Master's Thesis

Finite Element Methods for the Stokes Equations with Non-constant Viscosity

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Abstract

The accurate simulation of incompressible flow problems, particularly those involving non-constant viscosity, depends on the ability to account the complexities of this type of problems. This study focuses on a numerical study of the Stokes equations with varying viscosity functions, employing the non-conforming Crouzeix-Raviart (CR) finite element P_1^{nc}/P_0. Our research starts with the establishment of the general and continuous forms of the Stokes equations, accompanied by the introduction of relevant properties and assumptions crucial for subsequent analysis. Furthermore, a detailed errors analysis is conducted for both conforming and non-conforming finite element spaces examining how the method's accuracy depends on the underlying viscosity parameters. However, the finite element error estimates for this element remains incomplete due to challenges in proving the consistency error estimate. The numerical studies for various examples and cases using the CR element are presented in the fourth Section. These numerical studies shed light on the convergence behavior of this method and reveal how the considered errors are influenced by different values of viscosity parameters.

Keywords: Stokes equations, non-constant viscosity, Crouzeix-Raviart finite element, consistency error estimate, dependency on viscosity parameters.

Contents

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Statutory Declaration

I, Hassan Karanbash, hereby declare that this work is entirely my own, unless otherwise acknowledged. I certify that I have read and understood the University's policies on plagiarism. I have clearly referenced any and all external sources used in the creation of this work. I also declare that I have not previously submitted this work for any other course or assessment.

Signature: ______________________________ Date: _______________

1 Introduction

Consider first the dimensionless Navier-Stokes equations given by

$$\begin{aligned} \partial_t u - 2\nu\nabla\cdot\mathbb{D}(u) + (u\cdot\nabla)u + \nabla p = f \text{ in } (0,T]\times\Omega, \\ \nabla\cdot u = 0 \text{ in } (0,T]\times\Omega, \end{aligned} \tag{1.1}$$

where u is the velocity field, $\nu = Re^{-1}$ the dimensionless viscosity, $\mathbb{D}(u) = (\nabla u + \nabla u^T)/2$ the velocity deformation tensor, p the pressure and f represents the exterior forces acting on the fluid.

The first equation of (1.1) can be rewritten in another form by reformulating the viscous term with the help of the divergence constraint, the definition of $\mathbb{D}(u)$, and the second equation of (1.1). One can have

$$\nabla\cdot(\nabla u) = \Delta u, \quad \nabla\cdot(\nabla u^T) = \nabla(\nabla\cdot u) = 0,$$

under the assumption that u is sufficiently smooth and using the Theorem of Schwarz.

The first assumption involved in the formulation of Stokes equations is that the flow is stationary, implying that $\partial_t u = 0$. Moreover, the flow is assumed to be very slow which means that the Reynolds number is very small, and as a consequence the convective term can be neglected due to the domination of the viscous term $Re^{-1}\Delta u$ on the convective term $(u\cdot\nabla)u$. As a result, after scaling the resulting equation along with defining a new pressure and right-hand side, one gets the so-called Stokes equations as follows

$$\begin{aligned} -\Delta u + \nabla p = f \text{ in } \Omega, \\ \nabla\cdot u = 0 \text{ in } \Omega, \end{aligned} \tag{1.2}$$

which is associated with appropriate boundary conditions, and in particular, homogeneous Dirichlet boundary conditions $u = 0$ on Γ.

2 The Stokes Equations with Variable Viscosity

In this chapter, the idea of of the Stokes equations with variable kinetic viscosity will be introduced, which is considered a special type of them and of interest to study the finite element analysis of such equations. The presentation of this chapter and Section 3.1 follows [1].

2.1 The general form

Let $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$, be a bounded domain associated with a Lipschitz boundary $\partial\Omega$. Assume that the kinetic viscosity is variable with $\nu(x) \geq \nu_{min} > 0$ almost everywhere in Ω, then the incompressible Stokes equations become

$$\begin{aligned} -2\nabla \cdot (\nu \mathbb{D}(u)) + \nabla p &= f \quad \text{in } \Omega, \\ \nabla \cdot u &= 0 \quad \text{in } \Omega. \end{aligned} \tag{2.1}$$

For the analysis, these equations will be associated with homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

It will be assumed that $\nu \in L^\infty(\Omega)$ in the analysis of (2.1). Moreover, the viscous term can be reformulated as follows

$$\begin{aligned}-2\nabla \cdot (\nu \mathbb{D}(u)) &= -2\nu \nabla \cdot \mathbb{D}(u) - 2\mathbb{D}(u)\nabla \nu \\ &= -\nu \Delta u - 2\mathbb{D}(u)\nabla \nu,\end{aligned}$$

under the assumption that $\nu \in H^1(\Omega)$.

This implies that an additional first order term of the velocity term appears in (2.1), namely $2\mathbb{D}(u)\nabla\nu$, compared to (1.2).

2.2 The continuous problem

First of all, one has to define the standard velocity and pressure spaces that will be used in the analysis of the Stokes equations. In the case of a connected and bounded domain $\Omega \subset \mathbb{R}^d$, the velocity spaces will be defined as follows

$$V = H_0^1(\Omega) = \{v : v \in H^1(\Omega) \text{ with } v|_\Gamma = 0\},$$

where the restriction of u to the boundary is considered in the sense of traces i.e, u has a vanishing trace. For the pressure space, it is defined as

$$Q = L_0^2(\Omega) = \{q : q \in L^2(\Omega) \text{ with } \int_\Omega q(x)dx = 0\},$$

that is q has a zero integral mean value.

The weak formulation of (2.1) and (2.2) can be obtained by using the usual way with the help of the following steps: multiplying the equation by the test functions, then integrating these equations on Ω, and finally applying integration by parts to transfer the derivatives from the solution to the test functions. Indeed, the problem reads as follows: find $(u, p) \in (V, Q)$ such that

$$\begin{aligned}2(\nu \mathbb{D}(u), \mathbb{D}(v)) - (\nabla \cdot v, p) &= \langle f, v\rangle_{V',V} \quad \forall v \in V, \\ -(\nabla \cdot u, q) &= 0 \quad \forall q \in Q,\end{aligned} \tag{2.3}$$

with $V' = (H^{-1})^d$ denoting the dual space of V.

Definition 2.1 (Space of weakly divergence-free functions). *Let* $b(v, q) = (\nabla \cdot v, q)$. *Define*

$$V_{div} = \{v \in V : b(v,q) = 0 \quad \forall q \in Q\},$$

the space of weakly divergence-free functions.

The proof of the uniqueness of (2.3) is clearly based on the application of the classical theory for the linear saddle point problems due to the uniform positivity and the boundedness of ν. To perform this analysis, one can simply equip V with an appropriate norm with two possibilities, the first one is the standard norm

$$|v|_1 = \|\nabla v\|_{L^2(\Omega)},$$

and the second one is the induced norm for the bilinear form of the viscous term, that is

$$\|\nu\|_\nu = \|\nu^{1/2}\mathbb{D}(v)\|_{L^2(\Omega)}.$$

The following lemma shows that the above two norms are equivalent.

Lemma 2.2 (Norm equivalence)**.** *Let* $\nu_{max} = \|\nu\|_{L^\infty(\Omega)}$*, the following inequality*

$$\nu_{max}^{-1/2}\|v\|_\nu \le \|\nabla v\|_{L^2(\Omega)} \le C_K \nu_{min}^{-1/2}\|v\|_\nu \quad \forall v \in V, \tag{2.4}$$

with C_K *is the constant from Korn's inequality, holds.*

Proof. Korn's inequality (A.4) gives the estimate

$$\|\nabla v\|_{L^2(\Omega)} \le C_K \|\mathbb{D}(v)\|_{L^2(\Omega)}.$$

We have

$$\|\mathbb{D}(v)\|_{L^2(\Omega)} \le \frac{1}{2}(\|\nabla v\|_{L^2(\Omega)} + \|\nabla v^T\|_{L^2(\Omega)}) = \|\nabla v\|_{L^2(\Omega)}.$$

The above estimates yields

$$\begin{aligned}
\underline{\|v\|_\nu} &= \|\nu^{1/2}\mathbb{D}(v)\|_{L^2(\Omega)} \le \nu_{max}^{1/2}\|\mathbb{D}(v)\|_{L^2(\Omega)} \\
&\le \underline{\nu_{max}^{1/2}\|\nabla v\|_{L^2(\Omega)}} \le C_K \nu_{max}^{1/2}\|\mathbb{D}(v)\|_{L^2(\Omega)} \\
&\le C_k \nu_{max}^{1/2}\nu_{min}^{-1/2}\|\nu^{1/2}\mathbb{D}(v)\|_{L^2(\Omega)} \\
&= \underline{C_k \nu_{max}^{1/2}\nu_{min}^{-1/2}\|v\|_\nu},
\end{aligned}$$

combining the underlined terms in one estimate and dividing it with $\nu_{max}^{1/2}$ gives (2.4). $\square$

3 Finite Element Error Analysis

3.1 Conforming finite element spaces

In this section, the conforming discretization of (2.3) will be considered where the discrete inf-sup condition is satisfied. In other words, it is $V^h \subset V$, $Q^h \subset Q$ and

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{v^h \in V^h \setminus \{0\}} \frac{b(v^h, q^h)}{\|\nabla v^h\|_{L^2(\Omega)} \|q^h\|_{L^2(\Omega)}} \geq \beta_{is} > 0. \tag{3.1}$$

This inf-sup condition can be also written the form

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{v^h \in V^h \setminus \{0\}} \frac{b(v^h, q^h)}{\|v^h\|_{\nu} \|q^h\|_{L^2(\Omega)}} \geq \beta_{is,\nu} > 0, \tag{3.2}$$

since $\|v^h\|_\nu$ and $\|\nabla v^h\|_{L^2(\Omega)}$ are equivalent in V^h.

One gets from (2.4) and (3.1)

$$\beta_{is} \|q^h\|_{L^2(\Omega)} \leq \sup_{v^h \in V^h \setminus \{0\}} \frac{b(v^h, q^h)}{\|\nabla v^h\|_{L^2(\Omega)}} \leq \frac{1}{\nu_{max}^{-1/2}} \sup_{v^h \in V^h \setminus \{0\}} \frac{b(v^h, q^h)}{\|v^h\|_\nu} \quad \forall q^h \in Q^h,$$

which implies that

$$\beta_{is,\nu} = \nu_{max}^{-1/2} \beta_{is}. \tag{3.3}$$

Denote

$$V_{div}^h = \{v^h \in V^h : b(v^h, q^h) = 0 \quad \forall q^h \in Q^h\}$$

the space of discretely divergence-free functions.

Throughout the finite element error analysis, an estimate of the best approximation error will be given. This will be possible for pairs of inf-sup stable finite element spaces by constructing a sequence of discretely divergence-free functions which have the optimal order of convergence. However, this could not be doable in some cases, and one needs to get an estimate with the best approximation error in V^h. This estimate reads as follows

$$\inf_{v^h \in V^h \setminus \{0\}} \|\nabla(v - v^h)\|_{L^2(\Omega)} \leq \left(1 + \frac{1}{\beta_{is}}\right) \inf_{w^h \in V^h \setminus \{0\}} \|\nabla(v - w^h)\|_{L^2(\Omega)}. \quad (3.4)$$

There is also a possibility to get an estimate for the ν-weighted norm by using (2.4) and (3.4). The next lemma gives this estimate.

3.1.1 The pair of finite element spaces with the property $V_{div}^h \not\subset V_{div}$.

The majority of the pairs of inf-sup stable finite element spaces has this property.

Lemma 3.1 (Estimate of the best approximation in V_{div}^h in the ν-weighted norm)**.** *Let $v \in V_{div}$ be arbitrary, then it holds*

$$\inf_{v^h \in V_{div}^h \setminus \{0\}} \|v - v^h\|_\nu \leq \left(1 + \frac{C_K}{\beta_{is,\nu}\nu_{min}^{1/2}}\right) \inf_{w^h \in V^h \setminus \{0\}} \|v - w^h\|_\nu. \quad (3.5)$$

Proof. Let the operator $B^h \in \mathcal{L}(V^h, (Q^h)')$ be defined by

$$\langle B^h v, q^h \rangle_{(Q^h)', Q^h} = b(v, q^h), \quad \forall v \in V, \forall q^h \in Q^h.$$

The operator B^h is an isomorphism from $(V_{div})^\perp$(with respect to inner product in V) onto $(Q^h)'$ and $\|B^h z^h\|_{(Q^h)'} \geq \beta_{is} \|\nabla z^h\|$ for all $z^h \in (V_{div})^\perp$.

Let w^h be an arbitrary element of V^h. Since $B^h(v - w^h) \in (Q^h)'$, there exists a unique $z^h \in (V_{div}^h)^\perp$ such that

$$B^h z^h = B^h(v - w^h). \quad (3.6)$$

Using the estimate of $L^2(\Omega)$ norm of the divergence by the same norm of the gradient, it reveals that

$$\|\nabla z^h\|_{L^2(\Omega)} \leq \frac{1}{\beta_{is}} \|B^h(v - w^h)\|_{(Q^h)'} \leq \frac{1}{\beta_{is}} \|\nabla(v - w^h)\|_{L^2(\Omega)}.$$

Applying now the norm equivalence (2.4) and (3.3) gives

$$\begin{aligned} \|z^h\|_\nu &\leq \nu_{max}^{1/2} \|\nabla z^h\|_{L^2(\Omega)} \\ &\underbrace{\leq \frac{1}{\beta_{is,\nu}} \|\nabla(v - w^h)\|_{L^2(\Omega)}}_{\text{by (3.3)}} \\ &\underbrace{\leq \frac{1}{\beta_{is,\nu}} C_K \nu_{min}^{-1/2} \|v - w^h\|_\nu\,.}_{\text{by (2.4)}} \end{aligned}$$

Setting $v^h = z^h + w^h$, one gets with (3.6) that

$$b(v^h, q^h) = b(v - w^h, q^h) + b(w^h, q^h) = b(v, q^h) \underbrace{=}_{v \in V_{div}} 0,$$

for all $q^h \in Q^h$, which implies that $v^h \in V_{div}^h$. The definition of v^h, the triangular inequality and (3.4) yield

$$\begin{aligned} \|v - v^h\|_\nu &\leq \|v - w^h\|_\nu + \|z^h\|_\nu \\ &\leq \left(1 + \frac{C_K}{\beta_{is,\nu} \nu_{min}^{1/2}}\right) \|v - w^h\|_\nu. \end{aligned}$$

The statement of the lemma is valid, as w^h is arbitrarily chosen. $\square$

Consider now the Galerkin finite element formulation of (2.3) with the given problem: find $(u^h, p^h) \in V^h \times Q^h$ such that

$$\begin{aligned} 2(\nu \mathbb{D}(u^h), \mathbb{D}(v^h)) - (\nabla \cdot v^h, p^h) &= \langle f, v^h \rangle_{V',V} \quad \forall v^h \in V^h, \\ -(\nabla \cdot u^h, q^h) &= 0 \quad \forall q^h \in Q^h. \end{aligned} \tag{3.7}$$

For the existence and uniqueness of a solution of (3.7), it can be obviously proved by the general theory of linear saddle point problems, the uniform positivity of ν and its boundedness.

Theorem 3.2 (Finite element error estimate for the velocity in the ν-weighted norm)**.** *Let $\Omega \in \mathbb{R}^d$ be a bounded domain with polyhedral Lipschitz boundary and let $(u, p) \in V \times Q$ be the unique solution of the Stokes problem* (2.3)*. Given a discretization with inf-sup stable conforming finite element spaces $V^h \times Q^h$ and Let $v^h \in V^h_{div}$ be the finite element solution for the velocity field. Then the following error estimate holds:*

$$\begin{aligned} \|u - u^h\|_\nu \leq & 2\left(1 + \frac{C_K}{\beta_{is,\nu}\nu_{min}^{1/2}}\right) \inf_{v^h \in V^h} \|u - v^h\|_\nu \\ & + \frac{C_K}{2\nu_{min}^{1/2}} \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)}, \end{aligned} \tag{3.8}$$

where $\beta_{is,\nu}$ depends on ν_{max} like in (3.3)*.*

Proof. The proof follows the classical way. Taking $v^h \in V^h_{div}$ as a test function in (2.3) and (3.7), noticing that normally $V^h_{div} \not\subset V_{div}$, subtracting these equations and using $(\nabla \cdot v^h, q^h) = 0$ for all $q^h \in Q^h$ gives

$$2(\nu\mathbb{D}(u - u^h), \mathbb{D}(v^h)) - (\nabla \cdot v^h, p - q^h) = 0 \quad \forall (v^h, q^h) \in V^h_{div} \times Q^h. \tag{3.9}$$

The error is now split into an approximation error in V^h_{div} and finite element remainder

$$u - u^h = (u - I^h u) - (u^h - I^h u) = \eta - \phi^h,$$

with $I^h u \in V^h_{div}$ is an interpolant of u in V^h_{div}. Choosing $\phi^h \in V^h_{div}$ as test function in (3.9) leads to the estimate

$$\|\phi^h\|_\nu^2 \leq |(\nu\mathbb{D}(\eta^h), \mathbb{D}(\phi^h))| + \frac{1}{2}|(\nabla \cdot \phi^h, p - q^h)|.$$

The terms on the right-hand side of the last inequality can be estimated using Cauchy-Schwarz inequality and for the second term also with norm equivalence (2.4) to get

$$|(\nu\mathbb{D}(\eta), \mathbb{D}(\phi^h))| \leq \|\nu^{1/2}\mathbb{D}(\eta)\|_{L^2(\Omega)}\|\nu^{1/2}\mathbb{D}(\phi^h)\|_{L^2(\Omega)} = \|\eta\|_\nu\|\phi^h\|_\nu,$$

and

$$\begin{aligned} |(\nabla \cdot \phi^h, p - q^h)| &\leq \|\nabla \cdot \phi^h\|_{L^2(\Omega)}\|p - q^h\|_{L^2(\Omega)} \\ &\leq \|\nabla\phi^h\|_{L^2(\Omega)}\|p - q^h\|_{L^2(\Omega)} \\ &\leq C_K\nu_{min}^{-1/2}\|\phi^h\|_\nu\|p - q^h\|_{L^2(\Omega)}. \end{aligned}$$

Then the estimate of $\|\phi^h\|_\nu$ becomes

$$\|\phi^h\|_\nu \leq \|\eta\|_\nu + \frac{C_K}{2\nu_{min}^{1/2}}\|p - q^h\|_{L^2(\Omega)}.$$

Since these estimates hold for all $I^h u$ and for all q^h, one obtains with the triangle inequality

$$\begin{aligned}
\|u - u^h\|_\nu &= \|\eta - \phi^h\|_\nu \\
&\leq \|\eta\|_\nu + \|\phi^h\|_\nu \\
&\leq 2\|\eta\|_\nu + \frac{C_K}{2\nu_{min}^{1/2}}\|p - q^h\|_{L^2(\Omega)} \\
&\leq 2 \inf_{I^h u \in V_{div}^h} \|u - I^h u\|_\nu + \frac{C_K}{2\nu_{min}^{1/2}} \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, applying the estimate (3.5) gives the desired estimate. □

Using (2.4) and (3.3), it is obvious to have another formula of (3.8) in a way that one can notice the dependency of the error bound on the viscosity. Inequality (3.8) becomes

$$\begin{aligned}
\|u - u^h\|_\nu \leq & 2\nu_{max}^{1/2} \left(1 + \frac{C_K}{\beta_{is}} \left(\frac{\nu_{max}}{\nu_{min}}\right)^{1/2}\right) \inf_{v^h \in V^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} \\
& + \frac{C_K}{2\nu_{min}^{1/2}} \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)},
\end{aligned}$$

and then one finds directly by (2.4) the estimate

$$\begin{aligned}
&\|\nabla(u - u^h)\|_{L^2(\Omega)} \\
\leq & 2C_K \left(\frac{\nu_{max}}{\nu_{min}}\right)^{1/2} \left(1 + \frac{C_K}{\beta_{is}} \left(\frac{\nu_{max}}{\nu_{min}}\right)^{1/2}\right) \inf_{v^h \in V^h} \|\nabla(u - v^h)\|_{L^2(\Omega)} \\
& + \frac{C_K^2}{2\nu_{min}} \inf_{q^h \in Q^h} \|p - q^h\|_{L^2(\Omega)}.
\end{aligned} \tag{3.10}$$

One can proceed analogously to the proof of Theorem 3.2 to get an error estimate to $\|\nabla(u - u^h)\|_{L^2(\Omega)}$ directly. The only difference in the error bound appears in the factor of the best approximation error of the viscosity term, which is

$$\left(1 + C_K^2 \frac{\nu_{max}}{\nu_{min}}\right)\left(1 + \frac{1}{\beta_{is}}\right).$$

Obviously, small values of ν_{min} or a large value of the ratio $\frac{\nu_{max}}{\nu_{min}}$ would result in a larger error bound of $\|\nabla(u - u^h)\|_{L^2(\Omega)}$.

Theorem 3.3 (Finite element error estimate for the pressure in the $L^2(\Omega)$ norm.)**.** *Let the assumptions of Theorem 3.2 hold, then it is*

$$\begin{aligned}\|p - p^h\|_{L^2(\Omega)} \le & \left(1 + \frac{2C_K}{\beta_{is}}\left(\frac{\nu_{max}}{\nu_{min}}\right)^{1/2}\right)\inf_{q^h \in Q^h}\|p - q^h\|_{L^2(\Omega)} \\ & + \frac{4\nu_{max}}{\beta_{is}}\left(1 + \frac{C_K}{\beta_{is}}\left(\frac{\nu_{max}}{\nu_{min}}\right)^{1/2}\right)\inf_{v^h \in V\|^h}\nabla(u - v^h)\|_{L^2(\Omega)}.\end{aligned} \tag{3.11}$$

Proof. It is

$$\|p - p^h\|_{L^2(\Omega)} \le \|p - q^h\|_{L^2(\Omega)} + \|p^h - q^h\|_{L^2(\Omega)},$$

where q^h is arbitrary. The finite element problem (3.7) can be rewritten as follows

$$b(v^h, p^h - q^h) = \langle f, v^h\rangle_{V',V} - 2(\nu\mathbb{D}(u^h), \mathbb{D}(v^h)) - b(v^h, q^h). \tag{3.12}$$

Now, all v^h can be used as test function in the continuous problem (3.9), because conforming finite element spaces are used. One gets by replacing $\langle f, v^h\rangle_{V',V}$ in (3.14) with the left-hand side of the continuous problem the following

$$b(v^h, p^h - q^h) = 2(\nu(\mathbb{D}(u) - \mathbb{D}(u^h)), \mathbb{D}(v^h)) - b(v^h, p - q^h) \quad \forall q^h \in Q^h, v^h \in Q^h.$$

With the discrete inf-sup condition (3.2), the Cauchy-Schwarz inequality, and the norm equivalence (2.4), one obtains

$$
\begin{aligned}
\|p-q^h\|_0 &\le \frac{1}{\beta_{is,\nu}} \sup_{v^h\in V^h\setminus\{0\}} \frac{2(\nu(\mathbb{D}(u)-\mathbb{D}(u^h)),\mathbb{D}(v^h)) - b(v^h,p-q^h)}{\|v^h\|_\nu} \\
&\le \frac{1}{\beta_{is,\nu}} \sup_{v^h\in V^h\setminus\{0\}} \frac{2\|u-u^h\|_\nu\|v^h\|_\nu + \|\nabla v^h\|_{L^2(\Omega)}\|p-q^h\|_{L^2(\Omega)}}{\|v^h\|_\nu} \\
&\le \frac{1}{\beta_{is,\nu}} \sup_{v^h\in V^h\setminus\{0\}} \frac{2\|u-u^h\|_\nu\|v^h\|_\nu + C_K\nu_{min}^{-1/2}\|v^h\|_\nu\|p-q^h\|_{L^2(\Omega)}}{\|v^h\|_\nu} \\
&\le \frac{1}{\beta_{is,\nu}}(2\|u-u^h\|_\nu\| + C_K\nu_{min}^{-1/2}\|p-q^h\|_{L^2(\Omega)}).
\end{aligned}
$$

The estimate now follows by inserting (3.8) and using (2.4) and (3.3). □

It is clear that the error bound (3.11) becomes larger with large values of ν_{max} or large ratios ν_{max}/ν_{min}.

Now, the error $L^2(\Omega)$ norm of the velocity will be explored. Consider the dual Stokes problem: Find $(\phi_{\hat f},\xi_{\hat f}) \in V\times Q$ such that for given $\hat f\in L^2(\Omega)$

$$
\begin{aligned}
-2\nabla\cdot(\nu\mathbb{D}(\phi_{\hat f})) + \phi_{\hat f} &= \hat f \quad \text{in } \Omega, \\
(-\nabla\cdot\phi_{\hat f}) &= 0 \quad \text{in } \Omega,
\end{aligned}
\tag{3.13}
$$

with homogeneous Dirichlet boundary conditions and its weak form

$$
\begin{aligned}
2(\nu\mathbb{D}(\phi_{\hat f}),\mathbb{D}(v)) - (\nabla\cdot v,\xi_{\hat f}) &= (\hat f, v) \quad \forall v\in V, \\
-(\nabla\cdot\phi_{\hat f}, q) &= 0 \qquad\quad \forall q\in Q.
\end{aligned}
\tag{3.14}
$$

The mapping $(\phi_{\hat f},\xi_{\hat f}) \mapsto -2\nabla\cdot(\nu\mathbb{D}(\phi_{\hat f})) + \nabla\xi_{\hat f}$ is assumed to be an isomorphism from $(H^2(\Omega)^d\cap V)\times(H^1(\Omega)\cap Q)$ to $(L^2(\Omega))^d$. Therefore, $(\phi_{\hat f},\xi_{\hat f})$ is called a regular solution of (3.13).

Theorem 3.4 (Finite element error estimate for the velocity in the $L^2(\Omega)$ norm.)**.** *With the assumptions of Theorem 3.2 and $(\phi_{\hat f},\xi_{\hat f})$ is a regular solution of* (3.13)*, the following error estimate for the $L^2(\Omega)$ norm of the velocity* *holds*

$$
\begin{aligned}
&\|u-u^h\|_{L^2(\Omega)} \\
&\leq \left(2\|\nabla(u-u^h)\|_{L^2(\Omega)} + \frac{1}{\nu_{max}} \inf_{q^h\in Q^h} \|p-q^h\|_{L^2(\Omega)}\right) \\
&\quad \times \sup_{\hat{f}\in L^2(\Omega)\setminus\{0\}} \frac{1}{\|\hat{f}\|_{L^2(\Omega)}} \left[\left(1+\frac{1}{\beta_{is}}\right)\nu_{max} \inf_{\phi^h\in V^h} \|\nabla(\phi_{\hat{f}}-\phi^h)\|_{L^2(\Omega)}\right. \\
&\quad \left. + \frac{1}{2}\inf_{r^h\in Q^h} \|\xi_{\hat{f}} - r^h\|_{L^2(\Omega)}\right].
\end{aligned} \tag{3.15}
$$

Proof. To begin with, the $L^2(\Omega)$ norm is defined as follows

$$
\|u-u^h\|_{L^2(\Omega)} = \sup_{\hat{f}\in L^2(\Omega)\setminus\{0\}} \frac{(\hat{f}, u-u^h)}{\|\hat{f}\|_{L^2(\Omega)}}. \tag{3.16}
$$

Setting $v = u - u^h$ in (3.12) yields

$$
(\hat{f}, u-u^h) = 2(\nu\mathbb{D}(\phi_{\hat{f}}), \mathbb{D}(u-u^h)) - (\nabla\cdot(u-u^h), \xi_{\hat{f}}). \tag{3.17}
$$

Using the weak form of the Stokes problem (2.3) and the corresponding finite element problem (3.7), one finds for $\phi^h \in V^h_{div} \subset V$ and $q^h \in Q^h$ arbitrary

$$
2(\nu\mathbb{D}(\phi_h), \mathbb{D}(u-u^h)) = (\nabla\cdot\phi^h, p) = (\nabla\cdot\phi^h, p-q^h).
$$

Inserting this identity into (3.17) and adding some zero terms to (3.17) gives

$$
\begin{aligned}
(\hat{f}, u-u^h) =& 2(\nu\mathbb{D}(\phi_{\hat{f}}-\phi_h), \mathbb{D}(u-u^h)) - (\nabla\cdot(u-u^h), \xi_{\hat{f}}) \\
&+ (\nabla\cdot(\phi^h-\phi_{\hat{f}}), p-q^h)
\end{aligned}
$$

for all $\phi^h \in V^h_{div}$ and $q^h, r^h \in Q^h$. Applying now the Cauchy-Schwarz inequality, the estimate of the norm of the divergence by the the norm of the gradient, and the norm equivalence (2.4) leads to

$$
\begin{aligned}
&|(\hat{f}, u-u^h)| \\
&\leq \|\phi_{\hat{f}}-\phi_h\|_\nu \|u-u^h\|_\nu + \|\nabla(u-u^h)\|_{L^2(\Omega)}\|\xi_{\hat{f}}-r^h\|_{L^2(\Omega)} \\
&\quad + \|\nabla(\phi^h-\phi_{\hat{f}})\|_{L^2(\Omega)}\|p-q^h\|_{L^2(\Omega)} \\
&\leq \left(2\|\nabla(u-u^h)\|_{L^2(\Omega)} + \frac{1}{\nu_{max}}\|p-q^h\|_{L^2(\Omega)}\right)\left(\nu_{max}\|\nabla(\phi^h-\phi_{\hat{f}})\|_{L^2(\Omega)}\right. \\
&\quad \left. + \frac{1}{2}\|\xi_{\hat{f}}-r^h\|_{L^2(\Omega)}\right)
\end{aligned}
$$

for all $\phi^h \in V_{div}^h$ and $q^h, r^h \in Q^h$. Estimate (3.15) is obtained obviously by inserting this estimate into (3.16). $\square$

The velocity error estimates obtained depends on the pressure term, which means that they are not pressure-robust estimates. However, the case where $V_{div}^h \subset V_{div}$ reveals a vanishing pressure term of the left-hand side of (3.9), (3.10) and (3.15), see Section 3.1.2.

The studying of the dependency of the error bound on the right-hand of (3.15) on the viscosity is of interest only if all the norms depend on the viscosity. It was supposed that (u, p) is independent of the viscosity as shown in the results of Theorems 3.2 and 3.3, implying that only f depends on the viscosity, but f doesn't appear in the estimates. Thus, one of the norms $\|\hat{f}\|_{L^2(\Omega)}$,$\|\nabla(\phi_{\hat{f}} - \phi^h)\|_{L^2(\Omega)}$ and $\|\xi_{\hat{f}} - r^h\|_{L^2(\Omega)}$ has to depend on the viscosity. In this case, one can assume that $\hat{f}$ is independent of the viscosity. One can have

$$\|\nabla\phi_{\hat{f}}\|_{L^2(\Omega)} \leq \frac{C}{\nu_{min}}\|\hat{f}\|_{L^2(\Omega)}$$

using $\phi_{\hat{f}}$ as a test function in (3.14) and then applying Cauchy-Schwarz inequality, Poincaré inequality, and the norm equivalence (2.4) which means that the term $\|\nabla(\phi_{\hat{f}} - \phi^h)\|_{L^2(\Omega)}$ can be estimated to a scale of ν_{min}^{-1}. Now, $\hat{f}$ can be decomposed using Helmholtz decomposition into $\hat{f} = w + \nabla r$ where w is divergence-free and ∇r is orthogonal to w with respect to the $L^2(\Omega)$ inner product. Inserting this decomposition in (3.13) yields

$$-2\nabla \cdot (\nu\mathbb{D}(\phi_{\hat{f}})) + \nabla\xi_{\hat{f}} = w + \nabla r.$$

Therefore, $\xi_{\hat{f}}$ is independent of ν as it is balanced by r and $\hat{f}$ is assumed to be independent of ν. One deduces that the only term depending on the viscosity is $\|\nabla(\phi_{\hat{f}} - \phi^h)\|_{L^2(\Omega)}$, and the term in (3.15) with the dual velocity can be scaled like ν_{max}/ν_{min}.

3.1.2 The pair of finite element spaces with the property $V_{div}^h \subset V_{div}$.

In the previous section, the study of the finite element error analysis was investigated for inf-sup stable pairs of FE spaces with the condition that $V_{div}^h \not\subset V_{div}$. The aim of this section is to consider inf-sup stable FE pairs with $V_{div}^h \subset V_{div}$ and discover how the error estimates will change in this case. One will find that some terms will disappear from the error estimates compared to those in the Section 3.1.1, and there will also be no dependency anymore of the error estimate for the velocity on the best approximation errors of the pressure finite element space. A well-known example of such pairs is the Scott-Vogelius pairs of the finite element spaces P_k/P_{k-1}^{disc}, $k \geq d$, on special grids, in particular barycentric-refined meshes. It is shown that these pairs satisfy the inf-sup condition, see [2] for $k \in \{2, 3\}$ and [3] for $k \geq 3$.

Corollary 3.5 (Finite element error estimate for the velocity in the ν-weighted norm for inf-sup stable pairs of FE spaces with $V_{div}^h \subset V_{div}$.)**.** *Let the assumptions of Theorem 3.2 hold and consider an inf-sup stable pair of finite element spaces with $V_{div}^h \subset V_{div}$. Then, the following estimate holds*

$$\|u - u^h\|_\nu \leq 2 \left(1 + \frac{C_K}{\beta_{is,\nu} \nu_{min}^{1/2}} \right) \inf_{v^h \in V^h} \|u - v^h\|_\nu. \tag{3.18}$$

Proof. The proof of (3.18) follows the same steps in the proof of Theorem 3.2. However, the fact that $V_{div}^h \subset V_{div}$ results in vanishing of the term $-(\nabla \cdot v^h, p - q^h)$ in (3.9). □

One can proceed analogously to the part on obtaining the estimate (3.10) in Section 3.1.1 to obtain the estimate

$$\begin{aligned} &\|\nabla(u - u^h)\|_{L^2(\Omega)} \\ \leq &2C_K \left(\frac{\nu_{max}}{\nu_{min}} \right)^{1/2} \left(1 + \frac{C_K}{\beta_{is}} \left(\frac{\nu_{max}}{\nu_{min}} \right)^{1/2} \right) \inf_{v^h \in V^h} \|\nabla(u - v^h)\|_{L^2(\Omega)}, \end{aligned} \tag{3.19}$$

from the estimate (3.18).

Corollary 3.6 (Finite element error estimate for the velocity in the $L^2(\Omega)$ norm for inf-sup stable pairs of FE spaces with $V_{div}^h \subset V_{div}$.)**.** *Let the assumptions of Theorem 3.4 be satisfied and consider an inf-sup stable pair of finite element spaces with $V_{div}^h \subset V_{div}$. Then the following estimate holds*

$$\begin{aligned} \|u - u^h\|_{L^2(\Omega)} &\leq 2\|\nabla(u - u^h)\|_{L^2(\Omega)} \\ &\times \sup_{\hat{f} \in L^2(\Omega)\setminus\{0\}} \frac{1}{\|\hat{f}\|_{L^2(\Omega)}} \left[\left(1 + \frac{1}{\beta_{is}} \right) \nu_{max} \inf_{\phi^h \in V^h} \|\nabla(\phi_{\hat{f}} - \phi^h)\|_{L^2(\Omega)} \right]. \end{aligned} \tag{3.20}$$

Proof. The proof is the same as the proof of Theorem 3.4. One should notice that $\nabla \cdot (u - u^h) = 0$ and $\nabla \cdot (\phi^h - \phi_{\hat{f}}) = 0$ in the weak sense. $\square$

The finite element error estimates for the velocity (3.18), (3.19) and (3.20) do not depend on the pressure term, which reveals that these estimates are pressure-robust estimates in-contrast to the estimates obtained in Section 3.1.1. Pressure-robustness is very important and essential in numerous practical problems of the Navier-Stokes equations. Some results show that pressure-robustness could avoid high velocity errors in some cases, see [4, 5].

3.2 Non-conforming finite element spaces.

The finite element error analysis for non-conforming FE pairs of spaces will be considered in this section, in particular Crouzeix-Raviart finite element pairs $P_k^{nc}/P_{k-1}, k = 1$. First of all, such pairs satisfy the discrete inf-sup condition (3.1), see [6, Section 3.6.5] for $k = 1$ and [7] for $k = 2, 3, 4$ and $k \geq 5$, k odd. However, they have the property that $V^h \not\subset V$, strictly speaking the properties of V are not transferred to V^h. As a consequence, a so-called consistency error needs to be estimated. In the framework of this section, the CR finite element pair of piecewise linear velocity functions P_1^{nc} and piecewise constant pressure functions P_0 will be typically used.

For the analysis of the non-conforming FE space, the finite element problem will be formulated using mesh cell by mesh cell definitions of the bilinear forms. In other words, the problem reads: given $f \in L^2(\Omega)$, find $(u^h, p^h) \in V^h \times Q^h = P_1^{nc} \times P_0$ such that

$$\begin{aligned} a_\nu^h(u^h, v^h) + b^h(v^h, p^h) &= (f, v^h) \ \forall v^h \in V^h, \\ b^h(u^h, q^h) &= 0 \ \forall q^h \in Q^h, \end{aligned} \tag{3.21}$$

with the bilinear forms

$$a_\nu^h(u^h, v^h) = \sum_{K \in \mathcal{T}^h} (\nu \nabla u^h, \nabla v^h)_K, \ b^h(v^h, q^h) = -\sum_{K \in \mathcal{T}^h} (\nabla \cdot v^h, q^h)_K. \tag{3.22}$$

The above formulation (3.21) is not equivalent to the deformation tensor representation (3.7) in Section 3.1.1 in the case of non-constant viscosity. For the Crouzeix-Raviart finite element pair, there is no direct simplification for the viscous term in terms of the rate of deformation tensor, see Remark 1 below.

One can define the ν-norm in this way

$$\|v^h\|_{V^h,\nu} = (a_\nu^h(v^h, v^h))^{1/2} = \left(\sum_{K \in \mathcal{T}^h} (\nu \nabla v^h, \nabla v^h)_K \right)^{1/2}$$

or, equivalently

$$\|v^h\|_{V^h,\nu} = \left(\sum_{K \in \mathcal{T}^h} \|\nu^{1/2} \nabla v^h\|^2_{L^2(K)} \right)^{1/2}.$$

Remark 1 (On the satisfaction of Korn's inequality for the Crouzeix-Raviart element). For the non-conforming finite element methods, Korn's inequality doesn't hold because the necassary continuity for the gradiant field is not ensured for the CR finite element, see [8]. Thus, the consideration of the deformation tensor is ignored for such method.

Corollary 3.7 (Uniqueness of the solution of (3.21))**.** *The solution* $(u^h, p^h) \in V^h \times Q^h$ *of the finite element problem* (3.21) *is unique.*

Proof. See [6, Corollary 4.47]. □

The finite element problem (3.18) will be reduced for the sake of error analysis by considering functions from discretely divergence-free space V^h_{div}. Then the problem becomes: find $u^h \in V^h = P_1^{nc}$ such that

$$a_\nu^h(u^h, v^h) = (f, v^h) \; \forall v^h \in V^h_{div}. \tag{3.23}$$

Let $\{\mathcal{T}^h\}$ be a family of regular simplicial triangulations and consider $v^h \in V^h$, then a Poincaré inequality

$$\|v^h\|_{L^2(\Omega)} \leq C \|v^h\|_{V^h} \tag{3.24}$$

with

$$\|v^h\|_{V^h} = \left(\sum_{K \in \mathcal{T}^h} \|\nabla v^h\|^2_{L^2(K)} \right)^{1/2}$$

holds.

Clearly, the spaces V^h and Q^h must satisfy the discrete inf-sup condition of the form

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{v^h \in V^h \setminus \{0\}} \frac{b^h(v^h, q^h)}{\|v^h\|_{V^h,\nu} \|q^h\|_{Q^h}} \geq \beta^h_{is,\nu} > 0, \tag{3.25}$$

where $b^h : V^h \times Q^h \to \mathbb{R}$ is the bilinear form defined in (3.22), and $\beta_{is,\nu}$ defined (3.3). In other words, there is a $\beta^h_{is,\nu} > 0$ such that

$$\sup_{v^h \in V^h \setminus \{0\}} \frac{b^h(v^h, q^h)}{\|v^h\|_{V^h,\nu}} \geq \beta^h_{is,\nu} \|q^h\|_{Q^h} \quad \forall q^h \in Q^h. \tag{3.26}$$

Remark 2 (On the discrete inf-sup condition for the P_1^{nc}/P_0 pair). For the Crouzeix-Raviart pair of finite elements spaces P_1^{nc}/P_0, it holds that $\beta^h_{is} = \beta_{is}$, where β^h_{is}, β_{is} denote the discrete, continuous inf-sup condition, respectively. For more details, see [6, Theorem 3.151].

For the norm equivalence (2.4), this form will be formulated in this case as follows

$$\nu_{max}^{-1/2} \|v^h\|_{V^h,\nu} \leq \|v^h\|_{V^h} \leq \nu_{min}^{-1/2} \|v^h\|_{V^h,\nu}, \tag{3.27}$$

where $\|v^h\|_{V^h,\nu}$ and $\|v^h\|_{V^h}$ are defined above.

Lemma 3.8 (Stability of the finite element solution)**.** *Assume that* $f \in L^2(\Omega)$*. Then, the following estimates for the solution of* (3.21)

$$\|u^h\|_{V^h,\nu} \leq C \nu_{min}^{-1/2} \|f\|_{L^2(\Omega)}, \quad \|p^h\|_{L^2(\Omega)} \leq C \left(\frac{1 + \nu_{min}^{-1/2}}{\beta^h_{is,\nu}} \right) \|f\|_{L^2(\Omega)},$$

where C *is the constant from Poincaré inequality* (3.24)*, hold.*

Proof. For the velocity, let us consider first the solution of (3.23) as a test function and then use the Cauchy-Schwarz inequality (A.2), the Poincaré inequality (3.24) and (3.27) to get

$$\begin{aligned} \|u^h\|^2_{V^h,\nu} &= a^h_\nu(u^h, u^h) = (f, u^h) \\ &\leq \|f\|_{L^2(\Omega)} \|u^h\|_{L^2(\Omega)} \\ &\leq C \|f\|_{L^2(\Omega)} \|u^h\|_{V^h} \\ &\leq C \nu_{min}^{-1/2} \|f\|_{L^2(\Omega)} \|u^h\|_{V^h,\nu}. \end{aligned}$$

In addition, the estimate for the pressure could be found using the discrete inf-sup condition (3.26), (3.21), the Cauchy-Schwarz inequality (A.2), the Poincaré inequality (3.24), and the stability estimate for the velocity

$$
\begin{aligned}
\|p^h\|_{L^2(\Omega)} &\leq \frac{1}{\beta^h_{is,\nu}} \sup_{v^h \in V^h\setminus\{0\}} \frac{b^h(v^h, q^h)}{\|v^h\|_{V^h,\nu}} \\
&= \frac{1}{\beta^h_{is,\nu}} \sup_{v^h \in V^h\setminus\{0\}} \frac{(f, v^h) - a^h_\nu(u^h, v^h)}{\|v^h\|_{V^h,\nu}} \\
&\leq \frac{1}{\beta^h_{is,\nu}} \sup_{v^h \in V^h\setminus\{0\}} \frac{C\|f\|_{L^2(\Omega)}\|v^h\|_{V^h,\nu} + \|u^h\|_{V^h,\nu}\|v^h\|_{V^h,\nu}}{\|v^h\|_{V^h,\nu}} \\
&\leq \frac{1}{\beta^h_{is,\nu}} (C\|f\|_{L^2(\Omega)} + \|u^h\|_{V^h,\nu}) \\
&\leq C \left(\frac{1 + \nu_{min}^{-1/2}}{\beta^h_{is,\nu}} \right) \|f\|_{L^2(\Omega)}.
\end{aligned}
$$

□

Lemma 3.9 (Abstract error estimate)**.** *Let* $u \in V$ *be the solution of* (2.3) *and* $u^h \in V^h$ *be the solution of* (3.23)*. Then, the following error estimate*

$$
\begin{aligned}
\|u - u^h\|_{V^h,\nu} \leq & 2 \inf_{v^h \in V^h_{div}} \|u - v^h\|_{V^h,\nu} \\
&+ \inf_{v^h \in V^h_{div}, \|v^h\|_{V^h,\nu}=1} |a^h_\nu(u, v^h) - (f, v^h)|
\end{aligned}
\tag{3.28}
$$

holds.

Proof. Consider first $v^h \in V^h_{div}$ arbitrary and rewrite $u - u^h$ as follows

$$
u - u^h = u - v^h - (u^h - v^h) = u - v^h - \phi^h
$$

with $\phi^h \in V^h_{div}$. Then, one gets, using (3.23) and the Cauchy-Schwarz inequality

$$
\begin{aligned}
\|\phi^h\|^2_{V^h,\nu} &= a^h_\nu(\phi^h, \phi^h) = a^h_\nu(u^h - v^h, \phi^h) \\
&= a^h_\nu(u - v^h, \phi^h) + a^h_\nu(u^h, \phi^h) - a^h_\nu(u, \phi^h) \\
&= a^h_\nu(u - v^h, \phi^h) + (f, \phi^h) - a^h_\nu(u, \phi^h) \\
&\leq \|u - u^h\|_{V^h,\nu}\|\phi^h\|_{V^h,\nu} + |(f, \phi^h) - a^h_\nu(u, \phi^h)|.
\end{aligned}
$$

Dividing the obtained estimate with $\|\phi^h\|_{V^h,\nu}$, using the triangle inequality

$$\|u - u^h\|_{V^h,\nu} \le \|u - v^h\|_{V^h,\nu} + \|u^h - v^h\|_{V^h,\nu}$$

and noticing that v^h and ϕ^h are arbitrarily chosen, yields the estimate (3.27). If $\|\phi^h\|_{V^h,\nu} = 0$, estimate (3.27) could be obtained directly from the decomposition of the error. $\square$

Lemma 3.10 (Best approximation error estimate in V^h_{div})**.** *Let $\{\mathcal{T}^h\}$ be a quasi-uniform family of triangulations and let $u \in H^2(\Omega)$. Then, the estimate of the best approximation error of u in V^h_{div} has the following form*

$$\inf_{v^h \in V^h_{div}} \|u - v^h\|_{V^h,\nu} \le Ch|u|_{H^2(\Omega)}. \tag{3.29}$$

Proof. see [6, Lemma 4.53]. The proof is exactly the same for the norm $\|\cdot\|_{V^h,\nu}$. $\square$

Now, one proceeds to find the consistency error estimate as in the constant viscosity case, see [6, Lemma 4.55]. However, it will turn out that one encounters technical difficulties that could not be overcome in the considered case. These difficulties will be explained below.

Initially, we assume that (u, p) solves the Stokes equation (2.3) with $u \in C^1(\overline{\Omega}) \cap V$ and $p \in C(\overline{\Omega}) \cap Q$, and ν is continuous and sufficiently smooth with $\nu \in H^1(\overline{\Omega})$. Moreover, a family of quasi-uniform triangulations will be considered.

Let $v \in V_{div} \oplus V^h_{div}$ be arbitrary. Using the momentum equation of the Stokes problem (2.3), one gets

$$\begin{aligned} a^h_\nu(u, v) - (f, v) &= \sum_{K \in \mathcal{T}^h} (\nu\nabla u, \nabla v)_K - (f, v)_K \\ &= \sum_{K \in \mathcal{T}^h} (\nu\nabla u, \nabla v)_K - (-\nu\Delta u + \nabla p, v)_K. \end{aligned} \tag{3.30}$$

Velocity term. Applying integration by parts, mesh cell by mesh cell, gives

$$\begin{aligned} &\sum_{K \in \mathcal{T}^h} (\nu\nabla u, \nabla v)_K + (\nu\Delta u, v)_K \\ &\quad = \sum_{E \in \mathcal{E}^h} \int_E \nu[|(\nabla u n_E) \cdot v|]_E \, ds + \sum_{E \in \overline{\mathcal{E}^h} \setminus \mathcal{E}^h} \int_E \nu(\nabla u n_E) \cdot v \, ds \end{aligned} \tag{3.31}$$

where n_E are the unit normals of $E \in \mathcal{E}$ chosen arbitrarily but fixed and the normals for $E \in \overline{\mathcal{E}^h} \setminus \mathcal{E}^h$ are the outward pointing unit normals. One should notice that changing the normal for an interior face changes the both, the sign of the normal and the sign of the jump, such that one obtains the same result as with the other normal. Two cases will be considered now depending on how one would define the integral mean of v on E (either defined with the appearance of ν in front of the integral or without).

Case 1: Let

$$\overline{v}_E = \frac{1}{|E|} \int_E v(s)\, ds, \quad E \in \mathcal{E}^h,$$

the integral mean of v on E. Note that the integral mean value is well defined for functions on V^h, since it is the nodal functional for defining the space P_1^{nc} and this functional has to be continuous. Then, one has

$$\int_E (v - \overline{v}_E)(s)\, ds = 0. \tag{3.32}$$

Moreover, let $I^h : V \cap H^2(\Omega) \to P_1 \subset V^h$ be the Lagrangian interpolation operator to the space of continuous, piecewise linear functions. Then, $(\nabla I^h u) n_E$ is constant for each mesh cell and each face E of the mesh cell. The analog to the constant viscosity case would be that the left-hand side of (3.31) is equivalent to

$$\begin{aligned} &\sum_{K \in \mathcal{T}^h} (\nu \nabla u, \nabla v)_K + (\nu \Delta u, v)_K \\ &\quad = \sum_{E \in \mathcal{E}^h} \int_E \nu [|(\nabla (u - I^h u) n_E) \cdot (v - \overline{v}_E)|]_E(s)\, ds \\ &\qquad + \sum_{E \in \overline{\mathcal{E}^h} \setminus \mathcal{E}^h} \int_E \nu (\nabla (u - I^h u) n_E) \cdot (v - \overline{v}_E)(s)\, ds. \end{aligned} \tag{3.33}$$

However, this equality doesn't hold for the non-constant viscosity case, since for the integral on the boundary faces, one can get using $\overline{v}_E = 0$

$$
\begin{aligned}
&\int_E \nu(\nabla(u - I^h u) n_E) \cdot (v - \overline{v}_E)(s)\, ds \\
&= \int_E \nu(\nabla u n_E) \cdot v(s)\, ds - \int_E \nu(\nabla u n_E) \cdot \overline{v}_E(s)\, ds \\
&\quad - \nabla(I^h u) n_E \cdot \int_E \nu(v - \overline{v}_E)(s)\, ds \\
&= \int_E \nu(\nabla u n_E) \cdot v(s)\, ds - \nabla(I^h u) n_E \cdot \int_E \nu(v - \overline{v}_E)(s)\, ds.
\end{aligned}
$$

One can also get for the interior edges

$$
\begin{aligned}
&\int_E \nu[|(\nabla(u - I^h u) n_E) \cdot (v - \overline{v}_E)|]_E(s)\, ds \\
&= \int_E \nu[|(\nabla u n_E) \cdot v|]_E(s)\, ds - \int_E \nu[|(\nabla u n_E) \cdot \overline{v}_E|]_E(s)\, ds \\
&\quad - \int_E \nu[|(\nabla(I^h u) n_E) \cdot (v - \overline{v}_E)|]_E(s)\, ds \\
&= \int_E \nu[|(\nabla u n_E) \cdot v|]_E(s)\, ds - \int_E \nu[|(\nabla(I^h u) n_E) \cdot (v - \overline{v}_E)|]_E(s)\, ds.
\end{aligned}
$$

The second term vanishes, since the jump is zero almost everywhere due to the continuity of the jump function and the viscosity function ν.

One can notice that we get an extra term for the boundary faces and for the interior faces. In this case, one could not proceed as in the case of constant viscosity and it is even very difficult to estimate (3.31).

Case 2: Let

$$\overline{v}_E = \frac{1}{|E|} \int_E \nu v(s)\, ds, \quad E \in \mathcal{E}^h,$$

the integral mean of v on E. Note that the integral mean value is well defined for functions on V^h, since it is the nodal functional for defining the space P_1^{nc} and this functional has to be continuous. Then, one has

$$\int_E (\nu v - \overline{v}_E)(s)\, ds = 0. \tag{3.34}$$

Moreover, let $I^h : V \cap H^2(\Omega) \to P_1 \subset V^h$ be the Lagrangian interpolation operator to the space of continuous, piecewise linear functions. Then,

$(\nabla I^h u)n_E$ is constant for each mesh cell and each face E of the mesh cell. One can get for equation (3.31)

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^h} (\nu\nabla u, \nabla v)_K + (\nu\Delta u, v)_K \\
&\quad = \sum_{E\in\mathcal{E}^h} \int_E \nu[|(\nabla(u - I^h u)n_E)\cdot(v - \frac{\overline{v}_E}{\nu})|]_E(s)\,ds \\
&\quad\quad + \sum_{E\in\overline{\mathcal{E}^h}\setminus\mathcal{E}^h} \Bigg[\int_E \nu(\nabla(u - I^h u)n_E)\cdot(v - \frac{\overline{v}_E}{\nu})(s)\,ds \\
&\quad\quad + \int_E (\nabla u n_E)\cdot\overline{v}_E(s)\,ds\Bigg],
\end{aligned} \tag{3.35}$$

since for the integral on the boundary faces, one can get using (3.34)

$$\begin{aligned}
&\int_E \nu(\nabla(u - I^h u)n_E)\cdot(v - \frac{\overline{v}_E}{\nu})(s)\,ds \\
&\quad = \int_E \nu(\nabla u n_E)\cdot v(s)\,ds - \int_E \nu(\nabla u n_E)\cdot\frac{\overline{v}_E}{\nu}(s)\,ds \\
&\quad\quad - \nabla(I^h u)n_E \cdot \int_E \nu(v - \frac{\overline{v}_E}{\nu})(s)\,ds \\
&\quad\quad = \int_E \nu(\nabla u n_E)\cdot v(s)\,ds - \int_E (\nabla u n_E)\cdot\overline{v}_E(s)\,ds.
\end{aligned}$$

One can also get for the interior edges

$$\begin{aligned}
&\int_E \nu[|(\nabla(u - I^h u)n_E)\cdot(v - \frac{\overline{v}_E}{\nu}|]_E(s)\,ds \\
&= \int_E \nu[|(\nabla u n_E)\cdot v|]_E(s)\,ds - \int_E \nu[|(\nabla u n_E)\cdot\frac{\overline{v}_E}{\nu}|]_E(s)\,ds \\
&\quad - \int_E \nu[|(\nabla(I^h u)n_E)\cdot(v - \frac{\overline{v}_E}{\nu})|]_E(s)\,ds \\
&= \int_E \nu[|(\nabla u n_E)\cdot v|]_E(s)\,ds - \int_E \nu[|(\nabla u n_E)\cdot\frac{\overline{v}_E}{\nu}|]_E(s)\,ds \\
&\quad - \nabla(I^h u)n_E|_{K_1} \int_E \nu(v - \frac{\overline{v}_E}{\nu})(s)\,ds + \nabla(I^h u)n_E|_{K_2} \int_E \nu(v - \frac{\overline{v}_E}{\nu})(s)\,ds \\
&= \int_E \nu[|(\nabla u n_E)\cdot v|]_E(s)\,ds.
\end{aligned}$$

As before, the second term vanishes, since the jump is zero almost everywhere due to the continuity of the jump function and the viscosity function ν. The vanishing of the last two terms is due to (3.34).
There are two difficulties in this case:

- We need to estimate the jump terms mentioned in (3.35), which depend on viscosity. The estimation of these terms for the constant viscosity case, where ν is not shown, has been demonstrated in [6, Lemma 4.58]. We can proceed in a similar manner to the proof presented there. However, the primary challenge lies in estimating these terms in the presence of non-constant viscosity.

- The second difficulty reveals in estimating the term for edges at the boundary
$$\int_E (\nabla u n_E) \cdot \overline{v}_E(s)\, ds,$$
where $\overline{v}_E(s)$ doesn't vanish in the considered case.

Pressure term. For the last term in the right-hand side of (3.31). One can obtain using integration by parts

$$\begin{aligned}\sum_{K\in\mathcal{T}^h}(\nabla p, v)_K = &\sum_{E\in\mathcal{E}^h}\int_E [|pv\cdot n_E|]_E\, ds + \sum_{E\in\overline{\mathcal{E}^h}\setminus\mathcal{E}^h}\int_E pv\cdot n_E\, ds \\ &- \sum_{K\in\mathcal{T}^h}(\nabla\cdot v, p)_K.\end{aligned} \tag{3.36}$$

Let $P^h_{L^2} : Q \to Q^h$ be the $L^2(\Omega)$ projection of pressure functions to the piecewise constant finite element pressure. Using an analogous way for estimating the integrals for the velocity, one finds

$$\begin{aligned}&\sum_{E\in\mathcal{E}^h}\int_E [|pv\cdot n_E|]_E\, ds + \sum_{E\in\overline{\mathcal{E}^h}\setminus\mathcal{E}^h}\int_E pv\cdot n_E\, ds \\ &= \sum_{E\in\mathcal{E}^h}\int_E [|(p - P^h_{L^2}p)(v-\overline{v}_E)\cdot n_E|]_E\, ds \\ &\quad + \sum_{E\in\overline{\mathcal{E}^h}\setminus\mathcal{E}^h}\int_E (p-P^h_{L^2}p)(v-\overline{v}_E)\cdot n_E\, ds \\ &\le Ch\|\nabla p\|_{L^2(\Omega)}\|v\|_{V^h}.\end{aligned} \tag{3.37}$$

The last term on the right-hand side of (3.34) would disappear, if $v \in V_{div}$. Then, let $v \in V_{div}^h$, one gets using the definition of V_{div}^h, the Cauchy-Schwarz inequality (A.2), the Cauchy-Schwarz inequality for sums , estimate (B.1), and the estimate for the $L^2(\Omega)$ projection (B.2)

$$\begin{aligned}\sum_{K\in\mathcal{T}^h}(\nabla\cdot v,p)_K &= \sum_{K\in\mathcal{T}^h}(\nabla\cdot v,p-P_{L^2}^h p)_K \\ &\leq \sum_{K\in\mathcal{T}^h}\|\nabla\cdot v\|_{L^2(K)}\|p-P_{L^2}^h p\|_{L^2(K)} \\ &\leq \|p-P_{L^2}^h p\|_{L^2(\Omega)}\left(\sum_{K\in\mathcal{T}^h}\|\nabla\cdot v\|_{L^2(K)}\right)^{1/2} \\ &\leq Ch\|\nabla p\|_{L^2(\Omega)}\|v\|_{V^h}.\end{aligned} \tag{3.38}$$

The proof of the pressure remains the same as in the case of constant viscosity since it is independent of viscosity.

4 Numerical Studies

The numerical studies aim at the discovering if the errors depend on the maximum and minimum values of viscosity. It should be noted that the error estimates are the worst case, but there might be some cases where the behavior would be better.

The example used for the numerical studies was introduced in [1]. The analytical solutions for both velocity and pressure fields don't depend on the viscosity. Let $\Omega = (0,1)^2$ and

$$\phi(x,y) = 1000x^2(1-x)^4y^3(1-y)^2,$$

therefore the velocity solution is given by

$$u = (\partial_y\phi, -\partial_x\phi)^T.$$

Indeed,

$$\nabla \cdot u = \partial_x(\partial_y\phi) + \partial_y(-\partial_x\phi) = \partial_{xy}\phi - \partial_{yx}\phi = 0,$$

thus u is divergence-free. It also satisfies homogeneous Dirichlet boundary conditions. The pressure is define as follows:

$$p(x,y) = \pi^2(xy^2\cos(2\pi x^2 y) - x^2 y\sin(2\pi xy)) + \frac{1}{8}.$$

Three different functions for the viscosity were considered:

$$\begin{aligned}
\nu_1(x,y) &= \nu_{min} + (\nu_{max} - \nu_{min})x^2(1-x)y^2(1-y)\cdot\frac{721}{16},\\
\nu_2(x,y) &= \nu_{min} + (\nu_{max} - \nu_{min})\exp(-10^{13}((x-0.5)^{10} + (y-0.5))^{10})\\
\nu_3(x,y) &= \nu_{min} + (\nu_{max} - \nu_{min})(1 - \exp(-10^{13}((x-0.5)^{10} + (y-0.5)^{10})).
\end{aligned}$$

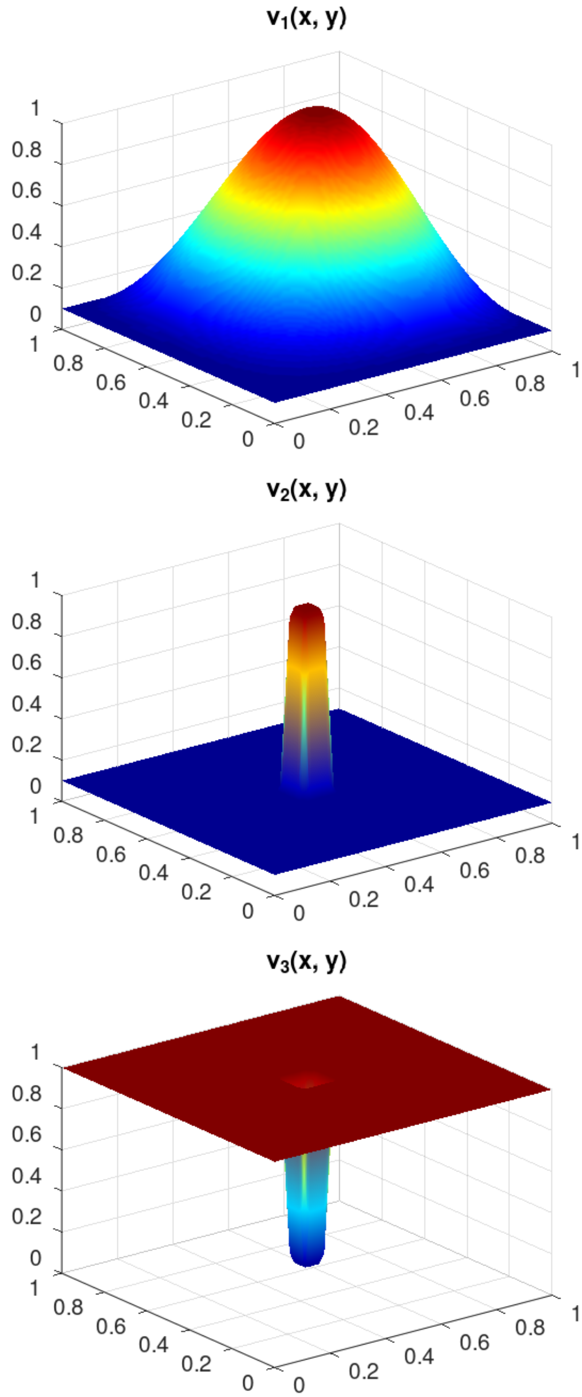

Figure 4.1: The 3D graphs of the functions of viscosity ν_1, ν_2 and ν_3 with $\nu_{min} = 0.1$ and $\nu_{max} = 1$.

The viscosity ν_1 is smoothly varying between the maximum and minimum value of the viscosity. However, ν_2 and ν_3 contain steep layers between ν_{max} and ν_{min}, as shown in Figure 4.1. Furthermore, ν_2 attains most of its values at ν_{min}, and ν_3 attains most of its values at ν_{max}.

We will start our study by performing simulations for one case of viscosity using two different formulations for the weak problem: the deformation tensor and the gradient. The example will be also studied for the above three viscosity functions along with the consideration of two cases for the values of ν_{min} and ν_{max} for each function. The first case is based on fixing the value ν_{min} and varying the value of ν_{max}, and the other case is based on fixing the value ν_{max} and varying the value of ν_{min}.

The simulations were carried out with the Crouzeix-Raviart pair of finite element spaces P_1^{nc}/P_0 on uniform triangular grids depicted in Figure 4.2. The level of refinement ranges between 1 and 8. The degrees of freedom for each level of refinement for computing the velocity and pressure fields are shown in Table 4.1 below.

Table 4.1: The degrees of freedom for the Crouzeix-Raviart finite element pair P_1^{nc}/P_0.

Level	d.o.f. velocity	d.o.f. pressure
1	32	8
2	112	32
3	416	128
4	1600	512
5	6272	2048
6	24832	8192
7	98816	32768
8	394240	131072

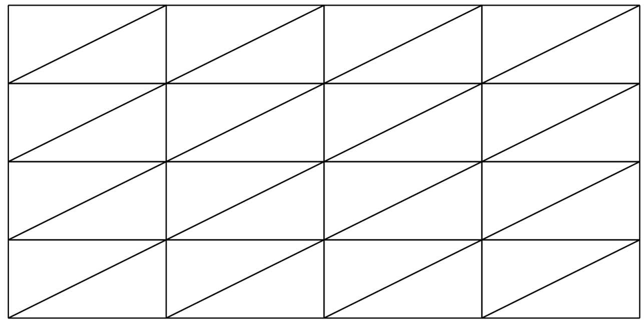

Figure 4.2: Uniform triangular mesh (level 2).

The linear system of equations were solved using the direct solver UMFPACK. All simulations were performed with the code ParMooN [9].

4.1 Comparing the error results between the deformation tensor and gradient formulation for the weak problem

In this section, we will compare between the numerical results for the two different weak formulations for the Crouzeix-Raviart element, see equations (3.7) and (3.21). The simulations were performed for the case with viscosity parameters $\nu_{min} = \nu_{max} = 1$. This indicates that the considered viscosity functions are all constant with value 1. Different levels of grid refinements were considered. The results of different measured errors are presented in Figure 4.3.

For the deformation tensor formulation, we notice that the velocity and the pressure errors don't show any aspect of convergence. In the case of $L^2(\Omega)$-norm of the gradient of the velocity error (equivalently $H^1(\Omega)$-seminorm of the velocity error), the value of the error has an increasing behavior on finer grids.

In contrast, the gradient formulation has very logical and acceptable results. We can observe that the $L^2(\Omega)$-norm of the gradient of the velocity error and of the pressure error have a decreasing behavior of order of convergence 1. Similarly for the $L^2(\Omega)$-norm of the velocity error, we obtain a second order of convergence for the velocity solution.

As a result, the use of the deformation tensor formulation for the Crouzeix-Raviart elements P_1^{nc}/P_0 gives very weird results for the numerical simulations. There is clearly an absence of convergence of the velocity and the pressure errors in the different norms used. As a consequence, the simulations using the gradient formulation will be performed for the rest of the cases of viscosity.

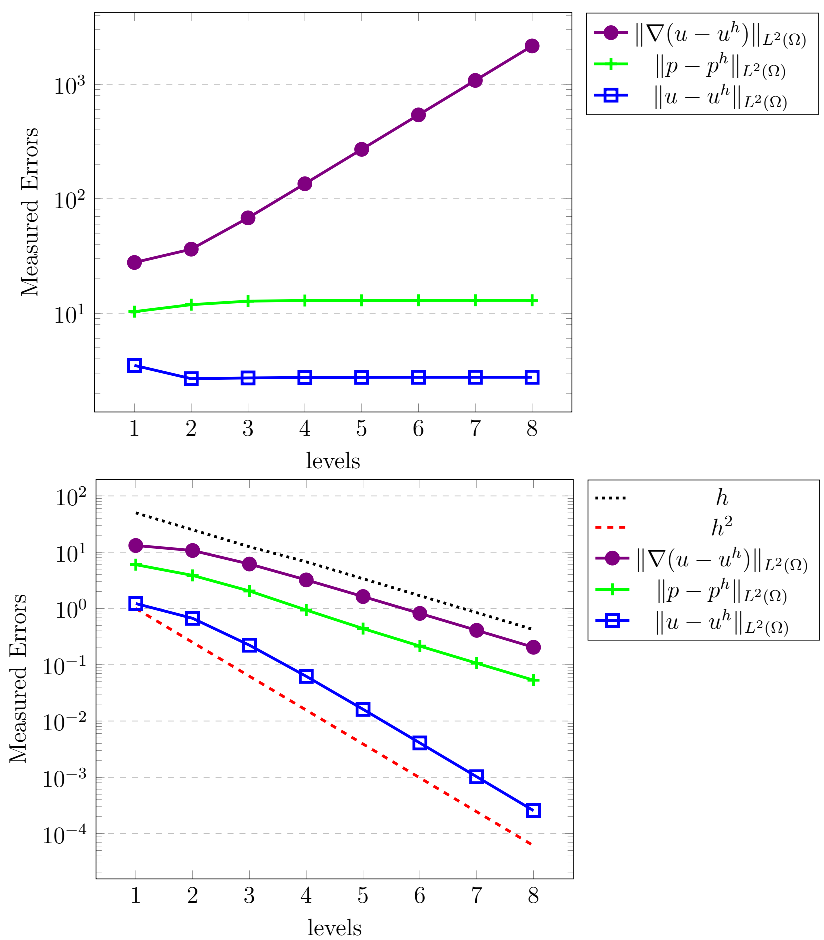

Figure 4.3: Different error plots for constant viscosity $\nu = 1$ for the weak formulations of CR element using the deformation tensor (upper) and the gradient (lower) of the viscous term.

4.2 The error study for the different viscosity functions

This section will investigate the dependency of the error on the values of viscosity for the three given types of viscosity ν_1, ν_2 and ν_3. The $L^2(\Omega)$-norm of the error for the velocity and pressure is calculated for several levels of refinement, mainly from 1 to 8. The simulations are performed using two cases for the viscosity parameters ν_{min} and ν_{max}. In the first case, we fix the value of ν_{max} into 1 and vary the value of ν_{min} in a decreasing manner between 1 and 10^{-6} following a second order of magnitude. In the other case, we set $\nu_{min} = 1$ and vary the values of ν_{max} in an increasing manner between 1 and 10^6 again with an order of magnitude of 2. As we discussed in Section 4.1, setting $\nu_{min} = \nu_{max} = 1$ implies the case of constant viscosity.

4.2.1 The smoothly varying ν_1

The results of the simulations are depicted in the Figures 4.4 and 4.5. For the first case (varying ν_{min}), one can observe that there is a first order of convergence for the $L^2(\Omega)$-norm of the gradient of the velocity and of the pressure. However, it can be noticed that the errors are getting higher values as we decrease the value of ν_{min} for the former norm. It is also not clear if the convergence would be consistent for the values 10^{-4} and 10^{-6} as we refine the mesh. Moreover, there is no observable effect on the pressure error for this case. It was not possible to use finer grids for the considered solver because this was interrupted by memory issues.

For the case of varying ν_{max}, we get the same order of convergence as the previous case for the velocity error. However, there is a significant increase in the error norm of the velocity as the values of ν_{max} become higher, but not as much as in the former case. Regarding the pressure error, we can notice that the value of the error increases by a second order of magnitude between $\nu_{max} = 10^2$ and $\nu_{max} = 10^6$. The advantage reveals in finding out that ν_{max} has notable effect on the pressure for the chosen values of viscosity parameters.

The behavior of the velocity and pressure errors is almost identical for both the conforming (mainly Taylor-Hood pair P_2/P_1) and non-conforming case for varying ν_{min} case of this example, see [1, Figure 3] . We also have the same results for the pressure error for varying ν_{max}. However, this is not true for the velocity errors, as one can notice some impact of the value of ν_{max} for the non-conforming case unlike the conforming case.

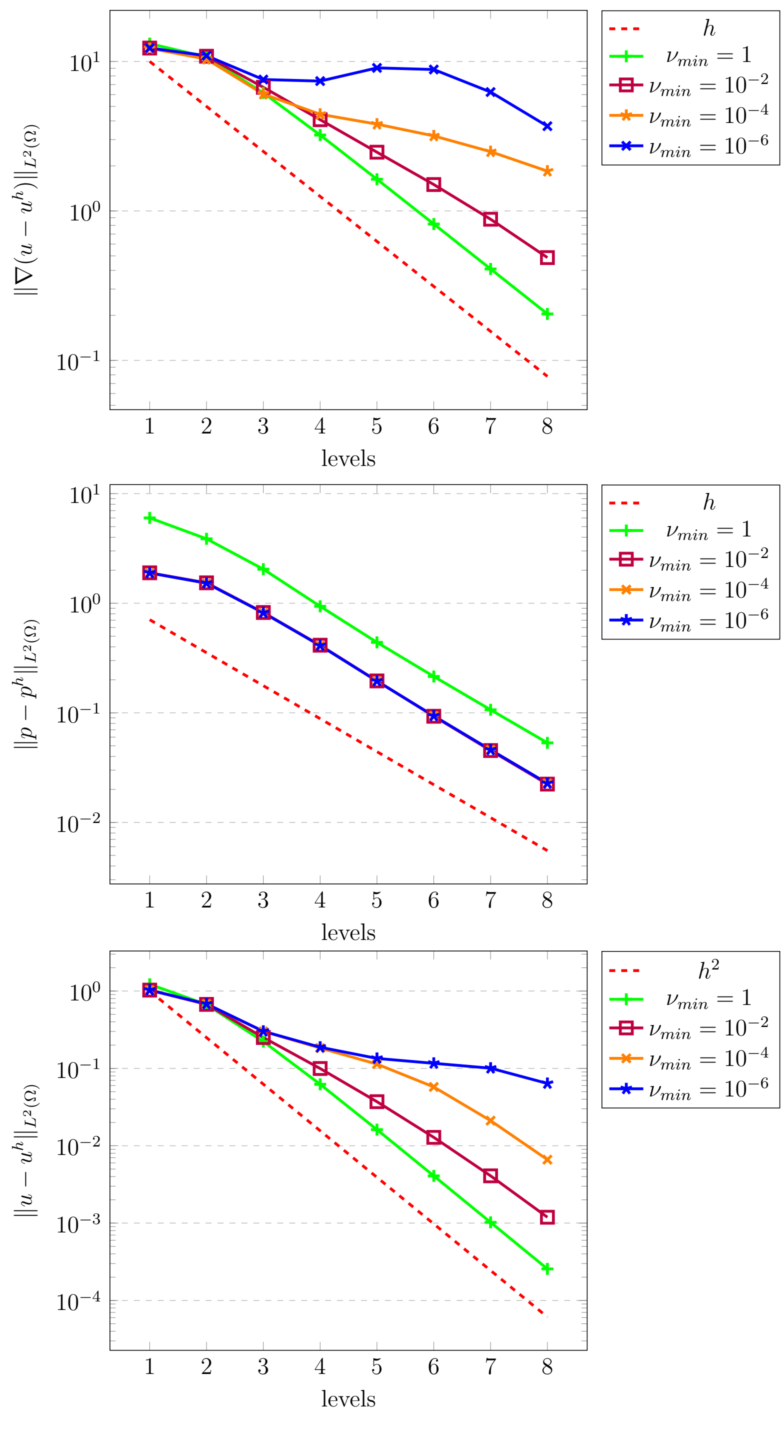

Figure 4.4: Different error plots for smoothly varying viscosity ν_1 with $\nu_{max} = 1$ and different values of ν_{min}.

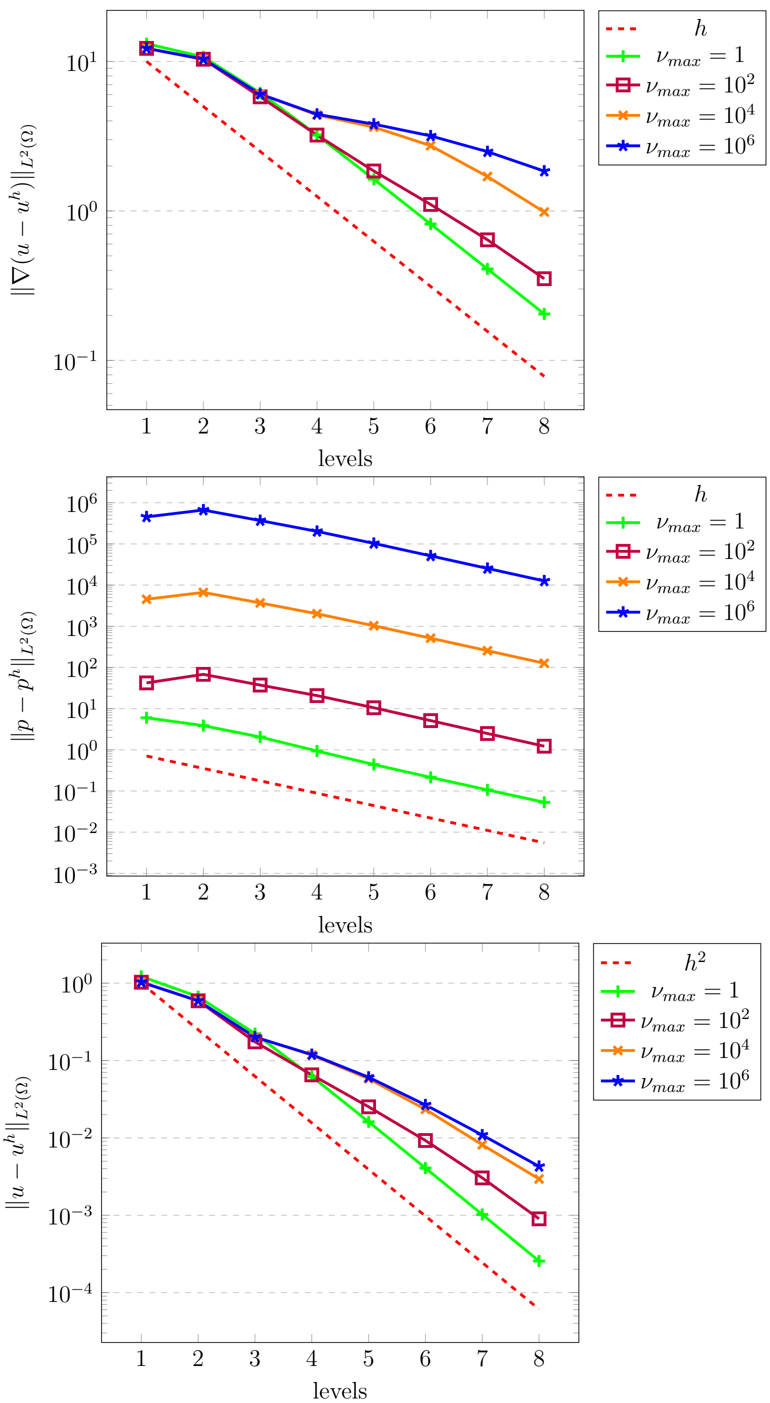

Figure 4.5: Different error plots for smoothly varying viscosity ν_1 with $\nu_{min} = 1$ and different values of ν_{max}.

4.2.2 The viscosity with steep layers ν_2

Figures 4.6 and 4.7 illustrate the obtained error results for the viscosity with steep layers ν_2.

We will first discuss the case of varying ν_{min}. The velocity error norms $\|\nabla(u - u^h)\|_{L^2(\Omega)}$ and $\|u - u^h\|_{L^2(\Omega)}$ tend to increase with varying the values of ν_{min}. However, these errors exhibit consistent convergence across all the tested ν_{min} values, unlike the behavior observed in the previous example. The pressure error shows also not to have any effect in this case.

The velocity errors have totally different behavior for the case of varying ν_{max}. For coarser level of grid refinements (level < 4), the error indicates an increase for $\nu_{max} = 10^4$ and $\nu_{max} = 10^6$. Then it starts to decrease again as the mesh becomes finer demonstrating consistent convergence regardless of the ν_{max} values. The reason behind this aspect is the effect of using the quadrature rule to solve the finite element problem. The pressure error has the same way of behaving (increasing) as in the example of ν_1.

For the conforming case, even though the errors are large on coarser grids, they behave on finer grids independently of ν_{max} and ν_{min}, see [1, Figure 4]. This is not the case for the CR element where the behavior of the errors appears to be independent of the grids as shown in Figure 4.6.

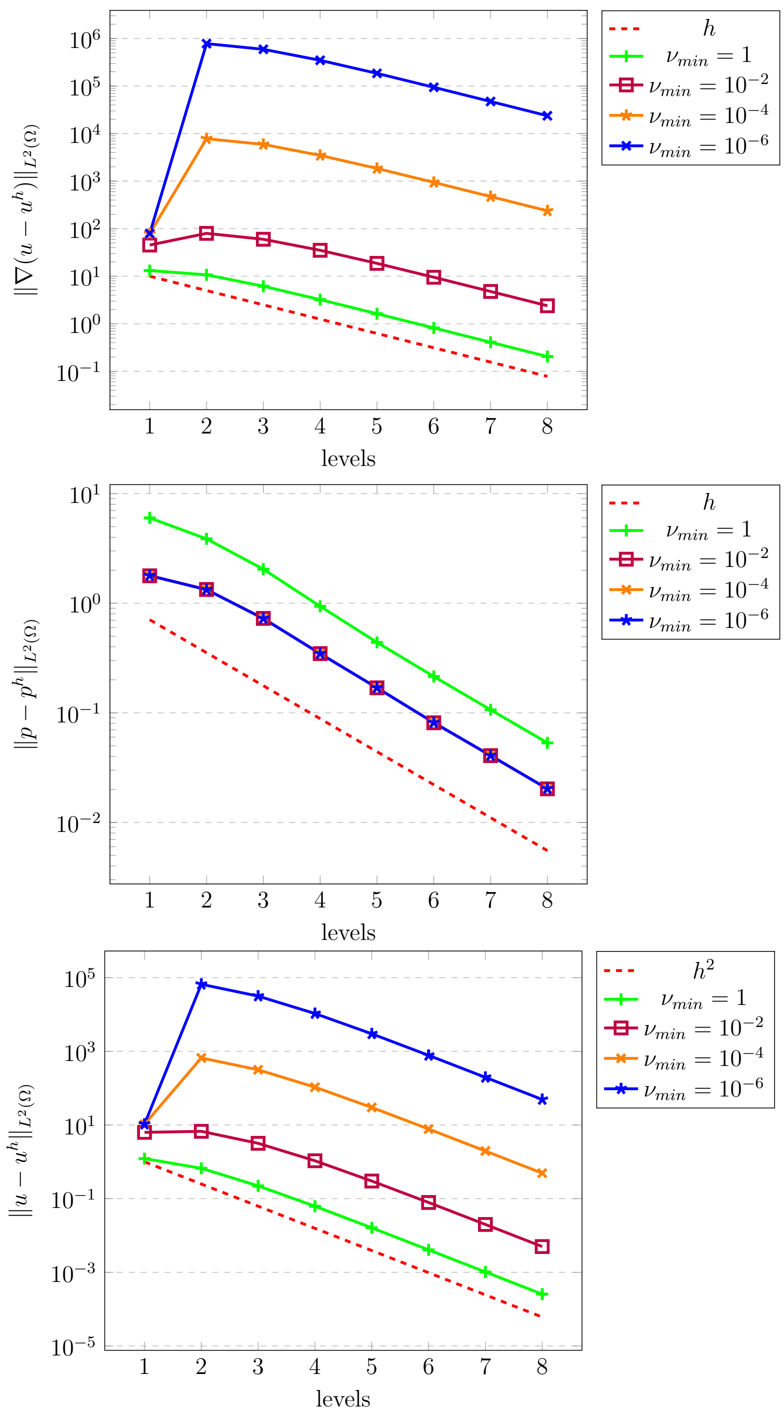

Figure 4.6: Different error plots for viscosity function ν_2 with $\nu_{max}=1$ and different values of ν_{min}.

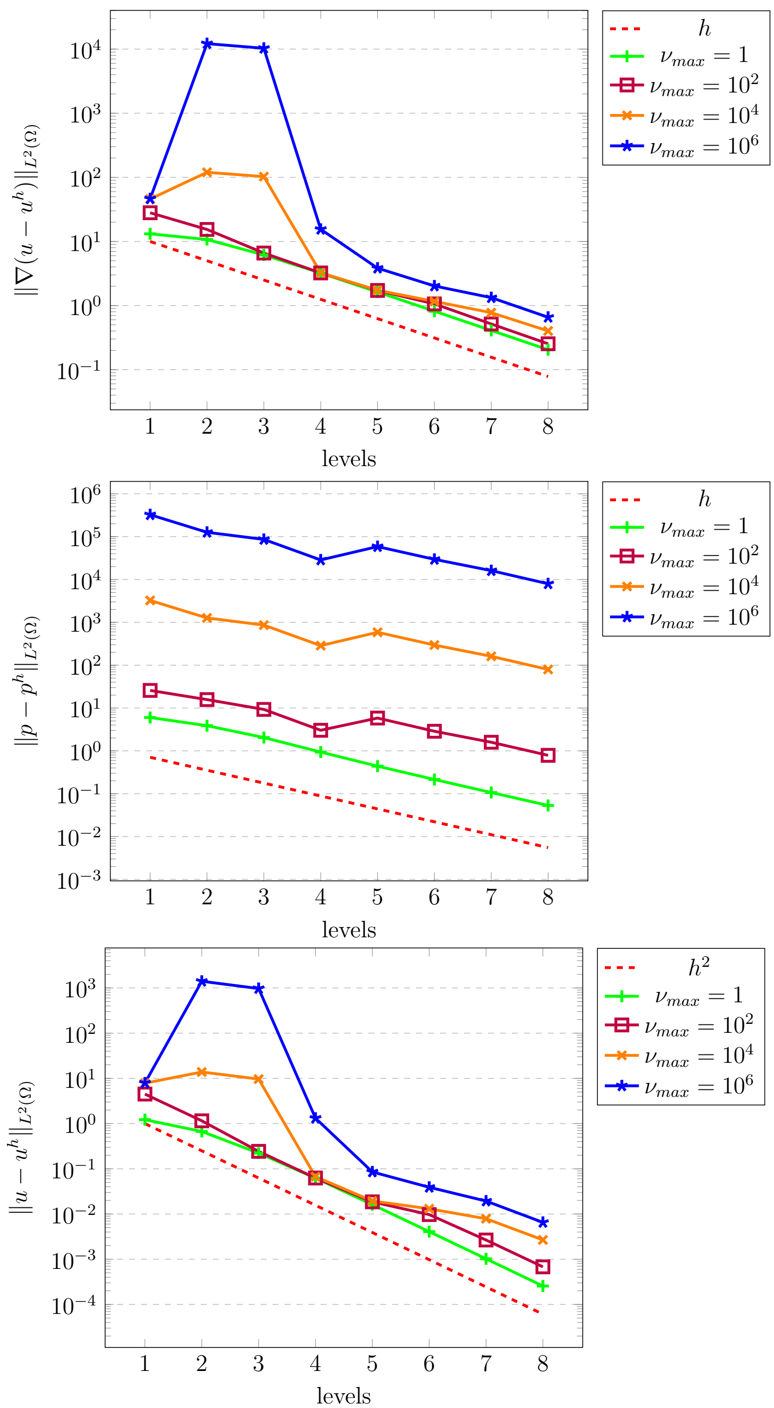

Figure 4.7: Different error plots for viscosity function ν_2 with $\nu_{min} = 1$ and different values of ν_{max}.

4.2.3 The viscosity with steep layers ν_3

The results are shown in Figures 4.8 and 4.9 for this example. When examining the graphs for the varying ν_{min} case, we observe an opposite behavior of the velocity errors compared to ν_2. The errors do not attain any change in their values for the coarser level of grid refinements. At level 5, the errors exhibit a sudden increase (a jump) and then decrease as the grid becomes finer for $\nu_{min} = 10^{-4}$ and $\nu_{min} = 10^{-6}$. However, $\nu_{min} = 10^{-2}$ does not have a significant impact on these errors. The dependency of the pressure error on ν_{max} closely resembles that seen in the cases of ν_1 and ν_2.

On the other hand, the velocity errors are not influenced for large values of ν_{max} for ν_3. The dependency of the pressure error on ν_{max} is almost the same as the examples of ν_1 and ν_2.

For this example, the impact of ν_{min} for fixed ν_{max} on the errors is smaller on finer grids for the conforming finite element and bigger for the non-conforming one (for $\nu_{min} = 10^{-4}$ and $\nu_{min} = 10^{-6}$) , see Figure 4.8 and [1, Figure 5 (left)]. However, we notice nearly the same results for both conforming and non-conforming finite element for fixed ν_{min}, compare Figure 4.9 and [1, Figure 5 (right)].

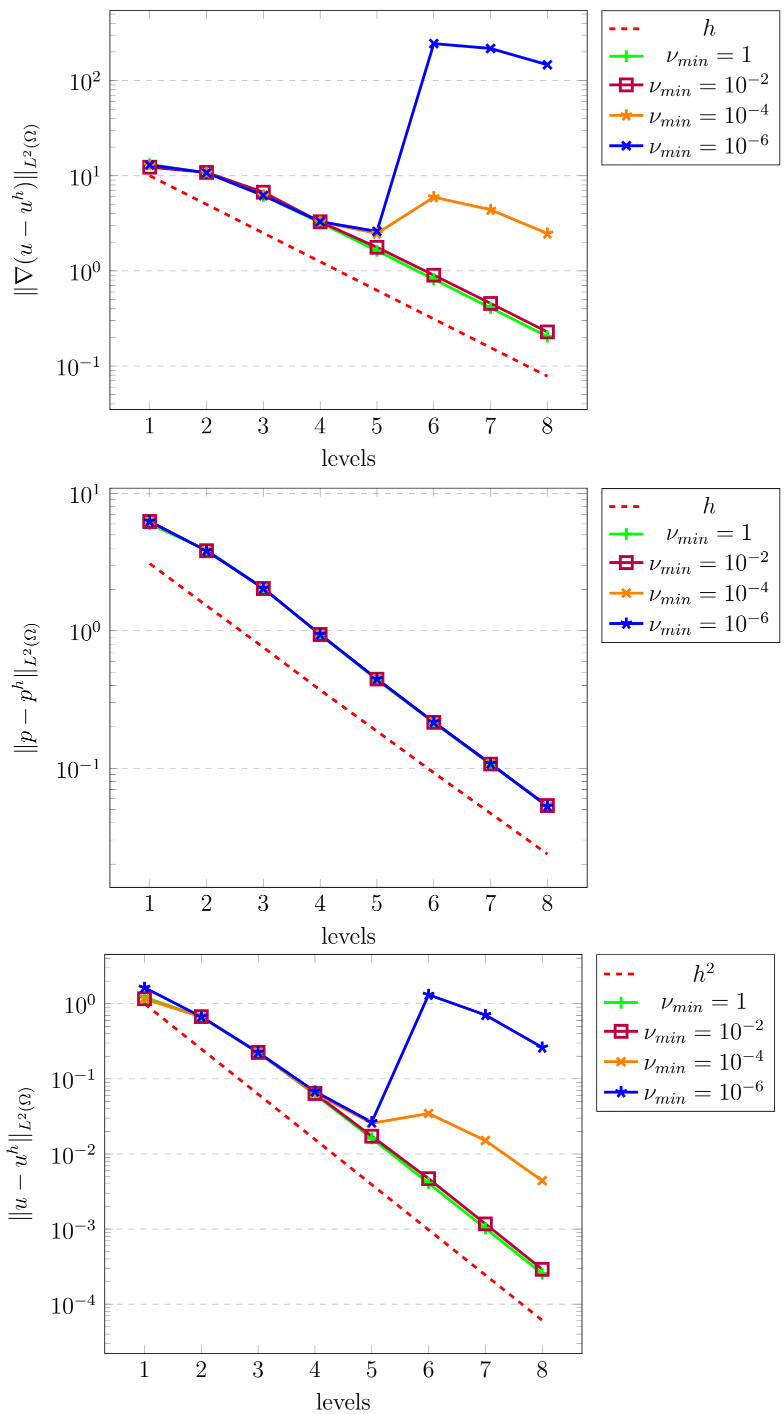

Figure 4.8: Different error plots for viscosity function with steep layer ν_3 with $\nu_{max} = 1$ and different values of ν_{min}.

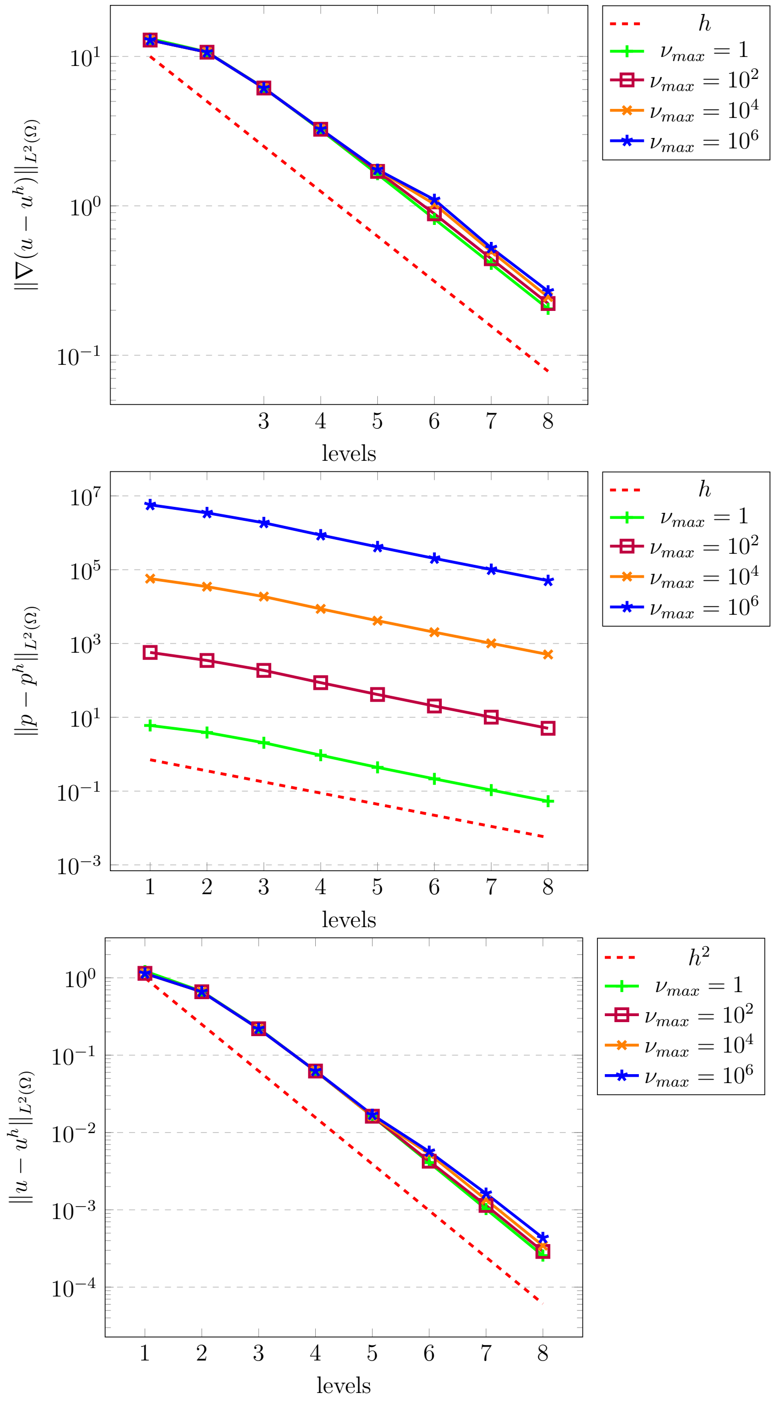

Figure 4.9: Different error plots for viscosity function with steep layer ν_3 with $\nu_{min}=1$ and different values of ν_{max}.

5 Summary and Outlook

This research delves into finite element methods for the Stokes equations with non-constant viscosity, mainly Crouzeix-Raviart finite element P_1^{nc}/P_0. During the error analysis of this element, we encountered challenges in providing a comprehensive consistency error estimate for the Crouzeix-Raviart element and, more importantly, in establishing the necessary error bounds for our numerical investigations. Additionally, we identified certain limitations of this finite element when compared to others in terms of its weak formulation.

The numerical studies show that the gradient formulation should be chosen for further simulations due to shortcoming of the deformation tensor formulation, as detailed in Section 4.1. This observation could be considered a drawback of using the Crouzeix-Raviart finite element for addressing the associated problem. Furthermore, we also notice the dependency of the error bounds on the viscosity parameters in some cases. The behavior of pressure error remains similar for both conforming and non-conforming finite elements, offering insights into how the pressure error estimate might appear. However, for velocity error, the situation is markedly different. In the non-conforming case, the errors exhibit poor behavior with respect to mesh refinement, showing a similar order of magnitude increase across all levels of refinement, including finer grids, as illustrated in Figure 4.5. In contrast, the velocity error curves for the conforming case, representing various values of the viscosity parameter, tend to converge as we refine the mesh, as depicted in [1, Figure 4].

Moving forward, we deduce that the Crouzeix-Raviart finite element method is not the best choice for solving Stokes equations with non-constant viscosity. This result opens paths for further exploration and improvement in the field of finite element methods for solving this type of problems. Addressing the challenges we encountered in consistency error estimation and error bounds estimation for the corresponding finite element will be crucial for enhancing the reliability and accuracy of numerical studies. Additionally, investigating alternative finite elements may help mitigate the observed limitations, especially in cases where non-constant viscosity plays a significant role. Moreover, gaining a deeper understanding of the dependence of error bounds on viscosity parameters will facilitate more informed choices in numerical simulations. Finally, one could consider alternatives for CR finite element:

- Non-conforming finite element methods with higher order (e.g, $k = 2$).
- Conforming finite element methods, which show better results (e.g, Taylor-Hood finite element pair) compared to non-conforming case.

These two suggestions warrant further investigation if they align with the specific requirements and goals of our problem.

A Functional Analysis

Theorem A.1 (Cauchy-Schwarz inequality and Hölder's inequality)**.** *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, with $p, q \in [1, \infty]$ and $1/p + 1/q = 1$. Then it is $fg \in L^1(\Omega)$ and the Hölder's inequality hold*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}. \tag{A.1}$$

For the case where $p = q = 2$, one gets the so-called Cauchy-Schwarz inequality

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \tag{A.2}$$

Theorem A.2 (Korn's inequality)**.** *Let $\mathbb{D}(u)$ be the deformation tensor of u and $p \in (1, \infty)$. Then, there is a constant C_K such that*

$$\|u\|^p_{W^{1,p}(\Omega)} \leq C_k(\|u\|^p_{L^p(\Omega)} + \|\mathbb{D}(u)\|^p_{L^p(\Omega)}), \quad \forall u \in W^{1,p}(\Omega). \tag{A.3}$$

Denote by $|\cdot|$ a seminorm on $L^p(\Omega)$. Then it is

$$\|u\|_{L^p(\Omega)} \leq C_k(|u|_{L^p(\Omega)} + \|\mathbb{D}(u)\|^p_{L^p(\Omega)}), \quad \forall u \in W^{1,p}(\Omega). \tag{A.4}$$

Proof. See [10]. □

B Finite Element Analysis

Lemma B.1 (Estimating the $L^2(\Omega)$ norm of the divergence by the $L^2(\Omega)$ norm of the gradient for functions from $H^1(\Omega)$)**.** *Let* $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, *and let* $v \in H^1(\Omega)$, *then it holds*

$$\|\nabla \cdot v\|_{L^2(\Omega)} \leq \sqrt{d}\|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega). \tag{B.1}$$

This estimate is sharp.

Proof. See [6, Lemma 3.34]. □

Theorem B.2 (Estimate for the $L^2(\Omega)$ projection)**.** *Consider the finite element spaces* $V^h = P_k$ *or* $V^h = Q_k$. *Let* $k \geq 0$ *and* $0 \leq l \leq k$, *then there is a constant* C, *independent of* h *such that*

$$\|v - P^h_{L^2} v\|_{L^2(\Omega)} \leq C h^{l+1} |v|_{H^{l+1}(\Omega)} \quad \forall\, v \in H^{l+1}(\Omega), \forall\, h. \tag{B.2}$$

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