interpolation inequality)**.** *[10, Cor. B.7]. Let $1 \le p \le q \le +\infty$ and $0 \le \alpha \le 1$. Let r be such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{(1-\alpha)}{q}$. Then*

$$\forall f \in L^p(\Omega) \cap L^q(\Omega) : ||f||_{L^r(\Omega)} \le ||f||^\alpha_{L^p(\Omega)} ||f||^{1-\alpha}_{L^q(\Omega)}.$$

A very important Lebesgue space is the space of square-integrable functions $L^2(\Omega)$. The following theorem is one reason for the important role of this space.

Theorem 1.4.6 (The space of square-integrable functions)**.** *[10, Thm. B.9]. The space of square-integrable functions $L^2(\Omega)$ is a Hilbert space when equipped with the scalar product*

$$(f, g)_{L^2(\Omega)} := \int_\Omega fg\, dx$$

and the corresponding norm

$$||f||_{L^2(\Omega)} = \left(\int_\Omega f^2 dx\right)^{\frac{1}{2}}.$$

[31] See definition A.2.1 on p.544 in [18].

[32] Otto Hölder (1859-1937) was a German mathematician (cf. [37]).

Proof. This is a consequence of the Fischer[33]-Riesz[34] Theorem. □

Remark 1.4.7 (Cauchy-Schwarz inequality)**.** In $L^2(\Omega)$ the Hölder inequality becomes the so-called Cauchy[35]-Schwarz[36] inequality:

$$\forall f, g \in L^2(\Omega) : (f, g)_{L^2(\Omega)} \leq ||f||_{L^2(\Omega)}||g||_{L^2(\Omega)}.$$

1.4.2 Sobolev Spaces

Definition 1.4.8 (Sobolev spaces)**.** [15, Def. A.30]. Let $s \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty)$, then the Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \forall \alpha : |\alpha| \leq s\}.$$

This space is equipped with the norm

$$||u||_{W^{s,p}(\Omega)} := \sum_{|\alpha| \leq s} ||D^\alpha u||_{L^p(\Omega)}.$$

We also use the notation $H^s(\Omega) := W^{s,2}(\Omega)$, which is a Hilbert[37] space[38] with the inner product[39]

$$(u, v)_{H^s(\Omega)} = \sum_{|\alpha| \leq s} \int_\Omega D^\alpha u(x) D^\alpha v(x) \mathrm{d}x$$

and the induced[40] norm

$$||u||_{H^s(\Omega)} = (u, u)^{1/2}_{H^k(\Omega)}.$$

Example 1.4.9 ($H^1(\Omega)$)**.** [10, Example B.28]. The space $H^1(\Omega) = \{u \in L^2(\Omega) \mid \partial_i u \in L^2(\Omega), 1 \leq i \leq d\}$ equipped with the scalar product

$$(u, v)_{H^1(\Omega)} = \int_\Omega uv \ \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x = \int_\Omega uv \ \mathrm{d}x + \sum_{i=1}^{d} \int_\Omega \partial_i u \partial_i v \ \mathrm{d}x$$

is a Hilbert space.

The corresponding norm is given by

$$||u||^2_{H^1(\Omega)} = \int_\Omega u^2 \ \mathrm{d}x + \int_\Omega (\nabla u)^2 \ \mathrm{d}x = ||u||_{L^2(\Omega)} + |u|^2_{H^1(\Omega)}.$$

[33]Ernst Sigismund Fischer (1875-1954) was an Austrian mathematician (cf. [28]).

[34]Frigyes Riesz (1880-1956) was a Hungarian mathematician (cf. [31]).

[35]Augustin-Louis Cauchy (1789-1857) was a French mathematician (cf. [25]).

[36]Hermann Amandus Schwarz (1843-1921) was a German mathematician (cf. [33]).

[37]David Hilbert (1862-1943) was a German mathematician and one of the most influential mathematicians of the 19th and early 20th century (cf. [27]).

[38]See definition on p.810 in [11].

[39]See definition A.5. on p.464 in [10].

[40]See example (1.5) on p.6 in [16].

Definition 1.4.10 (Fractional Sobolev spaces)**.** [10, Def. B.30]. For $0 < s < 1$ and $1 < p < +\infty$ the *Sobolev space with fractional exponent* is defined as

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid \frac{u(x) - u(y)}{||x-y||^{s+\frac{d}{p}}} \in L^p(\Omega \times \Omega) \right\}$$

Furthermore, if $s > 1$ is not an integer and letting $\sigma = s - [s]$ with $[s]$ denoting the integer part of s, then one defines the space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) = \{u \in W^{[s],p}(\Omega) \mid D^\alpha u \in W^{\sigma,p}(\Omega), \forall \alpha : |\alpha| = [s]\}.$$

If $p = 2$ we will write $H^s(\Omega) = W^{s,2}(\Omega)$ again.

1.5 Embedding

In some occasions we will need to make use of the embedding of Sobolev spaces. Here we recall some important results. The content of this section is taken from appendix B in [10].

Proposition 1.5.1 (Embedding of Lebesgue spaces)**.** *[10, Prop. B.39]. Let Ω be an open bounded set. Then, for $1 \leq p < q \leq +\infty$ the embedding $L^q(\Omega) \subset L^p(\Omega)$ is continuous.*

Proof. This statement follows from Hölder's inequality. □

Theorem 1.5.2 (Sobolev inequality)**.** *[10, Thm. B.40]. Let $1 \leq p < d$ and denote by p^* the number such that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$ or equivalently $p^* = \frac{pd}{d-p}$. Then there exists $c = \frac{p^*}{1^*}$ with $1^* := \frac{d}{d-1}$ such that for all functions $u \in W^{1,p}(\mathbb{R}^d)$ the inequality*

$$||u||_{L^{p^*}(\mathbb{R}^d)} \leq c||\nabla u||_{L^p(\mathbb{R}^d)}$$

holds.

Proof. See page 32 in [7]. □

Corollary 1.5.3. *[10, Cor. B.41]. Let $1 \leq p, q \leq +\infty$. The following embeddings*

$$W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \;\; \textit{if} \;\; \begin{cases} \textit{either} \;\; 1 \leq p < d \;\; \textit{and} \;\; p \leq q \leq p^*, \\ \textit{or} \;\; p = d \;\; \textit{and} \;\; p \leq q < +\infty \end{cases}$$

are continuous.

Proof. A proof is given on page 34 in [7]. □

Theorem 1.5.4 (Morrey[41])**.** *[10, Thm. B.42]. Let $d < p \leq +\infty$ and $\alpha = 1 - \frac{d}{p}$. The embedding*

$$W^{1,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d) \cap C^{0,\alpha}(\mathbb{R}^d)$$

is continuous.

[41] Charles B. Morrey Jr. (1907-1984) was an American mathematician (cf. [26]).

Proof. A proof can be found on page 37 in [7]. □

Corollary 1.5.5. *[10, Cor. B.43]. Let $1 \leq p, q \leq +\infty$. Let $s \geq 1$ be an integer. Let Ω be a bounded open set having the $(1, p)$-extension property. The embeddings*

$$W^{s,p}(\Omega) \subset \begin{cases} L^q(\Omega) & \text{if } 1 \leq p < \frac{d}{s} \text{ and } p \leq q \leq p^*, \\ L^q(\Omega) & \text{if } p = \frac{d}{s} \text{ and } p \leq q < +\infty, \\ L^\infty(\Omega) \cap C^{0,\alpha}(\overline{\Omega}) & \text{if } p > \frac{d}{s} \text{ and } \alpha = 1 - \frac{d}{sp}. \end{cases}$$

are continuous.

Remark 1.5.6 (Continuity of functions in $H^1(\Omega)$)**.** A consequence of this theorem is that in one dimension functions in $H^1(\Omega)$ are continuous. However, one should note that in dimension 2 or in dimension 3 this is not true any longer. This remark will be quite important for some proofs in the other chapters of this thesis. Nevertheless, this result will also limit some investigations as we will see in the section about the weighted L^2 projection.

1.6 Error Bounds for Interpolation in Sobolev Spaces

The objective of this section is to state local as well as global error estimates on affine meshes for scalar-valued functions belonging to Sobolev spaces, which we defined previously (cf. def. 1.4.8). We will refer to the literature for their proofs. The results given are taken from section 1.5.1 in [10].

Theorem 1.6.1 (Local Interpolation)**.** *[10, Thm.1.103]. Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a local finite element with associated normed vector space $V(\widehat{K})$. Let $1 \leq p \leq \infty$ and assume that there exists an integer k such that*

$$\mathbb{P}_k \subset \widehat{P} \subset W^{k+1,p}(\widehat{K}) \subset V(\widehat{K}). \tag{1}$$

Furthermore, let $T_K : \widehat{K} \longrightarrow K$ be an affine bijective mapping from the reference mesh cell to another mesh cell and let $\mathcal{I}_K^k$ be the local interpolation operator on K. Let l be chosen in such a way that $0 \leq l \leq k$ and $W^{l+1,p}(\widehat{K}) \subset V(\widehat{K})$ with continuous embedding. Then, setting $\sigma_K = \frac{h_K}{\rho_K}$ where ρ_K denotes the diameter of the largest ball that can be inscribed in K, there exists $c > 0$ such that for all $m \in \{0, ..., l+1\}$ we have

$$\forall K, \forall v \in W^{l+1,p}(K) : |v - \mathcal{I}_K^k v|_{W^{m,p}(K)} \leq c h_K^{l+1-m} \sigma_K^m |v|_{W^{l+1,p}(K)}.$$

Proof. A proof of this result is given on page 59 in [10]. □

Definition 1.6.2 (Degree of a finite element)**.** [10, Def.1.104]. The largest integer k such that the embedding (1) holds is called the *degree* of the finite element $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$.

Remark 1.6.3 (Optimal error estimate)**.** If the function we want to interpolate lies in $W^{k+1,p}(K)$ we may set $l = k$ in the previous theorem. Then one obtains an error estimate, which is optimal. In particular, one gets for $m \in \{0, ..., k+1\}$ that

$$\forall K, \forall W^{k+1,p}(K) : |v - \mathcal{I}_K^k v|_{W^{m,p}(K)} \leq ch_K^{k+1-m} \sigma_K^m |v|_{W^{l+1,p}(K)}.$$

Theorem 1.6.4 (Global interpolation)**.** *[10, Cor. 1.109]. Let p, k and l be such that they satisfy the assumptions of the previous theorem. Furthermore, let Ω be a polyhedron and let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine meshes of Ω. With V_h^k we denote the approximation space based on $\{\mathcal{T}_h\}$ and the finite element $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$. Let $\mathcal{I}_h^k$ be the corresponding global interpolation operator. Then, there exists a constant c such that for all h and $v \in W^{l+1,p}(\Omega)$ one finds*

$$||v - \mathcal{I}_h^k v||_{L^p(\Omega)} + \sum_{m=1}^{l+1} h^m \left(\sum_{K \in \mathcal{T}_h} |v - \mathcal{I}_h^k v|_{W^{m,p}(K)}^p \right)^{\frac{1}{p}} \leq ch^{l+1} |v|_{W^{l+1,p}(\Omega)},$$

for $p < \infty$, and for $p = \infty$ one has

$$||v - \mathcal{I}_h^k v||_{L^\infty(\Omega)} + \sum_{m=1}^{l+1} h^m \max_{K \in \mathcal{T}_h} |v - \mathcal{I}_h^k v|_{W^{m,\infty}(K)} \leq ch^{l+1} |v|_{W^{l+1,\infty}(\Omega)}.$$

Proof. A proof can be found on page 61 in [10]. □

Corollary 1.6.5 (Interpolation in $W^{s,p}(\Omega)$)**.** *[10, Cor.1.110]. Let the conditions of the previous theorem hold. We assume that V_h^k is $W^{1,p}$-conformal, which means that $V_h^k \subset W^{1,p}(\Omega)$. Then there exists a constant c such that for all h and $v \in W^{l+1,p}(\Omega)$ the estimate*

$$|v - \mathcal{I}_h^k v|_{W^{1,p}(\Omega)} \leq ch^l |v|_{W^{l+1,p}(\Omega)}$$

holds.

2 The Lagrange Interpolation Operator $\mathcal{I}_K^L$

In this section the Lagrange interpolation operator will be introduced, which is a popular choice for the interpolation of smooth functions. The main idea is to define the interpolation operator using the values of the function to be interpolated at the nodes that are distributed over the proposed mesh. That means in particular, following the definition of a general interpolation operator (cf. def. 1.3.1), that for a set of points $\{a_i\}_{i=1}^N$ in some mesh cell K the general Lagrange interpolation operator is defined by

$$\mathcal{I}_K^L v := \sum_{i \in N} v(a_i)\vartheta_i$$

where ϑ_i are suitable shape functions and v is the function to be interpolated (cf. p.54 in [19]). We will make this definition more accessible in the following sections. For the sake of simplicity the one-dimensional case will be presented first. Later the idea of this operator will be generalized to higher dimensions. Throughout this section we will derive some estimates on the interpolation error, which will help us to understand how interpolation estimates are proven. This chapter follows closely the presentation in chapter 1.1 in [10].

2.1 The One-Dimensional Case

2.1.1 Interpolation Using Polynomials of Degree 1

In one dimension a domain is an open, bounded interval (cf. def. 1.1.1). So we will consider the open interval $\Omega = (a, b)$, which will be decomposed by a mesh. In this scenario, following def. 1.1.6 and def. 1.1.7 as well as remark 1.1.8, a mesh is given by an indexed set of subintervals with non-zero measure $\{I_i = [x_{1,i}, x_{2,i}]\}_{0 \le i \le N}$ that form a *partition* of Ω, which means that

$$\overline{\Omega} = \bigcup_{i=0}^{N} I_i \text{ and } \overset{\circ}{I}_i \cap \overset{\circ}{I}_j = \emptyset \text{ for } i \neq j.$$

In order to construct such a mesh we pick $(N+2)$ points in $\overline{\Omega}$ such that

$$a = x_0 < x_1 < \dots < x_N < x_{N+1} = b,$$

where we set $x_{1,i} = x_i$ and $x_{2,i} = x_{i+1}$ for $0 \le i \le N$. Using the introduced terminology from definition 1.1.7 we call the points $\{x_0, \dots, x_{N+1}\}$ the *vertices* of the mesh. Furthermore we allow the mesh to have variable step size, i.e.,

$$h_i := x_{i+1} - x_i, \;\; 0 \le i \le N,$$

and define h to be the maximum step size, i.e.,

$$h := \max_{0 \le i \le N} h_i.$$

According to remark 1.1.8 the intervals will be called *mesh cells*. The mesh will be denoted as $\mathcal{T}_h := \{I_i\}_{0 \le i \le N}$.
As an approximation space we will consider the vector space of continuous, piecewise linear functions

$$P_h^1 = \{v_h \in \mathcal{C}^0(\overline{\Omega}) \mid \forall i \in \{0, ..., N\} : v_{h|I_i} \in \mathbb{P}_1\},$$

where $\mathcal{C}^0(\overline{\Omega})$ means the space of continuous functions on $\overline{\Omega} \subset \mathbb{R}$ and $\mathbb{P}_1$ denotes the vector space composed of univariate polynomial functions of degree at most 1. We define for $i \in \{0, ..., N+1\}$ the functions

$$\varphi_i(x) := \begin{cases} \frac{1}{h_{i-1}} \cdot (x - x_{i-1}) = \frac{(x-x_{i-1})}{(x_i-x_{i-1})} & \text{if } x \in I_{i-1}, \\ \frac{1}{h_i} \cdot (x_{i+1} - x) = \frac{(x_{i+1}-x)}{(x_{i+1}-x_i)} & \text{if } x \in I_i, \\ 0 & \text{otherwise,} \end{cases}$$

with suitable modifications if $i = 0$ or $i = N+1$ (cf. p.4 in [10]).

Remark 2.1.1 (Hat functions)**.** One easily sees that the φ_i are continuous, piecewise linear functions, i.e., $\varphi_i \in P_h^1$. With respect to the shape of their graphs the defined functions are also often called *hat functions* (cf. figure 5).

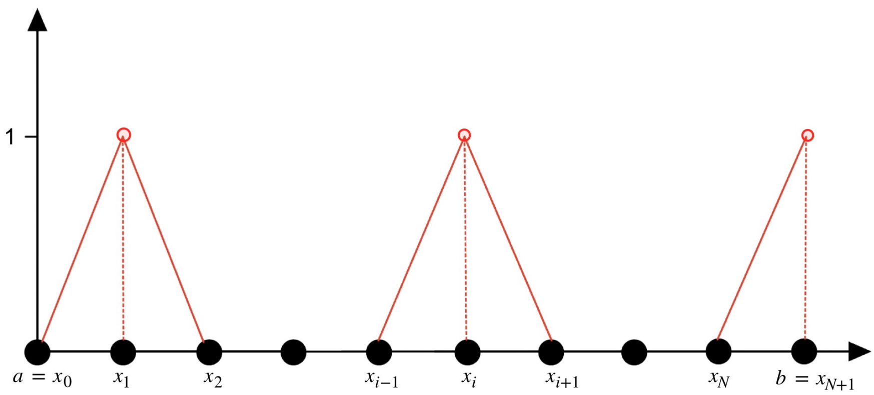

Figure 5: The graphs of hat functions (based on figure 1.1 in [10]).

Proposition 2.1.2 (Basis of P_h^1)**.** *[10, Prop.1.1]. The set of hat functions* $\{\varphi_0, ..., \varphi_{N+1}\}$ *is a basis*[42] *for the space of continuous, piecewise linear functions* P_h^1.

Proof. From the definition it follows directly that

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

[42] See definition on p.204 in [11].

for $0 \leq i, j \leq N+1$ with δ_{ij} denoting the Kronecker symbol. This fact will be of importance for our argumentation. In order to prove the linear independence[43] let $(\lambda_0, ..., \lambda_{N+1})^T \in \mathbb{R}^{N+2}$. We make the assumption that the continuous function $w = \sum_{i=0}^{N+1} \lambda_i \varphi_i$ vanishes for all $x \in \Omega$. Thus, it holds by the observation from the beginning that

$$w(x_i) = \sum_{i=0}^{N+1} \lambda_i \varphi_i(x_i) = \lambda_i = 0 \text{ for } 0 \leq i \leq N+1.$$

That proves the linear independence.
Now we need to show that the set $\{\varphi_0, ..., \varphi_{N+1}\}$ spans P_h^1. Given that on each mesh cell I_i the functions v_h and $\sum_{i=0}^{N+1} v_h(x_i)\varphi_i$ are affine and coincide at the two points x_i and x_{i+1}, we know that for all $v_h \in P_h^1$ it holds that $v_h = \sum_{i=0}^{N+1} v_h(x_i)\varphi_i$. This completes the proof. □

Having found appropriate shape functions, we still need linear forms to properly define an interpolation operator in the sense of definition 1.3.1. As we have already stated in the beginning of this chapter, the essence of Lagrange interpolation is that it uses the values of the function to be interpolated. So consider the linear forms that evaluate a function at the i-th node, i.e.,

$$\begin{aligned} \sigma_i : \mathcal{C}^0(\overline{\Omega}) &\longrightarrow \mathbb{R} \\ v &\longmapsto \sigma_i(v) = v(x_i), \quad i \in \{0, ..., N+1\}. \end{aligned}$$

Using the gathered information we are able to define the Lagrange interpolation operator.

Definition 2.1.3 (Lagrange interpolation operator)**.** [10, p.5]. The interpolation operator

$$\begin{aligned} \mathcal{I}_h^1 : \mathcal{C}^0(\overline{\Omega}) &\longrightarrow P_h^1 \\ v &\longmapsto \sum_{i=0}^{N+1} \sigma_i(v)\varphi_i \end{aligned}$$

is called the *Lagrange interpolation operator* of degree 1.

Remark 2.1.4 (On the defintion of the Lagrange interpolation operator)**.** Looking at the definition of this interpolation operator it is clear that for a function v that belongs to $\mathcal{C}^0(\overline{\Omega})$, the Lagrange interpolant of v, i.e., $\mathcal{I}_h^1 v$, is the unique continuous, piecewise linear function that takes the same value as v at all vertices within the mesh.

Having defined the Lagrange interpolation operator we want to discuss its properties in Sobolev spaces.

Lemma 2.1.5 (Subspace of $H^1(\Omega)$)**.** *[10, Lemma 1.3]. It holds that $P_h^1 \subset H^1(\Omega) = W^{1,2}(\Omega)$.*

[43] See definition on p.201 and the used criterion for linear independence on p.203 in [11].

Proof. Let v_h be a function in P_h^1. It is clear by definition that v_h also lies in $L^2(\Omega)$. Recall that functions in P_h^1 are continuous by definition. Thus, by the continuity of v_h we see that its first-order distributional derivative[44] is the piecewise constant function u_h such that for the restriction of u_h to each mesh cell I_i it holds that

$$\forall I_i \in \mathcal{T}_h : u_{h|I_i} = \frac{v_h(x_{i+1}) - v_h(x_i)}{h_i}.$$

Then, by the same argumentation as for v_h, we know that the function u_h lies in $L^2(\Omega)$ too and hence $v_h \in H^1(\Omega)$. □

Proposition 2.1.6 (H^1-stability of $\mathcal{I}_h^1$)**.** *[10, Prop. 1.4]. $\mathcal{I}_h^1$ is a linear continuous mapping from $H^1(\Omega)$ to $H^1(\Omega)$, and there exists a positive constant C such that*

$$||\mathcal{I}_h^1 v||_{H^1(\Omega)} \leq C||v||_{H^1(\Omega)}.$$

Proof. For the first part of the proposition we know by the embedding theorems that in dimension 1, a function in $H^1(\Omega)$ is continuous (cf. remark 1.5.6). Let $v \in H^1(\Omega)$ and $x, y \in \overline{\Omega}$. By the Cauchy-Schwarz inequality, which we stated in remark 1.4.7, one finds

$$|v(y) - v(x)| \leq \left| \int_x^y |v'(s)| \mathrm{d}s \right| = \left| \int_x^y |v'(s)| \cdot 1 \mathrm{d}s \right| \leq |y - x|^{\frac{1}{2}} \left(\int_\Omega |v'(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}} = |y - x|^{\frac{1}{2}} |v|_{H^1(\Omega)}. \tag{2}$$

Let x be a point where $|v|$ attains its minimum in $\overline{\Omega}$. By inequality (2) we can argue that

$$||v||_{L^\infty(\Omega)} \leq |b - a|^{-\frac{1}{2}} ||v||_{L^2(\Omega)} + |b - a|^{\frac{1}{2}} |v|_{H^1(\Omega)}, \tag{3}$$

because $|v(x)| \leq |b - a|^{-\frac{1}{2}} ||v||_{L^2(\Omega)}$. A detailed proof of a more general version of this statement can be found on p.24 in [20]. Thus, the Lagrange interpolant $\mathcal{I}_h^1 v$ is well-defined for $v \in H^1(\Omega)$. By the previous lemma we know that $P_h^1 \subset H^1(\Omega)$, which implies that the Lagrange interpolant lies in $H^1(\Omega)$, i.e., $\mathcal{I}_h^1 v \in H^1(\Omega)$. This means $\mathcal{I}_h^1$ maps $H^1(\Omega)$ to $H^1(\Omega)$, which proves the first part of the statement.

For the second part of the statement we consider a mesh cell $I_i \in \mathcal{T}_h$ for $0 \leq i \leq N$. By the proof of the previous lemma we know that $(\mathcal{I}_h^1 v)'_{|I_i} = h_i^{-1}(v(x_{i+1}) - v(x_i))$. We get

$$\begin{aligned}
|\mathcal{I}_h^1 v|_{H^1(I_i)} = \left(\int_{x_i}^{x_{i+1}} |h_i^{-1}(v(x_{i+1}) - v(x_i))|^2 \mathrm{d}s \right)^{\frac{1}{2}} &= h_i^{-1} |v(x_{i+1}) - v(x_i)| \left(\int_{x_i}^{x_{i+1}} 1 \mathrm{d}s \right)^{\frac{1}{2}} \\
&= h_i^{-1} |v(x_{i+1}) - v(x_i)| |x_{i+1} - x_i|^{\frac{1}{2}} \\
&= h_i^{-\frac{1}{2}} |v(x_{i+1}) - v(x_i)| \\
&= h_i^{-\frac{1}{2}} \int_{x_i}^{x_{i+1}} |v'(s)| \mathrm{d}s \\
&\leq h_i^{-\frac{1}{2}} |x_{i+1} - x_i|^{\frac{1}{2}} |v|_{H^1(I_i)} = |v|_{H^1(I_i)},
\end{aligned}$$

[44] See definition B.19 on p. 481 in [10].

using estimate (2) in the last step. It immediately follows that $|\mathcal{I}_h^1 v|_{H^1(\Omega)} \leq |v|_{H^1(\Omega)}$. It further holds by the definition of the L^∞-norm (cf. def. 1.4.1) that

$$||\mathcal{I}_h^1 v||_{L^2(\Omega)} = \left(\int_\Omega |\mathcal{I}_h^1 v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq \left(\int_\Omega ||\mathcal{I}_h^1 v||^2_{L^\infty(\Omega)} \mathrm{d}x\right)^{\frac{1}{2}} \leq |b-a|^{\frac{1}{2}} ||\mathcal{I}_h^1 v||_{L^\infty(\Omega)}.$$

Because the function and its linear interpolant coincide at the vertices we also have $||\mathcal{I}_h^1 v||_{L^\infty(\Omega)} \leq ||v||_{L^\infty(\Omega)}$. We further assume that h is bounded. Now using these estimates we infer by (3) that

$$||\mathcal{I}_h^1 v||_{L^2(\Omega)} \leq C||v||_{H^1(\Omega)},$$

where C is a constant that is independent of h. This completes the proof. □

Proposition 2.1.7 (Interpolation error)**.** *[10, Prop. 1.5]. For all h and $v \in H^2(\Omega)$,*

$$||v - \mathcal{I}_h^1 v||_{L^2(\Omega)} \leq h^2 ||v''||_{L^2(\Omega)} \text{ and } ||(v - \mathcal{I}_h^1 v)'||_{L^2(\Omega)} \leq h||v''||_{L^2(\Omega)}.$$

Proof. Let $I_i \in \mathcal{T}_h$ be a mesh cell and let u be a function in $H^1(\Omega)$ which takes the value zero at some point η in the mesh cell I_i. Then, by inequality (2) we obtain

$$||u||_{L^2(I_i)} \leq h_i ||u'||_{L^2(I_i)}.$$

Next, let $v \in H^2(\Omega)$ and let $i \in \{0, ..., N\}$. Furthermore, we set $u_i = (v - \mathcal{I}_h^1 v)'_{|I_i}$. Note that u_i lies in $H^1(I_i)$. Observe that Rolle[45]'s theorem tells us that u_i vanishes at some point η in the mesh cell I_i. By the application of the inequality $||u||_{L^2(I_i)} \leq h_i||u'||_{L^2(I_i)}$ to u_i and by using the fact that $(\mathcal{I}_h^1 v)''$ becomes zero for all arguments $x \in I_i$ we come to the conclusion that

$$||v' - (\mathcal{I}_h^1 v)'||_{L^2(I_i)} \leq h_i ||v''||_{L^2(I_i)}.$$

Now summation over the mesh cells yields the second estimate in the proposition.
For proving the first estimate, one needs to apply the estimate $||u||_{L^2(I_i)} \leq h_i||u'||_{L^2(I_i)}$ to $(v - \mathcal{I}_h^1 v)_{|I_i}$. Using that one finds

$$||v - \mathcal{I}_h^1 v||_{L^2(I_i)} \leq h_i ||v' - (\mathcal{I}_h^1 v)'||_{L^2(I_i)} \leq h_i^2 ||v''||_{L^2(I_i)}.$$

Finally, one concludes again by summing over the mesh cells. □

Remark 2.1.8 (On the proven error bounds)**.** [10, p. 7]. (i) Looking at the proven error bound one sees that the bound on the interpolation error involves second-order derivatives of v. This results from the fact that a large second derivative corresponds to a high deviation of the graph of v from the piecewise linear interpolant.
(ii) If the function, which we seek to interpolate lies in $H^1(\Omega)$ only, one can verify that

$$\forall h : ||v - \mathcal{I}_h^1 v||_{L^2(\Omega)} \leq h||v'||_{L^2(\Omega)} \text{ and } \lim_{h \longrightarrow 0} ||(v - \mathcal{I}_h^1 v)'||_{L^2(\Omega)} = 0.$$

(iii) The proof we have given for the last proposition indicates that the Lagrange interpolation operator $\mathcal{I}_h^1$ possesses numerous local interpolation properties. In other words, one observes that the interpolation error is controlled elementwise before it is controlled on a global level. This observation serves as a motivation for the introduction of local interpolation operators.

[45] Michel Rolle (1652-1719) was a French mathematician (cf. [36]).

2.1.2 Interpolation with Higher-Degree Polynomials

The interpolation technique from the previous section can be generalized to higher-degree polynomials. This has the advantage that we can interpolate smooth functions to high-order accuracy. This section follows closely the material from section 1.1.3 in [10]. In order to do this this generalization process we again consider the mesh $\mathcal{T}_h = \{I_i\}$ from the beginning. Different to the previous part is the choice of the approximation space. Now we consider the space

$$P_h^k = \{v_h \in \mathcal{C}^0(\overline{\Omega}); \forall i \in \{0, ..., N\}, v_{h|I_i} \in \mathbb{P}_k\},$$

with $\mathbb{P}_k$ denoting the vector space that consists of univariate polynomials of degree at most k, i.e., $p \in \mathbb{P}_k$ if $p(t) = \sum_{i=0}^{k} \alpha_i t^i$ for all $t \in \mathbb{R}$, with $\alpha_i \in \mathbb{R}$ for every integer $i \in \{0, ..., k\}$. Before defining the corresponding interpolation operator we need to recall the notion of Lagrange polynomials.

Definition 2.1.9 (Lagrange polynomials)**.** [10, Def. 1.7]. Let $k \geq 1$ and let $\{s_0, ..., s_k\}$ be $(k+1)$ distinct numbers. The Lagrange polynomials $\{\ell_0^k, ..., \ell_k^k\}$ associated with the nodes $\{s_0, ..., s_k\}$ are defined as

$$\ell_m^k(t) := \frac{\prod_{l \neq m}(t - s_l)}{\prod_{l \neq m}(s_m - s_l)}, \;\; 0 \leq m \leq k.$$

Remark 2.1.10 (Kronecker delta)**.** The Lagrange polynomials satisfy the property

$$\ell_m^k(s_l) = \delta_{ml}, \;\; 0 \leq m, l \leq k,$$

by definition.

Example 2.1.11 (Computation of Lagrange polynomials)**.** Consider the unit interval $[0, 1]$ using equi-distributed nodes. One computes for the degree $k = 1$ the Lagrange polynomials

$$\ell_0^1(t) = \frac{t-1}{0-1} = \frac{t-1}{(-1)} = -t + 1 \;\text{ and }\; \ell_1^1(t) = \frac{t-0}{1-0} = t.$$

For $k = 2$ one finds

$$\begin{aligned}
\ell_0^2(t) &= \frac{t-\frac{1}{2}}{0-\frac{1}{2}} \cdot \frac{t-1}{0-1} = 2t^2 - 3t + 1, \\
\ell_1^2(t) &= \frac{t-0}{\frac{1}{2}-0} \cdot \frac{t-1}{\frac{1}{2}-1} = -4t^2 + 4t \;\text{ and} \\
\ell_2^2(t) &= \frac{t-0}{1-0} \cdot \frac{t-\frac{1}{2}}{1-\frac{1}{2}} = 2t^2 - t.
\end{aligned}$$

Remark 2.1.12 (Nodes for different degrees k)**.** The example illustrates that for the degree $k = 1$ the nodes coincide with the vertices of the proposed mesh. For a degree of $k = 2$ one sees that the nodes include the midpoint of the interval. For higher k it generalizes in the obvious way.

Since we want to define the Lagrange interpolation operator for arbitrary degrees k we will proceed in a similar way as we did in section 2.1. Hence, we will now define an analogon to the hat functions from the previous section where we looked at $k = 1$. Therefore, we will define the *nodes* $\xi_{i,m} := x_i + \frac{m}{k} h_i, 0 \leq m \leq k$ in the mesh cell I_i for $i \in \{0, ..., N\}$. Next we will use the Lagrange polynomials, which we defined above and associate those to the nodes $\xi_{i,m}$. So let $\{\ell^k_{i,0}, ..., \ell^k_{i,k}\}$ be the Lagrange polynomials associated with these nodes. Now for $j \in \{0, ..., k(N+1)\}$ with $j = ki+m$ and $0 \leq m \leq k-1$, we define for $1 \leq m \leq k-1$ the functions

$$\varphi_{ki+m}(x) = \begin{cases} \ell^k_{i,m}(x) & \text{if } x \in I_i, \\ 0 & \text{otherwise}, \end{cases}$$

and for $m = 0$,

$$\varphi_{ki}(x) = \begin{cases} \ell^k_{i-1,k}(x) & \text{if } x \in I_{i-1}, \\ \ell^k_{i,0}(x) & \text{if } x \in I_i, \\ 0 & \text{otherwise}, \end{cases}$$

with certain modifications if $i = 0$ or $i = N + 1$ (cf. p.8 in [10]).

Lemma 2.1.13 (Hat functions lie in P^k_h)**.** *[10, Lemma 1.8]. The hat functions φ_j lie in P^k_h.*

Proof. Let $j \in \{0, ..., k(N + 1)\}$ with $j = ki + m$. For $1 \leq m \leq k - 1$ we find that

$$\varphi_j(x_i) = \varphi_j(x_{i+1}) = 0.$$

This leads to the conclusion that φ_j lies in the space $\mathcal{C}^0(\overline{\Omega})$. Also note that if we restrict the φ_j to each mesh cell that these restrictions will lie in $\mathbb{P}_k$ of the considered mesh cell, which is due to their construction. It follows that $\varphi_j \in P^k_h$. Next we make the assumption that $m = 0$ and $0 < i < N+1$. It is clear that φ_{ki} is continuous at x_i by construction. Furthermore it holds that

$$\varphi_{ki}(x_{i-1}) = \varphi_{ki}(x_{i+1}) = 0.$$

We conclude that φ_{ki} is an element of P^k_h. For the cases $i = 0$ and $i = N + 1$ one follows a similar procedure. This ends the proof. □

As in the previous section we will again need linear forms for the definition of the interpolation operator. We introduce the set of nodes $\{a_j\}_{0 \leq j \leq k(N+1)}$ such that $a_j = \xi_{i,m}$ where $j = ik + m$. For $j \in \{0, ..., k(N + 1)\}$ we consider the linear forms

$$\begin{aligned} \sigma_j : \mathcal{C}^0(\overline{\Omega}) &\longrightarrow \mathbb{R} \\ v &\longmapsto \sigma_j(v) = v(a_j), \end{aligned}$$

which are defined by evaluating the function v at the nodes a_j.

Proposition 2.1.14 (Bases for P_h^k and $\mathcal{L}(P_h^k, \mathbb{R})$)**.** *[10, Prop. 1.9]. The set $\{\varphi_0, ..., \varphi_{k(N+1)}\}$ is a basis for P_h^k and $\{\sigma_0, ..., \sigma_{k(N+1)}\}$ is a basis for $\mathcal{L}(P_h^k; \mathbb{R})$.*

Proof. Using the fact that for the linear forms it holds that $\sigma_j(\varphi_{j'}) = \delta_{jj'}$ for $0 \le j, j' \le k(N+1)$, one can prove this statement similarly to the proof of proposition 2.1.2. □

We are now ready to define the interpolation operator $\mathcal{I}_h^k$.

Definition 2.1.15 (Lagrange interpolant of degree k)**.** [10, p.10].

The interpolation operator

$$\begin{aligned} \mathcal{I}_h^k : \mathcal{C}^0 &\longrightarrow P_h^k \\ v &\longmapsto \sum_{j=0}^{k(N+1)} \sigma_j(v)\varphi_j \end{aligned}$$

is called the *Lagrange interpolation operator* of degree k.

Remark 2.1.16 (Properties of the Lagrange interpolant of degree k)**.** [10, p.10]. (i) It is clear that the Lagrange interpolation operator of degree k is linear.
(ii) The Lagrange interpolant $\mathcal{I}_h^k v$ is the unique function in P_h^k for which its values at all the nodes of the mesh coincide with the values of the function v.

Lemma 2.1.17 (Subspace of $H^1(\Omega)$)**.** *[10, Lemma 1.10]. The space P_h^k is a subset of $H^1(\Omega)$.*

Proof. Similar to the proof for P_h^1 (cf. Lemma 2.1.5). □

We want to take a closer look at the properties of $\mathcal{I}_h^k$. The following construction is taken from p.10 in [10]. We look at the i-th interval $I_i = [x_i, x_{i+1}] \in \mathcal{T}_h$ and consider the $(k+1)$ linear forms $\{\sigma_{i,0}, ..., \sigma_{i,k}\}$ defined by the point evaluations at the nodes $\xi_{i,m}$, i.e.,

$$\begin{aligned} \sigma_{i,m} : \mathbb{P}_k &\longrightarrow \mathbb{R} \\ p &\longmapsto \sigma_{i,m}(p) = p(\xi_{i,m}), \;\; 0 \le m \le k. \end{aligned}$$

As the local shape functions $\{\vartheta_{i,0}, ..., \vartheta_{i,k}\}$ we choose the $(k+1)$ Lagrange polynomials associated with the nodes $\{\xi_{i,0}, ..., \xi_{i,k}\}$, i.e.,

$$\vartheta_{i,m} = \ell_{i,m}^k \;\text{ for }\; 0 \le m \le k.$$

A familiy $\{\mathcal{I}_{I_i}^k\}_{I_i \in \mathcal{T}_h}$ of local interpolation operators for $i \in \{0, ..., N\}$ is now defined by

$$\begin{aligned} \mathcal{I}_{I_i}^k : \mathcal{C}^0(I_i) &\longrightarrow P_h^k \\ v &\longmapsto \sum_{m=0}^{k} \sigma_{i,m}(v)\vartheta_{i,m} = \sum_{m=0}^{k} v(\xi_{i,m})\ell_{i,m}^k. \end{aligned}$$

This means for all $0 \leq i \leq N$ and $v \in \mathcal{C}^0(\overline{\Omega})$ we have by definition that

$$(\mathcal{I}_h^k v)_{|I_i} = \mathcal{I}_{I_i}^k(v_{|I_i}).$$

We want to show that a reference interpolation operator can generate the family $\{\mathcal{I}_{I_i}^k\}_{I_i \in \mathcal{T}_h}$. Therefore we consider the unit interval $\widehat{K} = [0,1]$ as a reference cell. We further set $\widehat{P} = \mathbb{P}_k$ and define on the reference mesh cell, analogously to the prior discussion, the $(k+1)$ linear forms $\{\widehat{\sigma}_0, ..., \widehat{\sigma}_k\}$ by

$$\begin{aligned} \widehat{\sigma}_m : \mathbb{P}_k &\longrightarrow \mathbb{R} \\ \widehat{p} &\longmapsto \widehat{\sigma}_m(\widehat{p}) = \widehat{p}(\widehat{\xi}_m), \ \ 0 \leq m \leq k, \end{aligned}$$

where the m-th node on the reference cell is given by $\widehat{\xi}_m = \frac{m}{k}$. Let $\{\widehat{\ell}_0^k, ..., \widehat{\ell}_k^k\}$ be the Lagrange polynomials that we associate with the reference nodes $\{\widehat{\xi}_0, ..., \widehat{\xi}_k\}$.
Next let $\widehat{\vartheta}_m = \widehat{\ell}_m^k, 0 \leq m \leq k$, such that $\widehat{\sigma}_m(\widehat{\vartheta}_n) = \delta_{mn}$ for $0 \leq m, n \leq k$. Now the interpolation operator on the reference cell is defined as

$$\begin{aligned} \mathcal{I}_{\widehat{K}}^k : \mathcal{C}^0(\widehat{K}) &\longrightarrow \mathbb{P}_k \\ \widehat{v} &\longmapsto \sum_{m=0}^{k} \widehat{\sigma}_m(\widehat{v})\widehat{\vartheta}_m = \sum_{m=0}^{k} \widehat{v}(\widehat{\xi}_m)\widehat{\ell}_m^k. \end{aligned}$$

We want to transform the reference mesh cell to the other mesh cells. Hence, according to remark 1.1.10, we use the affine transformations

$$\begin{aligned} T_i : \widehat{K} &\longrightarrow I_i \\ t &\longmapsto x = x_i + th_i \ \text{ for } \ i \in \{0, ..., N\}. \end{aligned}$$

Observe that $T_i(\widehat{K}) = I_i$. That means in particular that we can construct the mesh by applying the affine transformations T_i to the reference mesh cell $\widehat{K}$. Also note that the transformations T_i map the reference nodes to the nodes of the i-th interval, i.e., $T_i(\widehat{\xi}_m) = \xi_{i,m}$ for $0 \leq m \leq k$. So we have that $\vartheta_{i,m} \circ T_i = \widehat{\vartheta}_m$ and for all continuous functions $v \in \mathcal{C}^0(I_i)$ we find $\sigma_{i,m}(v) = \widehat{\sigma}_m(v \circ T_i)$. With the computation

$$\begin{aligned} \mathcal{I}_{I_i}^k(v)(T_i(\widehat{x})) &\overset{\text{def.}}{=} \sum_{m=0}^{k} \sigma_{i,m}(v)\vartheta_{i,m}(T_i(\widehat{x})) \\ &= \sum_{m=0}^{k} \sigma_{i,m}(v)\widehat{\vartheta}_m(\widehat{x}) && (\text{using } \ \vartheta_{i,m} \circ T_i = \widehat{\vartheta}_m) \\ &= \sum_{m=0}^{k} \widehat{\sigma}_m(v \circ T_i)\widehat{\vartheta}_m(\widehat{x}) && (\text{using } \ \sigma_{i,m}(v) = \widehat{\sigma}_m(v \circ T_i)) \\ &\overset{\text{def.}}{=} \mathcal{I}_{\widehat{K}}^k(v \circ T_i)(\widehat{x}) \end{aligned}$$

we conclude that for all functions continuous functions $v \in \mathcal{C}^0(I_i)$ it holds that

$$\mathcal{I}_{I_i}^k(v) \circ T_i = \mathcal{I}_{\widehat{K}}^k(v \circ T_i). \tag{4}$$

In conclusion, the above construction shows that combining the reference interpolation operator with the transformations T_i we can completely generate the family $\{\mathcal{I}_{I_i}^k\}_{I_i \in \mathcal{T}_h}$.

Proposition 2.1.18 (H^1-stability of $\mathcal{I}_h^k$)**.** *[10, Prop. 1.11]. $\mathcal{I}_h^k$ is a linear continuous mapping from $H^1(\Omega)$ to $H^1(\Omega)$, and $||\mathcal{I}_h^k||_{\mathcal{L}(H^1(\Omega);H^1(\Omega))}$ is uniformly bounded with respect to h.*

Proof. To prove that $\mathcal{I}_h^k$ is a linear continuous mapping from $H^1(\Omega)$ to $H^1(\Omega)$ one can just repeat the proof for P_h^1 (cf. lemma 2.1.6).
For the other part let v be a function in $H^1(\Omega)$ and let $I_i \in \mathcal{T}_h$ be a mesh cell. We make use of the fact that $\sum_{m=0}^k \vartheta'_{i,m} = 0$. This is true because looking at $\sum_{m=0}^k \vartheta_{i,m}$ we notice that this sum is a polynomial in $\mathbb{P}_k$ and hence $\sum_{m=0}^k \vartheta_{i,m} - 1$ lies in $\mathbb{P}_k$ also. Now using the property from remark 2.1.10 it follows that

$$\sum_{m=0}^{k} \vartheta_{i,m}(\xi_{i,n}) - 1 = 0 \text{ for } n = 0, 1, ..., k.$$

This means that this polynomial has $k+1$ zeros and hence must be the zero polynomial since a polynomial of degree k can only have up to k zeros.
So using the just proven result we have

$$(\mathcal{I}_{I_i}^k v)' = \sum_{m=0}^{k} [v(\xi_{i,m}) - v(x_i)]\vartheta'_{i,m}.$$

By inequality (2) we have

$$|v(\xi_{i,m}) - v(x_i)| \leq h_i^{\frac{1}{2}} ||v'||_{L^2(I_i)} \text{ for } 0 \leq m \leq k. \tag{5}$$

Now doing a change of variables in the integral leads to

$$||\vartheta'_{i,m}||_{L^2(I_i)} = h_i^{-\frac{1}{2}} ||\widehat{\vartheta}'_m||_{L^2(\widehat{K})}. \tag{6}$$

We set the constant $c_k = \max_{0 \leq m \leq k} ||\widehat{\vartheta}'_m||_{L^2(\widehat{K})}$. Note that this constant is independent of the given mesh.
Using the derived expression for $(\mathcal{I}_{I_i}^k)'$ from above, applying the triangle inequality and then the inequality from above, we have

$$\begin{aligned}
||(\mathcal{I}_{I_i}^k v)'||_{L^2(I_i)} &= \left|\left| \sum_{m=0}^{k} [v(\xi_{i,m}) - v(x_i)]\vartheta'_{i,m} \right|\right|_{L^2(I_i)} && \left(\text{since } (\mathcal{I}_{I_i}^k v)' = \sum_{m=0}^{k} [v(\xi_{i,m}) - v(x_i)]\vartheta'_{i,m}\right) \\
&\leq \sum_{m=0}^{k} ||[v(\xi_{i,m}) - v(x_i)]\vartheta'_{i,m}||_{L^2(I_i)} && \text{(triangle inequality)} \\
&= \sum_{m=0}^{k} (|v(\xi_{i,m}) - v(x_i)| \cdot ||\vartheta'_{i,m}||_{L^2(I_i)}) \\
&\leq \sum_{m=0}^{k} h_i^{\frac{1}{2}} ||v'||_{L^2(I_i)} \cdot h_i^{-\frac{1}{2}} ||\widehat{\vartheta}_m||_{L^2(\widehat{K})} && \text{(by (5) and (6))} \\
&\leq (k+1)c_k ||v'||_{L^2(I_i)} && \text{(summing over all } m \text{ and using } c_k).
\end{aligned}$$

It follows that $||(\mathcal{I}_{I_i}^k v)'||_{L^2(I_i)}$ is controlled by $||v'||_{L^2(\Omega)}$ uniformly with respect to h.
Moreover, using the observation $\sum_{m=0}^k \vartheta_{i,m} = 1$ we have

$$\mathcal{I}_{I_i}^k v - v(x_i) = \sum_{m=0}^{k} [v(\xi_{i,m}) - v(x_i)]\vartheta_{i,m}.$$

It follows that for an argument from the i-th interval $x \in I_i$ one gets

$$|\mathcal{I}_{I_i}^k v(x)| \leq ||v||_{L^\infty(\Omega)} + (k+1)M_k h_i^{\frac{1}{2}} ||v'||_{L^2(I_i)}$$

with the mesh-independent constant $M_k = \max_{0\leq m\leq k} ||\widehat{\vartheta}_m||_{L^\infty(\widehat{K})}$.
Then by (3) it follows that $||\mathcal{I}_h^k v||_{L^\infty(\Omega)}$ is controlled by $(||v||_{L^2(\Omega)}^2 + ||v'||_{L^2(\Omega)}^2)^{\frac{1}{2}}$ uniformly with respect to h. Finally, using that

$$||\mathcal{I}_h^k v||_{L^2(\Omega)} \leq |b-a|^{\frac{1}{2}} ||\mathcal{I}_h^k v||_{L^\infty(\Omega)}$$

finishes the proof. $\square$

Proposition 2.1.19 (Interpolation error)**.** *[10, Prop. 1.12]. Let $0 \leq l \leq k$. Then, there exists a constant c such that, for all h and $v \in H^{l+1}(\Omega)$ we have*

$$||v - \mathcal{I}_h^k v||_{L^2(\Omega)} + h|v - \mathcal{I}_h^k v|_{H^1(\Omega)} \leq ch^{l+1}|v|_{H^{l+1}(\Omega)},$$

and for $l \geq 1$,

$$\sum_{m=2}^{l+1} h^m \left(\sum_{i=0}^{N} |v - \mathcal{I}_h^k v|_{H^m(I_i)}^2 \right)^{\frac{1}{2}} \leq ch^{l+1}|v|_{H^{l+1}(\Omega)}.$$

Proof. A proof of this proposition is given on pages 12 and 13 in [10]. $\square$

Remark 2.1.20. [10, Rmk. 1.13]. (i) One obtains optimal error estimates for smooth enough functions, i.e., $v \in H^{k+1}(\Omega)$. By proposition 2.1.19 one finds

$$\forall h, \forall v \in H^{k+1}(\Omega) : ||v - \mathcal{I}_h^k v||_{L^2(\Omega)} + h|v - \mathcal{I}_h^k v|_{H^1(\Omega)} \leq ch^{k+1}|v|_{H^{k+1}(\Omega)}.$$

Nevertheless, the interpolation error is not necessarily of optimal order if the function to be interpolated is a non-smooth function. That is the case if the function v lies in $H^s(\Omega)$ but not in $H^{s+1}(\Omega)$ for $s \geq 2$. Then using polynomials of degree larger than $s-1$ will not improve the interpolation error.
(ii) We further note that for a function v that lies only in $H^1(\Omega)$ the interpolation error in the H^1-seminorm still goes to zero for $h \to 0$, i.e., $\lim_{h\longrightarrow 0} |v - \mathcal{I}_h^k v|_{H^1(\Omega)} = 0$.

2.2 On the L^p-Stability of the Lagrange interpolant

We will now investigate the L^p-stability of the Lagrange interpolant. The following construction was taken from the pages 57 to 58 in [20]. Let $\alpha \in (0,1)$. We consider the Lagrange $\mathbb{P}_1$ shape functions $\vartheta_1(x) := 1 - x$ and $\vartheta_2(x) := x$. Furthermore we consider the sequence of continuous functions $\{v_n\}_{n\in\mathbb{N}\setminus\{0\}}$, which are defined over the unit interval $K := [0,1]$ as

$$v_n(x) := \begin{cases} n^\alpha - 1 & \text{if } 0 \leq x \leq \frac{1}{n}, \\ x^{-\alpha} - 1 & \text{otherwise.} \end{cases}$$

In order to get a better understanding of the behavior of this sequence of functions we have plotted the graphs for different n and the case of $\alpha = \frac{1}{2}$ (cf. figure 6).

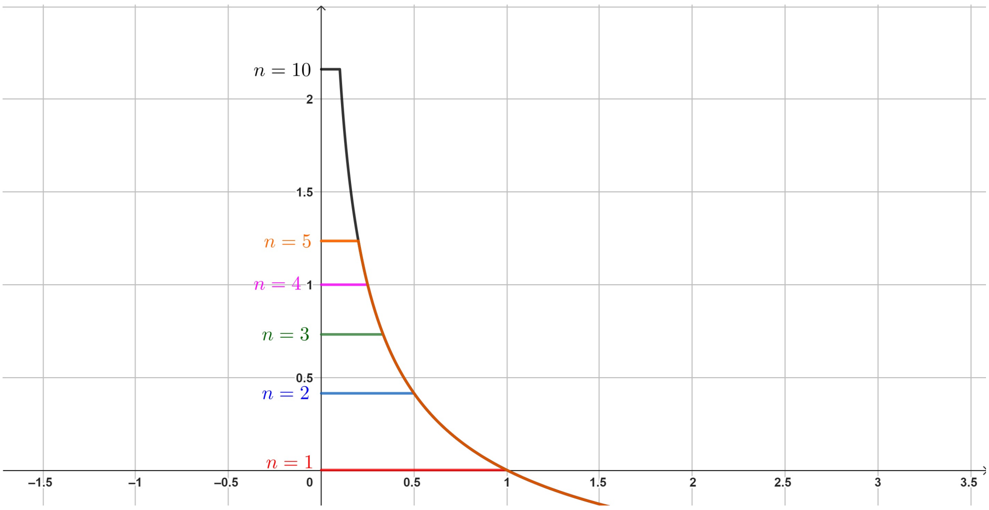

Figure 6: Graphs of v_n for $\alpha = \frac{1}{2}$ and $n = 1, 2, 3, 4, 5$ as well as 10.

Firstly, we will show that this sequence is uniformly bounded in $L^p(0,1)$ for all p such that $p\alpha < 1$. Using the definition of v_n, applying the triangle inequality and computing the L^p norm we have

$$||v_n||_{L^p(0,1)} \leq ||x^{-\alpha} - 1||_{L^p(0,1)} \leq 1 + ||x^{-\alpha}||_{L^p(0,1)} = 1 + \frac{1}{(1-p\alpha)^{\frac{1}{p}}} < \infty$$

because $p\alpha < 1$.

Next we compute the Lagrange interpolant $\mathcal{I}_K^1(v_n)$ using the defined shape functions. Using that by definition it holds that $v_n(1) = 0$ gives us

$$\mathcal{I}_K^1 v_n(x) = v_n(0)\vartheta_1(x) + v_n(1)\vartheta_2(x) = (n^\alpha - 1)(1 - x).$$

Plugging the computed expression for the interpolant into the L^p-norm implies

$$||\mathcal{I}_K^1 v_n||_{L^p(0,1)} = (n^\alpha - 1)||(1-x)||_{L^p(0,1)} \geq (n^\alpha - 1)\left|\left|\frac{1}{2}\right|\right|_{L^p\left(0,\frac{1}{2}\right)} \geq \frac{1}{4}(n^\alpha - 1).$$

To see that the first inequality holds, one computes the integrals

$$\int_0^1 (1-x)^p \mathrm{d}x = -\left[\frac{1}{p+1}(1-x)^{p+1}\right]_0^1 = \frac{1}{p+1}$$

and

$$\int_0^{\frac{1}{2}} \left(\frac{1}{2}\right)^p = \frac{1}{2} \cdot \frac{1}{2^p} = \frac{1}{2^{p+1}}.$$

Then, by performing a simple induction on p, one shows that

$$\frac{1}{p} \geq \frac{1}{2^{p+1}}.$$

This verifies the first inequality.
For the second inequality one just needs to look at the second integral. There one easily sees that this implies that $\left|\left|\frac{1}{2}\right|\right|_{L^p\left(0,\frac{1}{2}\right)} \geq \frac{1}{4}$. Hence, the second inequality is also true.
Now one finds that

$$||\mathcal{I}_K^1 v_n||_{L^p(0,1)} \geq \frac{1}{4}(n^\alpha - 1)\gamma^{-1}||v_n||_{L^p(0,1)}$$

with $\gamma := 1 + \frac{1}{(1-p\alpha)^{\frac{1}{p}}}$.
This proves that for all $n \in \mathbb{N}$ we have

$$||\mathcal{I}_K^1||_{\mathcal{L}(L^p;L^p)} \geq \frac{1}{4}(n^\alpha - 1)\gamma^{-1}.$$

Thus,

$$||\mathcal{I}_K^1||_{\mathcal{L}(L^p;L^p)} = \infty,$$

which means that $\mathcal{I}_K^1$ is not L^p-stable for all $p < \frac{1}{\alpha}$. Note that since α was arbitrary in $(0,1)$ one can conclude that the interpolation operator $\mathcal{I}_K^1$ is not L^q stable for all $q \in [1,\infty)$ in dimension 1.

Remark 2.2.1 (On the L^2-stability of the Lagrange interpolant)**.** In fact, the above construction shows that the Lagrange interpolant is not L^2-stable in general.

2.3 Extension of the Lagrange Interpolation Operator to Higher Dimensions

In order to extend the idea of the Lagrange interpolation operator to higher dimensions one can reformulate the notions from the previous sections with the help of barycentric coordinates. The information presented in this section were taken from section 1.2.3 in [10] and the appendix B in [15].

Definition 2.3.1 (Barycentric coordinates)**.** [10, p.21]. Let K be a simplex in $\mathbb{R}^d$. For $0 \leq i \leq d$ let F_i be the face of K that is opposite to the node a_i. The associated *barycentric coordinates* $\{\lambda_0, ..., \lambda_d\}$ are defined in the following way:
For $0 \leq i \leq d$ we define the map

$$\begin{aligned} \lambda_i : \mathbb{R}^d &\longrightarrow \mathbb{R} \\ x &\longmapsto \lambda_i(x) = 1 - \frac{(x - a_i) \cdot n_i}{(a_j - a_i) \cdot n_i} \end{aligned}$$

with n_i denoting the outward normal to F_i and a_j being an arbitrary vertex in the face F_i

Remark 2.3.2. [10, p.22]. (i) The barycentric coordinate λ_i is an affine function that takes the value 1 at the node a_i and which is equal to zero at F_i.
(ii) The barycenter B of K has barycentric coordinates $(\frac{1}{d+1}, ..., \frac{1}{d+1})$ (cf. remark B.32 in [15]).
(iii) The barycentric coordinates fulfill the following properties:
For all $x \in K, 0 \leq \lambda_i \leq 1$ and for all $x \in \mathbb{R}^d$ we have

$$\sum_{i=1}^{d+1} \lambda_i(x) = 1 \text{ and } \sum_{i=1}^{d+1} \lambda_i(x)(x - a_i) = 0.$$

Example 2.3.3. (i) In dimension 1, the barycentric coordinates on $K = [x_0, x_1]$ are given by

$$\lambda_0(x) = 1 - \frac{x - x_0}{x_1 - x_0} = \frac{x_1 - x}{x_1 - x_0} \text{ and } \lambda_1(x) = 1 - \frac{x - x_1}{x_0 - x_1} = \frac{x - x_0}{x_1 - x_0}$$

which coincides with the results for the linear shape functions from the beginning in the one dimensional case.
(ii) In the unit simplex in dimension 2 one obtains the barycentric coordinates $\lambda_0 = 1 - x_1 - x_2, \lambda_1 = x_1$ and $\lambda_2 = x_2$.
(iii) In dimension 3 one finds $\lambda_0 = 1 - x_1 - x_2 - x_3, \lambda_1 = x_1, \lambda_2 = x_2$ and $\lambda_3 = x_3$.

Of course there is also a generalization of the polynomial space for higher dimensions. We will denote it by $\mathbb{P}_k^d$.

Definition 2.3.4 (The polynomial space $\mathbb{P}_k^d$)**.** [10, p.22]. Let $x = (x_1, ..., x_d)$ and let $\mathbb{P}_k^d$ denote the space of polynomials in the variables $x_1, ..., x_d$ with real coefficients and of global degree at most k, i.e.,

$$\mathbb{P}_k^d = \left\{ p(x) = \sum_{0 \leq i_1, ..., i_d \leq k, i_1 + ... + i_d \leq k} \alpha_{i_1 ... i_d} x_1^{i_1} ... x_d^{i_d} \mid \alpha_{i_1, ..., i_d} \in \mathbb{R} \right\}.$$

Remark 2.3.5 (Local shape functions)**.** [10, p.22]. (i) For $k = 1$ the local shape functions ϑ_i are given through the barycentric coordinates, i.e.,

$$\vartheta_i = \lambda_i, \;\; 0 \leq i \leq d.$$

(ii) In the situation of $k = 2$ the local shape functions are given by

$$\lambda_i(2\lambda_i - 1), \;\; 0 \leq i \leq d, \;\; \text{and } 4\lambda_i\lambda_j, \;\; 0 \leq i < j \leq d.$$

2.4 Error Bounds

In the end of this chapter we want to give multiple error estimates for the Lagrange interpolation operator. They are based on the error estimates which we stated earlier in sections 1.3 and 1.6. We follow the presentation in section 1.5.1 in [10].

2.4.1 Error Bound Involving the Lebesgue Constant

We have already proven that the interpolation error of an interpolation operator is bounded by an expression involving the Lebesgue constant (cf. theorem 1.3.7). Hence, we obtain the following result for the Lagrange interpolation operator.

Theorem 2.4.1 (Error bound for the Lagrange interpolation operator involving the Lebesgue constant)**.** *Consider the general Lagrange interpolation operator $\mathcal{I}_K^L$ (cf. p.15) with the space $V(K) = \mathcal{C}^0(K)$ and denote the Lebesgue constant by Λ. It holds*

$$||v - \mathcal{I}_K^L(v)||_{\mathcal{C}^0(K)} \leq (1 + \Lambda) \inf_{p \in P} ||v - p||_{\mathcal{C}^0(K)}.$$

Proof. This is an application of theorem 1.3.7. □

Remark 2.4.2 (On the error bound)**.** (i) One can show that the Lebesgue constant in the case of the Lagrange interpolation operator is given by

$$\Lambda = ||\lambda||_{\mathcal{C}^0(K)}$$

with the Lebesgue function $\lambda(x) := \sum_{i \in N} |\vartheta_i(x)|$ for all $x \in K$ (cf. example 5.1.5 in [19]).
(ii) Note that only the first inequality of the theorem 1.3.7 holds since $\mathcal{C}^0(K)$ together with the L^2 scalar product is only a pre-Hilbert space.

2.4.2 Error Bounds for Interpolation in $W^{s,p}(\Omega)$

Now we give some error estimates for the Lagrange interpolation operator when interpolating in Sobolev spaces. We consider the Lagrange finite element of degree k and set $V(\widehat{K}) = \mathcal{C}^0(\widehat{K})$. It follows that under these circumstances the condition on l in the local error estimate (cf. theorem 1.6.1) becomes $\frac{d}{p} - 1 < l \leq k$. Concretely, by the Rellich[46]-Kondrachov[47] theorem (cf. theorem B.46 in [10].) it holds that $W^{l+1,p}(\widehat{K}) \subset V(\widehat{K})$ if $l+1 > \frac{d}{p}$. Using this observation on can formulate the following local error estimate (cf. example 1.106 (i) in [10]).

Corollary 2.4.3 (Local error estimate)**.** *In the setting of theorem 1.6.1 and using the condition on l from above it holds*

$$\forall K, \forall v \in W^{l+1,p}(K) : |v - \mathcal{I}_K^k v|_{W^{m,p}(K)} \leq c h_K^{l+1-m} \sigma_K^m |v|_{W^{l+1,p}(K)}.$$

.

[46]Franz Rellich (1906-1955) was an Austrian-German mathematician (cf. [30]).
[47]Vladimir Iosifovich Kondrashov (1909-1971) was a Soviet mathematician (cf. [40]).

Proof. See theorem 1.6.1. □

Corollary 2.4.4 (Global error estimate)**.** *[10, Example 1.111]. Considering a Lagrange finite element of degree k with $p = 2$ and $d \leq 3$. One can take by the conditions on l from above $1 \leq l \leq k$ and using the global interpolation estimate (cf. theorem 1.6.4) one finds for all functions $v \in H^{l+1}(\Omega)$ that*

$$||v - \mathcal{I}_h^k v||_{L^2(\Omega)} + h|v - \mathcal{I}_h^k|_{H^1(\Omega)} \leq ch^{l+1}|v|_{H^{l+1}(\Omega)}.$$

Proof. See theorem 1.6.4. □

Remark 2.4.5 (Optimal error estimate)**.** [10, Example 1.111]. (i) Note that this error estimate is optimal if $v \in H^{k+1}(\Omega)$.
(ii) [10, Rmk.1.112]. The estimate also holds true for l not being an integer. Let $k \geq 1$ and $d \leq 3$. We can apply the derived error estimate with $l = k - \frac{d}{2}$ and $p = \infty$ because $W^{k+1-\frac{d}{2},\infty}(\widehat{K}) \subset \mathcal{C}^0(\widehat{K})$ with continuous embedding, i.e., $k + 1 - \frac{d}{2} > 0$. We obtain for functions $v \in W^{k+1-\frac{d}{2},\infty}$ that

$$||v - \mathcal{I}_h^k v||_{L^\infty(\Omega)} \leq ch^{k+1-\frac{d}{2}}|v|_{W^{k+1-\frac{d}{2},\infty}(\Omega)}.$$

Using the continuous embedding $H^{k+1}(\Omega) \subset W^{k+1-\frac{d}{2},\infty}$ yields for all h and for all $v \in H^{k+1}(\Omega)$ that

$$||v - \mathcal{I}_h^k v||_{L^\infty(\Omega)} \leq ch^{k+1-\frac{d}{2}}|v|_{H^{k+1}(\Omega)}.$$

Moreover, if $v \in W^{k+1,\infty}(\Omega)$ we obtain the sharper estimate

$$||v - \mathcal{I}_h^k v||_{L^\infty(\Omega)} \leq ch^{k+1}|v|_{W^{k+1,\infty}(\Omega)}$$

using theorem 1.6.4.

3 The Clément Interpolation Operator $\mathcal{I}_h^{\text{Cle}}$

Often one faces the problem of interpolating non-smooth functions. This means that a function may be too rough and hence is not in the domain of the Lagrange interpolation operator anymore. Recall that we concluded in remark 1.5.6 that functions which lie in $L^2(\Omega)$ or in $H^1(\Omega)$ in dimension $d \geq 2$ are discontinuous in general. Then the Lagrange interpolation operator, which evaluates functions at a finite number of given points, is not well-defined anymore. Clément proposed in his paper from 1975 (cf. [1]) an interpolation operator in order to fix this problem. His idea is based on "local regularization" and is therefore a combination of the L^2 projection, which we will discuss in more detail in chapter 5, and the Lagrange interpolation from chapter 2. We will first present the construction of this interpolation operator and then discuss its properties. In the end of this chapter we will comment on the advantages and disadvantages of this operator. The construction presented here closely follows section 1.6.1 in [10].

3.1 Construction of the Clément Interpolation Operator

Let Ω be a polyhedron and let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine, simplicial, geometrically conformal meshes (cf. chapter 1).
We will use the H^1-conformal approximation space

$$P_{c,h}^k := \{v_h \in \mathcal{C}^0(\overline{\Omega}) \mid \forall K \in \mathcal{T}_h, v_h \circ T_K \in \mathbb{P}_k\},$$

where c refers to the continuity across mesh interfaces. Recall that H^1-conformal means in this case that $P_{c,h}^k \subset H^1$. We denote by $\{a_1, ..., a_N\}$ the Lagrange nodes. Further, we consider a set $\{\varphi_1, ..., \varphi_n\}$ of global shape functions in $P_{c,h}^k$. The Clément interpolation operator uses collections of simplices that share a common node.

Definition 3.1.1 (Macroelements)**.** [10, p.68]. Let a_i be a node. A *macroelement associated with* a_i consists of all simplices in the mesh that contain the node a_i.

We will write M_i for a macroelement associated with a node a_i. Examples for macroelements are given in figure 7.

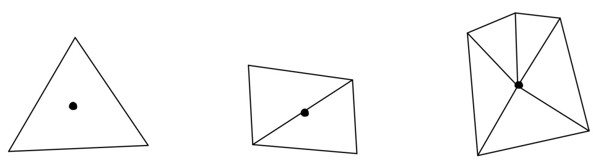

Figure 7: Different macroelements (inspired by figure 1.24 in [10]).

Those macroelements M_i can be configurated in only a finite number of ways. We will denote this number of configurations by n_{cf}. Let $\{\widehat{M}_n\}_{1 \leq n \leq n_{\text{cf}}}$ be the list of reference macroelements. Next we define the map

$$j : \{1, ..., N\} \longrightarrow \{1, ..., n_{\text{cf}}\}$$

which maps the index i of a macroelement M_i to the index of a reference configuration. Moreover, let

$$F_{M_i} : \widehat{M}_{j(i)} \to M_i$$

be a diffeomorphism from $\widehat{M}_{j(i)}$ to M_i such that for all reference mesh cells $\widehat{K}$ that lie in the reference configuration $\widehat{M}_{j(i)}$, i.e., $\widehat{K} \in \widehat{M}_{j(i)}$, the restriction $F_{M_i|\widehat{K}}$ to the reference mesh cell is affine.
Now for a reference macroelement $\widehat{M}_n$ and a function $\widehat{v} \in L^1(\widehat{M}_n)$, we let $\widehat{\pi}_n\widehat{v}$ be the unique polynomial in $\mathbb{P}_k(\widehat{M}_n)$ such that for all polynomials $\widehat{q} \in \mathbb{P}_k(\widehat{M}_n)$ one has

$$\int_{\widehat{M}_n} (\widehat{\pi}_n\widehat{v} - \widehat{v})\widehat{q} = 0$$

or equivalently

$$(\widehat{\pi}_n\widehat{v} - \widehat{v}, \widehat{q})_{L^2(\widehat{M}_n)} = 0 \;\; \forall \widehat{q} \in \mathbb{P}_k.$$

In particular, $\widehat{\pi}_n\widehat{v}$ is the local $L^2(\widehat{M}_n)$-projection of $\widehat{v} \in L^1(\widehat{M}_n)$. Finally, the Clément interpolation operator is of the form

$$\begin{aligned} \mathcal{I}_h^{\text{Cle}} : L^1(\Omega) &\longrightarrow P_{c,h}^k \\ v &\longmapsto \mathcal{I}_h^{\text{Cle}} v = \sum_{i=1}^{N} \widehat{\pi}_{j(i)}(v \circ F_{M_i})(F_{M_i}^{-1}(a_i))\varphi_i. \end{aligned}$$

3.2 Stability and Approximation Estimates

Lemma 3.2.1 (Stability)**.** *[10, Lemma 1.127]. Let the assumptions from section 3.1 hold and let $1 \leq p < \infty$ and $0 \leq m \leq 1$. There exists a constant c such that for all h and for all $v \in W^{m,p}(\Omega)$ it holds that*

$$||\mathcal{I}_h^{Cle} v||_{W^{m,p}(\Omega)} \leq c||v||_{W^{m,p}(\Omega)}.$$

Lemma 3.2.2 (Approximation)**.** *[10, Lemma 1.127]. For $K \in \mathcal{T}_h$ we denote by Δ_K the set of elements in $\mathcal{T}_h$ sharing at least one vertex with the mesh cell K. Let F be an interface between two elements of $\mathcal{T}_h$, and denote by Δ_F the set of elements in $\mathcal{T}_h$ sharing at least one vertex with F. Let l, m and p satisfy $1 \leq p \leq \infty$ and $0 \leq m \leq l \leq k+1$. Then there exists a constant c such that for all h, for all mesh cells $K \in \mathcal{T}_h$ and for all $v \in W^{l,p}(\Delta_K)$ one has*

$$||v - \mathcal{I}_h^{Cle} v||_{W^{m,p}(K)} \leq c h_K^{l-m} ||v||_{W^{l,p}(\Delta_K)}.$$

Similarly, if $m + \frac{1}{p} \leq l \leq k+1$*, then for all* h *for all mesh cells* $K \in \mathcal{T}_h$ *and for all* $v \in W^{l,p}(\Delta_F)$ *the estimate*

$$||v - \mathcal{I}_h^{Cle} v||_{W^{m,p}(F)} \leq c h^{l-m-\frac{1}{p}} ||v||_{W^{l,p}(\Delta_F)}$$

holds.

Corollary 3.2.3. *[10, Cor. 1.128]. Let the assumptions from the previous lemma hold. Furthermore let* $0 \leq l \leq k+1$ *and let* $0 \leq m \leq \min(1, l)$*. Then there exists a constant* c *such that for all* h *and for all* $v \in W^{l,p}(\Omega)$ *it holds that*

$$\inf_{v \in P^k_{c,h}} ||v - v_h||_{W^{m,p}(\Omega)} \leq c h^{l-m} ||v||_{W^{l,p}(\Omega)}.$$

Remark 3.2.4 (On the Clément interpolation operator)**.** [10, Rmk.1.129]. (i) The Clément interpolation operator is not a projection, i.e. $\mathcal{I}_h^{\mathrm{Cle}} \mathcal{I}_h^{\mathrm{Cle}} v \neq v$.
(ii) Another disadvantage of the Clément interpolation operator relies on the fact that this operator does not preserve homogenous boundary conditions. In particular, if v vanishes on the boundary, then in general the interpolant $\mathcal{I}_h^{\mathrm{Cle}} v$ does not need to vanish on the boundary as well. Usually one approaches this problem by setting the boundary nodal values to zero. In fact, one can show that the Clément interpolation operator, modified this way, still fulfills the lemmas 3.2.1 and 3.2.2.
(iii) For the treatment of domains with curved boundaries there exist generalizations of the Clément interpolation operator. One can also adjust the operator to other finite elements. References on these topics are [2] and [8].

Remark 3.2.5 (Motivating the Scott-Zhang interpolation operator)**.** The disadvantages adressed in the previous remark motivated the construction of the Scott-Zhang interpolation operator, which tries to fix those issues and which we are going to introduce in the next chapter.

4 The Scott-Zhang Interpolation Operator $\mathcal{I}_h^{\mathsf{SZ}}$

The disadvantages of the discussed Clément interpolation operator are that it does not preserve homogenous boundary conditions and that it does not satisfy the projection property (cf. remark 3.2.4). In order to solve the problem of not preserving homogenous boundary conditions one can set the boundary nodal values to zero. However, this procedure cannot be easily generalized to the case of a nonhomogenous boundary.
Scott and Zhang constructed in their paper from 1990 (cf. [3]) another interpolation operator which is a projection and preserves homogenous boundary conditions in a natural way. We will again explain its construction and then state its stability and approximation properties. This chapter is based on section 1.6.2 in [10] and on the paper [3].

4.1 Construction of the Scott-Zhang Interpolation Operator

Again we will consider a polyhedron Ω and a shape-regular family of affine, simplicial, geometrically conformal meshes $\{\mathcal{T}_h\}_{h>0}$. As before we will use the approximation space $P_{c,h}^k$ and consider a set of Lagrange nodes $\{a_i\}_{i=1}^N$. To each of these nodes we will assign either a d-simplex or a $(d-1)$-simplex, which we denote by ζ_i. In particular we need to differ between different cases.

Case 1: If the node a_i lies in the interior of a d-simplex, i.e., $a_i \in \mathring{K}$, we set $\zeta_i = K$.
Case 2: If the node a_i lies in the interior of a face F, which is a $(d-1)$-simplex one chooses $\zeta_i = F$. For the other nodes a_i that lie on a $(d-2)$-simplex one is allowed to associate this node with any $(d-1)$-simplex F such that $a_i \in \overline{F}$, i.e., $\zeta_i = F$.
One exception to the freedom of choice in case 2 is the third case.
Case 3: In the case that a_i is at the boundary and lies in the intersection of a number of different faces one chooses the face that belongs to the boundary, i.e., $F \subset \partial\Omega$.

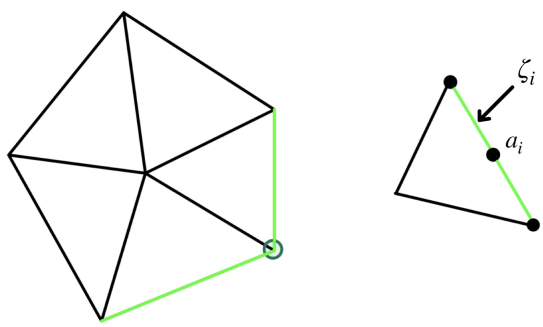

Figure 8: Visualizations of cases 2 and 3 (the right-hand side was inspired by figure 1.26 in [10]).

Figure 8 visualizes case 2 and case 3. The left-hand side shows the situation that a node lies on the boundary of the triangulated domain. Hence, as described in case 3, one needs to choose one of the green faces. The right-hand side shows a node a_i, which is on a face. Thus, one associates the green face with the node a_i.
For the number of nodes being contained in ζ_i we will write n_i. For example, the chosen face on the righ-hand side of figure 8 contains three nodes. That means for this case we would have $n_i = 3$.
By $\{\varphi_{i,q}\}_{1\leq q\leq n_i}$ we denote the restrictions to ζ_i of the local shape functions associated with the nodes that lie in ζ_i. We will follow the convention of setting $\varphi_{i,1} = \varphi_i$.
Finally, let t be an integer with $1 \leq t \leq n$ we define the function $\Psi_{i,t} \in \mathrm{span}\{\varphi_{i,1}, ..., \varphi_{i,n_i}\}$ as the unique function for which

$$\int_{\zeta_i} \Psi_{i,t}\varphi_{i,r} = \delta_{tr}, \quad 1 \leq t, r \leq n_i,$$

holds. Note that δ_{tr} denotes the Kronecker delta.
Using this, we define the Scott-Zhang interpolation operator as

$$\begin{aligned} \mathcal{I}_h^{\mathrm{SZ}} : W^{l,p}(\Omega) &\longrightarrow P_{c,h}^k \\ v &\longmapsto \mathcal{I}_h^{\mathrm{SZ}} v = \sum_{i=1}^{N} \varphi_i \int_{\zeta_i} \Psi_{i,1} v. \end{aligned}$$

Remark 4.1.1 (Properties of the Scott-Zhang interpolation operator)**.** From $v_{|\partial\Omega} = 0$ it follows by the definition of the operator that $\mathcal{I}_h^{\mathrm{SZ}} v_{|\partial\Omega} = 0$, which means that that the Scott-Zhang interpolation operator preserves homogenous boundary conditions. Also note that by the definition of $\Psi_{i,t}$ it is true that for all functions $v_h \in P_{c,h}^k$ it holds that $\mathcal{I}_h^{\mathrm{SZ}} v_h = v_h$. In fact, one computes for all basis functions φ_k that

$$\mathcal{I}_h^{\mathrm{SZ}} \varphi_k = \sum_{i=1}^{N} \varphi_i \int_{\zeta_i} \Psi_{i,1}\varphi_k = \sum_{i=1}^{N} \varphi_i \cdot \delta_{ik} = \varphi_k$$

with δ_{ik} denoting the Kronecker delta. Hence, the Scott-Zhang interpolation operator is a projection. We see that the disadvantages that we identified in the case of the Clément interpolation operator are cured.

4.2 Stability and Approximation Properties

We will now state the the stability and approximation properties of the interpolation operator we have just constructed.

Lemma 4.2.1 (Stability and approximation properties)**.** *[10, Lemma 1.130]. Let p and l satisfy $1 \leq p < +\infty$ and $l \geq 1$ if $p = 1$, and $l > \frac{1}{p}$ otherwise. Then there exists a constant c such that*
(i) for all $0 \leq m \leq \min(1, l)$ it holds that

$$\forall h, \forall v \in W^{l,p}(\Omega) : ||\mathcal{I}_h^{SZ} v||_{W^{m,p}(\Omega)} \leq c||v||_{W^{l,p}(\Omega)}.$$

(ii) Provided $l \leq k+1$*, for all* $0 \leq m \leq l$*, one has*

$$\forall h, \forall K \in \mathcal{T}_h, \forall v \in W^{l,p}(\Delta_K) : ||v - \mathcal{I}_h^{SZ} v||_{W^{m,p}(K)} \leq c h_K^{l-m} |v|_{W^{l,p}(\Delta_K)}.$$

Proof. The corresponding proofs are given in [3]. □

Remark 4.2.2 (Disadvantages of the Scott-Zhang interpolation operator)**.** Even though, the problems of the Clément interpolation operator get fixed, the Scott-Zhang operator comes with its own disadvantages. First, one should note that this operator is only defined for Lagrange finite elements. A second issue, one should care about, is the fact that for the projection over the facets one needs that the function one wants to project is smooth enough because one needs to make sure that its trace over any facet is well-defined (cf. p.2 in [14]).

5 L^2-Orthogonal Projection π_{L^2}

Instead of interpolation operators one can use so-called projection operators. An often considered choice is the L^2 projection, which we are going to present in this chapter. We will first introduce the ordinary L^2 projection following the structure and material from section 1.6.3 in [10]. Afterwards we will present a more special case of the L^2 projection, namely the weighted L^2 projection. We will see that the weighted version of the L^2 projection leads to a more complicated situation where error and stability estimates are not derived as easily as in the case of the ordinary L^2 projection.

5.1 The Ordinary L^2 Projection

Let $P^k_{c,h}$ again denote the H^1-conformal space used previously in the chapters 3 and 4.
We consider the orthogonal projection operator

$$\pi_{L^2} : L^2(\Omega) \longrightarrow P^k_{c,h}$$

with the scalar product $(u,v)_{L^2(\Omega)} = \int_\Omega uv \ \mathrm{d}x$.
Now the L^2-projection $\pi_{L^2} v \in P^k_{c,h}$ for a function $v \in L^2(\Omega)$ is defined by the equation

$$(v - \pi_{L^2} v, u)_{L^2(\Omega)} = 0 \ \ \forall u \in P^k_{c,h}.$$

Lemma 5.1.1 (Stability)**.** *[10, Lemma 1.131]. Let $k \geq 1$. It holds that*

$$\forall v \in L^2(\Omega) : ||\pi_{L^2} v||_{L^2(\Omega)} \leq ||v||_{L^2(\Omega)}.$$

Moreover, if the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, there exists a constant c such that

$$\forall h, \forall v \in H^1(\Omega) : ||\pi_{L^2} v||_{H^1(\Omega)} \leq c||v||_{H^1(\Omega)}.$$

Proof. The first stability estimate follows directly from the definition by using the Pythagoras identity, which results in

$$||v||^2_{L^2} = ||\pi_{L^2} v||^2_{L^2(\Omega)} + ||v - \pi_{L^2} v||^2_{L^2(\Omega)}.$$

For the second estimate let $v \in H^1(\Omega)$. We consider its Scott-Zhang interpolant $\mathcal{I}^{\mathrm{SZ}}_h v$, which we defined in the previous section. One finds

$$\begin{aligned}
||\pi_{L^2} v||_{H^1(\Omega)} &\leq ||\pi_{L^2} v - \mathcal{I}^{\mathrm{SZ}}_h v||_{H^1(\Omega)} + ||\mathcal{I}^{\mathrm{SZ}}_h v||_{H^1(\Omega)} \ \text{ (triangle inequality)} \\
&= ||\pi_{L^2}(v - \mathcal{I}^{\mathrm{SZ}}_h v)||_{H^1(\Omega)} + ||\mathcal{I}^{\mathrm{SZ}}_h v||_{H^1(\Omega)} \ \text{ (using } \pi_{L^2}(\mathcal{I}^{\mathrm{SZ}}_h v) = \mathcal{I}^{\mathrm{SZ}}_h v) \\
&\leq ch^{-1}||\pi_{L^2}(v - \mathcal{I}^{\mathrm{SZ}}_h v)||_{L^2(\Omega)} + ||\mathcal{I}^{\mathrm{SZ}}_h v||_{H^1(\Omega)} \ \text{ (inverse inequality)} \\
&\leq ch^{-1}||v - \mathcal{I}^{\mathrm{SZ}}_h v||_{L^2(\Omega)} + ||\mathcal{I}^{\mathrm{SZ}}_h v||_{H^1(\Omega)} \ \text{ (} L^2\text{-stability of } \pi_{L^2}) \\
&\leq c'||v||_{H^1(\Omega)} \ \text{ (} H^1\text{-stability and approximation properties of } \mathcal{I}^{\mathrm{SZ}}_h).
\end{aligned}$$

This completes the proof. □

Definition 5.1.2 (Negative-norm)**.** [10, p.72]. For $s \geq 1$ and $v \in L^2(\Omega)$ the *negative-norm* is defined by

$$||v||_{H^{-s}(\Omega)} = \sup_{w \in H^s(\Omega) \cap H_0^1(\Omega)} \frac{(v, w)_{L^2(\Omega)}}{||w||_{H^s(\Omega)}}.$$

Remark 5.1.3 (About this norm)**.** [10, p.72]. This norm is not the norm, which is used used to construct the dual space $H^{-s}(\Omega)$ except for $s = 1$.

Having defined this norm, we are able to write down an error bound for the L^2-projection operator.

Proposition 5.1.4 (Error bound in the negative-norm)**.** *[10, Prop. 1.133]. Let $k \geq 1$ and $1 \leq s \leq k+1$. Then there exists a constant c such that*

$$\forall h, \forall v \in L^2(\Omega) : ||v - \pi_{L^2} v||_{H^{-s}(\Omega)} \leq ch^s \inf_{v_h \in P^k_{c,h}} ||v - v_h||_{L^2(\Omega)}.$$

Proof. Let $v \in L^2(\Omega)$ and $w \in H^s(\Omega) \cap H_0^1(\Omega)$. We know that $s \leq k+1$. Hence, we can argue by the lemma about the approximation properties of the Clément interpolation operator (cf. lemma 3.2.2) that

$$||w - \mathcal{I}_h^{\text{Cle}} w||_{L^2(\Omega)} \leq ch^s |w|_{H^s(\Omega)}.$$

Next we note that $v - \pi_{L^2} v$ is L^2-orthogonal to $P^k_{c,h}$. Thus, we may conclude that

$$(v - \pi_{L^2} v, w)_{L^2(\Omega)} = (v - \pi_{L^2} v, w - \mathcal{I}_h^{\text{Cle}} w)_{L^2(\Omega)} \leq ch^s ||v - \pi_{L^2} v||_{L^2(\Omega)} ||w||_{H^s(\Omega)}.$$

This completes the proof. □

In the end we also want to give the following local and global error bounds for functions in Sobolev spaces.

Lemma 5.1.5 (Local error estimate)**.** *[19, Lemma 11.18]. Let $p \in [1, \infty]$. There exists a constant c such that for every integers $r \in \{0, ..., k+1\}$ and $m \in \{0, ...r\}$, all functions $v \in W^{r,p}(K)$, all mesh cells $K \in \mathcal{T}_h$, and all h the error estimate*

$$|v - \pi_{L^2}(v)|_{W^{m,p}(K)} \leq ch_K^{r-m} |v|_{W^{r,p}(K)}$$

holds.

Proof. A proof is given in on page 134 in [19]. □

Proposition 5.1.6 (Global error estimates)**.** *[10, Prop. 1.134]. Let $1 \leq l \leq k$. There exists a constant c such that, for all h and all functions $v \in H^{l+1}(\Omega)$ it holds that*

$$||v - \pi_{L^2}(v)||_{L^2(\Omega)} \leq ch^{l+1} |v|_{H^{l+1}(\Omega)}.$$

Furthermore, if the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, there exists a constant c such that for all h and for a function $v \in H^{l+1}(\Omega)$ one has

$$||v - \pi_{L^2}(v)||_{H^1(\Omega)} \leq ch^l |v|_{H^{l+1}(\Omega)}.$$

Proof. A sketch of the proof is given on page 80 in [10]. $\square$

Remark 5.1.7 (On the L^2 projection). (i) An advantage of the presented L^2 projection is that, contrary to the introduced Lagrange interpolation operator, the L^2 projection does not require that the functions we want to approximate are continuous.
(ii) An algorithm for computing the L^2 projection is given in section 1.3.3 in [13].

5.2 The Weighted L^2 Projection

The study of solving second-order elliptic boundary value problems with discontinuous coefficients, where the discretized version appears harder to be solved, motivated a modification of the ordinary L^2 projection by introducing weights in the definition of the L^2 projection. In this section we will introduce these kinds of weighted L^2-projections and investigate L^2-error estimates as well as their H^1-stability. This section follows the presentation in the papers [4] and [6].
In this section let $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, be a domain, which will be decomposed such that

$$\overline{\Omega} = \bigcup_{i=1}^{J} \overline{\Omega}_i$$

with the Ω_i being mutually disjoint, i.e., for any pair of subdomains $\{\Omega_m, \Omega_n\}$ it holds that $\Omega_m \cap \Omega_n = \emptyset$ for $m \neq n$. We further assume that $\{\mathcal{T}_h\}_{h>0}$ with $h \in (0,1)$ is a quasi-uniform (cf. def. 1.1.20) family of triangulations of $\overline{\Omega}$ such that the triangulations line up with the subdomains. As a consequence of this assumption, a restriction of $\mathcal{T}_h$ to a subdomain Ω_i will be a triangulation of the subdomain itself. We will denote the set of interfaces by Γ, i.e., $\Gamma := \bigcup_{i=1}^{J} \partial\Omega_i \setminus \partial\Omega$ and make the assumption that Γ consists only of segments $(d = 2)$ or plane polygons $d = 3$. Let $\{\omega_i\}_{i=1}^{J}$ be a set of positive constants. We define the weighted inner products by

$$(u, v)_{L^2_\omega(\Omega)} := \sum_{i=1}^{J} \omega_i (u, v)_{L^2(\Omega_i)}$$

and

$$(u, v)_{H^1_\omega(\Omega)} := \sum_{i=1}^{J} \omega_i \int_{\Omega_i} \nabla u \cdot \nabla v dx$$

with the induced norms $|| \cdot ||_{L^2_\omega(\Omega)}$ and $| \cdot |_{H^1_\omega(\Omega)}$. Additionally, we define

$$|| \cdot ||^2_{H^1_\omega(\Omega)} = || \cdot ||^2_{L^2_\omega(\Omega)} + | \cdot |^2_{H^1_\omega(\Omega)}$$

to be the full weighted H^1 norm.

Definition 5.2.1 (The weighted L^2 projection). [4, p.470]. Let $P^1_{0,h} \subset H^1_0(\Omega)$ be the finite element space consisting of piecewise linear polynomials that vanish on the boundary of the domain $\partial\Omega$. The *weighted L^2 projection*

$$\pi_h^\omega : L^2(\Omega) \longrightarrow P^1_{0,h}$$

is defined by the property

$$\forall u \in L^2(\Omega), v \in P^1_{0,h} : (\pi^\omega_h u, v)_{L^2_\omega(\Omega)} = (u, v)_{L^2_\omega(\Omega)}.$$

In dimension 1, one easily derives the following error estimate.

Proposition 5.2.2 (Error estimate for $d = 1$)**.** *[4, Prop. 4.1]. For $d = 1$, we have that for all functions $u \in H^1_0(\Omega)$ there exist constants c_1, c_2 such that*

$$||u - \pi^\omega_h u||_{L^2_\omega(\Omega)} \leq c_1 h |u|_{H^1_\omega(\Omega)}$$

and

$$|\pi^\omega_h u|_{H^1_\omega(\Omega)} \leq c_2 |u|_{H^1_\omega(\Omega)}.$$

Proof. Since we are just in one dimension it holds by the Sobolev embedding theorems that $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ (cf. remark 1.5.6). In particular, we know that the Lagrange interpolation operator $\mathcal{I}^1_h : C(\overline{\Omega}) \to P^1_{0,h}$ is well-defined for functions from $H^1(\Omega)$. Moreover, we know that for any $K \in \mathcal{T}_h$ it holds that for all $u \in H^1(K)$ we have

$$||u - \mathcal{I}^1_h u||_{L^2(K)} \leq ch|u|_{H^1(K)}.$$

Now by summing over all mesh cells $K \in \mathcal{T}_h$ with using proper weights we find

$$||u - \mathcal{I}^1_h u||_{L^2_\omega(\Omega)} \leq ch|u|_{H^1_\omega(\Omega)}$$

for all functions $u \in H^1_0(\Omega)$. Finally, we note that the first inequality is now obtained by observing that $||u - \pi^\omega_h u||_{L^2_\omega(\Omega)} \leq ||u - \mathcal{I}^1_h u||_{L^2_\omega(\Omega)}$. The second inequality follows just as in the case of the ordinary L^2 projection. $\square$

Remark 5.2.3 (Generalization to higher dimensions)**.** One should note that one cannot use the same approach in order to find similar error estimates in higher dimensions. The reason for that is that for $d > 1$ functions in $H^1(\Omega)$ do not need to be continuous any longer (cf. remark 1.5.6).

5.2.1 No Internal Cross Points

Definition 5.2.4 (Internal cross points)**.** [4, p. 471]. A point on Γ that belongs to more than two subdomains $\overline{\Omega}_i$ is called an *internal cross point.*

We introduce a weighted inner product on $L^2(\Gamma)$. Therefore we define

$$(u, v)_{L^2_\omega(\Gamma)} := \sum_{i=1}^{J} \int_{\partial\Omega_i \setminus \partial\Omega} \omega_i uv \ \mathrm{d}s.$$

Also define $P^1_{0,h}(\Gamma) := \{v_{|\Gamma} \mid v \in P^1_{0,h}\}$. Let $P_\Gamma : L^2(\Gamma) \longrightarrow P^1_{0,h}(\Gamma)$ be the orthogonal projection with respect to the just defined inner product $(\cdot, \cdot)_{L^2_\omega(\Gamma)}$.

Lemma 5.2.5. *[4, Lemma 4.2]. For all $u \in H_0^1(\Omega)$ it holds that*

$$||u - \pi_h^\omega u||_{L^2_\omega(\Omega)} \leq C(h|u|_{H^1_\omega(\Omega)} + h^{\frac{1}{2}}||u - P_\Gamma u||_{L^2_\omega(\Gamma)}).$$

Proof. [48] On each subdomain Ω_i there exists $w_i \in P^1_{0,h}(\Omega_i)$ such that

$$||u - w_i||^2_{L^2(\Omega_i)} + h^2||u - w_i||^2_{H^1(\Omega_i)} \leq ch^2||u||^2_{H^1(\Omega_i)}.$$

A justification for this is given by proposition 3.5 in [4]. Next, define $w \in P^1_{0,h}$ by

$$w := \begin{cases} w_i & \text{at the nodes in } \Omega_i, \\ P_\Gamma u & \text{on } \Gamma. \end{cases}$$

Then, using the well known fact that for constants C_1 and C_2 as well as functions $v \in P^1_{0,h}$ one has

$$C_1 h^d \sum_{x \in N} v^2(x) \leq ||v||_{L^2(\Omega)} \leq C_2 h^d \sum_{x \in N} v^2(x) \tag{7}$$

with N denoting the set of vertices of the triangulation, one obtains

$$\begin{aligned}
||u - w||^2_{L^2_\omega(\Omega)} &= ||u - w_i + w_i - w||^2_{L^2_\omega(\Omega)} \\
&= ||u - w_i||^2_{L^2_\omega(\Omega)} + 2(u - w_i, w_i - w)_{L^2_\omega(\Omega)} + ||w_i - w||^2_{L^2_\omega(\Omega)} \\
&= \sum_{i=1}^J \omega_i ||u - w_i||^2_{L^2(\Omega_i} + \sum_{i=1}^J \omega_i ||w_i - w||^2_{L^2(\Omega_i)} \\
&\overset{(7)}{\leq} \sum_{i=1}^J \omega_i ||u - w_i||^2_{L^2(\Omega_i)} + C_1 h^2 \sum_{i=1}^J \omega_i \sum_{p \in \mathcal{T}_h} |(w_i - w)(p)|^2 \\
&= \sum_{i=1}^J \omega_i ||u - w_i||^2_{L^2(\Omega_i)} + C_1 h^2 \sum_{i=1}^J \omega_i \left(\sum_{p \in \Omega_i} |(w_i - w_i)(p)|^2 + \sum_{p \in \Gamma_i} |(w_i - P_\Gamma u)(p)|^2 \right) \\
&= \sum_{i=1}^J ||u - w_i||^2_{L^2(\Omega_i)} + C_1 h^2 \sum_{i=1}^J \sum_{p \in \Gamma_i} |(w_i - P_\Gamma u)(p)|^2 \\
&\leq \sum_{i=1}^J \omega_i ||u - w_i||^2_{L^2(\Omega_i)} + C_1 h ||w_i - P_\Gamma u||_{L^2_\omega(\Gamma)} \\
&\leq C_2 \sum_{i=1}^J \omega_i h^2 ||u||_{H^1(\Omega_i)} + C_1 h ||u - P_\Gamma u||^2_{L^2_\omega(\Gamma)} \\
&= C_2 h^2 |u|^2_{H^1_\omega(\Omega)} + C_1 h ||u - P_\Gamma u||^2_{L^2_\omega(\Gamma)}.
\end{aligned}$$

Note that for establishing the inequality in the seventh line we have used the fact that for a polynomial f of degree 1, the inequality $|f(0)|^2 + |f(h)|^2 < Ch^{-1} \int_0^h f^2 dx$ holds. That is the reason why the exponent of h drops in two dimensions. In three dimensions the factor h would vanish.
For the last inequality we have used the inequality stated at the beginning of the proof.
Finally, using that $||u - \pi_h^\omega u||_{L^2_\omega(\Omega)} \leq ||u - w||_{L^2_\omega(\Omega)}$ we obtain the desired estimate. □

[48] Note that there is a typing error in the original paper. When bounding $||u - w||^2_{L^2_\omega(\Omega)}$ the index i for the w in the last term of the first estimate is missing.

Remark 5.2.6 (Independence of cross points)**.** [4, p.471]. Taking a look at the proof, one sees that the proof does not depend on the existence of cross points. However, the only known application of it is the application to the case that the interface does not have any internal cross points.

Under the assumption that the decomposition of the domain Ω does not have any internal cross points, one can prove the following theorem.

Theorem 5.2.7. *[4, Thm.4.3]. Assume that the decomposition of the domain Ω has no internal cross points. Then for all functions $u \in H_0^1(\Omega)$ one has*

$$||u - \pi_h^\omega u||_{L^2_\omega(\Omega)} \leq h|u|_{H^1_\omega(\Omega)}$$

and

$$|\pi_h^\omega u|_{H^1_\omega(\Omega)} \leq c|u|_{H^1_\omega(\Omega)}.$$

Proof. First, we define a function $\varphi \in P_0^1(\Gamma)$ as the orthogonal L^2-projection from $L^2(\Gamma_i)$ to the restriction from P_0^1 to Γ_i, i.e., $\varphi = P_{\Gamma_i}u$. Since we know that Γ has no internal cross points we conclude that the defined function φ is well-defined.
Observe that it also holds that for all Γ_i one finds

$$||u - \varphi||_{L^2(\Gamma_i)} \leq ||u - w_i||_{L^2(\Gamma_i)}.$$

Due to lemma 2.1 in [4] we have

$$||u - w_i||^2_{L^2(\Gamma_i)} \leq \frac{1}{h}||u - w_i||^2_{L^2(\Omega_i)} + h||u - w_i||^2_{H^1(\Omega_i)}.$$

It follows

$$h||u - w_i||^2_{L^2(\Gamma_i)} \leq c(||u - w_i||^2_{L^2(\Omega_i)} + h^2||u - w_i||^2_{H^1(\Omega_i)}) \leq ch^2|u|_{H^1(\Omega_i)}.$$

This implies

$$h||u - P_\Gamma u||^2_{L^2_\omega(\Gamma_i)} \leq ch\sum_{i=1}^{J} \omega_i||u - \varphi||^2_{L^2(\Gamma_i)} \leq ch^2\sum_{i=1}^{J} \omega_i|u|^2_{H^1(\Omega_i)} = h^2|u|_{H^1_\omega(\Omega)}.$$

Now using lemma 5.2.4 leads to the desired estimate. Moreover, note that the other inequality follows similarly to the proof for the ordinary L^2 projection (cf. lemma 5.1.1). □

5.2.2 Error Estimates Using "Finer" Finite Elements

The whole situation becomes more complicated if one allows internal cross points. We will now present some estimates under special circumstances. Even though the embedding $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ does not hold in $d = 2$ any longer, one could say that it is almost true for functions that lie in finite element subspaces (cf. p.472 in [4]). Based on this observation we will now state some further results for the more general situation.

Theorem 5.2.8. *[4, Thm. 4.5]. For any function $u \in P^1_{0,\underline{h}}$ with $\underline{h} < h$ one has*

$$||u - \pi_h^\omega||_{L^2_\omega(\Omega)} \leq \begin{cases} ch(\log \frac{h}{\underline{h}})\frac{1}{2}|u|_{H^1_\omega(K)}, & d = 2, \\ ch\left(\frac{h}{\underline{h}}\right)^{\frac{1}{2}}|u|_{H^1_\omega(K)} & d = 3, \end{cases}$$

and

$$|\pi_h^\omega u|_{H^1_\omega(\Omega)} \leq \begin{cases} c(\log \frac{h}{\underline{h}})\frac{1}{2}|u|_{H^1_\omega(K)}, & d = 2, \\ c\left(\frac{h}{\underline{h}}\right)^{\frac{1}{2}}|u|_{H^1_\omega(K)} & d = 3, \end{cases}$$

with K being a mesh cell.

Proof. This is a direct consequence of lemma 4.4 in [4]. □

5.2.3 Error Estimates for Functions in H^1

In this section we will state some estimates for functions in H^1. For the proofs we will refer to the original paper [4].
The first lemma, we will present, tells us that we can find nearly optimal estimates if we use the full weighted H^1 norms.

Lemma 5.2.9. *[4, Lemma 4.6]. For all functions $u \in H^1_0(\Omega)$ one has*

$$||u - \pi_h^\omega u||_{L^2_\omega(\Omega)} \leq ch|\log h|^{\frac{1}{2}}||u||_{H^1_\omega(\Omega)}.$$

Proof. A proof of this statement is given on page 473 in [4]. □

From this lemma one obtains the following theorem.

Theorem 5.2.10. *[4, Thm. 4.7]. If for all i the $(d-1)$-dimensional Lebesgue measure of $(\partial\Omega_i \cap \partial\Omega)$ is positive, then for all functions $u \in H^1_0(\Omega)$ it holds that*

$$||u - \pi_h^\omega u||_{L^2_\omega(\Omega)} \leq ch|\log h|^{\frac{1}{2}}|u|_{H^1_\omega(\Omega)}$$

and

$$|\pi_h^\omega u|_{H^1_\omega(\Omega)} \leq c|\log h|^{\frac{1}{2}}|u|_{H^1_\omega(\Omega)}.$$

Proof. This is a direct consequence of the previous lemma. □

Remark 5.2.11. Note that the assumption on the measure is necessary and cannot be excluded. For more details the interested reader may consult [6].

6 The Elliptic Projector π_{H^1}

In this section we are going to present another orthogonal projection operator, which can be used for functions in $H^1(\Omega)$. This operator is called the elliptic projection operator or the Riesz projector. The material presented here is taken from section 1.6.3 in [10].
As before we use the H^1-conformal space $P^k_{c,h}$ (cf. section 3.1). We consider the map

$$\pi_{H^1} : H^1(\Omega) \longrightarrow P^k_{c,h}$$

together with the scalar product

$$(u, v)_{H^1(\Omega)} = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v.$$

Then the elliptic projection of a function $v \in H^1(\Omega)$ denoted by $\pi_{H^1} v$ is defined by the property

$$(\pi_{H^1} v - v, u)_{H^1(\Omega)} = 0 \ \ \forall u \in P^k_{c,h}.$$

Lemma 6.0.1 (Stability). *[10, Lemma 1.131]. Let $k \geq 1$. It holds that*

$$\forall v \in H^1(\Omega) : ||\pi_{H^1}||_{H^1(\Omega)} \leq ||v||_{H^1(\Omega)}.$$

Proof. This estimate follows just as in the case for the L^2 projection by using the Pythagoras identity (cf. lemma 5.1.1). □

Proposition 6.0.2 (Global error estimate). *[10, Prop. 1.134]. Let $1 \leq l \leq k$.*
(i) There exists a constant c such that

$$\forall h, \forall v \in H^{l+1} : ||v - \pi_{H^1} v||_{H^1(\Omega)} \leq ch^l |v|_{H^{l+1}(\Omega)}.$$

(ii) If Ω is convex, there exists a constant c such that,

$$\forall h \forall v \in H^{l+1}(\Omega) : ||v - \pi_{H^1} v||_{L^2(\Omega)} \leq ch^{l+1} |v|_{H^{l+1}(\Omega)}.$$

Proof. This can be proven in conjunction with the global error estimate for the L^2 projection. A sketch of the proof is given on page 80 in [10]. □

7 Conclusion

Throughout the previous discussions we have seen that even though the Lagrange interpolation is a popular choice when it comes to approximating functions, its use is accompanied by some difficulties like the lack of L^2-stability and the fact that it cannot handle functions that are not smooth enough in the sense that point evaluations are not well-defined.

We have then explained the construction of an interpolation operator invented by Clément, which addresses the problems that we identified while working with the Lagrange interpolation operator. Nevertheless, the Clément operator is not a good choice when it comes to the treatment of homogenous boundary conditions. Of course, one can follow the approach by Clément in setting the boundary nodal values to zero but this strategy cannot be generalized easily to the case of non-homogenous boundaries conditions.

Hence, we introduced the Scott-Zhang interpolation operator, which fixes this problem and satisfies in addition the projection property.

Next to the interpolation operators, we have also looked at projection operators, namely the L^2 projection and a weighted version of it as well as the elliptic projection, which have the advantage that they are very simple and therefore easy to implement. However, we have seen that in the case of the weighted L^2 projection, error estimates are a lot harder to derive as in the case for the ordinary L^2 projection. After presenting this variety of different interpolation and projection operators, we have collected some of their most important stability and approximation properties in the table on the pages 46 and 47. This table is supposed to give the reader a good overview over the results we have seen in our discussions.

Obviously, we have not considered all interpolation operators available and this thesis just focuses on a choice of the most important ones. For example the interpolation operators by Crouzeix-Raviart or the Raviart-Thomas interpolation operator have not been adressed. The interested reader may consult the sections 1.2.6 and 1.2.7 in [10] for the treatment of those operators. Additionally, there are a lot more specializations of the interpolation operators we have presented. Some of them are presented in [19].

Moreover, we have limited ourselves to a quite basic setting regarding the domain, the properties of the meshes and the choice of the finite element spaces. Hence, the considerations that were undertaken in this thesis can be extended to way more general settings. For a more complete discussion of these topics we recommend the book [19].

Interpolation/ Projection Operator	Stability	Error Estimates
Lagrange Interpolation	H^1-stable and L^∞-stable but not L^p-stable for $p \in [1, \infty)$.	**Local error estimate:** Under the assumptions of corollary 2.4.3: $\forall K, \forall v \in W^{l+1,p}(K) : \lvert v - \mathcal{I}_K^k\rvert_{W^{m,p}(K)} \le ch_K^{l+1-m}\sigma_K^m \lvert v\rvert_{W^{l+1,p}}$. **Global error estimate**: Under the assumptions of corollary 2.4.4: $\forall v \in H^{l+1} : \lvert\lvert v - \mathcal{I}_h^k v\rvert\rvert_{L^2(\Omega)} + h\lvert v - \mathcal{I}_h^k\rvert_{H^1(\Omega)} \le ch^{l+1}\lvert v\rvert_{H^{l+1}(\Omega)}$.
Clément Interpolation	$W^{m,p}$-stable for $1 \le p \le \infty$ and $0 \le m \le 1$.	**Local error estimate:** Under the assumptions of lemma 3.2.2: $\forall K, \forall v \in W^{l,p}(\Delta_K) : \lvert\lvert v - \mathcal{I}_h^{\mathrm{Cle}} v\rvert\rvert_{W^{m,p}(K)} \le ch_K^{l-m}\lvert\lvert v\rvert\rvert_{W^{l,p}(\Delta_K)}$. $\forall K, \forall v \in W^{l,p}(\Delta_F) : \lvert\lvert v - \mathcal{I}_h^{\mathrm{Cle}} v\rvert\rvert_{W^{m,p}(F)} \le ch^{l-m-\frac{1}{p}}\lvert\lvert v\rvert\rvert_{W^{l,p}(\Delta_F)}$.
Scott-Zhang Interpolation	Under the assumptions of lemma 4.2.1: $\forall h, \forall v \in W^{l,p}(\Omega)$: $\lvert\lvert \mathcal{I}_h^{\mathrm{SZ}} v\rvert\rvert_{W^{m,p}(\Omega)} \le c\lvert\lvert v\rvert\rvert_{W^{l,p}(\Omega)}$.	**Local error estimate**: Under the assumptions of lemma 4.2.1: $\forall K, \forall v \in W^{l,p}(\Delta_K) : \lvert\lvert v - \mathcal{I}_h^{\mathrm{SZ}} v\rvert\rvert_{W^{m,p}(K)} \le ch_K^{l-m}\lvert v\rvert_{W^{l,p}(\Delta_k)}$.
Ordinary L^2 Projection	L^2-stable and if the family $\{T_h\}_{h>0}$ is quasi-uniform also H^1-stable.	**Local error estimate**: Under the assumptions of lemma 5.1.5: $\lvert v - \pi_{L^2}(v)\rvert_{W^{m,p}(K)} \le ch_K^{r-m}\lvert v\rvert_{W^{r,p}(K)}$. **Global error estimate**: $\forall v \in H^{l+1} : \lvert\lvert v - \pi_{L^2}(v)\rvert\rvert_{L^2(\Omega)} \le ch^{l+1}\lvert v\rvert_{H^{l+1}(\Omega)}$. For a quasi-uniform family $\{T_h\}_{h>0}$ one has $\forall v \in H^{l+1} : \lvert\lvert v - \pi_{L^2}(v)\rvert\rvert_{H^1(\Omega)} \le ch^l\lvert v\rvert_{H^{l+1}(\Omega)}$.

Interpolation/ Projection Operator	Stability	Error Estimates
Weighted L^2 Projection	For $u \in P^1_{0,\underline{h}}$: $$\lvert\pi_h^\omega u\rvert_{H^1_\omega(\Omega)} \le \begin{cases} c(\log \frac{h}{\underline{h}})^{\frac{1}{2}} \lvert u\rvert_{H^1_\omega(K)}, & d=2, \\ c\left(\frac{h}{\underline{h}}\right)^{\frac{1}{2}} \lvert u\rvert_{H^1_\omega(K)} & d=3. \end{cases}$$ Under the assumptions of theorem 5.2.10: $\forall u \in H^1_0(\Omega) : \lvert\pi_h^\omega u\rvert_{H^1_\omega(\Omega)} \le c\lvert \log h\rvert^{\frac{1}{2}} \lvert u\rvert_{H^1_\omega(\Omega)}$.	For $u \in P^1_{0,\underline{h}}$: $$\lvert\lvert u - \pi_h^\omega u\rvert\rvert_{L^2_\omega(\Omega)} \le \begin{cases} ch(\log \frac{h}{\underline{h}})^{\frac{1}{2}} \lvert u\rvert_{H^1_\omega(K)}, & d=2, \\ ch\left(\frac{h}{\underline{h}}\right)^{\frac{1}{2}} \lvert u\rvert_{H^1_\omega(K)} & d=3. \end{cases}$$ Under the assumptions of theorem 5.2.10: $\forall u \in H^1_0(\Omega) : \lvert\lvert u - \pi_h^\omega u\rvert\rvert_{L^2_\omega(\Omega)} \le ch\lvert \log h\rvert^{\frac{1}{2}} \lvert u\rvert_{H^1_\omega(\Omega)}$.
Elliptic Projection	H^1-stable	**Global error estimate** $\forall v \in H^{l+1}(\Omega) : \lvert\lvert v - \pi_{H^1} v\rvert\rvert_{H^1(\Omega)} \le ch^l \lvert v\rvert_{H^{l+1}(\Omega)}$. If Ω is convex, then $\forall v \in H^{l+1}(\Omega) : \lvert\lvert v - \pi_{H^1} v\rvert\rvert_{L^2(\Omega)} \le ch^{l+1} \lvert v\rvert_{H^{l+1}(\Omega)}$.

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