Masterarbeit

Finite Element Methods for the Incompressible Stokes Equations with Non-Constant Viscosity

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1. Introduction

"Kazakh mathematician may have solved $1 million puzzle."

On 22th January 2014, this is the headline of a New Scientist article [1] that reports on Mukhtarbay Otelbayev's claim to have proved the existence and the smoothness of a solution for the Navier-Stokes equations in three dimensions.

Describing numerous physical phenomena like for example the breaking of a wave in the ocean ([2]) as well as the air flow past an airplane wing ([3]) this system of equations plays an fundamental role in modeling and solving problems of academic and economic interest.

Albeit their wide range of applications the existence of a solution for the equations in three dimensions has not been proven yet. In the year 2000, the Clay Mathematics Institute ranked the problem among its seven Millennium Prize problems, promising one million dollars for a correct proof. It is, however, questionable whether the proof by Otelbayev, once translated into English, will withstand public scrutiny.

In most of the applications where the Navier-Stokes equations provide the mathematical model, instead of finding an analytical solution one is satisfied with solving the equations numerically and their solution on computers therefore has a long history dating back several decades.

The methods presented in most of the work in this field consider homogeneous fluids with spatially constant viscosity. The viscosity is one of the key determinants of the flow properties of a fluid and one might ask whether the assumption of it to be constant allows for a realistic simulation in all applications.

The simulation of processes in the Earth's mantle with the aim to understand its dynamics, composition, and interaction with the Earth's crust and core is one exemplary application of fluid mechanics where a model with constant viscosity reaches its limits. This is due to the fact that the viscosity of magma can be expected to depend strongly on the different temperatures occurring in the Earth's mantle ([4], [5]).

In this work, finite element methods for the incompressible Stokes problem with variable viscosity will be examined. The existing theory and methods presented in [6] will be extended to a setting with arbitrary viscosity function $\nu(\mathbf{x}) \in \mathcal{L}^\infty$ and the results will be interpreted with respect to the characteristics of this function.

In Section 2, a derivation of the Navier-Stokes equations can be found before Section 3 presents the functional analysis of an abstract problem with the same characteristic properties as the Navier-Stokes equations.

The subsequent sections discuss the case of a slow fluid where the viscous transport dominates the convection. The corresponding simplified equations, called Stokes equations are introduced in Section 4. Also the main result of the work, the finite element error estimates are presented in this section.

In an attempt to verify the developed theory, an exemplary velocity and a pressure field have been implemented and simulated for two dimensions. The results can be found in Section 5.

The work concludes with a summary of the results and an outlook on promising directions for future research.

2. Derivation of the Navier-Stokes Equations

2.1. Conservation Laws

For deriving the Navier-Stokes equations there are numerous approaches. In the following, the classical way of continuum mechanics will be chosen.
A fluid is described by a velocity field and a pressure field. These quantities generally vary both in space and in time.
The flow variables
- $\rho(t, \mathbf{x})$: density in [kg/m^3],
- $\mathbf{v}(t, \mathbf{x}) = \begin{pmatrix} v_1(t, \mathbf{x}) \\ v_2(t, \mathbf{x}) \\ v_3(t, \mathbf{x}) \end{pmatrix}$: velocity in [m/s],
- $P(t, \mathbf{x})$: pressure in [N/m^2]

are assumed to be sufficently smooth functions in $[0, T] \times \Omega \subset \mathbb{R}^3$.

Remark 2.1. **Conservation of mass.**

$$(\partial_t \rho + \nabla \cdot (\rho \mathbf{v}))(t, \mathbf{x}) = 0 \tag{2.1}$$

For incompressible fluids ρ is constant. Thus, the equation simplyfies to $\nabla \cdot \mathbf{v}(t, \mathbf{x}) = 0$.

Remark 2.2. **Conservation of linear momentum.**
The acceleration can be approximated as follows

$$\frac{d\mathbf{v}}{dt}(t, \mathbf{x}) \approx (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v})(t, \mathbf{x}).$$

For the net force which is the product of mass and acceleration, this gives the approximation

$$\int_V \rho(t, \mathbf{x})(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v})(t, \mathbf{x}) d\mathbf{x} = \underbrace{\int_V \mathbf{F}(t, \mathbf{x}) d\mathbf{x}}_{\text{ext. forces}} + \underbrace{\int_{\partial V} \mathbf{t}(t, \mathbf{s}) d\mathbf{s}}_{\text{int. forces}}.$$

With the help of the divergence theorem and the stress principle of Cauchy $\mathbf{t} = \mathbb{S}\mathbf{n}$, on which fluid mechanics is based, this gives,

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = \nabla \cdot \mathbb{S} + \mathbf{F}, \, \forall t, \mathbf{x}, \tag{2.2}$$

where the divergence of a matrix is defined as $\nabla \cdot \mathbb{S} = \begin{pmatrix} (s_{11})_x + (s_{12})_y + (s_{13})_z \\ (s_{21})_x + (s_{22})_y + (s_{23})_z \\ (s_{31})_x + (s_{32})_y + (s_{33})_z \end{pmatrix}$.

Furthermore, it is $\mathbb{S} = \mathbb{V} - P\mathbb{I}$, where P describes the pressure field and $\mathbb{V}$ is the viscous stress tensor

$$\mathbb{V} = 2\mu \mathbb{D}(\mathbf{v}) + (\zeta - 2\mu/3)(\nabla \cdot \mathbf{v})\mathbb{I}, \tag{2.3}$$

where the velocity deformation tensor

$$\mathbb{D}(\mathbf{v}) = \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^T}{2} = \frac{1}{2}\begin{pmatrix} 2v_{1x} & v_{1y}+v_{2x} & v_{1z}+v_{3x} \\ v_{1y}+v_{2x} & 2v_{2y} & v_{3y}+v_{2z} \\ v_{1z}+v_{3x} & v_{3y}+v_{2z} & 2v_{3z} \end{pmatrix},$$

is the symmetric part of the gradient and ν, ζ are the viscosities of first and second order. This linear equation is of course only an approximation of real fluids which in general are described by non-linear equations. Fluids described by the linear relation are called *Newtonian fluids*.
With (2.3) and the decomposition of $\mathbb{S}$ equation (2.2) becomes:

$$\begin{aligned} \rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - 2\nabla\cdot(\mu\mathbb{D}(\mathbf{v})) \\ -\nabla\cdot((\zeta - 2\mu/3)(\nabla\cdot\mathbf{v})\mathbb{I}) + \nabla P = \mathbf{F}, \qquad (0,T]\times\Omega. \end{aligned} \tag{2.4}$$

Together with the equation for the conservation of mass,

$$(\partial_t \rho + \nabla\cdot(\rho\mathbf{v})) = 0, \qquad (0,T]\times\Omega, \tag{2.5}$$

this system of equations is called Navier-Stokes equations.

For incompressible fluids, i.e., $\rho = \rho_0$, the equations simplify to

$$\begin{aligned} \partial_t \mathbf{v} - 2\nabla\cdot(\nu\mathbb{D}(\mathbf{v})) + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\frac{P}{\rho_0} &= \frac{\mathbf{F}}{\rho_0}, \qquad (0,T]\times\Omega, \\ \nabla\cdot\mathbf{v} &= 0, \qquad (0,T]\times\Omega, \end{aligned} \tag{2.6}$$

where $\nu(\mathbf{x}) = \mu(\mathbf{x})/\rho_0$. The viscous term can be rewritten as follows

$$\begin{aligned} -2\nabla\cdot(\nu\mathbb{D}(\mathbf{v})) &= -2\nabla\nu\cdot\mathbb{D}(\mathbf{v}) - 2\nu\nabla\cdot\mathbb{D}(\mathbf{v}) \\ &= -2\nabla\nu\cdot\mathbb{D}(\mathbf{v}) - \nu\nabla\cdot\nabla\mathbf{v} - \nu\nabla\cdot(\nabla\mathbf{v})^T \\ &= -2\nabla\nu\cdot\mathbb{D}(\mathbf{v}) - \nu\Delta\mathbf{v}. \end{aligned} \tag{2.7}$$

Note that

$$\nabla\cdot\nabla\mathbf{v} = \nabla\cdot\begin{pmatrix} v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \\ v_{3x} & v_{3y} & v_{3z} \end{pmatrix} = \begin{pmatrix} v_{1xx}+v_{1yy}+v_{1zz} \\ v_{2xx}+v_{2yy}+v_{2zz} \\ v_{3xx}+v_{3yy}+v_{3zz} \end{pmatrix} = \Delta\mathbf{v}$$

and

$$\nabla\cdot(\nabla\mathbf{v})^T = \nabla\cdot\begin{pmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{pmatrix} = \begin{pmatrix} v_{1xx}+v_{2xy}+v_{3xz} \\ v_{1yx}+v_{2yy}+v_{3yz} \\ v_{1zx}+v_{2zy}+v_{3zz} \end{pmatrix} = \begin{pmatrix} (\nabla\cdot\mathbf{v})_x \\ (\nabla\cdot\mathbf{v})_y \\ (\nabla\cdot\mathbf{v})_z \end{pmatrix} = 0.$$

Remark 2.3. **Special case: ν is constant.**
As already mentioned, in the literature the Navier-Stokes equations are usually considered for a constant viscosity ν. There, it is

$$-2\nabla\cdot(\nu\mathbb{D}(\mathbf{v})) = -\nu\Delta\mathbf{v}.$$

Consequently, in this case one gets rid of the deformation tensor which appears in the formulation of the Navier-Stokes equations for non-constant ν and will also enter the analysis and the numerics considered in this work.

2.2. Dimensionless Equations

For the analysis and the numerical simulation of the Navier-Stokes equation it is covenient to consider dimensionless equations.
After introducing the characteristic scales

- L [m], characteristic length,
- U [m/s] characteristic velocity,
- T^* [s] characteristic timescale,
- N^* [Ns/m^2] characteristic viscosity

transforming $\tilde{\mathbf{x}} = \mathbf{x}/L$, $\tilde{\mathbf{v}} = \mathbf{v}/U$, $\tilde{t} = t/T^*$, $\tilde{\nu} = \nu/N^*$ in (2.6) gives

$$\begin{aligned}
\frac{\mathbf{F}}{\rho_0} &= \partial_t \mathbf{v} - 2\nabla \cdot (\nu \mathbb{D}(\mathbf{v})) + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla \frac{P}{\rho_0} \\
\iff \frac{\mathbf{F}}{\rho_0} &= \frac{1}{T^*}\partial_{\tilde{t}}(U\tilde{\mathbf{v}}) - 2\frac{1}{L}\nabla_{\tilde{\mathbf{x}}} \cdot \left(N^*\tilde{\nu}\frac{1}{L}\mathbb{D}_{\tilde{\mathbf{x}}}(U\tilde{\mathbf{v}})\right) + \left(U\tilde{\mathbf{v}} \cdot \frac{1}{L}\nabla_{\tilde{\mathbf{x}}}\right) U\tilde{\mathbf{v}} + \frac{1}{L}\nabla_{\tilde{\mathbf{x}}}\frac{P}{\rho_0} \\
\iff \mathbf{f} &= St\,\partial_{\tilde{t}}\tilde{\mathbf{v}} - \frac{2}{Re}\nabla_{\tilde{\mathbf{x}}} \cdot (\tilde{\nu}\mathbb{D}_{\tilde{\mathbf{x}}}(\tilde{\mathbf{v}})) + (\tilde{\mathbf{v}} \cdot \nabla_{\tilde{\mathbf{x}}})\,\tilde{\mathbf{v}} + \nabla_{\tilde{\mathbf{x}}} p, \qquad (2.8)
\end{aligned}$$

where Strouhal- and Reynoldsnumber are definded as

$$St = \frac{L}{T^* U}, \quad Re = \frac{LU}{N^*}$$

and we indroduce the scaled pressure and right-hand side

$$\mathbf{f} = \frac{L}{U^2\rho_0}\mathbf{F}, \quad p = \frac{1}{U^2\rho_0}P.$$

The divergence-free condition for $\mathbf{v}$ does not change

$$\begin{aligned}
& 0 = \nabla \cdot \mathbf{v}, \\
\iff\ & 0 = \frac{1}{L}\nabla \cdot U\tilde{\mathbf{v}} \\
\iff\ & 0 = \nabla \cdot \tilde{\mathbf{v}}. \qquad (2.9)
\end{aligned}$$

For simplicity of notation the variables are renamed again. With $\tilde{\mathbf{x}} = \mathbf{x}$, $\tilde{t} = t, \tilde{\nu} = \nu$, $\tilde{\mathbf{v}} = \mathbf{u}$ one gets the dimensionless Navier-Stokes equations

$$\begin{aligned}
St\,\partial_t \mathbf{u} - \frac{2}{Re}\nabla \cdot (\nu\mathbb{D}(\mathbf{u})) + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p &= \mathbf{f}, && (0,T]\times\Omega, \qquad (2.10) \\
\nabla \cdot \mathbf{u} &= 0, && (0,T]\times\Omega.
\end{aligned}$$

In order to simplify further the equations one chooses the characteristic time to be $T^* = L/U = 1\,\mathrm{s}$ and $UL/N^* = 1$ and gets

$$\begin{aligned}
\partial_t \mathbf{u} - \underbrace{2\nabla \cdot (\nu\mathbb{D}(\mathbf{u}))}_{\text{viscous term}} + \underbrace{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}_{\text{convection}} + \nabla p &= \mathbf{f}, && (0,T]\times\Omega, \qquad (2.11) \\
\nabla \cdot \mathbf{u} &= 0, && (0,T]\times\Omega. \qquad (2.12)
\end{aligned}$$

Remark 2.4. **Reformulating the equations** (2.10) **and** (2.11)**.**
Making use of the manipulation in (2.7) the first equation in (2.10) becomes

$$St\,\partial_t \mathbf{u} - \frac{1}{Re}\left(2\nabla\nu \cdot \mathbb{D}(\mathbf{u}) + \nu\Delta\mathbf{u}\right) + \left(\mathbf{u}\cdot\nabla\right)\mathbf{u} + \nabla p = \mathbf{f}, \qquad (0,T]\times\Omega$$

and in (2.11) one finds

$$\partial_t \mathbf{u} - 2\nabla\nu \cdot \mathbb{D}(\mathbf{u}) + \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad (0,T]\times\Omega.$$

Remark 2.5. **Intricacy of the Navier-Stokes equations.**
From looking at the different versions of the derived Navier-Stokes equations one can assume that the simulation as well as the analysis are rather intricate because of

1. the coupling of $\mathbf{u}$ and p,
2. the nonlinearity of the convective term,
3. the convective term is dominating the viscous term if ν is small.

3. Functional Analysis for Linear Saddle Point Problems

For the purpose of tackling the first of the aforementioned problems (see Remark 2.5), namely the coupling of velocity and pressure, one can interpret the continuity equation (2.12) as a constraint for the velocity $\mathbf{u}$ in the momentum equation (2.11) and the pressure p as a Lagrange multiplier.
Instead of the non-linear Navier-Stokes equations an abstract linear saddle point problem is studied that is characterized by the same coupling behavior.
Note that the solution of a linear saddle point problem is of utmost interest also for the nonlinear Navier-Stokes equations. After discretizing those with respect to time one has to solve a nonlinear saddle point problem for each time step. This is done iteratively by solving a linear saddle point problem for the velocity and the pressure in each iteration step.
This section therefore deals with

1. a necessary and sufficient condition for the unique solvability of an abstract saddle point problem,
2. appropriate function spaces for the continuous, incompressible flow problem and
3. choosing appropriate finite element spaces.

3.1. Abstract Linear Saddle Point Problem

In order to formulate an abstract linear saddle point problem, one needs to define appropriate spaces together with their dual spaces, bilinear forms and the corresponding linear operators. This framework will now be introduced.
Let V, Q be real Hilbert spaces and V', Q' the corresponding dual spaces with dual pairing $\langle \cdot, \cdot \rangle_{V',V}$ and norm

$$\|\phi\|_{V'} := \sup_{v \in V, v \neq 0} \frac{\langle \phi, v \rangle_{V',V}}{\|v\|_V}.$$

One defines continuous bilinear forms: $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ and $b(\cdot, \cdot) : V \times Q \to \mathbb{R}$ with norms

$$\|a\| = \sup_{v,w \in V, v,w \neq 0} \frac{a(v,w)}{\|v\|_V \|w\|_V} \quad \text{and} \quad \|b\| = \sup_{v \in V, q \in Q, v,q \neq 0} \frac{b(v,q)}{\|v\|_V \|q\|_Q}.$$

Now, the linear saddle point problem can be formulated.
It consists in finding $(u,p) \in V \times Q$ for $(f,r) \in V' \times Q'$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v \rangle_{V',V}, \ \forall v \in V, \\ b(u,q) &= \langle r, q \rangle_{Q',Q}, \ \forall q \in Q. \end{aligned} \tag{3.1}$$

There is another way of formulating the saddle point problem using linear operators instead of bilinear forms.
Let $A \in \mathcal{L}(V, V')$ and $B \in \mathcal{L}(V, Q')$ be defined by

$$\langle Au, v \rangle_{V',V} = a(u,v), \ \forall u,v \in V \quad \text{resp.} \quad \langle Bu, q \rangle_{Q',Q} = b(u,q), \ \forall u \in V, \forall q \in Q.$$

For the norms one finds

$$\|A\|_{\mathcal{L}(V,V')} = \sup_{v\in V, v\neq 0} \frac{\|Av\|_{V'}}{\|v\|_{V'}} = \sup_{v,w\in V, v,w\neq 0} \frac{|a(v,w)|}{\|v\|_{V'}\|w\|_{V'}} = \|a\|$$

and similarily, $\|B\|_{\mathcal{L}(V,Q')} = \|b\|$.
Let $B' \in \mathcal{L}(Q, V')$ be the dual operator of B defined by

$$\langle B'q, v\rangle_{V',V} = \langle Bv, q\rangle_{Q',Q} = b(v,q), \quad \forall v \in V, \forall q \in Q.$$

One can now rewrite (3.1) in operator form:

$$\begin{aligned} Au + B'p &= f \in V', \\ Bu &= r \in Q'. \end{aligned} \tag{3.2}$$

The problem (3.2) is well-posed if $\Phi \in \mathcal{L}(V\times Q, V'\times Q')$ with $\Phi(v,q) = (Av + B'p, Bv)$ is an isomorphism. Then a unique solution of (3.2) exists.
Define $V(r) = \{v \in V : Bv = r\}$, $\quad V_0 = V(0) = \ker(B)$. Note that V_0 is a closed subspace of V.
The space of functionals that vanish on V_0 will also be of importance. One therefore defines

$$\tilde{V}' = \{\phi \in \mathbb{V}' : \langle \phi, v\rangle_{V',V} = 0, \forall v \in V_0\} \subset V'.$$

The problem (3.1) resp. (3.2) can be associated with a linear problem in the subspace V_0:
Find $u \in V(r)$ such that

$$a(u,v) = \langle f, v\rangle_{V',V}, \quad \forall v \in V_0. \tag{3.3}$$

A solution of (3.1) resp. (3.2) is obviously also a solution of (3.3). The goal consists now in finding a condition such that also the reverse is true, i.e., given a solution $u \in V(r)$ of (3.3) one can find $p \in Q$ uniquely, such that (u,p) is the unique solution of (3.1) resp. (3.2).

Lemma 3.1. **The inf-sup condition.**
The following three properties are equivalent

(i) $\exists \beta_{\text{is}} > 0$ such that

$$\inf_{q\in Q} \sup_{v\in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta_{\text{is}}. \tag{3.4}$$

(ii) $B' : Q \to \tilde{V}'$ is an isomorphism and

$$\|B'q\|_{V'} \geq \beta_{\text{is}}\|q\|_Q, \forall q \in Q.$$

(iii) $B : V_0^{\perp} \to Q'$ is an isomorphism and

$$\|Bv\|_{Q'} \geq \beta_{\text{is}}\|v\|_V \quad \forall v \in V_0^{\perp}.$$

Proof. A proof can be found in [6] on pp. 26-28. □

Remark 3.2. **V(r) is not empty.**
The inf-sup condition implies $V(r) \neq \emptyset$ since according to Lemma 3.1 iii) for $r \in Q'$ there is a $v \in V_0^\perp$ with $Bv = r$.

In order to state the theorem on existence and uniqueness of a solution to the saddle point problem, one defines the embedding operator $P_0 \in \mathcal{L}(V', V_0')$ as

$$\langle P_0 \phi, v\rangle_{V',V} = \langle \phi, v\rangle_{V',V}, \ \forall \phi \in V', \forall v \in V_0,$$

and it is $\|P_0\phi\|_{V_0'} = \|\phi\|_{V'}$.

Theorem 3.3. **Existence and uniqueness of a solution of** (3.2)**.**
The saddle point problem (3.2) has a unique solution if and only if
(i) $P_0 \circ A : V_0 \to V_0'$ is an isomorphism.
(ii) $b(\cdot,\cdot)$ satisfies the inf-sup condition (3.4).

Proof. The proof will be omitted here and can be found in [6] on pp. 30-32. □

Lemma 3.4. **Sufficient condition on** $a(\cdot,\cdot)$ **for existence of a unique solution of** (3.2)**.**
Assume $\exists\, \alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|_V^2, \ \forall v \in V_0, \tag{3.5}$$

($a(\cdot,\cdot)$ is V_0-elliptic or coercive) then there is a unique solution for (3.2) if and only if $b(\cdot,\cdot)$ satisfies the inf-sup condition (3.4).

Proof. A proof can be found in [6] on pp. 32. □

3.2. Appropriate Function Spaces for Continuous Incompressible Flow Problems with Variable Viscosity

The abstract theory developed in Section 3.1 will now be applied to incompressible flow problems. First of all, a velocity and a pressure space are defined.

Remark 3.5. **The spaces** $H_0^1(\Omega)$ **and** $L_0^2(\Omega)$**.**
For a connected and bounded domain $\Omega \subset \mathbb{R}^d$ the velocity space is defined as

$$V = H_0^1(\Omega) = \{\mathbf{v} : \mathbf{v} \in H^1(\Omega) \text{ with } \mathbf{v}\,|_\Gamma = \mathbf{0}\}.$$

Only problems with homogeneous Dirichlet boundary conditions, which enter the definition of V as essential boundary conditions, are considered in the analysis. Note that the restriction of $\mathbf{v}$ on the boundary has to be understood in the sense of traces.
The pressure space is defined as

$$Q = L_0^2(\Omega) = \{q : q \in L^2(\Omega) \text{ with } \int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0\}.$$

Since the boundary conditions are chosen to be of Dirichlet type the pressure is unique only up to an additive constant and thus, one needs a further condition in order to determine this constant. Here, one requires that the mean integral value of the pressure vanishes.
The Hilbert space Q is equipped with the standard inner product

$$(q, r) = \int_\Omega (qr)(\mathbf{x})\, d\mathbf{x}, \tag{3.6}$$

and induced norm $\|q\|_Q = \|q\|_{L^2(\Omega)}$.
In the following, we will use the notation $\|\cdot\|_0 := \|\cdot\|_{L^2(\Omega)}$.
For the velocity space V we will distinguish two choices of inner products. First, in order to develop a ν-independent theory, the velocity space V will be equipped with the inner product

$$(\nabla\mathbf{v}, \nabla\mathbf{w}) = \int_\Omega (\nabla\mathbf{v} \cdot \nabla\mathbf{w})(\mathbf{x})\, d\mathbf{x}, \tag{3.7}$$

which defines the H_0^1-seminorm $|\mathbf{v}|_{H_0^1(\Omega)} = \|\nabla\mathbf{v}\|_0$.
The H_0^1-seminorm will be denoted by $|\cdot|_1 := |\cdot|_{H_0^1(\Omega)}$ in the following.
Later, we will consider the Hilbert space V together with the ν-dependent inner product[1]

$$(\nu\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{w})) = \int_\Omega (\nu\mathbb{D}(\mathbf{v}) \cdot \mathbb{D}(\mathbf{w}))(\mathbf{x})\, d\mathbf{x}, \tag{3.8}$$

where

$$\mathbb{D}(\mathbf{v}) = \frac{\nabla\mathbf{v} + (\nabla\mathbf{v})^T}{2}$$

is the deformation tensor (i.e., the symmetric part of the gradient) and induced norm $\|\mathbf{v}\|_\nu = \|\nu^{1/2}\mathbb{D}(\mathbf{v})\|_0$.
One can show with the Poincaré inequality that the choices (3.7) and (3.8) define indeed norms $|\cdot|_1$ resp. $\|\cdot\|_\nu$ where one assumes the viscosity ν to be positive in the latter case. Furthermore, those norms are equivalent.

Lemma 3.6. Equivalence of the ν-norm and the $H_0^1(\Omega)$-seminorm.
Let $\nu(\mathbf{x}) > 0 \in L^\infty$, $\quad \nu(\mathbf{x}) \geq \nu_{\min} > 0$ a.e. in Ω and $\nu_{\max} = \|\nu\|_{L^\infty(\Omega)}$.
The norms $\|\mathbf{v}\|_\nu = \|\nu^{1/2}\mathbb{D}(\mathbf{v})\|_0$ and $|\mathbf{v}|_1 = \|\nabla\mathbf{v}\|_0$ are equivalent, i.e.,

$$\nu_{\max}^{-1/2}\|\mathbf{v}\|_\nu \leq |\mathbf{v}|_1 \leq C_K\nu_{\min}^{-1/2}\|\mathbf{v}\|_\nu, \tag{3.9}$$

$$C_K^{-1}\nu_{\min}^{1/2}\,|\mathbf{v}|_1 \leq \|\mathbf{v}\|_\nu \leq \nu_{\max}^{1/2}\,|\mathbf{v}|_1\,, \tag{3.10}$$

where C_K is the constant from the Korn inequality, Theorem A.1 in the appendix.

Proof. In order to prove the estimates one has to find estimates for the norm of the deformation tensor $\mathbb{D}(\mathbf{v})$. Obviously, one can estimate the norm of the deformation tensor by the norm of the gradient, i.e.,

$$\|\mathbb{D}(\mathbf{v})\| \leq \frac{1}{2}\left(\|\nabla\mathbf{v}\| + \|(\nabla\mathbf{v})^T\|\right) = \|\nabla\mathbf{v}\|. \tag{3.11}$$

[1] A motivation for this choice can be found at the beginning of Section 4.1 where the weak form of the Stokes problem suggests a definition of the bilinear form $a(\cdot,\cdot)$ matching that inner product.

This is true for any norm $\|\cdot\|$, in particular for the choice $\|\cdot\| = \|\cdot\|_0$.
For functions in $H_0^1(\Omega)$, Korn's inequality (A.2) says

$$\|\nabla \mathbf{v}\|_0 \leq C_K \|\mathbb{D}(\mathbf{u})\|_0, \tag{3.12}$$

since the seminorm mentioned in the Theorem is defined here as the trace of the function $\mathbf{v}$ on the boundary which vanishes for $\mathbf{v} \in H_0^1(\Omega)$.
Now, both estimates can be derived from the following chain of inequalities:

$$\begin{aligned}
\|\mathbf{v}\|_\nu &= \|\nu^{1/2}\mathbb{D}(\mathbf{v})\|_0 \leq \nu_{\max}^{1/2}\|\mathbb{D}(\mathbf{v})\|_0 \\
&\leq \nu_{\max}^{1/2} \underbrace{\|\nabla \mathbf{v}\|_0}_{|\mathbf{v}|_1} \overset{(3.12)}{\leq} C_K \nu_{\max}^{1/2}\|\mathbb{D}(\mathbf{v})\|_0 \\
&\leq C_K \nu_{\max}^{1/2}\nu_{\min}^{-1/2}\|\nu^{1/2}\mathbb{D}(\mathbf{v})\|_0 = C_K \nu_{\max}^{1/2}\nu_{\min}^{-1/2}\|\mathbf{v}\|_\nu,
\end{aligned}$$

where one has to divide by $\nu_{\max}^{1/2}$ in order to get (3.9) and by $C_K \nu_{\max}^{1/2}\nu_{\min}^{-1/2}$ for the first inequality in (3.10). □

Lemma 3.7. **Estimating the L^2-norm of the divergence by the L^2-norm of the gradient for H^1-functions.**
Let $\Omega \subset \mathbb{R}^d$ and $\mathbf{v} \in H^1(\Omega)$, then it is

$$\|\nabla \cdot \mathbf{v}\|_0 \leq \sqrt{d}\|\nabla \mathbf{v}\|_0, \quad \forall \mathbf{v} \in H^1(\Omega) \tag{3.13}$$

and this estimate is sharp.

Remark 3.8. **An estimate for $H_0^1(\Omega)$-functions.**
One can even show that

$$\|\nabla \cdot \mathbf{v}\|_0 \leq \|\nabla \mathbf{v}\|_0, \quad \forall \mathbf{v} \in H_0^1(\Omega). \tag{3.14}$$

The proof in [6], p. 105 ff., is based on the fact that for functions from $H_0^1(\Omega)$ it holds

$$\|\nabla \cdot \mathbf{v}\|_0^2 = \|\nabla \mathbf{v}\|_0^2 + \|\nabla \times \mathbf{v}\|_0^2, \tag{3.15}$$

where $\nabla \times \mathbf{v}$ denotes the rotation resp. the curl of the vector field $\mathbf{v}$.
To see that the estimate (3.14) is sharp one needs to find a function in $H_0^1(\Omega)$ which is irrotational but not divergence-free.
For $\Omega = [0, 2\pi]^2$ the vector field

$$\mathbf{v}(x,y) = \begin{pmatrix} \sin(x)(\cos(y) - 1) \\ \sin(y)(\cos(x) - 1) \end{pmatrix}$$

is such a function since

$$\nabla \times \mathbf{v} = \partial_x \mathbf{v}_2 - \partial_y \mathbf{v}_1 = -\sin(y)\sin(x) + \sin(x)\sin(y) = 0,$$

and

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \partial_x \mathbf{v}_1 + \partial_y \mathbf{v}_2 = \cos(x)(\cos(y)-1) + \cos(y)(\cos(x)-1) \\ &= 2\cos(x)\cos(y) - \cos(x) - \cos(y) \neq 0.\end{aligned}$$

Thus, the estimate (3.14) is sharp.

Lemma 3.9. Estimating the weighted L^2-norm of the divergence by the ν-norm for H^1-functions.
Let $\Omega \subset \mathbb{R}^d$ and $\mathbf{v} \in H^1(\Omega)$, then it is

$$\|\nu^{1/2}\, \nabla \cdot \mathbf{v}\|_0 \leq \sqrt{d} \|\nu^{1/2}\, \mathbb{D}(\mathbf{v})\|_0. \tag{3.16}$$

Proof. It is

$$\begin{aligned}\|\nu^{1/2}\, \nabla \cdot \mathbf{v}\|_0^2 &= \int_\Omega \left(\nu^{1/2}(\mathbf{x}) \left(\sum_{i=1}^d \mathbf{v}_{ix_i}(\mathbf{x})\right)\right)^2 d\mathbf{x} \\ &\leq \int_\Omega \nu(\mathbf{x}) \left(\sum_{i=1}^d 1^2\right)\left(\sum_{i=1}^d \mathbf{v}_{ix_i}^2(\mathbf{x})\right) d\mathbf{x} \\ &\leq d \int_\Omega \nu(\mathbf{x}) \left(\sum_{i,j=1}^d \left(\frac{\mathbf{v}_{ix_j}(\mathbf{x}) + \mathbf{v}_{jx_i}(\mathbf{x})}{2}\right)^2\right) d\mathbf{x} \\ &= d \int_\Omega \sum_{i,j=1}^d \left(\nu(\mathbf{x})^{1/2} \left(\frac{\mathbf{v}_{ix_j}(\mathbf{x}) + \mathbf{v}_{jx_i}(\mathbf{x})}{2}\right)\right)^2 d\mathbf{x} \\ &= d\|\nu^{1/2}\, \mathbb{D}(\mathbf{v})\|_0^2.\end{aligned}$$

□

Now that those spaces are determined, one can define the bilinear form $b(\cdot,\cdot)$ that couples velocity and pressure in the inf-sup condition (3.4). By looking at the weak formulation of the Navier-Stokes equations this definition comes naturally. For the continuity equation in (2.11) one gets

$$\int_\Omega (\nabla \cdot \mathbf{u})\, q \, d\mathbf{x} = (\nabla \cdot \mathbf{u}, q) = 0, \ \forall q \in Q$$

and for the pressure term in the momentum equation one applies integration by parts which yields

$$\int_\Omega \nabla p \cdot \mathbf{v} \, d\mathbf{x} = -\int_\Omega (\nabla \cdot \mathbf{v})\, p \, d\mathbf{x} = -(\nabla \cdot \mathbf{v}, p), \quad \forall \mathbf{v} \in V.$$

One therefore defines the bilinear form to be

$$b(\mathbf{v}, q) = -\int_\Omega (\nabla \cdot \mathbf{v})\, q \, d\mathbf{x} = -(\nabla \cdot \mathbf{v}, q), \ \forall \mathbf{v} \in V, q \in Q. \tag{3.17}$$

Note that with $\mathbf{v} \in V$ it is, independent of the norm of V,

$$\|\nabla \mathbf{v}\|_0 < \infty.$$

This is clear if V is equipped with $|\cdot|_1$ by definition of the $H_0^1(\Omega)$-seminorm and for the ν-norm one has, making use of the norm equivalence (3.9),

$$\|\nabla \mathbf{v}\|_0 \leq C_K \nu_{\min}^{-1/2} \|\mathbf{v}\|_\nu < \infty.$$

Therefore, it is $\nabla \mathbf{v} \in L^2(\Omega)$ which, according to Lemma 3.7, implies $\nabla \cdot \mathbf{v} \in L^2(\Omega)$ such that the bilinear form is well-defined for functions from V and Q.

Lemma 3.10. **Properties of $b(\cdot,\cdot)$.**
The bilinear form $b(\cdot,\cdot)$ from (3.17) is bounded by

$$|b(\mathbf{v}, q)| \leq |\mathbf{v}|_1 \, \|q\|_Q, \tag{3.18}$$

$$|b(\mathbf{v}, q)| \leq C_K \nu_{\min}^{-1/2} \|\mathbf{v}\|_\nu \|q\|_Q, \tag{3.19}$$

and therefore continuous for both, the ν-dependent and the H_0^1-seminorm.

Proof. Using the Cauchy-Schwarz inequality and (3.9) one gets

$$\begin{aligned} |b(\mathbf{v}, q)| &= \left| - \int_\Omega (\nabla \cdot \mathbf{v})\, q \, d\mathbf{x} \right| \leq \|\nabla \cdot \mathbf{v}\|_0 \|q\|_0 \\ &\leq \underbrace{\|\nabla \mathbf{v}\|_0}_{|\mathbf{v}|_1} \|q\|_0 \leq C_K \nu_{\min}^{-1/2} \|\mathbf{v}\|_\nu \|q\|_Q. \end{aligned} \tag{3.20}$$

□

Remark 3.11. **Norm of $b(\cdot,\cdot)$.**
For the H_0^1-seminorm of $b(\cdot,\cdot)$ it holds

$$|b|_1 = 1$$

since

$$|b|_1 = \sup_{\mathbf{v}\in V,\, q\in Q,\, \mathbf{v},q\neq 0} \frac{b(\mathbf{v}, q)}{|\mathbf{v}|_1 \, \|q\|_Q} \leq \frac{|\mathbf{v}|_1 \, \|q\|_Q}{|\mathbf{v}|_1 \, \|q\|_Q} = 1 \tag{3.21}$$

and the estimate in Remark 3.8 is sharp.
For the ν-norm of $b(\cdot,\cdot)$ one finds, using the definition and (3.20),

$$\|b\|_\nu = \sup_{\mathbf{v}\in V,\, q\in Q,\, \mathbf{v},q\neq 0} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_\nu \|q\|_Q} \leq C_K \nu_{\min}^{-1/2}. \tag{3.22}$$

Remark 3.12. **$\|b\|_\nu$ for the special case of constant ν.**
For the special case that ν is constant, one finds

$$\|b\|_\nu = \sup_{\mathbf{v}\in V,\, q\in Q,\, \mathbf{v},q\neq 0} \frac{b(\mathbf{v}, q)}{\nu^{1/2} \|\mathbb{D}(\mathbf{v})\|_0 \|q\|_Q} \leq \frac{C_K}{\nu^{1/2}} \sup_{\mathbf{v}\in V,\, q\in Q,\, \mathbf{v},q\neq 0} \frac{b(\mathbf{v}, q)}{\|\nabla \mathbf{v}\|_0 \|q\|_Q}.$$

Since Remark 3.11 just proved

$$\sup_{\mathbf{v}\in V,\, q\in Q,\, \mathbf{v},q\neq 0} \frac{b(\mathbf{v},q)}{\|\nabla \mathbf{v}\|_0 \|q\|_Q} = 1$$

the supremum in (3.22) is achieved if the constant C_K is chosen such that Korn's inequality (3.12) is sharp. Thus, for the special case $\nu = \nu_{\min} = \text{const.}$ one finds

$$\|b\|_\nu = \frac{C_K}{\nu^{1/2}}.$$

In any other case, it is not to be expected that this upper bound for the norm of $b(\cdot,\cdot)$ is achieved since in the proof the norm equivalence was used.[2]

The aim consists now in finding the operators $B \in \mathcal{L}(V, Q')$ and $B' \in \mathcal{L}(Q, V')$ from the foregoing Section 3.1 that correspond to the bilinear form $b(\cdot,\cdot)$ for the particular choice of V and Q.

Remark 3.13. **The divergence operator.**
We define the divergence operator as

$$\operatorname{div} : V \to \operatorname{rg}(\operatorname{div}), \quad \mathbf{v} \mapsto \nabla \cdot \mathbf{v}.$$

As mentioned above, for $\mathbf{v} \in V$ it is, regardless of the chosen norm,

$$\nabla \cdot \mathbf{v} \in L^2(\Omega).$$

Furthermore, it is

$$\int_\Omega (\nabla \cdot \mathbf{v})(\mathbf{x})\, d\mathbf{x} = \int_{\partial\Omega} (\mathbf{v}\cdot\mathbf{n})(\mathbf{x})\, d\mathbf{s} - \int_\Omega (\mathbf{v}\cdot\nabla 1)(\mathbf{x})\, d\mathbf{x} = 0.$$

Consequently, $\operatorname{rg}(\operatorname{div}) \subset Q = Q'$ and one can even show $\operatorname{rg}(\operatorname{div}) = Q = Q'$.
The negative divergence operator is therefore identified with the operator $B \in \mathcal{L}(V, Q')$ from Section 3.1.

Remark 3.14. **The gradient operator.**
The gradient operator is defined as

$$\operatorname{grad} : Q \to \operatorname{rg}(\operatorname{grad}), \quad q \mapsto \nabla q.$$

It is $\operatorname{rg}(\operatorname{grad}) \subset V'$. Furthermore, it is

$$\langle -\operatorname{div}(\mathbf{v}), q\rangle_{Q',Q} = \langle \operatorname{grad}(q), \mathbf{v}\rangle_{V',V}.$$

Thus, $-\operatorname{div}$ and grad are dual operators, i.e., $\operatorname{grad} = B' \in \mathcal{L}(Q, V')$.

[2] One would need to construct a function with a gradient equal to one where $\nu = \nu_{\min}$ and otherwise zero. There are choices of ν for which this is not possible.

The kernel of the operator B is the space of weakly divergence-free functions

$$V_0 = V_{\text{div}} = \{\mathbf{v} \in V : (\nabla \cdot \mathbf{v}, q) = 0 \iff B\mathbf{v} = 0, \quad \forall q \in Q\}. \tag{3.23}$$

Lemma 3.15. **V_{div} is a closed subspace of V.**
The subspace of weakly divergence-free functions V_{div} is closed in V independently of the chosen norm.

Proof. The proof can be found in [6], p. 38. □

Lemma 3.16. **Isomorphism of the gradient operator.**
For every $\mathbf{f} \in V'$ with

$$\langle \mathbf{f}, \mathbf{v} \rangle_{V',V} = 0 \quad \mathbf{v} \in V_{\text{div}}$$

there exists a unique $q \in Q$ such that

$$\mathbf{f} = \text{grad}(q) = \nabla q,$$

i.e., $\text{rg}(\text{grad}) = \tilde{V}' = \{\mathbf{f} \in V' : \langle \mathbf{f}, \mathbf{v} \rangle_{V',V} = 0, \forall \mathbf{v} \in V_{\text{div}}\}$ and the gradient operator is an isomorphism from Q onto $\tilde{V}'$.

Proof. The proof in [6], p. 39, does not change for variable ν and new inner product (3.8). □

Remark 3.17. **Decomposition of V.**
Assume now that the velocity space V is equipped with the H_0^1-seminorm induced by (3.7). Then, it can be decomposed in V_{div} and its orthogonal complement

$$V = V_{\text{div}} \oplus V_{\text{div}}^{\perp}.$$

Of course, this can also be done when V is equipped with the norm defined by the inner product (3.8) but in this case the orthogonality and thus, the space $V_{\text{div}}^{\perp}$ depends on the viscosity ν.

Lemma 3.18. **Isomorphism of the divergence operator.**
Let V be equipped with the H_0^1-seminorm.
The divergence operator is an isomorphism from $V_{\text{div}}^{\perp}$ onto Q.

Proof. The proof given in [6], on p. 40, does not change for variable ν and new inner product (3.8). □

Corollary 3.19. **Each pressure is the divergence of a velocity field.**
For each pressure $q \in Q$, it exists a unique velocity field $\mathbf{v} \in V_{\text{div}}^{\perp} \subset V$ such that

$$\nabla \cdot \mathbf{v} = q \quad \text{and} \quad \|q\|_Q \leq C_K \nu_{\min}^{-1/2} \|\mathbf{v}\|_{\nu}, \quad \|\mathbf{v}\|_{\nu} \leq C_{\nu} \|q\|_Q,$$

where C_{ν} is independent of $\mathbf{v}, q$ but depends on ν.

Proof. The isomorphy of the divergence operator, see Lemma 3.18, guarantees the existence of a unique $\mathbf{v} \in V_{\text{div}}^{\perp}$ with $\nabla \cdot \mathbf{v} = q$ for all $q \in Q$. Using (3.14) and the norm equivalence (3.9), one gets

$$\|q\|_Q = \|\nabla \cdot \mathbf{v}\|_0 \le \|\nabla \mathbf{v}\|_0 \le C_K \nu_{\min}^{-1/2} \|\mathbf{v}\|_\nu.$$

The inverse map is an isomorphism, too and it is bounded (Theorem of Banach on the inverse operator). Thus, one finds

$$\|\nabla \mathbf{v}\|_0 = \|\nabla(\operatorname{div}^{-1} q)\|_0 = \left|\operatorname{div}^{-1} q\right|_1 \le C\|q\|_Q, \quad \forall q \in Q, \mathbf{v} \in V_{\text{div}}^{\perp}. \tag{3.24}$$

Now, one makes use of the norm equivalence (3.9) again to find

$$\nu_{\max}^{-1/2} \|\mathbf{v}\|_\nu \le \|\nabla \mathbf{v}\|_0$$

and thus,

$$\|\mathbf{v}\|_\nu \le \underbrace{\nu_{\max}^{1/2} C}_{C_\nu} \|q\|_Q. \tag{3.25}$$

$\square$

Remark 3.20. On Corollary 3.19.
The proof of Corollary 3.19 yields also the corresponding estimates for the H_0^1-seminorm,

$$\|q\|_Q \le \|\nabla \mathbf{v}\|_0, \tag{3.26}$$

$$\|\nabla \mathbf{v}\|_0 \le C\|q\|_Q, \tag{3.27}$$

where C does not depend on $\mathbf{v}, q$ and ν.

Theorem 3.21. The inf-sup condition for V and Q.
The Hilbert spaces $V = H_0^1(\Omega)$ and $Q = L_0^2(\Omega)$ equipped with the inner products (3.6) and (3.8) satisfy the inf-sup condition (3.4), i.e., $\exists\, \beta_{\text{is},\nu} > 0$ such that

$$\inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{(\nabla \cdot \mathbf{v}, q)}{\|\mathbf{v}\|_\nu \|q\|_Q} \ge \beta_{\text{is},\nu}. \tag{3.28}$$

Proof. From Corollary 3.19 one knows that for an arbitrary $q \in Q$ there is a unique $\mathbf{v} \in V_{\text{div}}^{\perp}$ such that

$$\nabla \cdot \mathbf{v} = q, \quad \|\mathbf{v}\|_\nu \le \nu_{\max}^{1/2} C \|q\|_Q.$$

Consequently,

$$\frac{(\nabla \cdot \mathbf{v}, q)}{\|\mathbf{v}\|_\nu} = \frac{(q, q)}{\|\mathbf{v}\|_\nu} = \frac{\|q\|_Q^2}{\|\mathbf{v}\|_\nu} \ge \frac{1}{\nu_{\max}^{1/2} C} \|q\|_Q.$$

This estimate holds of course if we replace $\mathbf{v}$ by the supremum

$$\sup_{\mathbf{v} \in V} \frac{(\nabla \cdot \mathbf{v}, q)}{\|\mathbf{v}\|_\nu} \ge \frac{1}{\nu_{\max}^{1/2} C} \|q\|_Q.$$

Since this holds for an arbitrary q one can conclude

$$\inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{(\nabla \cdot \mathbf{v}, q)}{\|\mathbf{v}\|_\nu \|q\|_Q} \ge \frac{1}{\nu_{\max}^{1/2} C} =: \beta_{\text{is},\nu}. \tag{3.29}$$

$\square$

Remark 3.22. $\beta_{\text{is},\nu}$ **depends on the viscosity.**
From the proof of Theorem 3.21 it is clear that the inf-sup constant $\beta_{\text{is},\nu}$ depends on the maximal value of the viscosity $\nu_{\max}$, i.e., for large values of the viscosity the inf-sup constant tends to zero. As the next section will reveal this can lead to problems since in the error estimates the factor $\beta_{\text{is},\nu}^{-1}$ will appear.

Lemma 3.23. **Estimating the gradient by the divergence for $V_{\text{div}}^{\perp}$-functions.**
For $\mathbf{v} \in V_{\text{div}}^{\perp}$ it is

$$\|\mathbf{v}\|_{\nu} \leq \frac{1}{\beta_{\text{is},\nu}} \|\nabla \cdot \mathbf{v}\|_0, \tag{3.30}$$

compare with Lemma 3.1 (iii).

Proof. In Lemma 3.19 the estimate

$$\|\mathbf{v}\|_{\nu} \leq C_{\nu} \|q\|_0$$

is proved, where $q = \nabla \cdot \mathbf{v}$. The proof of Theorem 3.21 reveals that one can choose $C_{\nu} = \nu_{\max}^{1/2} C = 1/\beta_{\text{is},\nu}$. □

Remark 3.24. **Two different inf-sup conditions.**
Note that the inf-sup condition where V is equipped with the ν-norm holds if and only if it holds when V is equipped with the H_0^1-seminorm. This is due to the norm equivalence. The ν-independent inf-sup condition can be proved also seperately (see [6] p. 41).
Thus, there is an β_{is} independent of ν such that

$$\inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{(\nabla \cdot \mathbf{v}, q)}{|\mathbf{v}|_1 \|q\|_Q} \geq \beta_{\text{is}}. \tag{3.31}$$

3.3. Finite Element Discretization: Function Spaces

When finite element methods are used, the infinite dimensional spaces V and Q are replaced by finite dimensional spaces V^h and Q^h such that the Galerkin Method can be applied.
In the following, the pair of velocity-pressure finite element spaces will be denoted by V^h/Q^h, where it should be clear that the elements in V^h are vector-valued.
The theory developed so far will now be applied to the Hilbert spaces V^h and Q^h.
To ensure the existence of a unique solution of the finite element discretization, the spaces V^h and Q^h have to fulfill a discrete inf-sup condition

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in V^h} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_V \|q^h\|_0} \geq \beta_{\text{is},\nu}^h > 0, \tag{3.32}$$

where the norm of V^h is either

$$\|\mathbf{v}^h\|_V = \|\mathbf{v}^h\|_{\nu} \quad \text{or} \quad \|\mathbf{v}^h\|_V = \left|\mathbf{v}^h\right|_1.$$

Note that for general finite element spaces one would need to consider a discrete bilinear form $b^h : V^h \times Q^h \to \mathbb{R}$ defined as

$$b^h(\mathbf{v}^h, q^h) := -\sum_{K\in\mathcal{T}^h} (\nabla\cdot\mathbf{v}^h, q^h)_K, \tag{3.33}$$

and discrete norms $\|\mathbf{v}^h\|_{V,h}$ with either

$$\|\mathbf{v}^h\|_{\nu,h} = \left(\sum_{K\in\mathcal{T}^h} (\nu\mathbb{D}(\mathbf{v}^h), \mathbb{D}(\mathbf{v}^h))_K\right)^{1/2} \tag{3.34}$$

or

$$\|\mathbf{v}^h\|_{1,h} = \left(\sum_{K\in\mathcal{T}^h} (\nabla\mathbf{v}^h, \nabla\mathbf{v}^h)_K\right)^{1/2}, \tag{3.35}$$

and

$$\|q^h\|_{0,h} = \left(\sum_{K\in\mathcal{T}^h} (q^h, q^h)_K\right)^{1/2}, \tag{3.36}$$

where $\mathcal{T}^h$ is a triangulation of the domain Ω and $K\in\mathcal{T}^h$ are the corresponding mesh cells.
For conforming finite element spaces ($V^h \subset V, Q^h \subset Q$) it is $\mathbf{v}^h \in V, q^h \in Q$ and therefore,

$$\|\mathbf{v}^h\|_{\nu,h} = \|\mathbf{v}^h\|_{\nu}, \quad \|\mathbf{v}^h\|_{\nu,1} = \left|\mathbf{v}^h\right|_1 \quad \text{and} \quad \|q^h\|_{0,h} = \|q^h\|_0.$$

Since in this work we will consider only conforming finite element spaces, we can use the norms from the original spaces V and Q for the ease of notation. The same is true for the bilinear form where we can omit the index h, i.e.,

$$b^h(\mathbf{v}^h, q^h) = b(\mathbf{v}^h, q^h),\ \forall\,(\mathbf{v}^h, q^h) \in (V^h \times Q^h) \subset (V \times Q).$$

Again a linear operator $B^h = B$ is associated with $b(\cdot,\cdot)$,

$$B : V^h \to (Q^h)', \quad \langle B\mathbf{v}^h, q^h\rangle_{(Q^h)',Q^h} = b(\mathbf{v}^h, q^h).$$

The kernel of B is the space of discretely divergence-free functions,

$$V^h_{\text{div}} = \{\mathbf{v}^h \in V^h : b(\mathbf{v}^h, q^h) = 0,\ \forall\, q^h \in Q^h\}.$$

The dual operator

$$B^T : Q^h \to (V^h)', \quad \langle B^T q^h, \mathbf{v}^h\rangle_{(V^h)',V^h} = b(\mathbf{v}^h, q^h)$$

is denoted by grad^h.

Remark 3.25. Discrete analogues of Corollary 3.19 and Lemma 3.23.
The discrete inf-sup condition yields discrete analogues to Corollary 3.19 and Lemma 3.23.
For each pressure $q^h \in Q^h$ there is a unique $\mathbf{v}^h \in (V_{\text{div}}^h)^\perp$ with $\nabla \cdot \mathbf{v}^h = q^h$ and

$$\|\mathbf{v}^h\|_\nu \leq \frac{1}{\beta_{\text{is},\nu}^h} \|\nabla \cdot \mathbf{v}^h\|_0, \tag{3.37}$$

where like in the continuous case (compare with (3.29)) it is

$$\beta_{\text{is},\nu}^h = C^{-1} \nu_{\max}^{-1/2}. \tag{3.38}$$

Lemma 3.26. Best approximation estimate for V_{div}^h and ν-Norm.
For the conforming velocity space $V^h \subset V$ and $\mathbf{v} \in V_{\text{div}}$, let the discrete inf-sup condition (3.32) with $\|\cdot\|_V = \|\cdot\|_\nu$ hold. Then, the following estimate holds

$$\inf_{\mathbf{v}^h \in V_{\text{div}}^h} \|\mathbf{v} - \mathbf{v}^h\|_\nu \leq \left(1 + \frac{C_K}{\beta_{\text{is},\nu}^h \nu_{\min}^{1/2}}\right) \inf_{\mathbf{w}^h \in V^h} \|\mathbf{v} - \mathbf{w}^h\|_\nu. \tag{3.39}$$

Proof. The discrete inf-sup condition (3.32) ensures that the space V_{div}^h is not empty. Hilbert space theory yields that an arbitrary $\mathbf{w}^h \in V^h$ can be decomposed like

$$\mathbf{w}^h = \mathbf{v}^h - \mathbf{z}^h, \quad \text{where} \quad \mathbf{v}^h \in V_{\text{div}}^h \quad \text{and} \quad -\mathbf{z}^h \in (V_{\text{div}}^h)^\perp.$$

Again, the decomposition according to the $|\cdot|_1$-orthogonality is considered, such that $\mathbf{z}^h$ does not depend on ν.
For $q^h \in Q^h$ it is $b(\mathbf{v}^h, q^h) = 0$ as $\mathbf{v}^h \in V_{\text{div}}^h$.
In addition, one finds $b(\mathbf{v}, q^h) = 0$ since

$$\mathbf{v} \in V_{\text{div}} \to b(\mathbf{v}, q) = 0, \, \forall q \in Q$$

and $q^h \in Q^h \subset Q$. Thus,

$$b(\mathbf{z}^h, q^h) = b(\mathbf{v}^h - \mathbf{w}^h, q^h) = b(\mathbf{v} - \mathbf{w}^h, q^h), \, \forall q^h \in Q^h. \tag{3.40}$$

According to Remark 3.25, the discrete inf-sup condition yields a discrete analogue to Corollary 3.19. Consequently, there is a $q^h = \nabla \cdot \mathbf{z}^h \in Q^h$. Inserting this into (3.40) gives

$$\begin{aligned} | \, b(\mathbf{z}^h, \nabla \cdot \mathbf{z}^h) \, | &= | \, b(\mathbf{v} - \mathbf{w}^h, \nabla \cdot \mathbf{z}^h) \, | \\ &= | -(\nabla \cdot (\mathbf{v} - \mathbf{w}^h), \nabla \cdot \mathbf{z}^h) \, | \\ &\leq \|\nabla \cdot (\mathbf{v} - \mathbf{w}^h)\|_0 \|\nabla \cdot \mathbf{z}^h\|_0 \\ &\leq \|\nabla (\mathbf{v} - \mathbf{w}^h)\|_0 \|\nabla \cdot \mathbf{z}^h\|_0. \end{aligned}$$

Since $| \, b(\mathbf{z}^h, \nabla \cdot \mathbf{z}^h) \, | = \|\nabla \cdot \mathbf{z}^h\|_0^2$ it follows

$$\|\nabla \cdot \mathbf{z}^h\|_0 \leq \|\nabla (\mathbf{v} - \mathbf{w}^h)\|_0. \tag{3.41}$$

Using the estimate (3.37), the discrete analogue of (3.30), one finds

$$\|\mathbf{z}^h\|_\nu \leq \frac{1}{\beta^h_{\mathrm{is},\nu}}\|\nabla\cdot\mathbf{z}^h\|_0 \leq \frac{1}{\beta^h_{\mathrm{is},\nu}}\|\nabla(\mathbf{v}-\mathbf{w}^h)\|_0 \leq \frac{1}{\beta^h_{\mathrm{is},\nu}} C_K \nu_{\min}^{-1/2}\|\mathbf{v}-\mathbf{w}^h\|_\nu .$$

This estimate yields, together with the triangle inequality,

$$\begin{aligned} \|\mathbf{v}-\mathbf{v}^h\|_\nu &\leq \|\mathbf{v}-\mathbf{w}^h\|_\nu + \|\mathbf{z}^h\|_\nu && (3.42)\\ &\leq \left(1+\frac{C_K}{\beta^h_{\mathrm{is},\nu}\nu_{\min}^{1/2}}\right)\|\mathbf{v}-\mathbf{w}^h\|_\nu . && (3.43)\end{aligned}$$

Since there is for every $\mathbf{w}^h$ a $\mathbf{v}^h$ such that this equation holds, it is true also for the infimum,

$$\inf_{\mathbf{v}^h\in V^h_{\mathrm{div}}}\|\mathbf{v}-\mathbf{v}^h\|_\nu \leq \left(1+\frac{C_K}{\beta^h_{\mathrm{is},\nu}\nu_{\min}^{1/2}}\right)\inf_{\mathbf{w}^h\in V^h}\|\mathbf{v}-\mathbf{w}^h\|_\nu .$$

□

For the sake of completeness this section concludes with the best approximation estimate for the ν-independent H^1_0-seminorm.

Lemma 3.27. Best approximation estimate for V^h_{div} and H^1_0-seminorm.
For the conforming velocity space $V^h \subset V$ and $\mathbf{v}\in V_{\mathrm{div}}$, let the discrete inf-sup condition (3.32) with $\|\cdot\|_V = |\cdot|_1$ hold. Then, the following estimate holds

$$\inf_{\mathbf{v}^h\in V^h_{\mathrm{div}}}\left|\mathbf{v}-\mathbf{v}^h\right|_1 \leq \left(1+\frac{1}{\beta^h_{\mathrm{is}}}\right)\inf_{\mathbf{w}^h\in V^h}\left|\mathbf{v}-\mathbf{w}^h\right|_1 . \tag{3.44}$$

Proof. A proof can be found in [6], pp. 46. □

3.4. Examples of Inf-Sup Stable Finite Element Spaces

Due to their comparatively easy implementation the inf-sup stable Taylor-Hood finite element spaces are very popular for the discretization of incompressible flow problems. On triangular and tetrahedral grids they are given by P_k/P_{k-1}, $k\geq 2$ and on quadrilateral and hexahedral grids by Q_k/Q_{k-1}, where P_k and Q_k denote the polynomial spaces (see B.1 and B.2).
The proof that the Taylor-Hood finite element pair Q_2/Q_1 fulfills the discrete inf-sup condition (3.32) as well as references to the proofs for other Taylor-Hood pairs and for the similar pair Q_2/P_1^{disc} can be found in [6].
Another interesting pair of finite element spaces that can easily be implemented is the Scott-Vogelius element $P_k/P_{k-1}^{\mathrm{disc}}$, where P_{k-1}^{disc} is the space of functions from P_{k-1} that are discontinuous.
This pair of finite element spaces has the desirable property $V^h_{\mathrm{div}} \subset V_{\mathrm{div}}$, i.e., discretely

divergence-free functions are divergence-free also in the weak sense. This is due to the fact that for $\mathbf{v}^h \in V_{\text{div}}^h \subset P_k$ it is $\nabla \cdot \mathbf{v}^h \in P_{k-1}^{\text{disc}} = Q^h$. From $\mathbf{v}^h \in V_{\text{div}}^h$ it follows

$$b(\mathbf{v}^h, q^h) = 0, \quad \forall q^h \in Q^h,$$

and thus

$$0 = b(\mathbf{v}^h, \nabla \cdot \mathbf{v}^h) = \|\nabla \cdot \mathbf{v}^h\|_0.$$

However, it has to be remarked that the Scott-Vogelius element is inf-sup stable only on special types of grids, e.g., on barycentric grids. For a description of the initial grid see Section 5.2.

4. The Stokes Equations

4.1. The Continuous Stokes Equations

In a steady flow ($\partial_t \mathbf{u} = \mathbf{0}$) the viscous transport dominates the convection which can then be neglected if the fluid moves sufficiently slowly. The momentum equation of the general Navier-Stokes equations (2.11)

$$\partial_t \mathbf{u} - \underbrace{2\nabla \cdot (\nu \mathbb{D}(\mathbf{u}))}_{\text{viscous term}} + \underbrace{(\mathbf{u} \cdot \nabla)\,\mathbf{u}}_{\text{convection}} + \nabla p = \mathbf{f}, \qquad (0,T] \times \Omega$$

becomes a linear equation and the resulting system,

$$\begin{aligned} -2\nabla \cdot (\nu \mathbb{D}(\mathbf{u})) + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega \end{aligned} \tag{4.1}$$

is called Stokes equations.
One can derive a weak formulation in the usual way by multiplying a test function $\mathbf{v}$ from $H_0^1(\Omega)$ and integrating over the domain Ω before integration by parts is applied:

$$\begin{aligned} \int_\Omega -2\nabla \cdot (\nu \mathbb{D}(\mathbf{u})) \cdot \mathbf{v} + \int_\Omega \nabla p \cdot \mathbf{v} &= \int_\Omega \mathbf{f} \cdot \mathbf{v}, \\ 2\int_\Omega \nu \mathbb{D}(\mathbf{u}) \nabla \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} &= \int_\Omega \mathbf{f} \cdot \mathbf{v}, \\ 2(\nu \mathbb{D}(\mathbf{u}), \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned}$$

Note that with $\mathbf{v} \in H_0^1(\Omega)$ the integral over the boundary, $\partial\Omega$, vanishes when integration by parts is applied.
Thus, a weak solution $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ solves

$$\begin{aligned} 2\,\underbrace{(\nu \mathbb{D}(\mathbf{u}), \nabla \mathbf{v})}_{\widetilde{a_\nu}(\mathbf{u},\mathbf{v})} \underbrace{-(\nabla \cdot \mathbf{v}, p)}_{b(\mathbf{v},p)} &= \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, && \forall \mathbf{v} \in H_0^1(\Omega), \\ \underbrace{-(\nabla \cdot \mathbf{u}, q)}_{b(\mathbf{u},q)} &= 0, && \forall q \in L_0^2(\Omega). \end{aligned} \tag{4.2}$$

Remark 4.1. **Manipulation of the bilinear form $\widetilde{a_\nu}(\cdot,\cdot)$.**
The weak formulation of the Stokes problem yields a natural choice for the bilinear form $a(\cdot,\cdot)$ that appears in the formulation of the abstract saddle point problem in Section 3.1, namely $\widetilde{a_\nu}(\mathbf{u}, \mathbf{v}) = (\nu \mathbb{D}(\mathbf{u}), \nabla \mathbf{v})$.
However, to emphasize the symmetry of the viscous term, the equivalent bilinear form $a_\nu(\mathbf{u}, \mathbf{v}) := (\nu \mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v}))$ is used in the analysis.

The equivalence can be seen easily

$$
\begin{aligned}
(\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v})) =& \sum_{ij}\int_\Omega \nu\left(\frac{1}{2}\frac{\partial u_i}{\partial x_j}+\frac{1}{2}\frac{\partial u_j}{\partial x_i}\right)\left(\frac{1}{2}\frac{\partial v_i}{\partial x_j}+\frac{1}{2}\frac{\partial v_j}{\partial x_i}\right)\,d\mathbf{x}\\
=&\frac{1}{4}\sum_{ij}\int_\Omega \nu\left(\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\frac{\partial v_i}{\partial x_j}+\frac{\partial u_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}+\frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\right)\,d\mathbf{x}\\
=&\frac{1}{4}\left(\sum_{ij}\int_\Omega \nu\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\mathbf{x}+\sum_{ij}\int_\Omega \nu\frac{\partial u_j}{\partial x_i}\frac{\partial v_i}{\partial x_j}\,d\mathbf{x}\right.\\
&\left.+\sum_{ij}\int_\Omega \nu\frac{\partial u_j}{\partial x_i}\frac{\partial v_i}{\partial x_j}\,d\mathbf{x}+\sum_{ij}\int_\Omega \nu\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\mathbf{x}\right)\\
=&\frac{1}{2}\sum_{ij}\int_\Omega \nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial v_i}{\partial x_j}\,d\mathbf{x}=(\nu\mathbb{D}(\mathbf{u}),\nabla\mathbf{v}).
\end{aligned}
$$

From now on, we will use the following formulation of the weak Stokes problem:
Find $(\mathbf{u},p)\in H_0^1(\Omega)\times L_0^2(\Omega)$ such that

$$
\begin{aligned}
2\underbrace{(\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))}_{a_\nu(\mathbf{u},\mathbf{v})}\underbrace{-(\nabla\cdot\mathbf{v},p)}_{b(\mathbf{v},p)} &= \langle\mathbf{f},\mathbf{v}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, && \forall\mathbf{v}\in H_0^1(\Omega),\\
\underbrace{-(\nabla\cdot\mathbf{u},q)}_{b(\mathbf{u},q)} &= 0, && \forall q\in L_0^2(\Omega).
\end{aligned}
\tag{4.3}
$$

Let now $V=H_0^1(\Omega)$ denote the velocity space and $Q=L_0^2(\Omega)$ the pressure space with inner products $\|\cdot\|_\nu$ and $\|\cdot\|_0$, as introduced in Section 3.2.
An equivalent formulation of the weak problem is

$$
2a_\nu(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)-b(\mathbf{u},q)=\langle\mathbf{f},\mathbf{v}\rangle_{V',V},\ \forall(\mathbf{v},q)\in V\times Q. \tag{4.4}
$$

This equivalence can be seen by choosing the test functions $(\mathbf{v},0)$ and $(\mathbf{0},q)$.
Let V_{div} be the space of weakly divergence-free functions. The problem that is associated to (4.1) (see (3.3)) is:
Find $\mathbf{u}\in V_{\text{div}}$ such that

$$
2(\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))=\langle\mathbf{f},\mathbf{v}\rangle_{V',V},\quad\forall\mathbf{v}\in V_{\text{div}}. \tag{4.5}
$$

Lemma 4.2. **Norm of the bilinear form $a_\nu(\cdot,\cdot)$.**
For the bilinear form $a_\nu(\cdot,\cdot)$ defined as in (4.3) it is $\|a_\nu\|=1$.

Proof. Applying the Cauchy-Schwarz inequality one gets

$$
\begin{aligned}
\|a_\nu\| &= \sup_{\mathbf{v},\mathbf{w}\in V,\mathbf{v},\mathbf{w}\neq 0}\frac{a_\nu(\mathbf{v},\mathbf{w})}{\|\mathbf{v}\|_\nu\,\|\mathbf{w}\|_\nu}\\
&= \sup_{\mathbf{v},\mathbf{w}\in V,\mathbf{v},\mathbf{w}\neq 0}\frac{(\nu\,\mathbb{D}(\mathbf{v}),\mathbb{D}(\mathbf{w}))_{L^2(\Omega)}}{\|\nu^{1/2}\mathbb{D}(\mathbf{v})\|_0\,\|\nu^{1/2}\mathbb{D}(\mathbf{w})\|_0}\\
&\le \sup_{\mathbf{v},\mathbf{w}\in V,\mathbf{v},\mathbf{w}\neq 0}\frac{\|\nu^{1/2}\mathbb{D}(\mathbf{v})\|_0\,\|\nu^{1/2}\mathbb{D}(\mathbf{w})\|_0}{\|\nu^{1/2}\mathbb{D}(\mathbf{v})\|_0\,\|\nu^{1/2}\mathbb{D}(\mathbf{w})\|_0}=1.
\end{aligned}
$$

Choosing now $\mathbf{w} = \mathbf{v}$ shows that the supremum 1 is achieved:

$$\frac{a_\nu(\mathbf{v}, \mathbf{v})}{\|\mathbf{v}\|_\nu^2} = \frac{(\nu\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{v}))}{\|\mathbf{v}\|_\nu^2} = \frac{\|\mathbf{v}\|_\nu^2}{\|\mathbf{v}\|_\nu^2} = 1.$$

□

Remark 4.3. **V_{div}-ellipticity.**
Obviously, the bilinear form $a_\nu(\cdot, \cdot)$ is V_{div}-elliptic, i.e.,

$$a_\nu(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|_\nu, \quad \forall \mathbf{v} \in V \supset V_{\mathrm{div}}, \tag{4.6}$$

such that (3.5) is fulfilled with $\alpha = 1$.

Theorem 4.4. **Existence and uniqueness of a solution of the Stokes equations.**
Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary Γ and let $\mathbf{f} \in H^{-1}(\Omega)$. There is a unique solution $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ of the weak formulation (4.3) of the Stokes equations.

Proof. The bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition (3.28) as proved in Theorem 3.21 and $a_\nu(\cdot, \cdot)$ is V_{div}-elliptic (see Remark 4.3).
The existence of a unique solution follows from Lemma 3.4. □

Theorem 4.5. **Stability of the solution.**
Let the assumptions of Theorem 4.4 hold. The weak solution of the Stokes equations (4.3) depends continuously on the right-hand side $\mathbf{f}$,

$$\|\mathbf{u}\|_\nu \leq \frac{1}{2}\|\mathbf{f}\|_{H^{-1}(\Omega)}, \tag{4.7}$$

$$\|p\|_0 \leq \frac{2}{\beta_{\mathrm{is},\nu}}\|\mathbf{f}\|_{H^{-1}(\Omega)}. \tag{4.8}$$

Proof. The weak solution $\mathbf{u}$ is in V_{div} and can therefore be inserted as test function in (4.5) and using (4.6) one finds

$$2\|\mathbf{u}\|_\nu^2 = 2a_\nu(\mathbf{u}, \mathbf{u}) = \langle \mathbf{f}, \mathbf{u} \rangle_{V',V}.$$

The right-hand side can be estimated by

$$\langle \mathbf{f}, \mathbf{u} \rangle_{V',V} \leq \|\mathbf{f}\|_{H^{-1}(\Omega)}\|\mathbf{u}\|_\nu \tag{4.9}$$

and consequently,

$$\|\mathbf{u}\|_\nu^2 \leq \frac{1}{2}\|\mathbf{f}\|_{H^{-1}(\Omega)}\|\mathbf{u}\|_\nu.$$

For the trivial case $\|\mathbf{u}\|_\nu = 0$ this inequality is satisfied. In any other case one divides by $\|\mathbf{u}\|_\nu$ to get an estimate for the velocity.

For the pressure field p one makes use of the inf-sup condition (3.28) and (4.3). This yields

$$\begin{aligned}
\|p\|_0 &\overset{(3.28)}{\leq} \frac{1}{\beta_{\mathrm{is},\nu}} \sup_{\mathbf{v}\in V} \frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_\nu} \\
&\overset{(4.3)}{=} \frac{1}{\beta_{\mathrm{is},\nu}} \sup_{\mathbf{v}\in V} \frac{\langle \mathbf{f},\mathbf{v}\rangle_{V',V} - 2(\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))}{\|\mathbf{v}\|_\nu} \\
&\leq \frac{1}{\beta_{\mathrm{is},\nu}} \sup_{\mathbf{v}\in V} \frac{\|\mathbf{f}\|_{H^{-1}(\Omega)}\|\mathbf{v}\|_\nu + 2\|\mathbf{u}\|_\nu\|\mathbf{v}\|_\nu}{\|\mathbf{v}\|_\nu} \\
&= \frac{1}{\beta_{\mathrm{is},\nu}} \left(\|\mathbf{f}\|_{H^{-1}(\Omega)} + 2\|\mathbf{u}\|_\nu\right).
\end{aligned}$$

Inserting the estimate for $\|\mathbf{u}\|_\nu$ one gets

$$\|p\|_0 \leq \frac{1}{\beta_{\mathrm{is},\nu}} \left(\|\mathbf{f}\|_{H^{-1}(\Omega)} + \|\mathbf{f}\|_{H^{-1}(\Omega)}\right) \leq \frac{2}{\beta_{\mathrm{is},\nu}}\|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

$\square$

Remark 4.6. Norm of the dual space $H^{-1}(\Omega)$.
Note that in the foregoing theorem, the space V is equipped with the ν-norm, such that the estimate (4.9) holds.
However, the consequence is that the norm of the dual space $V' = H^{-1}(\Omega)$ depends on ν. It is a reasonable objection that the norm of the dual space should be ν-independent since now it is not clear how to interpret the estimate (4.7).
If V is equipped with the $H_0^1(\Omega)$-seminorm, $|\cdot|_1$, estimate (4.9) becomes, using the norm equivalence (3.9),

$$\langle \mathbf{f},\mathbf{u}\rangle_{V',V} \leq \|\mathbf{f}\|_{H^{-1}(\Omega)}\,|\mathbf{u}|_1 \leq \|\mathbf{f}\|_{H^{-1}(\Omega)} C_k\, \nu_{\min}^{-1/2}\|\mathbf{v}\|_\nu, \tag{4.10}$$

where the norm $\|\cdot\|_{H^{-1}}$ is independent of ν.
This yields the stability estimate

$$\|\mathbf{u}\|_\nu \leq \frac{C_k}{2\sqrt{\nu_{\min}}}\|\mathbf{f}\|_{H^{-1}(\Omega)}. \tag{4.11}$$

Note that when applying the inf-sup constant in order to get an estimate for the pressure one gets

$$\|p\|_0 \leq \frac{1}{\beta_{\mathrm{is}}} \sup_{\mathbf{v}\in V} \frac{b(\mathbf{v},p)}{|\mathbf{v}|_1},$$

since (3.31) has to be applied.
It is then

$$\begin{aligned}
\|p\|_0 &\leq \frac{1}{\beta_{\mathrm{is}}} \sup_{\mathbf{v}\in V} \frac{\langle \mathbf{f},\mathbf{v}\rangle_{V',V} - 2(\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}))}{|\mathbf{v}|_1} \\
&\leq \frac{1}{\beta_{\mathrm{is}}} \sup_{\mathbf{v}\in V} \frac{\|\mathbf{f}\|_{H^{-1}(\Omega)}\,|\mathbf{v}|_1 + 2\|\mathbf{u}\|_\nu \nu_{\max}^{1/2}\,|\mathbf{v}|_1}{|\mathbf{v}|_1} \\
&= \frac{1}{\beta_{\mathrm{is}}} \left(\|\mathbf{f}\|_{H^{-1}(\Omega)} + 2\,\nu_{\max}^{1/2}\|\mathbf{u}\|_\nu\right).
\end{aligned}$$

Again, one can insert the estimate for $\|\mathbf{u}\|_\nu$ and finds

$$\|p\|_0 \leq \frac{1}{\beta_{\text{is}}}\left(1 + C_K\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\right)\|\mathbf{f}\|_{H^{-1}(\Omega)}, \tag{4.12}$$

for (4.8), where this time the inf-sup constant β_{is} does not depend on ν.

4.2. The Finite Element Problem

The finite element problem is: Find $(\mathbf{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{aligned} 2a_\nu(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) &= \langle \mathbf{f}, \mathbf{v}^h\rangle_{V',V}, \quad \forall \mathbf{v}^h \in V^h, \\ b(\mathbf{u}^h, q^h) &= 0, \quad \forall q^h \in Q^h. \end{aligned} \tag{4.13}$$

Remark 4.7. **Notation for conforming finite element spaces.**
Note that usually for the finite element problem one would write $a_\nu^h(\cdot,\cdot)$ and $b^h(\cdot,\cdot)$ instead of $a_\nu(\cdot,\cdot)$ and $b(\cdot,\cdot)$, where

$$a_\nu^h(\mathbf{v}^h, \mathbf{w}^h) = \sum_{K\in\mathcal{T}^h} (\nu\mathbb{D}(\mathbf{v}^h), \mathbb{D}(\mathbf{w}^h))_K, \quad b^h(\mathbf{v}^h, q^h) = -\sum_{K\in\mathcal{T}^h} (\nabla\cdot\mathbf{v}^h, q^h)_K.$$

As already mentioned in Section 3.3, we are considering the case of conforming finite element spaces only and thus it holds $a_\nu^h(\mathbf{v}^h, \mathbf{u}^h) = a_\nu(\mathbf{v}^h, \mathbf{u}^h)$ for all $\mathbf{v}^h, \mathbf{u}^h \in V^h$ since the conformity yields $\mathbf{v}^h, \mathbf{u}^h \in V$.
In addition, as shown in Section 3.3, it is $b^h(\cdot,\cdot) = b(\cdot,\cdot)$.
The notation used in (4.13) is therefore correct and will be used from now on.
If not explicitly mentioned otherwise the velocity space can be assumed to be equipped with the ν-norm $\|\cdot\|_{V,h} = \|\cdot\|_V = \|\cdot\|_\nu$ and "inf-sup stable" then means that the discrete inf-sup condition (3.32) with that norm is fulfilled.

Theorem 4.8. **Unique Solvability of the FE Problem.**
Let V^h and Q^h be inf-sup stable, conforming finite element spaces.
Then there is a unique and stable solution to the finite element problem (4.13).

Proof. The proof is analogous to those of Theorems 4.4 and 4.5. □

According to (4.5) there is an equivalent formulation of (4.13):
Find $\mathbf{u}^h \in V^h_{\text{div}}$ such that

$$2(\nu\mathbb{D}(\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) = \langle \mathbf{f}, \mathbf{v}^h\rangle_{V',V}, \quad \forall \mathbf{v}^h \in V^h_{\text{div}}. \tag{4.14}$$

4.3. Finite Element Error Analysis

Remark 4.9. **Estimates for different norms.**
This section deals with the problem of getting information on the order of convergence of the finite element solution to the solution of the weak problem. Pairs of finite element spaces corresponding to families of triangulations $\{\mathcal{T}^h\}$ will be considered.

First of all, estimates for the finite element errors in the different norms will be presented. As already mentioned, the ν-norm is the natural choice for the velocity space. However, the ν-dependence comes at a price: comparing finite element errors of problems with different viscosity is hardly possible in this norm. The reason is that the norm is weighted by the viscosity and consequently the error will appear larger if the viscosity takes large values.
Furthermore, in order to compute the orders of convergence we need the estimates to be ν-independent, as one will see later in this section, in Corollary 4.32.
Therefore different estimates will be presented. For every estimate where the ν-norm appears an alternative estimate depending only on the H_0^1-seminorm will be given.

Theorem 4.10. **Finite element error estimate for $\|\mathbf{u}-\mathbf{u}^h\|_\nu$.**
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with polyhedral Lipschitz boundary and let $(\mathbf{u}, p) \in V \times Q$ be the unique solution of the Stokes problem (4.3). Given a discretization with inf-sup stable conforming finite element spaces $V^h \times Q^h$, let $\mathbf{u}^h \in V_{\text{div}}^h$ be the solution for the velocity field.
Then the following error estimate holds:

$$\|\mathbf{u}-\mathbf{u}^h\|_\nu \leq 2\left(1+\frac{C_K}{\beta_{\text{is},\nu}^h\sqrt{\nu_{\min}}}\right)\inf_{\mathbf{v}^h\in V^h}\|\mathbf{u}-\mathbf{v}^h\|_\nu + \frac{C_K}{2\sqrt{\nu_{\min}}}\inf_{q^h\in Q^h}\|p-q^h\|_0, \tag{4.15}$$

where $\beta_{\text{is},\nu}^h$ depends on $\nu_{\max}$ like in (3.38).

Proof. First of all, the so-called error equation has to be formulated. As the finite element spaces are conforming one has $V_{\text{div}}^h \subset V$ and thus, functions from V_{div}^h can be chosen as test functions in the continuous Stokes problem (4.4) .
Consider the difference of (4.4) and (4.14)

$$2(\nu\mathbb{D}(\mathbf{u}-\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) - (\nabla\cdot\mathbf{v}^h, p) + \underbrace{(\nabla\cdot\mathbf{u}, q^h)}_{=0} = 0,\ \forall(\mathbf{v}^h, q^h) \in V_{\text{div}}^h \times Q^h. \tag{4.16}$$

Here, it was used that the solution $\mathbf{u}$ is in V_{div}^h since

$$\mathbf{u} \in V_{\text{div}} \iff (\nabla\cdot\mathbf{u}, q) = 0,\ \forall q \in Q \longrightarrow (\nabla\cdot\mathbf{u}, q^h) = 0,\ \forall q^h \in Q^h \subset Q.$$

The pressure term does not necessarily vanish because in general $p \notin Q^h$ since $Q^h \subset Q$ and not vice versa. Consequently it is $V_{\text{div}}^h \not\subset V_{\text{div}}$.
With $(\nabla\cdot\mathbf{v}^h, q^h) = 0,\ \forall q^h \in Q^h$ one gets

$$2(\nu\mathbb{D}(\mathbf{u}-\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) - (\nabla\cdot\mathbf{v}^h, p-q^h) = 0, \quad \forall(\mathbf{v}^h, q^h) \in V_{\text{div}}^h \times Q^h. \tag{4.17}$$

We will now split the error as follows

$$\mathbf{u}-\mathbf{u}^h = (\mathbf{u}-I_h\mathbf{u}) - (\mathbf{u}^h - I_h\mathbf{u}) = \boldsymbol{\eta} - \boldsymbol{\phi}^h. \tag{4.18}$$

Here $I_h\mathbf{u}$ is the interpolant of $\mathbf{u}$ in V_{div}^h. Consequently, $\boldsymbol{\eta}$ is the interpolation error which depends only on V_{div}^h.

The goal consists now in estimating $\boldsymbol{\phi}^h \in V^h_{\text{div}}$ by the interpolation error as well.
In order to do so, one combines (4.18) and (4.17) and chooses $\boldsymbol{\phi}^h \in V^h_{\text{div}} \subset V^h$ as test function:

$$\begin{aligned} & 2\left(\nu\mathbb{D}(\boldsymbol{\eta}-\boldsymbol{\phi}^h),\mathbb{D}(\boldsymbol{\phi}^h)\right) - \left(\nabla\cdot\boldsymbol{\phi}^h, p-q^h\right) = 0,\ \forall q^h \in Q^h \\ \iff\quad & 2\underbrace{\left(\nu\mathbb{D}(\boldsymbol{\phi}^h),\mathbb{D}(\boldsymbol{\phi}^h)\right)}_{a_\nu(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h)} = 2\left(\nu\mathbb{D}(\boldsymbol{\eta}),\mathbb{D}(\boldsymbol{\phi}^h)\right) - \left(\nabla\cdot\boldsymbol{\phi}^h, p-q^h\right),\ \forall q^h \in Q^h. \end{aligned}$$

Using that $a_\nu(\cdot,\cdot)$ corresponds with the inner product of V (see also Remark 4.3) one finds

$$\|\boldsymbol{\phi}^h\|_\nu^2 = a_\nu(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) \le \left|(\nu\mathbb{D}(\boldsymbol{\eta}),\mathbb{D}(\boldsymbol{\phi}^h))\right| + \frac{1}{2}\left|\left(\nabla\cdot\boldsymbol{\phi}^h, p-q^h\right)\right|. \tag{4.19}$$

Now, the terms on the right-hand side will be estimated by the Cauchy-Schwarz inequality

$$\left|(\nu\mathbb{D}(\boldsymbol{\eta}),\mathbb{D}(\boldsymbol{\phi}^h))\right| \le \|\nu^{1/2}\mathbb{D}(\boldsymbol{\eta})\|_0\|\nu^{1/2}\mathbb{D}(\boldsymbol{\phi}^h)\|_0 = \|\boldsymbol{\eta}\|_\nu\|\boldsymbol{\phi}^h\|_\nu.$$

With $\|\nabla\cdot\mathbf{v}\|_0 \le \|\nabla\mathbf{v}\|_0$ for $\mathbf{v} \in H^1_0(\Omega)$ (see Remark 3.8) and the norm equivalence (3.9) one finds for the second term

$$\begin{aligned} \left|(\nabla\cdot\boldsymbol{\phi}^h, p-q^h)\right| &\le \|\nabla\cdot\boldsymbol{\phi}^h\|_0\|p-q^h\|_0 \\ &\le \|\nabla\boldsymbol{\phi}^h\|_0\|p-q^h\|_0 \\ &\le C_K\nu_{\min}^{-1/2}\|\boldsymbol{\phi}^h\|_\nu\|p-q^h\|_0. \end{aligned} \tag{4.20}$$

These estimates can be applied in (4.19) and division by $\|\boldsymbol{\phi}^h\|_\nu \ne 0$ yields

$$\|\boldsymbol{\phi}^h\|_\nu \le \|\boldsymbol{\eta}\|_\nu + \frac{1}{2}C_K\nu_{\min}^{-1/2}\|p-q^h\|_0.$$

Trivially, this estimate still holds if $\|\boldsymbol{\phi}^h\|_\nu = 0$.
Applying the triangle inequality gives for all $I_h\mathbf{u} \in V^h_{\text{div}}$, $q^h \in Q^h$

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}^h\|_\nu &\le \|\boldsymbol{\eta}\|_\nu + \|\boldsymbol{\phi}^h\|_\nu \\ &\le 2\|\boldsymbol{\eta}\|_\nu + \frac{1}{2}C_K\nu_{\min}^{-1/2}\|p-q^h\|_0 \end{aligned}$$

and in particular,

$$\le 2\inf_{I_h\mathbf{u}\in V^h_{\text{div}}}\|\boldsymbol{\eta}\|_\nu + \frac{1}{2}C_K\nu_{\min}^{-1/2}\inf_{q^h\in Q^h}\|p-q^h\|_0.$$

Furthermore,

$$\begin{aligned} \inf_{I_h\mathbf{u}\in V^h_{\text{div}}}\|\boldsymbol{\eta}\|_\nu &= \inf_{I_h\mathbf{u}\in V^h_{\text{div}}}\|\mathbf{u}-I_h\mathbf{u}\|_\nu \\ &\overset{\text{Lemma 3.26}}{\le} \left(1+\frac{C_K}{\beta^h_{\text{is},\nu}\nu_{\min}^{1/2}}\right)\inf_{\mathbf{v}^h\in V^h}\|\mathbf{u}-\mathbf{v}^h\|_\nu. \end{aligned}$$

We have proved the statement of the Theorem

$$\|\mathbf{u}-\mathbf{u}^h\|_\nu \le 2\left(1+\frac{C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu_{\min}}}\right)\inf_{\mathbf{v}^h\in V^h}\|\mathbf{u}-\mathbf{v}^h\|_\nu + \frac{C_K}{2\sqrt{\nu_{\min}}}\inf_{q^h\in Q^h}\|p-q^h\|_0.$$

□

Remark 4.11. On the estimate (4.15).
The finite element error of the velocity in the ν-norm is estimated by best approximation errors for both, the pressure and the velocity, where the factors depend on the discrete inf-sup constant $\beta^h_{\mathrm{is},\nu}$ and on the maximal and minimal values of the viscosity function $\nu(\mathbf{x})$. All of the estimates in the remainder of this section will be of that form. Note that the shape of the viscosity function does not influence the estimate at all, i.e., viscosities with steep gradients like highly-oscillating ones do not lead to larger error estimates.

Remark 4.12. Estimate for $\|\mathbf{u}-\mathbf{u}^h\|_\nu$ with ν-independent norms.
In order to compare the order of convergence for examples with different viscosities we also need an estimate for $\|\mathbf{u}-\mathbf{u}^h\|_\nu$ where the right-hand side is ν-independent. This means one has to estimate the ν-norm on the right-hand side of (4.15) by the H^1_0-seminorm.
Using the norm equivalence (3.10) one gets

$$\|\mathbf{u}-\mathbf{u}^h\|_\nu \le 2\sqrt{\nu_{\max}}\left(1+\frac{C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu_{\min}}}\right)\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1 + \frac{C_K}{2\sqrt{\nu_{\min}}}\inf_{q^h\in Q^h}\|p-q^h\|_0. \tag{4.21}$$

Remark 4.13. Special Case: Estimates (4.15) and (4.21) for constant ν.
Assume that the viscosity ν is constant on the whole domain Ω, i.e.,

$$\nu(\mathbf{x}) = \nu_{\min} = \nu_{\max} = \nu, \quad \forall \mathbf{x}\in\Omega.$$

Then, in (4.15) one can write the constant in front of the norms. Using the knowledge from (3.38) on the ν-dependence of the inf-sup constant, $\beta^h_{\mathrm{is},\nu} = \nu_{\max}^{-1/2}C^{-1}$, and $\|\mathbb{D}(\mathbf{v})\|_0 \le \|\nabla\mathbf{v}\|_0$ yields

$$\begin{aligned}
\sqrt{\nu}\|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_0 \le& 2\left(1+\frac{C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu}}\right)\inf_{\mathbf{v}^h\in V^h}\sqrt{\nu}\|\mathbb{D}(\mathbf{u}-\mathbf{v}^h)\|_0\\
&+\frac{C_K}{2\sqrt{\nu}}\inf_{q^h\in Q^h}\|p-q^h\|_0\\
\iff \|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_0 \le& 2\left(1+C_K C\right)\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1 + \frac{C_K}{2\nu}\inf_{q^h\in Q^h}\|p-q^h\|_0.
\end{aligned} \tag{4.22}$$

One derives exactly the same estimate for (4.21).

Remark 4.14. Estimate for H^1_0-seminorm.
For a constant viscosity we have just derived an estimate for the finite element error in

a norm that does not depend on ν any longer.
As mentioned before, to ensure comparability of problems with different viscosities such an estimate is needed also for non-constant $\nu(\mathbf{x})$.
We use the norm equivalence (3.9) again to find

$$
\begin{aligned}
\left|\mathbf{u}-\mathbf{u}^h\right|_1 \leq & \frac{C_K}{\sqrt{\nu_{\min}}}\|\mathbf{u}-\mathbf{u}^h\|_\nu \\
\leq & \frac{C_K}{\sqrt{\nu_{\min}}}\Bigg[2\sqrt{\nu_{\max}}\left(1+\frac{C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu_{\min}}}\right)\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1 \\
& + \frac{C_K}{2\sqrt{\nu_{\min}}}\inf_{q^h\in Q^h}\|p-q^h\|_0\Bigg] \\
\leq & 2\,C_K\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\left(1+\frac{C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu_{\min}}}\right)\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1 \\
& + \frac{C_K^2}{2\,\nu_{\min}}\inf_{q^h\in Q^h}\|p-q^h\|_0.
\end{aligned}
$$

Theorem 4.15. Finite element error estimate for $\left|\mathbf{u}-\mathbf{u}^h\right|_1$, I.
Let the assumptions of Theorem 4.10 be fulfilled. For the $H_0^1(\Omega)$-seminorm of the error the following estimate holds.

$$
\begin{aligned}
\left|\mathbf{u}-\mathbf{u}^h\right|_1 \leq & 2\,C_K\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\left(1+\frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\mathrm{is},\nu}}\right)\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1 \\
& + \frac{C_K^2}{2\,\nu_{\min}}\inf_{q^h\in Q^h}\|p-q^h\|_0,
\end{aligned}
\tag{4.23}
$$

where $\beta^h_{\mathrm{is},\nu}$ depends on $\nu_{\max}$ like in (3.38).

Proof. See Remark 4.14. □

For the sake of completeness we prove a similar estimate based only on the theory with ν-independent norms, i.e., instead of using the norm equivalence on the result of Theorem 4.10, the proof of this theorem is revisited and modified.

Theorem 4.16. Finite Element error estimate $\left|\mathbf{u}-\mathbf{u}^h\right|_1$, II.
Let the assumptions of Theorem 4.10 be fulfilled and let in addition the discrete inf-sup condition (3.32) hold for V equipped with the ν-independent H_0^1-seminorm $\|\cdot\|_V = |\cdot|_1$. For the H_0^1-seminorm of the error the following estimate holds:

$$
\begin{aligned}
\left|\mathbf{u}-\mathbf{u}^h\right|_1 \leq & \left(1+\frac{C_K^2\nu_{\max}}{\nu_{\min}}\right)\left(1+\frac{1}{\beta^h_{\mathrm{is}}}\right)\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1 \\
& + \frac{C_K^2}{2\,\nu_{\min}}\inf_{q^h\in Q^h}\|p-q^h\|_0.
\end{aligned}
\tag{4.24}
$$

Note that here β^h_{is} is independent of ν (see Remark 4.17 for a discussion).

Proof. Analogously to the proof of Theorem 4.10 the error equation (4.17) is derived and one gets (4.19) by splitting the error again into the terms $\boldsymbol{\eta}$ and $\boldsymbol{\phi}^h$.
Using now $\|\nabla\boldsymbol{\phi}^h\|_0^2 \leq C_K^2 \nu_{\min}^{-1} \|\boldsymbol{\phi}^h\|_\nu^2$ yields

$$\|\nabla\boldsymbol{\phi}^h\|_0^2 \leq C_K^2 \nu_{\min}^{-1} \left(\left| \left(\nu \mathbb{D}(\boldsymbol{\eta}), \nabla\boldsymbol{\phi}^h \right) \right| + \frac{1}{2} \left| \left(\nabla \cdot \boldsymbol{\phi}^h, p - q^h \right) \right| \right). \tag{4.25}$$

Now, the terms on the left-hand side have to be estimated again, this time by

$$\left| \left(\nu \mathbb{D}(\boldsymbol{\eta}), \nabla\boldsymbol{\phi}^h \right) \right| \leq \|\nu \mathbb{D}(\boldsymbol{\eta})\|_0 \|\nabla\boldsymbol{\phi}^h\|_0 \leq \|\nu\nabla\boldsymbol{\eta}\|_0 \|\nabla\boldsymbol{\phi}^h\|_0. \tag{4.26}$$

With Remark 3.8 it is

$$\left| (\nabla \cdot \boldsymbol{\phi}^h, p - q^h) \right| \leq \|\nabla \cdot \boldsymbol{\phi}^h\|_0 \|p - q^h\|_0 \leq \|\nabla\boldsymbol{\phi}^h\|_0 \|p - q^h\|_0.$$

Thus, one finds

$$\begin{aligned} \|\nabla\boldsymbol{\phi}^h\|_0 &\leq C_K^2 \nu_{\min}^{-1} \left(\|\nu\nabla\boldsymbol{\eta}\|_0 + \frac{1}{2}\|p - q^h\|_0 \right) \\ &\leq C_K^2 \nu_{\min}^{-1} \left(\nu_{\max} \|\nabla\boldsymbol{\eta}\|_0 + \frac{1}{2}\|p - q^h\|_0 \right), \end{aligned}$$

which again also holds for $\|\nabla\boldsymbol{\phi}^h\|_0 = 0$. Proceeding like in the proof of Theorem 4.10 one finds

$$\begin{aligned} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_0 &\leq \|\nabla\boldsymbol{\eta}\|_0 + \|\nabla\boldsymbol{\phi}^h\|_0 \\ &\leq \left(1 + \frac{C_K^2 \nu_{\max}}{\nu_{\min}} \right) \inf_{I_h \mathbf{u} \in V_{\mathrm{div}}^h} \|\nabla\boldsymbol{\eta}\|_0 + \frac{C_K^2}{2\nu_{\min}} \inf_{q^h \in Q^h} \|p - q^h\|_0. \end{aligned}$$

We now use the best approximation property for the case that V is equipped with the $H_0^1(\Omega)$-seminorm formulated in Lemma 3.27, which gives

$$\begin{aligned} \inf_{I_h \mathbf{u} \in V_{\mathrm{div}}^h} \|\nabla\boldsymbol{\eta}\|_0 &= \inf_{I_h \mathbf{u} \in V_{\mathrm{div}}^h} \|\nabla(\mathbf{u} - I_h \mathbf{u})\|_0 \\ &\leq \left(1 + \frac{1}{\beta_{\mathrm{is}}^h} \right) \inf_{\mathbf{v}^h \in V^h} \|\nabla(\mathbf{u} - \mathbf{v}^h)\|_0. \end{aligned}$$

The resulting estimate, where all norms do not depend on the viscosity, is

$$\begin{aligned} \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_0 \leq &\left(1 + \frac{C_K^2 \nu_{\max}}{\nu_{\min}} \right) \left(1 + \frac{1}{\beta_{\mathrm{is}}^h} \right) \inf_{\mathbf{v}^h \in V^h} \|\nabla(\mathbf{u} - \mathbf{v}^h)\|_0 \\ &+ \frac{C_K^2}{2\,\nu_{\min}} \inf_{q^h \in Q^h} \|p - q^h\|_0. \end{aligned}$$

□

Remark 4.17. **The estimates in Theorems 4.15 and 4.16.**
The two estimates from Theorems 4.15 and 4.16 differ in the factor in front of the best approximation error in the velocity space $\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1$.
In (4.23) it is

$$2\,C_K\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\left(1+\frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\text{is},\nu}}\right)=2\,C_K\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}+\frac{2\,C_K^2\sqrt{\nu_{\max}}}{\nu_{\min}\beta^h_{\text{is},\nu}}$$

whereas in (4.24) one has

$$\left(1+\frac{C_K^2\nu_{\max}}{\nu_{\min}}\right)\left(1+\frac{1}{\beta^h_{\text{is}}}\right)=1+\frac{C_K^2\nu_{\max}}{\nu_{\min}}+\frac{1}{\beta^h_{\text{is}}}+\frac{C_K^2\nu_{\max}}{\nu_{\min}\beta^h_{\text{is}}}.$$

The main difference is that $\nu_{\max}$ seems to appear under the square root in the leading term of (4.23),

$$\frac{2\,C_K^2\sqrt{\nu_{\max}}}{\nu_{\min}\beta^h_{\text{is},\nu}}.$$

However, one has to keep in mind that the inf-sup constant $\beta^h_{\text{is},\nu}$ depends on $\nu_{\max}$ like $\beta^h_{\text{is},\nu}=\nu_{\max}^{-1/2}C^{-1}$ (see (3.38)). Thus, for the leading term in (4.23) one gets

$$\frac{2\,C_K^2\sqrt{\nu_{\max}}}{\nu_{\min}\beta^h_{\text{is},\nu}}=\frac{2\,C_K^2C\nu_{\max}}{\nu_{\min}}$$

and consequently there is no qualitative difference between both estimates. In this work the estimate from Theorem 4.15 will be used.

Remark 4.18. **Special case: Estimate** (4.23) **for constant** ν**.**
Consider again the case of a constant viscosity $\nu(\mathbf{x})=\nu_{\min}=\nu_{\max}=\nu,\quad\forall\mathbf{x}\in\Omega$.
For the estimate in Theorem 4.15 one finds

$$\left|\mathbf{u}-\mathbf{u}^h\right|_1\leq\tilde{C}\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1+\nu^{-1}\frac{C_K^2}{2}\inf_{q^h\in Q^h}\|p-q^h\|_0,\tag{4.27}$$

where again the ν-dependence of the inf-sup constant like in (3.38) has been used and $\tilde{C}=2\,C_K\,(1+C_KC)$ has been introduced for the sake of readability.
Note that this estimate differs from (4.22) only in the operator ∇ resp. $\mathbb{D}(\cdot)$ used in the norm and the constant C_K. Thus, the theory developed so far is consistent.

Corollary 4.19. **Finite element error estimate for** $\|\nabla\cdot\mathbf{u}^h\|_0$**.**
With the assumptions of Theorem 4.10 it is

$$\begin{aligned}\|\nabla\cdot\mathbf{u}^h\|_0\leq&2\,C_K\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\left(1+\frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\text{is},\nu}}\right)\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1\\&+\frac{C_K^2}{2\,\nu_{\min}}\inf_{q^h\in Q^h}\|p-q^h\|_0.\end{aligned}\tag{4.28}$$

Proof. The solution $\mathbf{u}$ is divergence-free, i.e., $\nabla \cdot \mathbf{u} = 0$. With $\|\nabla \cdot \mathbf{v}\|_0 \leq \|\nabla \mathbf{v}\|_0$ for $\mathbf{v} \in H_0^1(\Omega)$ (see Remark 3.8) one gets

$$\|\nabla \cdot \mathbf{u}^h\|_0 = \|\nabla \cdot (\mathbf{u} - \mathbf{u}^h)\|_0 \leq \|\nabla(\mathbf{u} - \mathbf{u}^h)\|_0.$$

Applying the estimate in Theorem 4.15 proves (4.28). □

Remark 4.20. Special case: Estimate (4.28) for constant ν.
Proceeding as in Remark 4.18 one derives for estimate (4.28), in case of a constant viscosity,

$$\|\nabla \cdot \mathbf{u}^h\|_0 \leq \tilde{C} \inf_{\mathbf{v}^h \in V^h} \left|\mathbf{u} - \mathbf{v}^h\right|_1 + \nu^{-1} \frac{C_K^2}{2} \inf_{q^h \in Q^h} \|p - q^h\|_0. \tag{4.29}$$

Remark 4.21. Scott-Vogelius space.
As mentioned in Section 3.4 there are pairs of spaces (e.g., the Scott-Voglius spaces) where in addition to conformity $V^h \subset V$ also the stronger condition $V_{\text{div}}^h \subset V_{\text{div}}$ is fulfilled. If this is the case one can make use of the fact that $\nabla \cdot \boldsymbol{\phi}^h = 0$ for $\boldsymbol{\phi}^h \in V_{\text{div}}^h$ which means that the second term in (4.19) vanishes. Thus, the estimates in Theorems 4.10 and 4.15 simplify as follows

$$\|\mathbf{u} - \mathbf{u}^h\|_\nu \leq 2 \left(1 + \frac{C_K}{\beta_{\text{is},\nu}^h \sqrt{\nu_{\min}}}\right) \inf_{\mathbf{v}^h \in V^h} \|\mathbf{u} - \mathbf{v}^h\|_\nu, \tag{4.30}$$

$$\left|\mathbf{u} - \mathbf{u}^h\right|_1 \leq 2\, C_K \sqrt{\frac{\nu_{\max}}{\nu_{\min}}} \left(1 + \frac{C_K}{\sqrt{\nu_{\min}} \beta_{\text{is},\nu}^h}\right) \inf_{\mathbf{v}^h \in V^h} \left|\mathbf{u} - \mathbf{v}^h\right|_1. \tag{4.31}$$

Furthermore, one observes that here a constant viscosity ν would not appear in the estimates any longer and leads to two estimates that are qualitatively the same and differ only in the deformation tensor:

$$\|\mathbb{D}(\mathbf{u} - \mathbf{u}^h)\|_0 \leq 2\,(1 + C_K C) \inf_{\mathbf{v}^h \in V^h} \|\mathbb{D}(\mathbf{u} - \mathbf{v}^h)\|_0, \tag{4.32}$$

$$\|\nabla(\mathbf{u} - \mathbf{u}^h)\|_0 \leq 2\, C_K\,(1 + C_K C) \inf_{\mathbf{v}^h \in V^h} \|\nabla(\mathbf{u} - \mathbf{v}^h)\|_0. \tag{4.33}$$

Theorem 4.22. Finite elemente error estimate for $\|p - p^h\|_0$.
Let the assumptions of Theorem 4.10 hold. The finite element error for the pressure p can be estimated by

$$\|p - p^h\|_0 \leq \left(1 + \frac{2\, C_K}{\beta_{\text{is},\nu}^h \sqrt{\nu_{\min}}}\right) \inf_{q^h \in Q^h} \|p - q^h\|_0 + \frac{4}{\beta_{\text{is},\nu}^h} \left(1 + \frac{C_K}{\beta_{\text{is},\nu}^h \sqrt{\nu_{\min}}}\right) \inf_{\mathbf{v}^h \in V^h} \|\mathbf{u} - \mathbf{v}^h\|_\nu. \tag{4.34}$$

Proof. It is

$$\|p - p^h\|_0 \leq \|p - q^h\|_0 + \|p^h - q^h\|_0,$$

where $q^h \in Q^h$ is arbitrary.
The finite element problem (4.13) can be rewritten as follows

$$b(\mathbf{v}^h, p^h - q^h) = \langle \mathbf{f}, \mathbf{v}^h \rangle_{V',V} - 2a_\nu(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, q^h). \tag{4.35}$$

The conformity of the finite element spaces guarantees that the continuous problem (4.3) is in particular fulfilled for all $\mathbf{v}^h \in V^h$, i.e.,

$$2a_\nu(\mathbf{u}, \mathbf{v}^h) + b(\mathbf{v}^h, p) = \langle \mathbf{f}, \mathbf{v}^h \rangle_{V',V}. \tag{4.36}$$

One can now insert the left-hand side of (4.36) in (4.35)

$$\begin{aligned} b(\mathbf{v}^h, p^h - q^h) &= 2a_\nu(\mathbf{u}, \mathbf{v}^h) + b(\mathbf{v}^h, p) - a_\nu(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, q^h) \\ &= 2a_\nu(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p - q^h), \quad \forall q^h \in Q^h, \mathbf{v}^h \in V^h. \end{aligned}$$

With the discrete inf-sup condition (3.32) with $\|\cdot\|_V = \|\cdot\|_\nu$ and (3.14) one gets

$$\begin{aligned} \|p^h - q^h\|_0 &\le \frac{1}{\beta^h_{\mathrm{is},\nu}} \sup_{\mathbf{v}^h \in V^h} \frac{b(\mathbf{v}^h, p^h - q^h)}{\|\mathbf{v}^h\|_\nu} \\ &= \frac{1}{\beta^h_{\mathrm{is},\nu}} \sup_{\mathbf{v}^h \in V^h} \frac{2a_\nu(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b^h(\mathbf{v}^h, p - q^h)}{\|\mathbf{v}^h\|_\nu} \\ &\le \frac{1}{\beta^h_{\mathrm{is},\nu}} \sup_{\mathbf{v}^h \in V^h} \frac{2\|\mathbf{u} - \mathbf{u}^h\|_\nu \|\mathbf{v}^h\|_\nu + \|\nabla \mathbf{v}^h\|_0 \|p - q^h\|_0}{\|\mathbf{v}^h\|_\nu} \\ &\le \frac{1}{\beta^h_{\mathrm{is},\nu}} \sup_{\mathbf{v}^h \in V^h} \frac{2\|\mathbf{u} - \mathbf{u}^h\|_\nu \|\mathbf{v}^h\|_\nu + C_K \nu_{\min}^{-1/2} \|\mathbf{v}^h\|_\nu \|p - q^h\|_0}{\|\mathbf{v}^h\|_\nu} \\ &\le \frac{1}{\beta^h_{\mathrm{is},\nu}} \left(2\|\mathbf{u} - \mathbf{u}^h\|_\nu + C_K \nu_{\min}^{-1/2} \|p - q^h\|_0 \right). \end{aligned}$$

Applying the estimate from Theorem 4.10 yields

$$\begin{aligned} \|p - p^h\|_0 &\le \inf_{q^h \in Q^h} \left(\|p - q^h\|_0 + \|p^h - q^h\|_0 \right) \\ &\le \inf_{q^h \in Q^h} \left(\|p - q^h\|_0 + \frac{1}{\beta^h_{\mathrm{is},\nu}} \left(2\|\mathbf{u} - \mathbf{u}^h\|_\nu + C_K \nu_{\min}^{-1/2} \|p - q^h\|_0 \right) \right) \\ &= \left(1 + \frac{C_K}{\beta^h_{\mathrm{is},\nu} \nu_{\min}^{1/2}} \right) \inf_{q^h \in Q^h} \|p - q^h\|_0 + \frac{2}{\beta^h_{\mathrm{is},\nu}} \|\mathbf{u} - \mathbf{u}^h\|_\nu \\ &\le \left(1 + \frac{C_K}{\beta^h_{\mathrm{is},\nu} \nu_{\min}^{1/2}} \right) \inf_{q^h \in Q^h} \|p - q^h\|_0 \\ &+ \frac{2}{\beta^h_{\mathrm{is},\nu}} \left[2 \left(1 + \frac{C_K}{\beta^h_{\mathrm{is},\nu} \nu_{\min}^{1/2}} \right) \inf_{\mathbf{v}^h \in V^h} \|\mathbf{u} - \mathbf{v}^h\|_\nu + \frac{1}{2} C_K \nu_{\min}^{-1/2} \inf_{q^h \in Q^h} \|p - q^h\|_0 \right] \\ &= \left(1 + \frac{2\, C_K}{\beta^h_{\mathrm{is},\nu} \nu_{\min}^{1/2}} \right) \inf_{q^h \in Q^h} \|p - q^h\|_0 + \frac{4}{\beta^h_{\mathrm{is},\nu}} \left(1 + \frac{C_K}{\beta^h_{\mathrm{is},\nu} \nu_{\min}^{1/2}} \right) \inf_{\mathbf{v}^h \in V^h} \|\mathbf{u} - \mathbf{v}^h\|_\nu. \end{aligned}$$

□

Remark 4.23. Estimate for $\|p - p^h\|_0$ with ν-independent norms.
Again, for the sake of comparability one needs the norms on the right-hand side of the estimate (4.34) to be independent of ν. Applying the norm equivalence (3.10) one gets

$$\begin{aligned}\|p - p^h\|_0 \leq &\left(1 + \frac{2\,C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu_{\min}}}\right) \inf_{q^h \in Q^h} \|p - q^h\|_0 \\ &+ \frac{4\sqrt{\nu_{\max}}}{\beta^h_{\mathrm{is},\nu}} \left(1 + \frac{C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu_{\min}}}\right) \inf_{\mathbf{v}^h \in V^h} \left|\mathbf{u} - \mathbf{v}^h\right|_1 . \end{aligned} \tag{4.37}$$

Remark 4.24. Scott-Vogelius space.
Considering spaces with $V^h_{\mathrm{div}} \subset V_{\mathrm{div}}$, the estimate for the finite element error of the pressure does not change qualitatively. It is

$$\|p - p^h\|_0 \leq \left(1 + \frac{C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu_{\min}}}\right)\left(\inf_{q^h \in Q^h} \|p - q^h\|_0 + \frac{4}{\beta^h_{\mathrm{is},\nu}} \inf_{\mathbf{v}^h \in V^h} \|\mathbf{u} - \mathbf{v}^h\|_\nu\right) \tag{4.38}$$

or respectively with ν-independent right-hand side

$$\|p - p^h\|_0 \leq \left(1 + \frac{C_K}{\beta^h_{\mathrm{is},\nu}\sqrt{\nu_{\min}}}\right)\left(\inf_{q^h \in Q^h} \|p - q^h\|_0 + \frac{4\sqrt{\nu_{\max}}}{\beta^h_{\mathrm{is},\nu}} \inf_{\mathbf{v}^h \in V^h} \left|\mathbf{u} - \mathbf{v}^h\right|_1\right). \tag{4.39}$$

The only change one observes, compared to the original estimates, is a factor in front of the Korn constant C_K.

Remark 4.25. Special case: Estimates (4.34), (4.37), (4.38) and (4.39) for constant ν.
For a constant viscosity $\nu(\mathbf{x}) = \nu$ the estimates (4.34), (4.37), (4.38) and (4.39) become

$$\|p - p^h\|_0 \leq C_1 \inf_{q^h \in Q^h} \|p - q^h\|_0 + \nu\, C_2 \inf_{\mathbf{v}^h \in V^h} \|\nabla(\mathbf{u} - \mathbf{v}^h)\|_0, \tag{4.40}$$

where $C_2 = 4\,C\,(1 + CC_K)$ and $C_1 = (1 + 2C_K C)$ in general, i.e., for the estimates (4.34) and (4.37), and $C_1 = (1 + C_k C)$ for Scott-Vogelius, i.e., the estimates (4.38) and (4.39).
Note that all estimates are qualitatively the same in the case of a constant viscosity.
If the best approximation error of the velocity is not zero, large values of ν result in large pressure error estimates which then scale linearly with ν as we will see later in the simulations in Section 5.

Remark 4.26. The dual Stokes problem.
We want to obtain an estimate for the error of the velocity field in the L^2-norm.
Of course, a simple estimate could be derived, e.g., from (4.23) with the Poincaré inequaltity,

$$\|\mathbf{u} - \mathbf{u}^h\|_0 \leq C \left|\mathbf{u} - \mathbf{u}^h\right|_1 .$$

However, this estimate is not optimal. Therefore we consider the dual Stokes problem.
Find $(\boldsymbol{\phi}_{\hat{\mathbf{f}}}, \xi_{\hat{\mathbf{f}}}) \in V \times Q$ such that for given $\hat{\mathbf{f}} \in L^2(\Omega)$

$$\begin{aligned} -2\nabla\cdot(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{\mathbf{f}}})) + \nabla\xi_{\hat{\mathbf{f}}} &= \hat{\mathbf{f}} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{\phi}_{\hat{\mathbf{f}}} &= 0 \quad \text{in } \Omega, \end{aligned} \tag{4.41}$$

with its weak form

$$
\begin{aligned}
2(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{\mathbf{f}}}),\mathbb{D}(\mathbf{v})) - (\nabla\cdot\mathbf{v},\xi_{\hat{\mathbf{f}}}) = (\hat{\mathbf{f}},\mathbf{v}), &\quad \forall \mathbf{v}\in V,\\
(\nabla\cdot\boldsymbol{\phi}_{\hat{\mathbf{f}}}, q) = 0, &\quad \forall q \in Q.
\end{aligned} \tag{4.42}
$$

For the problem to be regular we need the mapping

$$
(\boldsymbol{\phi}_{\hat{\mathbf{f}}},\xi_{\hat{\mathbf{f}}}) \mapsto -2\nabla\cdot(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{\mathbf{f}}})) + \nabla\xi_{\hat{\mathbf{f}}}
$$

to be an isomorphism from $H^2(\Omega)\cap V \times H^1(\Omega)\cap Q$ to $L^2(\Omega)$.

Theorem 4.27. **Finite error estimate for $\|\mathbf{u}-\mathbf{u}^h\|_0$.**
With the assumptions of Theorem 4.10 and $(\boldsymbol{\phi}_{\hat{\mathbf{f}}},\xi_{\hat{\mathbf{f}}})$ being the regular solution of (4.41) the following error estimate for the L^2 norm of the finite element error holds

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|_0 \le & \left(2\,\|\mathbf{u}-\mathbf{u}^h\|_\nu + \frac{C_K}{\sqrt{\nu_{\min}}}\inf_{q^h\in Q^h}\|p-q^h\|_0\right)\\
&\times \sup_{\hat{\mathbf{f}}\in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0}\left[\left(1+\frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\mathrm{is},\nu}}\right)\inf_{\boldsymbol{\phi}^h\in V^h}\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\|_\nu\right.\\
&\left.+\frac{C_K}{\sqrt{\nu_{\min}}}\inf_{r^h\in Q^h}\|\xi_{\hat{\mathbf{f}}}-r^h\|_0\right].
\end{aligned} \tag{4.43}
$$

Proof. First of all, we make use of the definition of the L^2-norm

$$
\|\mathbf{u}-\mathbf{u}^h\|_0 = \sup_{\hat{\mathbf{f}}\in L^2(\Omega)}\frac{(\hat{\mathbf{f}},\mathbf{u}-\mathbf{u}^h)}{\|\hat{\mathbf{f}}\|_0}. \tag{4.44}
$$

For the weak formulation of the regular Stokes problem (4.42) we can choose $\mathbf{v}=\mathbf{u}-\mathbf{u}^h$ as test function and get

$$
(\hat{\mathbf{f}},\mathbf{u}-\mathbf{u}^h) = 2\left(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{\mathbf{f}}}),\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\right) - \left(\nabla\cdot(\mathbf{u}-\mathbf{u}^h),\xi_{\hat{\mathbf{f}}}\right). \tag{4.45}
$$

Using the weak form of the Stokes problem (4.3) and the corresponding finite element problem formulation (4.13) one finds for $\boldsymbol{\phi}^h\in V^h_{\mathrm{div}}\subset V$ and $q^h\in Q^h$ arbitrary

$$
\begin{aligned}
2\left(\nu\mathbb{D}(\boldsymbol{\phi}^h),\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\right) =& 2\left(\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\boldsymbol{\phi}^h)\right) - 2\left(\nu\mathbb{D}(\mathbf{u}^h),\mathbb{D}(\boldsymbol{\phi}^h)\right)\\
=& \underbrace{(\nabla\cdot\boldsymbol{\phi}^h,p)}_{\neq 0,\quad \boldsymbol{\phi}^h\notin V_{\mathrm{div}}} + (\mathbf{f},\boldsymbol{\phi}^h) - \underbrace{(\nabla\cdot\boldsymbol{\phi}^h,p^h)}_{=0,\quad \boldsymbol{\phi}^h\in V^h_{\mathrm{div}}} - (\mathbf{f},\boldsymbol{\phi}^h)\\
=& (\nabla\cdot\boldsymbol{\phi}^h,p) = (\nabla\cdot\boldsymbol{\phi}^h,p-q^h).
\end{aligned} \tag{4.46}
$$

Furthermore, it is

$$
(\underbrace{\nabla\cdot\boldsymbol{\phi}_{\hat{\mathbf{f}}}}_{=0},p-q^h) = 0,\quad \forall\, q^h\in Q^h \tag{4.47}
$$

and

$$
(\nabla\cdot(\mathbf{u}-\mathbf{u}^h),r^h) = 0,\quad \forall\, r^h\in Q^h\subset Q. \tag{4.48}
$$

Extending now (4.45) by the help of those terms leads to

$$\begin{aligned}
(\hat{\mathbf{f}}, \mathbf{u}-\mathbf{u}^h) &\overset{(4.46)}{=} 2\left(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{\mathbf{f}}}), \mathbb{D}(\mathbf{u}-\mathbf{u}^h)\right) - \left(\nabla\cdot(\mathbf{u}-\mathbf{u}^h), \xi_{\hat{\mathbf{f}}}\right) \\
&\quad -2\left(\nu\mathbb{D}(\boldsymbol{\phi}^h), \mathbb{D}(\mathbf{u}-\mathbf{u}^h)\right) + (\nabla\cdot\boldsymbol{\phi}^h, p-q^h) \\
&= 2\left(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h), \mathbb{D}(\mathbf{u}-\mathbf{u}^h)\right) - \left(\nabla\cdot(\mathbf{u}-\mathbf{u}^h), \xi_{\hat{\mathbf{f}}}\right) \\
&\quad +(\nabla\cdot\boldsymbol{\phi}^h, p-q^h) \\
&\overset{(4.47),(4.48)}{=} 2\left(\nu\mathbb{D}(\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h), \mathbb{D}(\mathbf{u}-\mathbf{u}^h)\right) - \left(\nabla\cdot(\mathbf{u}-\mathbf{u}^h), \xi_{\hat{\mathbf{f}}}-r^h\right) \\
&\quad +(\nabla\cdot(\boldsymbol{\phi}^h-\boldsymbol{\phi}_{\hat{\mathbf{f}}}), p-q^h),
\end{aligned} \tag{4.49}$$

for $\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h$ and $q^h, r^h \in Q^h$. We now want to estimate this expression by the Cauchy-Schwarz inequality, (3.11), (3.14), and the norm equivalence (3.9)

$$\begin{aligned}
\left|(\hat{\mathbf{f}}, \mathbf{u}-\mathbf{u}^h)\right| \leq & 2\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\|_\nu \|\mathbf{u}-\mathbf{u}^h\|_\nu \\
& + \|\nabla(\mathbf{u}-\mathbf{u}^h)\|_0 \|\xi_{\hat{\mathbf{f}}}-r^h\|_0 + \|\nabla(\boldsymbol{\phi}^h-\boldsymbol{\phi}_{\hat{\mathbf{f}}})\|_0 \|p-q^h\|_0 \\
\leq & 2\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\|_\nu \|\mathbf{u}-\mathbf{u}^h\|_\nu \\
& + \frac{C_K}{\sqrt{\nu_{\min}}}\|\mathbf{u}-\mathbf{u}^h\|_\nu \|\xi_{\hat{\mathbf{f}}}-r^h\|_0 + \frac{C_K}{\sqrt{\nu_{\min}}}\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\|_\nu \|p-q^h\|_0 \\
\leq & \left(2\|\mathbf{u}-\mathbf{u}^h\|_\nu + \frac{C_K}{\sqrt{\nu_{\min}}}\|p-q^h\|_0\right) \times \left(\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\|_\nu + \frac{C_K}{\sqrt{\nu_{\min}}}\|\xi_{\hat{\mathbf{f}}}-r^h\|_0\right).
\end{aligned}$$

Note that in order to apply (3.14) we need the functions to be in $H_0^1(\Omega)$ which is the case since $\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h \subset V$, $\boldsymbol{\phi}_{\hat{\mathbf{f}}} \in H^2 \cap V$, $\mathbf{u} \in V$, $\mathbf{u}^h \in V^h \subset V$ and $V = H_0^1(\Omega)$.
The inequality in the last line holds since multiplying the brackets yields the non-negative extra term

$$\frac{C_K^2}{\nu_{\min}}\|p-q^h\|_0 \|\xi_{\hat{\mathbf{f}}}-r^h\|_0.$$

Now, the L^2-norm of the velocity can be estimated as follows

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|_0 \leq & \sup_{\hat{\mathbf{f}}\in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0}\left(2\|\mathbf{u}-\mathbf{u}^h\|_\nu + \frac{C_K}{\sqrt{\nu_{\min}}}\|p-q^h\|_0\right) \\
& \times \left(\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\|_\nu + \frac{C_K}{\sqrt{\nu_{\min}}}\|\xi_{\hat{\mathbf{f}}}-r^h\|_0\right).
\end{aligned}$$

Since q^h, $\boldsymbol{\phi}^h$ and r^h were chosen arbitrarily, this yields in particular

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|_0 \leq & \left(2\|\mathbf{u}-\mathbf{u}^h\|_\nu + \frac{C_K}{\sqrt{\nu_{\min}}}\inf_{q^h\in Q^h}\|p-q^h\|_0\right) \\
& \times \sup_{\hat{\mathbf{f}}\in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0}\left(\inf_{\phi^h\in V_{\mathrm{div}}^h}\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\|_\nu + \frac{C_K}{\sqrt{\nu_{\min}}}\inf_{r^h\in Q^h}\|\xi_{\hat{\mathbf{f}}}-r^h\|_0\right).
\end{aligned}$$

Using Lemma 3.26, i.e.,

$$\inf_{\phi^h \in V^h_{\mathrm{div}}} \|\boldsymbol{\phi}_{\hat{\mathbf{f}}} - \boldsymbol{\phi}^h\|_\nu \leq \left(1 + \frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\mathrm{is},\nu}}\right) \inf_{\phi^h \in V^h} \|\boldsymbol{\phi}_{\hat{\mathbf{f}}} - \boldsymbol{\phi}^h\|_\nu,$$

one finally gets

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}^h\|_0 \leq & \left(2\|\mathbf{u} - \mathbf{u}^h\|_\nu + \frac{C_K}{\sqrt{\nu_{\min}}} \inf_{q^h \in Q^h} \|p - q^h\|_0\right) \\
& \times \sup_{\hat{\mathbf{f}} \in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0} \left[\left(1 + \frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\mathrm{is},\nu}}\right) \inf_{\phi^h \in V^h} \|\boldsymbol{\phi}_{\hat{\mathbf{f}}} - \boldsymbol{\phi}^h\|_\nu \right. \\
& \left. + \frac{C_K}{\sqrt{\nu_{\min}}} \inf_{r^h \in Q^h} \|\xi_{\hat{\mathbf{f}}} - r^h\|_0 \right].
\end{aligned}$$

□

Remark 4.28. **The appearance of the right-hand side $\|\hat{\mathbf{f}}\|_0$ of the dual problem.**
When interpreting the estimate (4.43) it is of utmost importance to note that here the norm of the right-hand side $\hat{\mathbf{f}}$ of the dual problem appears.
All previous estimates were independent of the right-hand side $\mathbf{f}$ of the Stokes problem, i.e. $\mathbf{f}$ never appeared in the estimates.
This is important since the right-hand side $\hat{\mathbf{f}}$ depends on ν. A detailed discussion and the analysis for the case that ν is constant can be found in Remark 4.31.

Remark 4.29. **Estimate for $\|\mathbf{u} - \mathbf{u}^h\|_0$ with ν-independent norms.**
Using the norm equivalence (3.9), we can again derive an estimate where the norms on the right-hand side do not depend on the viscosity ν

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}^h\|_0 \leq & \left(2\sqrt{\nu_{\max}}\left|\mathbf{u} - \mathbf{u}^h\right|_1 + \frac{C_K}{\sqrt{\nu_{\min}}} \inf_{q^h \in Q^h} \|p - q^h\|_0\right) \\
& \times \sup_{\hat{\mathbf{f}} \in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0} \left[\left(1 + \frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\mathrm{is},\nu}}\right) \sqrt{\nu_{\max}} \inf_{\phi^h \in V^h} \left|\boldsymbol{\phi}_{\hat{\mathbf{f}}} - \boldsymbol{\phi}^h\right|_1 \right. \\
& \left. + \frac{C_K}{\sqrt{\nu_{\min}}} \inf_{r^h \in Q^h} \|\xi_{\hat{\mathbf{f}}} - r^h\|_0 \right].
\end{aligned} \tag{4.50}$$

However, also here $\|\hat{\mathbf{f}}\|_0$ is not independent of ν.

Remark 4.30. **Scott-Vogelius space.**
Considering again the Scott-Vogelius finite element space one obtains the estimates

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}^h\|_0 \leq & 2\left\|\mathbf{u} - \mathbf{u}^h\right\|_\nu \\
& \times \sup_{\hat{\mathbf{f}} \in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0} \left[\left(1 + \frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\mathrm{is},\nu}}\right) \inf_{\phi^h \in V^h} \left\|\boldsymbol{\phi}_{\hat{\mathbf{f}}} - \boldsymbol{\phi}^h\right\|_\nu \right]
\end{aligned} \tag{4.51}$$

and

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_0 \leq & 2\sqrt{\nu_{\max}}\left|\mathbf{u}-\mathbf{u}^h\right|_1 \\ & \times \sup_{\hat{\mathbf{f}}\in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0}\left[\left(1+\frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\mathrm{is},\nu}}\right)\sqrt{\nu_{\max}} \inf_{\boldsymbol{\phi}^h\in V^h}\left|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\right|_1\right]. \end{aligned} \tag{4.52}$$

To prove these estimates one performs the proof of Theorem 4.27 and uses in (4.49) that $\nabla\cdot\left(\boldsymbol{\phi}^h-\boldsymbol{\phi}_{\hat{\mathbf{f}}}\right)=0$ and $\nabla\cdot\left(\mathbf{u}-\mathbf{u}^h\right)=0$.

Remark 4.31. **Special case: Estimates** (4.43)**,** (4.50)**,** (4.51) **and** (4.52) **for constant viscosity and the role of** $\|\hat{\mathbf{f}}\|_0$**.**
As mentioned already in Remark 4.28, the fact that the right-hand side $\|\hat{\mathbf{f}}\|_0$ of the dual problem (4.41) appears in the estimates has to be considered carefully.
The crucial point here is to understand how the right-hand of the dual problem $\hat{\mathbf{f}}$, its solution $\boldsymbol{\phi}_{\hat{\mathbf{f}}}$ and the viscosity ν are connected.
Looking at the first equation in (4.41), there are two equivalent approaches to think about varying ν.

1. Assume the right-hand side $\hat{\mathbf{f}}$ is fixed. If ν is varied, this leads to different solutions $\boldsymbol{\phi}_{\hat{\mathbf{f}},\nu}$ and $\xi_{\hat{\mathbf{f}},\nu}$ that depend on ν.
2. Assume a solution $\boldsymbol{\phi}$ is prescribed. If ν is varied, this must lead to a new right-hand side $\hat{\mathbf{f}}_\nu$ (and possibly also to a new pressure $\xi_{\hat{\mathbf{f}},\nu}$).

This becomes clear if one considers the problem with a constant viscosity

$$\nu(\mathbf{x})=\nu_{\min}=\nu_{\max}=\nu.$$

Let's consider the second case, i.e., there is a prescribed solution $\boldsymbol{\phi}$, that solves (4.41) for arbitrary values of ν, in particular for $\nu=1$.

$$-2\nabla\cdot(\mathbb{D}(\boldsymbol{\phi}))+\nabla\xi_{\hat{\mathbf{f}},1}=\hat{\mathbf{f}}_1 \quad \text{in } \Omega. \tag{4.53}$$

For an arbitrary ν, the same $\boldsymbol{\phi}$ solves

$$-2\nabla\cdot(\nu\mathbb{D}(\boldsymbol{\phi}))+\nabla\xi_{\hat{\mathbf{f}},\nu}=\hat{\mathbf{f}}_\nu \quad \text{in } \Omega, \tag{4.54}$$

and since ν is constant, this is equivalent to

$$-2\nabla\cdot(\mathbb{D}(\boldsymbol{\phi}))+\nabla\left(\nu^{-1}\xi_{\hat{\mathbf{f}},\nu}\right)=\nu^{-1}\hat{\mathbf{f}}_\nu \quad \text{in } \Omega. \tag{4.55}$$

Comparing (4.53) and (4.55), this yields $\hat{\mathbf{f}}_\nu=\nu\,\hat{\mathbf{f}}_1$ and $\xi_{\hat{\mathbf{f}},\nu}=\nu\,\xi_{\hat{\mathbf{f}},1}$.
Consequently, the right-hand side $\hat{\mathbf{f}}_\nu$ of the dual Stokes problem as well as the solution for the pressure $\xi_{\hat{\mathbf{f}},\nu}$ scale with ν.

Using this knowledge for the estimates (4.50) and (4.52), one finds

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|_0 \leq & \left(2\sqrt{\nu_{\max}}\left|\mathbf{u}-\mathbf{u}^h\right|_1 + \frac{C_K}{\sqrt{\nu_{\min}}} \inf_{q^h\in Q^h} \|p-q^h\|_0\right) \\
& \times \sup_{\hat{\mathbf{f}}_\nu \in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}_\nu\|_0} \left[\left(1+\frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\mathrm{is},\nu}}\right)\sqrt{\nu_{\max}} \inf_{\phi^h\in V^h}\left|\boldsymbol{\phi}-\boldsymbol{\phi}^h\right|_1 \right. \\
& \left. + \frac{C_K}{\sqrt{\nu_{\min}}} \inf_{r^h\in Q^h} \|\xi_{\hat{\mathbf{f}},\nu} - r^h\|_0\right] \\
= & \left(2\sqrt{\nu}\left|\mathbf{u}-\mathbf{u}^h\right|_1 + \frac{C_K}{\sqrt{\nu}} \inf_{q^h\in Q^h} \|p-q^h\|_0\right) \\
& \times \sup_{\hat{\mathbf{f}}_1 \in L^2(\Omega)} \frac{1}{\|\nu\hat{\mathbf{f}}_1\|_0} \left[C_1\sqrt{\nu} \inf_{\phi^h\in V^h}\left|\boldsymbol{\phi}-\boldsymbol{\phi}^h\right|_1 + \frac{C_K}{\sqrt{\nu}} \inf_{r^h\in Q^h} \|\nu\xi_{\hat{\mathbf{f}},1} - r^h\|_0\right] \\
= & \left(2\left|\mathbf{u}-\mathbf{u}^h\right|_1 + \nu^{-1} C_K \inf_{q^h\in Q^h} \|p-q^h\|_0\right) \\
& \times \sup_{\hat{\mathbf{f}} \in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0} \left[C_1 \inf_{\phi^h\in V^h}\left|\boldsymbol{\phi}-\boldsymbol{\phi}^h\right|_1 + C_K \inf_{r^h\in Q^h} \|\xi_{\hat{\mathbf{f}}} - r^h\|_0\right]. \qquad (4.56)
\end{aligned}$$

where $C_1 = (1+CC_K)$. Note that the second factor does not depend on ν.
The estimate for the Scott-Vogelius finite element space simplifies to

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|_0 \leq & 2\sqrt{\nu}\left|\mathbf{u}-\mathbf{u}^h\right|_1 \times \sup_{\hat{\mathbf{f}}_1 \in L^2(\Omega)} \frac{1}{\|\nu\hat{\mathbf{f}}_1\|_0} \left[C_1\sqrt{\nu} \inf_{\phi^h\in V^h}\left|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\right|_1\right] \\
= & 2\left|\mathbf{u}-\mathbf{u}^h\right|_1 \times \sup_{\hat{\mathbf{f}} \in L^2(\Omega)} \frac{1}{\|\hat{\mathbf{f}}\|_0} \left[C_1 \inf_{\phi^h\in V^h}\left|\boldsymbol{\phi}_{\hat{\mathbf{f}}}-\boldsymbol{\phi}^h\right|_1\right]. \qquad (4.57)
\end{aligned}$$

Thus, there is no ν-dependence of the error for the Scott-Vogelius element.

However, if the viscosity is not constant it is not clear how the right-hand side $\hat{\mathbf{f}}$ behaves. The above considerations give only an idea of what can be expected.
In the following, we will use the notations

$$\frac{1}{\|\hat{f}\|_0} = \frac{C}{\zeta_1(\nu)} \quad \text{and} \quad \|\xi_{\hat{\mathbf{f}}}\|_{H^1(\Omega)} = \zeta_2(\nu)\|\xi_{\hat{\mathbf{f}},1}\|_{H^1(\Omega)},$$

in order to account for the fact that, in general, $\|\hat{\mathbf{f}}\|_0$ and $\xi_{\hat{\mathbf{f}}}$ grow in ν in a non-linear manner.

Corollary 4.32. Orders of Convergence for Certain Spaces.
With the estimates derived so far the orders of convergence for certain pairs of finite element spaces can be derived. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with polyhedral and Lipschitz boundary provided with a regular and quasi-uniform family of triangulations $\{\mathcal{T}^h\}$. If $(\mathbf{u}, p) \in H^{k+1}(\Omega) \cap V \times H^k(\Omega) \cap Q$ is the solution of the Stokes problem (4.3)

then for the inf-sup-stable pairs of Taylor-Hood element P_k/P_{k-1}, Q_k/Q_{k-1}, $k \geq 2$ introduced in Section 3.4 and the pairs finite element spaces $Q_k/P_{k-1}^{\mathrm{disc}}$, $k \geq 2$, the following estimates hold[3]:

$$\|\mathbf{u}-\mathbf{u}^h\|_\nu \leq Ch^k \left(\frac{\nu_{\max}}{\sqrt{\nu_{\min}}}\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\sqrt{\nu_{\min}}}\|p\|_{H^k(\Omega)}\right), \tag{4.58}$$

$$\left|\mathbf{u}-\mathbf{u}^h\right|_1 \leq Ch^k \left(\frac{\nu_{\max}}{\nu_{\min}}\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\nu_{\min}}\|p\|_{H^k(\Omega)}\right), \tag{4.59}$$

$$\|\nabla\cdot\mathbf{u}^h\|_0 \leq Ch^k \left(\frac{\nu_{\max}}{\nu_{\min}}\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\nu_{\min}}\|p\|_{H^k(\Omega)}\right), \tag{4.60}$$

$$\|p-p^h\|_0 \leq Ch^k \left(\nu_{\max}\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\|p\|_{H^k(\Omega)}\right). \tag{4.61}$$

For the Scott-Vogelius finite element space $P_k/P_{k-1}^{\mathrm{disc}}$, $k \geq d$, the orders of convergence are the same but the terms in brackets change and of course, the divergence of $\mathbf{u}^h$ is zero since $\mathbf{u}^h \in V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$. One gets

$$\|\mathbf{u}-\mathbf{u}^h\|_\nu \leq Ch^k \left(\frac{\nu_{\max}}{\sqrt{\nu_{\min}}}\|\mathbf{u}\|_{H^{k+1}(\Omega)}\right), \tag{4.62}$$

$$\left|\mathbf{u}-\mathbf{u}^h\right|_1 \leq Ch^k \left(\frac{\nu_{\max}}{\nu_{\min}}\|\mathbf{u}\|_{H^{k+1}(\Omega)}\right), \tag{4.63}$$

$$\|\nabla\cdot\mathbf{u}^h\|_0 = 0. \tag{4.64}$$

The estimate for the finite element error of the pressure is exactly the same as (4.61).

Proof. In order to derive the estimates (4.58) - (4.61) one has to estimate the best approximation errors occurring in the estimates of Remark 4.12, Theorem 4.15, Corollary 4.19, and Remark 4.23. Taking into account that the interpolation error cannot be better than the best approximation error, one can apply the interpolation error estimates for the finite element spaces P_k, P_{k-1}, Q_k, Q_{k-1} see Theorem B.4 in the appendix.[4]
Starting with estimate (4.21), this yields for the finite element error in the ν-norm

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|_\nu \leq & 2\,\nu_{\max}^{1/2}\left(1+\frac{C_K}{\beta_{\mathrm{is},\nu}^h\nu_{\min}^{1/2}}\right)\inf_{\mathbf{v}^h\in V^h}\|\nabla(\mathbf{u}-\mathbf{v}^h)\|_0 + \frac{1}{2}C_K\nu_{\min}^{-1/2}\inf_{q^h\in Q^h}\|p-q^h\|_0 \\
\leq & C_1\nu_{\max}^{1/2}\left(1+C_2\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\right)\|D^1(\mathbf{u}-I_K\mathbf{u})\|_0 + C_3\frac{1}{\sqrt{\nu_{\min}}}\|D^0(p-I_Kp)\|_0 \\
\overset{B.4}{\leq} & Ch^k\left(\nu_{\max}^{1/2}\left(1+C_2\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\right)\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\sqrt{\nu_{\min}}}\|p\|_{H^k(\Omega)}\right) \\
\leq & Ch^k\left(\frac{\nu_{\max}}{\sqrt{\nu_{\min}}}\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\sqrt{\nu_{\min}}}\|p\|_{H^k(\Omega)}\right).
\end{aligned}$$

[3]The estimates do also hold for the finite element spaces $P_k^{\mathrm{bubble}}/P_k$, $k=1$ and $P_k^{\mathrm{bubble}}/P_{k-1}^{\mathrm{disc}}$ defined in [6] on p. 55 and p. 74, that will not be introduced in this work.

[4]For the pair $Q_k/P_{k-1}^{\mathrm{disc}}$ proving the optimal interpolation error is not straightforward, see [6], pp. 88 for a discussion. In this work, only such grids will be considered that the optimality can be guaranteed.

Note that terms of lower order have been neglected, i.e., one uses the estimate

$$\nu_{\max}^{1/2}\left(1+C_2\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\right)\|\mathbf{u}\|_{H^{k+1}(\Omega)} \leq C\frac{\nu_{\max}}{\sqrt{\nu_{\min}}}\|\mathbf{u}\|_{H^{k+1}(\Omega)}$$

which holds asymptotically for large values of $\nu_{\max}$.
The other estimates (4.59) - (4.63) are derived in an analogous way. □

Corollary 4.33. **Order of convergence for the L^2-norm of the velocity.**
For determining the order of convergence for the L^2-error of the velocity, the considerations from Remark 4.31 have to be taken into account again.
If the dual Stokes problem (4.41) has a unique solution $\left(\boldsymbol{\phi}, \xi_{\hat{\mathbf{f}},\nu}\right)$ it is

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_0 \leq & Ch^{k+1}\frac{1}{\zeta_1(\nu)}\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\left(\frac{\nu_{\max}}{\nu_{\min}}\|\mathbf{u}\|_{H^{k+1}}+\frac{1}{\nu_{\min}}\|p\|_{H^k(\Omega)}\right)\\ & \times\left(\nu_{\max}\|\boldsymbol{\phi}\|_{H^2(\Omega)}+\zeta_2(\nu)\|\xi_{\hat{\mathbf{f}},1}\|_{H^1(\Omega)}\right), \end{aligned} \tag{4.65}$$

for the Taylor-Hood element Q_2/Q_1 and the finite element pair Q_2/P_1^{disc} and

$$\|\mathbf{u}-\mathbf{u}^h\|_0 \leq Ch^{k+1}\frac{1}{\zeta_1(\nu)}\left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{3/2}\nu_{\max}\|\mathbf{u}\|_{H^{k+1}}\times\|\boldsymbol{\phi}\|_{H^2(\Omega)}, \tag{4.66}$$

for the Scott-Vogelius element.

Proof. Let's write (4.50) again with explicit ν-dependence.

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_0 \leq & \left(2\sqrt{\nu_{\max}}\left|\mathbf{u}-\mathbf{u}^h\right|_1+\frac{C_K}{\sqrt{\nu_{\min}}}\inf_{q^h\in Q^h}\|p-q^h\|_0\right)\\ & \times\sup_{\hat{\mathbf{f}}_\nu\in L^2(\Omega)}\frac{1}{\|\hat{\mathbf{f}}_\nu\|_0}\left[\left(1+\frac{C_K}{\sqrt{\nu_{\min}}\beta^h_{\text{is},\nu}}\right)\sqrt{\nu_{\max}}\inf_{\phi^h\in V^h}\left|\boldsymbol{\phi}-\boldsymbol{\phi}^h\right|_1\right.\\ & \left.+\frac{C_K}{\sqrt{\nu_{\min}}}\inf_{r^h\in Q^h}\|\xi_{\hat{\mathbf{f}},\nu}-r^h\|_0\right]. \end{aligned} \tag{4.67}$$

For the second factor in (4.67) interpolation error estimates can be applied for the best approximation errors of $\boldsymbol{\phi}$ and $\xi_{\hat{\mathbf{f}},\nu}$.
Note that for the dual problem one requires $\boldsymbol{\phi}$ to be in $H^2(\Omega)$ and $\xi_{\hat{\mathbf{f}},\nu}$ to be in $H^1(\Omega)$ such that

$$\inf_{\phi^h\in V^h}\|\boldsymbol{\phi}-\boldsymbol{\phi}^h\|_\nu \leq \|\mathbb{D}^1(\boldsymbol{\phi}-I_K\boldsymbol{\phi}_{\hat{\mathbf{f}}})\|_0 \leq \tilde{C}h^{2-1}\|\boldsymbol{\phi}\|_{H^2(\Omega)} \leq Ch\|\boldsymbol{\phi}\|_{H^2(\Omega)}$$

and

$$\inf_{r^h\in Q^h}\|\xi_{\hat{\mathbf{f}},\nu}-r^h\| \leq \|\mathbb{D}^0(\xi_{\hat{\mathbf{f}},\nu}-I_K\xi_{\hat{\mathbf{f}},\nu})\|_0 \leq \tilde{C}h^{1-0}\|\xi_{\hat{\mathbf{f}},\nu}\|_{H^1(\Omega)} \leq Ch\|\xi_{\hat{\mathbf{f}},\nu}\|_{H^1(\Omega)}.$$

This results in the additional power of h in estimate (4.65) as the following transformations of (4.67) show:

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}^h\|_0 \leq & \left(2\sqrt{\nu_{\max}}\left|\mathbf{u}-\mathbf{u}^h\right|_1 + \frac{1}{\sqrt{\nu_{\min}}}Ch^k\|p\|_{H^k(\Omega)}\right) \\
& \times \frac{C}{\zeta_1(\nu)}\left[\left(1+C\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\right)\sqrt{\nu_{\max}}Ch\|\boldsymbol{\phi}\|_{H^2(\Omega)} + \frac{1}{\sqrt{\nu_{\min}}}Ch\|\xi_{\hat{\mathbf{f}},\nu}\|_{H^1(\Omega)}\right] \\
\leq & \left(2\sqrt{\nu_{\max}}\left(Ch^k\left(\frac{\nu_{\max}}{\nu_{\min}}\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\nu_{\min}}\|p\|_{H^k(\Omega)}\right)\right) + Ch^k\frac{1}{\sqrt{\nu_{\min}}}\|p\|_{H^k(\Omega)}\right) \\
& \times \frac{C}{\zeta_1(\nu)}\left[Ch\left(1+C\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\right)\sqrt{\nu_{\max}}\|\boldsymbol{\phi}\|_{H^2(\Omega)} + Ch\,\frac{\zeta_2(\nu)}{\sqrt{\nu_{\min}}}\|\xi_{\hat{\mathbf{f}},1}\|_{H^1(\Omega)}\right] \\
\leq & Ch^{k+1}\frac{1}{\zeta_1(\nu)}\left(\sqrt{\nu_{\max}}\frac{\nu_{\max}}{\nu_{\min}}\|\mathbf{u}\|_{H^{k+1}} + \frac{\sqrt{\nu_{\max}}}{\nu_{\min}}\|p\|_{H^k(\Omega)}\right) \\
& \times \left(\frac{\nu_{\max}}{\sqrt{\nu_{\min}}}\|\boldsymbol{\phi}\|_{H^2(\Omega)} + \frac{\zeta_2(\nu)}{\sqrt{\nu_{\min}}}\|\xi_{\hat{\mathbf{f}},1}\|_{H^1(\Omega)}\right) \\
= & Ch^{k+1}\frac{1}{\zeta_1(\nu)}\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\left(\frac{\nu_{\max}}{\nu_{\min}}\|\mathbf{u}\|_{H^{k+1}} + \frac{1}{\nu_{\min}}\|p\|_{H^k(\Omega)}\right) \\
& \times \left(\nu_{\max}\|\boldsymbol{\phi}\|_{H^2(\Omega)} + \zeta_2(\nu)\|\xi_{\hat{\mathbf{f}},1}\|_{H^1(\Omega)}\right).
\end{aligned}$$

Note that terms of lower order have been neglected again and it was made use of the notations

$$\frac{1}{\|\hat{f}\|_0} = \frac{C}{\zeta_1(\nu)} \quad \text{and} \quad \|\xi_{\hat{\mathbf{f}}}\|_{H^1(\Omega)} = \zeta_2(\nu)\|\xi_{\hat{\mathbf{f}},1}\|_{H^1(\Omega)}.$$

□

Remark 4.34. **On the estimates** (4.65) **and** (4.66)**.**
It is hardly possible to draw conclusions from the estimates (4.65) and (4.66) on how the error $\|\mathbf{u}-\mathbf{u}^h\|_0$ depends on different viscosity functions since the functions ζ_1 and ζ_2 are unknown.
However, for constant ν this is possible and gives an idea on the behavior for non-constant viscosities. With $\zeta_1 = \zeta_2 = \nu_{\min} = \nu_{\max} = \nu$, one has for the Talyor-Hood element and Q_2/P_1^{disc}

$$\|\mathbf{u}-\mathbf{u}^h\|_0 \leq Ch^{k+1}\left(\|\mathbf{u}\|_{H^{k+1}} + \frac{1}{\nu}\|p\|_{H^k(\Omega)}\right) \times \left(\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}\|_{H^2(\Omega)} + \|\xi_{\hat{\mathbf{f}},1}\|_{H^1(\Omega)}\right), \tag{4.68}$$

and for the Scott-Vogelius element,

$$\|\mathbf{u}-\mathbf{u}^h\|_0 \leq Ch^{k+1}\|\mathbf{u}\|_{H^{k+1}} \times \|\boldsymbol{\phi}_{\hat{\mathbf{f}}}\|_{H^2(\Omega)}. \tag{4.69}$$

Remark 4.35. **ν-dependence of the solution $(\mathbf{u}, p)$ in estimates (4.58) - (4.66) and (4.68) - (4.69).**
In general, in the estimates (4.58) - (4.66) and (4.68) - (4.69) there is an additional

implicit ν-dependence hidden in the solution $(\mathbf{u}, p)$ which, of course, is not independent of ν in general. However, in the simulations presented in this work, this can be ignored as the solution was prescribed and only the right-hand side changed for different viscosity functions (see Section 5.1 and Remark 4.31).

5. Numerical Studies

5.1. Implemented Examples

The theory of the Stokes problem with variable viscosity is tested with a steady-state example where the solution is prescribed in the unit square $(0,1)^2$ and homogeneous Dirichlet boundary conditions are imposed.
As the velocity field has to be divergence-free one defines it as

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix}, \tag{5.1}$$

where ψ is the stream function

$$\psi(x,y) = 100x^2(1-x)^2y^2(1-y)^2. \tag{5.2}$$

Thus, it is

$$\mathbf{u}(x,y) = 200 \begin{pmatrix} x^2(1-x)^2y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix}, \tag{5.3}$$

and as required

$$\nabla \cdot \mathbf{u} = \partial_x u_1 + \partial_y u_2 = \partial_{xy}\psi - \partial_{yx}\psi = 0,$$

by the Theorem of Schwarz. The equations are equipped with homogeneous Dirichlet boundary conditions, i.e., $\mathbf{u} = \mathbf{g} = \mathbf{0}$ on Γ such that the compatibility condition

$$0 = \int_\Omega \nabla \cdot \mathbf{u}(\mathbf{x})\, d\mathbf{x} = \int_\Gamma (\mathbf{u}\cdot\mathbf{n})(\mathbf{s})\, d\mathbf{s} = \int_\Gamma (\mathbf{g}\cdot\mathbf{n})(\mathbf{s})\, d\mathbf{s}, \tag{5.4}$$

is fulfilled. Figure 1 shows the velocity field $\mathbf{u}(x,y)$ of the example.

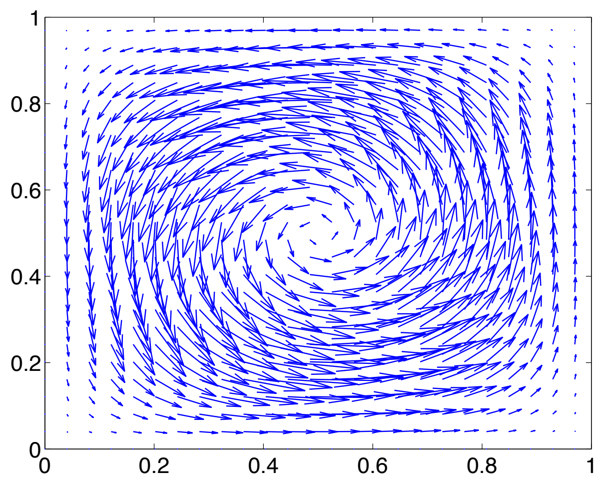


Figure 1: Velocity field $\mathbf{u}(x,y)$ on the unit square $(0,1)^2$.

Since the boundary conditions are chosen to be of Dirichlet type the pressure is unique only up to an additive constant and one, thus, needs a further condition in order to determine this constant. Here, one requires that the mean integral value of the pressure vanishes,

$$\int_\Omega p(\mathbf{x})\,d\mathbf{x} = 0,$$

i.e., the pressure is from $L_0^2(\Omega)$. In the present example it is chosen to be

$$p(x,y) = 10\left(\left(x-\frac{1}{2}\right)^3 y^2 + (1-x)^3\left(y-\frac{1}{2}\right)^3\right). \tag{5.5}$$

A surface and a contour plot of the pressure field are shown in Figure 2.
In the simulations, the solution does not depend on the viscosity, i.e., $\mathbf{u}$ and p are prescribed and if ν is varied this leads to a new right-hand side f. (This aspect was discussed in Remark 4.31.)

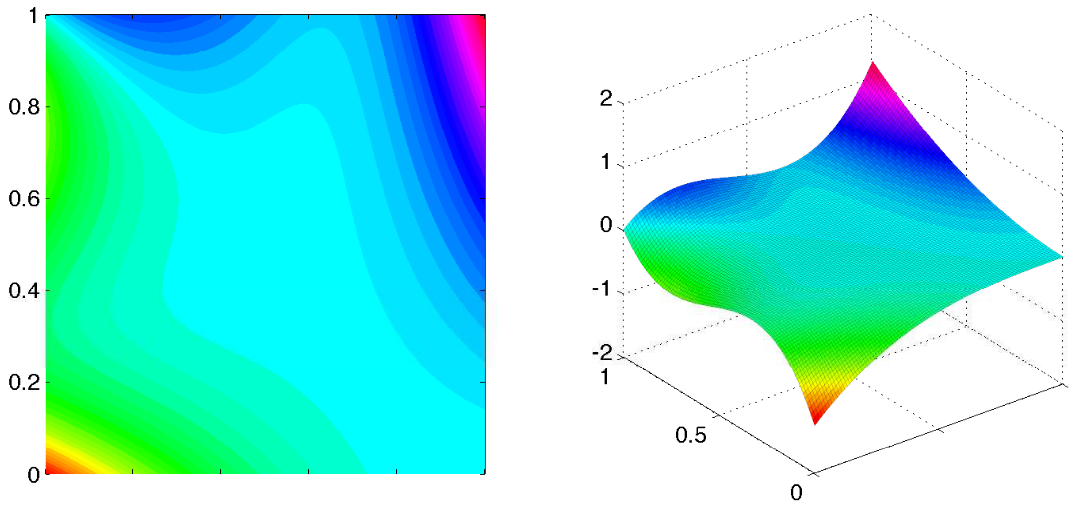


Figure 2: Pressure field.

In order to analyze the influence of the viscosity function $\nu(\mathbf{x})$ on the finite element solution of the Stokes problem (4.13), different viscosity functions will be considered in the following.
All functions are parametrized by two values, $\nu_{\min}$ and $\nu_{\max}$. The simplest one is a function that is linear in both arguments x and y and reaches its minimal value $\nu_{\min}$ in $(0,0)$ and $\nu_{\max}$ in $(1,1)$. It is given by

$$\nu_1(x,y) = \nu_{\min} + (\nu_{\max} - \nu_{\min})xy \tag{5.6}$$

and we will refer to it as **linear**.
Three more function types are defined which describe a fluid where the highest viscosity $\nu_{\max}$ is reached in the middle, i.e., at $(0.5, 0.5)$, and the lowest viscosity values can be found on the boundary of the unit square (see Figure 3). These viscosity functions are given by

$$\nu_2(x,y) = \nu_{\min} + (\nu_{\max} - \nu_{\min})16x(1-x)y(1-y), \tag{5.7}$$

referred to as **quadratic**. Another function type is given by

$$\begin{aligned}\nu_{3,4}(x,y) =&\nu_{\min} + (\nu_{\max} - \nu_{\min})16x(x-1)y(y-1)\times \\ &\times\left(0.5 + \arctan\left(\frac{\kappa\left(r - (x-0.5)^2 - (y-0.5)^2\right)}{\pi}\right)\right),\end{aligned} \tag{5.8}$$

where we set $r = 0.1$ for ν_3 (**atan**) and $r = 0.01$ for ν_4 (**atan(steep)**) and the scaling factor $\kappa = 2000$ resp. 200 for the function to take values in the range $[\nu_{\min}, \nu_{\max}]$.[5]
The last viscosity that decays exponentially is defined as

$$\nu_5(x,y) =\nu_{\min} + (\nu_{\max} - \nu_{\min})\exp\left(-10^{13}\left((x-0.5)^{10} + (y-0.5)^{10}\right)\right), \tag{5.9}$$

and referred to as **exp**.

[5] Note that the value $\nu_{\max}$ is taken only approximatively.

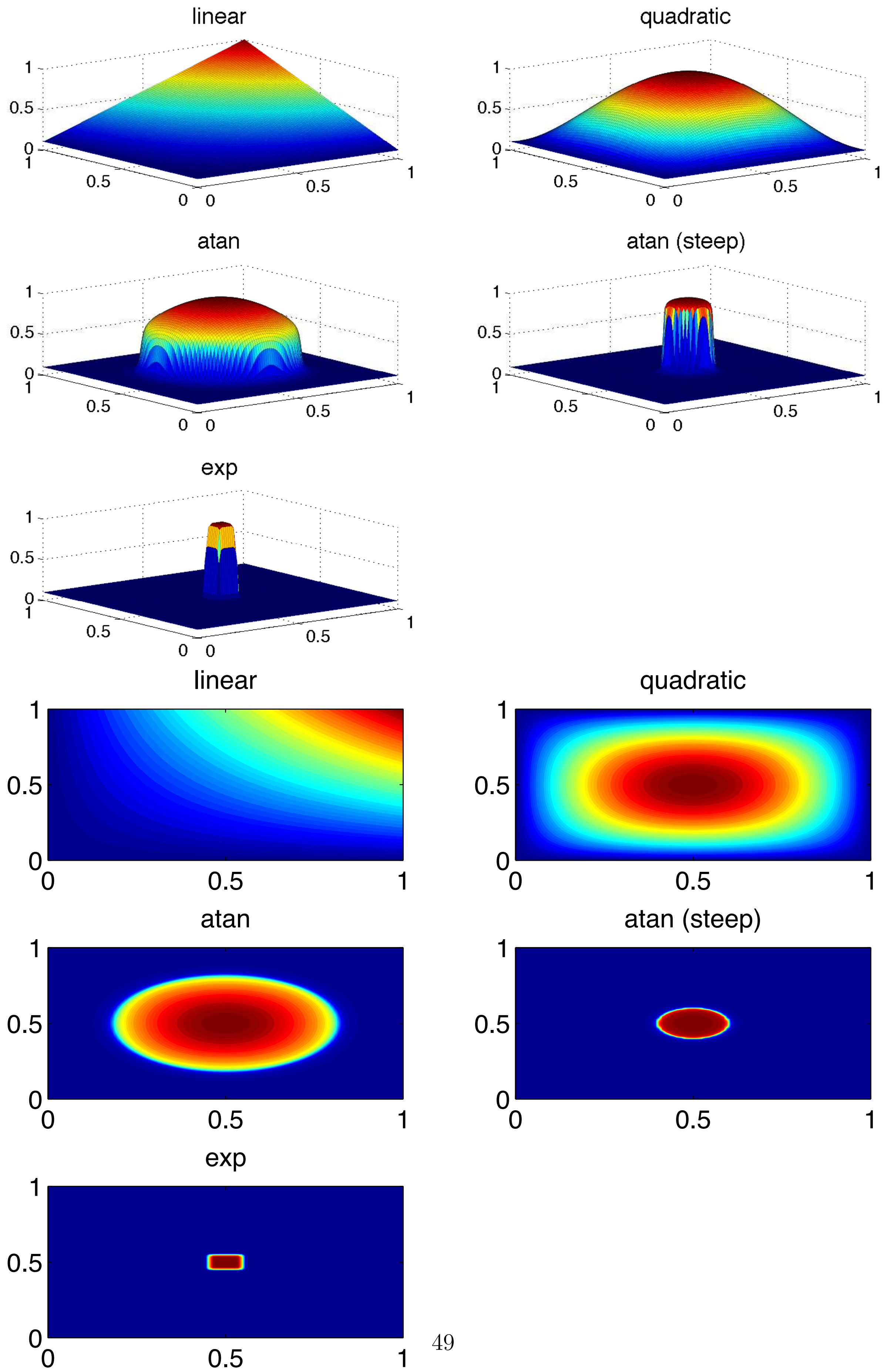


Figure 3: Surface plots (top) and contour plots (bottom) of the linear and mountain-shaped viscosity functions with $\nu_{\min} = 0.1$ and $\nu_{\max} = 1$.

5.2. The Error Estimates

In order to check the error estimates (4.58) - (4.65) derived in Section 4.3 one solves the finite element problem for the prescribed solution defined in Section 5.1 on a sequence of subsequently refined grids and measures the errors of those finite element solutions to the prescribed one. This is done for two different pairs of finite element spaces, namely
1. the element[6] Q_2/P_1^{disc} and
2. the Scott-Vogelius element P_2/P_1^{disc},
mentioned already in Section 3.4.
When implementing[7] those two types of finite element spaces one has to pay attention to the shape of the initial grid and the type of refinement that determines the number of degrees of freedom for different levels of grid refinement.
The table in Figure 4 gives an idea of the exponential growth of the degrees of freedom. For Q_2/P_1^{disc} the initial grid is the unit square. One regular refinement yields a 2×2 grid, depicted in Figure 5 (left), with 25 degrees of freedom for the two components of the finite element velocity $\mathbf{u}^h$, i.e, 50 degrees of freedom in total (compare to the leftmost column in the table in 4). The initial grid for P_2/P_1^{disc}, depicted in Figure 5 (right), is the unit square devided into two triangles that have then been refined barycentrically into six triangles.

	Q_2/P_1^{disc}		P_2/P_1^{disc}	
Level	d.o.f. velocity	d.o.f. pressure	d.o.f. velocity	d.o.f. pressure
1	50	12	114	72
2	162	48	418	288
3	578	192	1 602	1 152
4	2 178	768	6 274	4 608
5	8 450	3 072	24 834	18 432
6	33 282	12 288	98 818	73 728
7	132 098	49 152	394 242	294 912
8	526 338	196 608	1 574 914	1 179 648
9	2 101 250	786 432	6 295 554	4 718 592
10	8 396 802	3 145 728	25 174 018	18 874 368

Figure 4: Degrees of freedom for two different pairs of finite element spaces Q_2/P_1^{disc} (left) and P_2/P_1^{disc} (right).

[6]This element is similar to the popular Taylor-Hood element as far as exactness is concerned and the discontinuous pressure is advantageous for the implemented multgrid solver.
[7]Here, the implementation *Mathematics and object oriented Numerics in Magdeburg* (MooNMD) by Prof. V. John is used.

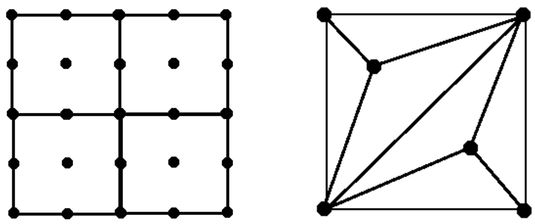


Figure 5: Initial grids for Q_2/P_1^{disc} (left) and P_2/P_1^{disc} (right).

5.3. Influence of Different Viscosity Functions on the Error $\|\mathbf{u}-\mathbf{u}^h\|_1$

As already mentioned, looking at the error estimates derived in Section 4.3 one does not expect the error to depend on the shape of the viscosity function, i.e., the gradient or higher derivatives of ν. This is also what can be observed in the simulations, as Figure 6 shows for $\left|\mathbf{u}^h-\mathbf{u}\right|_1$ and Level 7. Here, ν_{min} is set to 0.1 and $\nu_{\text{max}} = 1$ and the problem is solved for Q_2/P_1^{disc} .
The errors differ only by 10^{-7}.
However, one observes that the errors for problems where the viscosity functions behave rather roughly (**atan (steep)**) are slightly larger than those of problems with *mildly* varying viscosities like **linear**, in accordance with the intuition.
More functions have been tested, e.g.,one where ν_{max} is reached not in the center of the square but at $(0.8, 0.8)$ or one where the maximum is reached on the boundary and the minimum at $(0.5, 0.5)$. However, no significant consequences for the finite element errors could be observed in any of the norms. Therefore, only the five viscosity functions introduced in Section 5.1 will be considered in the remainder of this work.

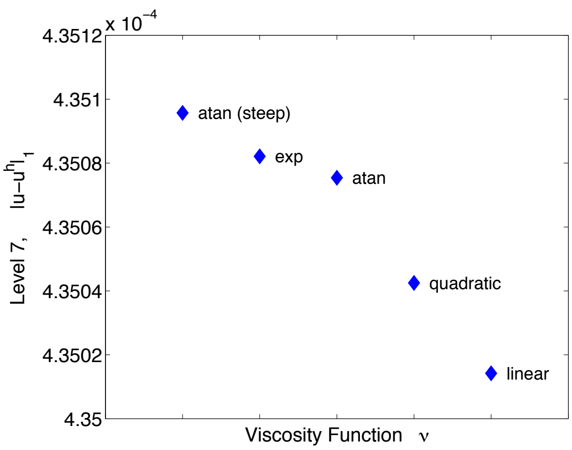


Figure 6: Finite element error in Level 7 of the velocity in the $H_0^1(\Omega)$-seminorm $\left|\mathbf{u}-\mathbf{u}^h\right|_1$ in finite element spaces Q_2/P_1^{disc} for $\nu_{\text{min}} = 0.1$ and $\nu_{\text{max}} = 1$.

5.4. Order of Convergence: $\|\mathbf{u} - \mathbf{u}^h\|_\nu$

As explained several times before, comparing the finite element error of the velocity field in the ν-norm for different viscosities does not make sense, since the norm of the error grows with the values $\nu(\mathbf{x})$ takes.
However, one can plot the errors to see the order of convergence given in (4.58), which is 2 since the used pair of finite element spaces is Q_2/P_1^{disc}, i.e., $k = 2$.
In Figure 7, the error in the ν-norm is depicted for different values of $\nu_{\max}$ and $\nu_{\min}$ in $\nu_3(x, y)$ (**atan**), while the corresponding other value has been set to $\nu_{\max} = 1$ resp. $\nu_{\min} = 0.1$.[8]
As expected the error takes large values when $\nu_{\max}$ is increased, it grows by the factor $\sqrt{\nu_{\max}}$.

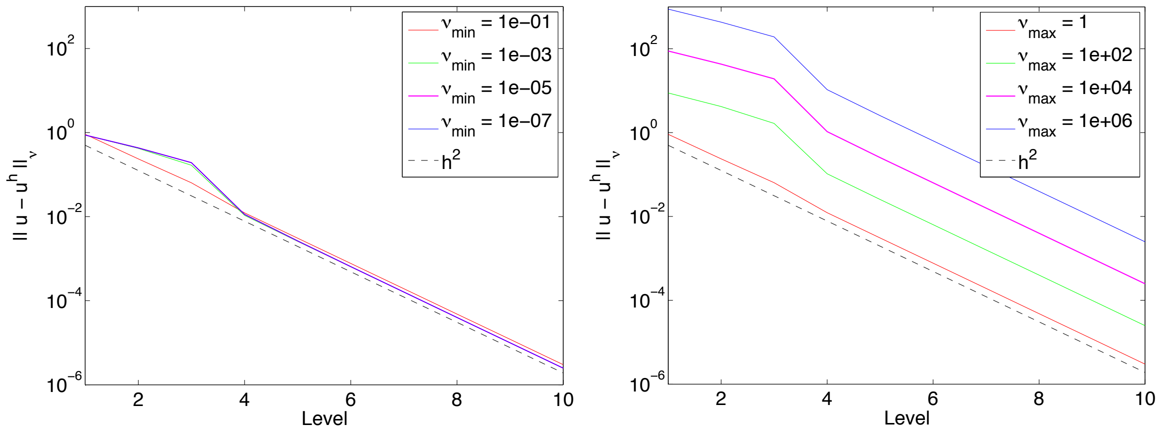
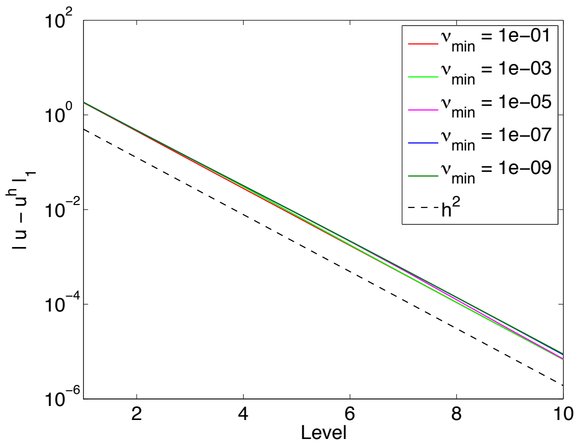


Figure 7: Order of convergence for the finite element error of the velocity in the ν-norm $\|\mathbf{u} - \mathbf{u}^h\|_\nu$ in finite element spaces Q_2/P_1^{disc} for varying $\nu_{\min}$ (left) and varying $\nu_{\max}$ (right) in $\nu_3(x, y)$ (**atan**).

5.5. Order of Convergence: $|\mathbf{u} - \mathbf{u}^h|_1$

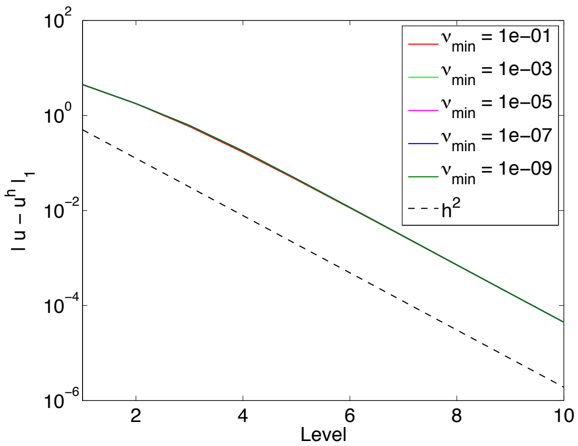
Figures 8 and 9 show the correct order of convergence for the error in the $H_0^1(\Omega)$-seminorm for the problem with the linear viscosity $\nu_1(x, y)$.
One cannot observe significant differences between the different finite element spaces Q_2/P_1^{disc} and Scott-Vogelius.
Furthermore, the dependence on $\nu_{\max}$ and $\nu_{\min}$ predicted in (4.59) cannot directly be observed. One can see however, that the variation of $\nu_{\min}$ has a larger influence on the solution in the spaces Q_2/P_1^{disc} than for the Scott-Vogelius space (right column in the Tables in Figures 8 and 9) which is consistent with the theory, i.e., the additional $\nu_{\min}$-dependence in (4.59) compared to (4.63).
This is illustrated also in the checkerboard plots in Figures 23 and 26, see Section 5.9.

[8] If not explicitly said otherwise, the values of $\nu_{\min}$ and $\nu_{\max}$ are always set to those values if fixed.



i	$\nu_{\min}$	Level 10 $\text{err}_i = \left\lvert \mathbf{u} - \mathbf{u}^h \right\rvert_1$	growth $\text{err}_{i+1}/\text{err}_i$
1	10^{-1}	$6.7968 \cdot 10^{-6}$	
2	10^{-3}	$6.7984 \cdot 10^{-6}$	1.0002
3	10^{-5}	$6.9821 \cdot 10^{-6}$	1.0270
4	10^{-7}	$8.5557 \cdot 10^{-6}$	1.2254
5	10^{-9}	$8.8498 \cdot 10^{-6}$	1.0344

Figure 8: Order of convergence for the finite element error of the velocity in the H_0^1-seminorm $\left\lvert \mathbf{u} - \mathbf{u}^h \right\rvert_1$ in finite element spaces Q_2/P_1^{disc} for different values $\nu_{\min}$ in the linear viscosity function $\nu_1(x, y)$ and $\nu_{\max} = 1$. The table on the right reveals a small growth in the errors for decreasing $\nu_{\min}$ and the corresponding error growth.

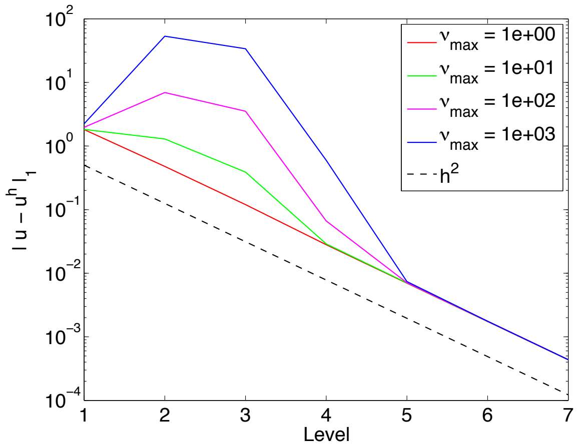


i	$\nu_{\min}$	Level 10 $\text{err}_i = \left\lvert \mathbf{u} - \mathbf{u}^h \right\rvert_1$	growth $\text{err}_{i+1}/\text{err}_i$
1	10^{-1}	$4.4465 \cdot 10^{-5}$	
2	10^{-3}	$4.4487 \cdot 10^{-5}$	1.0005
3	10^{-5}	$4.4561 \cdot 10^{-5}$	1.0017
4	10^{-7}	$4.4574 \cdot 10^{-5}$	1.0003
5	10^{-9}	$4.4575 \cdot 10^{-5}$	1.0000

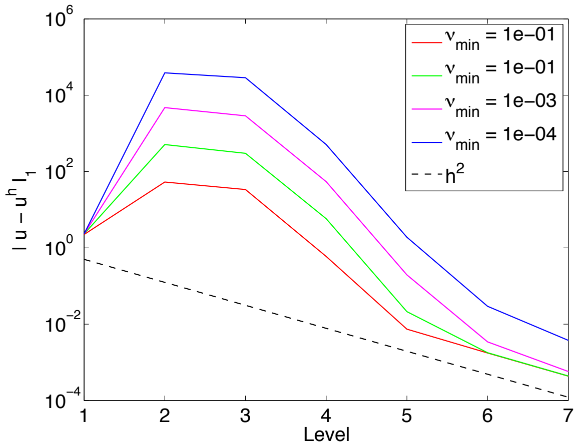
Figure 9: Order of convergence for the finite element error of the velocity in the H_0^1-seminorm $\left\lvert \mathbf{u} - \mathbf{u}^h \right\rvert_1$ for Scott-Vogelius finite element space for different values $\nu_{\min}$ in the linear viscosity function $\nu_1(x, y)$ and $\nu_{\max} = 1$. The table on the right reveals a small growth in the errors for decreasing $\nu_{\min}$ and the corresponding error growth.

That $\nu_{\min}$ has a stronger influence than $\nu_{\max}$ on the evolution of the error in Q_2/P_1^{disc} is illustrated in Figure 10 for the problem with exponential viscosity $\nu_5(x, y)$.
While changing $\nu_{\max}$ has only a small impact on the error $\left\lvert \mathbf{u} - \mathbf{u}^h \right\rvert_1$, (see table in Figure 10,) it increases visibly with decreasing $\nu_{\min}$. However it can be expected that the errors become rather independent of $\nu_{\min}$ for higher levels of grid refinement, such that one

observes a similar behavior as for varying $\nu_{\max}$.
It was not possible to do the simulations for finer grids since the computing capacity needed for those simulations was not available. Iterative solvers used for **atan**, **quadratic** and **linear** could not be used here, due to the difficulties they had with the rough shape of the function.
For a discussion of this issue it is referred to Section 6.1. However,



i	$\nu_{\max}$	Level 7 $\mathrm{err}_i = \left\|\mathbf{u} - \mathbf{u}^h\right\|_1$	growth $\mathrm{err}_{i+1}/\mathrm{err}_i$
1	1	$4.3508 \cdot 10^{-4}$	
2	10	$4.3528 \cdot 10^{-4}$	1.0005
3	10^2	$4.3621 \cdot 10^{-4}$	1.0021
4	10^3	$4.3715 \cdot 10^{-4}$	1.0022



i	$\nu_{\min}$	Level 7 $\mathrm{err}_i = \left\|\mathbf{u} - \mathbf{u}^h\right\|_1$	growth $\mathrm{err}_{i+1}/\mathrm{err}_i$
1	10^{-1}	$4.3508 \cdot 10^{-4}$	
2	10^{-2}	$4.3689 \cdot 10^{-4}$	1.0042
3	10^{-3}	$5.7581 \cdot 10^{-4}$	1.3180
4	10^{-4}	$3.7842 \cdot 10^{-3}$	6.5719

Figure 10: Order of convergence for the finite element error of the velocity in the H_0^1-seminorm $\left|\mathbf{u} - \mathbf{u}^h\right|_1$ in finite element spaces Q_2/P_1^{disc} for the exponential viscosity function $\nu_5(\mathbf{x})$ with parameters $\nu_{\min} = 0.1$ and variable $\nu_{\max}$ (top) and $\nu_{\max} = 1$ and variable $\nu_{\min}$ (bottom). The table in the top reveals that for the highest level there is a small growth also for increasing $\nu_{\max}$ whereas the error growth for decreasing $\nu_{\min}$ is visible already from the bottom left figure.

For the same problem with the Scott-Vogelius finite element space one can see in Figure 11 that the finite element errors $\left|\mathbf{u} - \mathbf{u}^h\right|_1$ grow linearly in $\nu_{\min}^{-1}$ resp. $\nu_{\max}$, as predicted in (4.63).
However, one can also observe a higher order of convergence 3 and not the predicted optimal order 2. This problem will be discussed in Section 6.1.

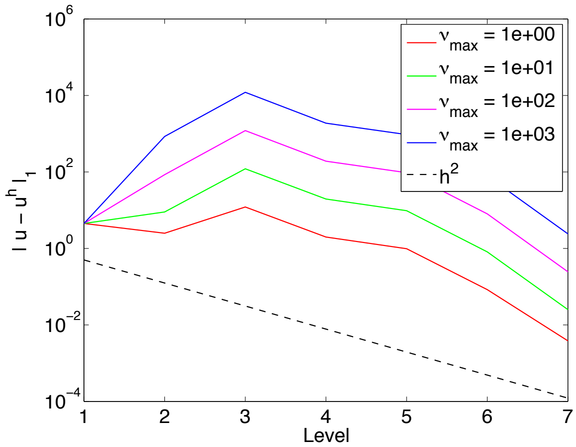


Figure 11: Order of convergence for the finite element error of the velocity in the H_0^1-seminorm $\left|\mathbf{u}-\mathbf{u}^h\right|_1$ in Scott-Vogelius finite element space for the exponential viscosity function $\nu_5(x,y)$ defined in (5.9) and increasing $\nu_{\max}$ with $\nu_{\min} = 10^{-4}$ (left) and decreasing $\nu_{\min}$ with $\nu_{\max} = 10^3$ (right).

For a constant viscosity and the finite element spaces Q_2/P_1^{disc} the error grows linearly in ν^{-1} like stated in (4.27) and (4.59) with $\nu_{\min} = \nu_{\max}$. For values $\nu > 1$ no effect can be observed which makes sense since then the first term in (4.27) determines the size of the error.
The left plot in Figure 12 as well as Figure 13 show these findings.
However, also in this case the optimal order of convergence cannot be seen but a higher one, like the gray dashed line illustrates in Figure 12 (left).
For the Scott-Vogelius space all errors converge with the optimal order 2 and no influence of ν is visible. This is consistent with the estimates (4.33) and (4.63).

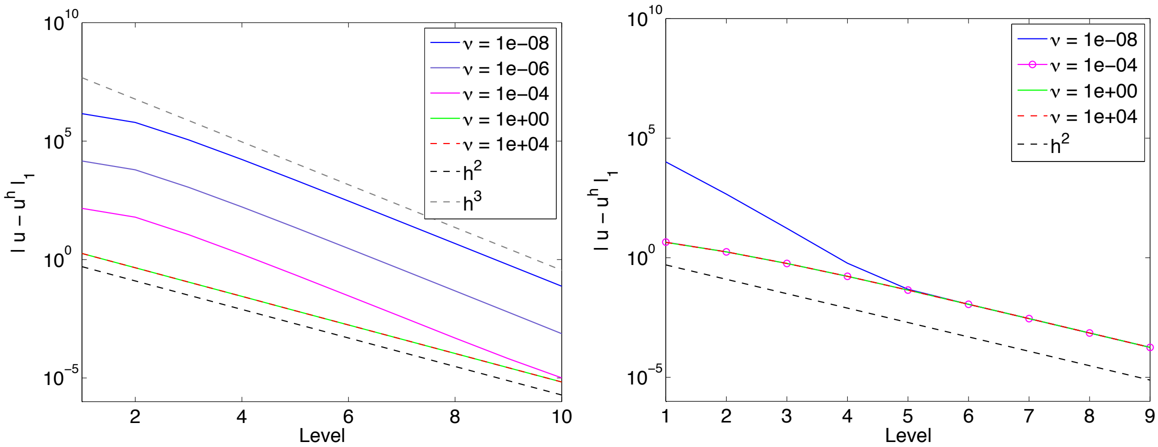


Figure 12: Order of convergence for the finite element error of the velocity in the H_0^1-seminorm $\left|\mathbf{u}-\mathbf{u}^h\right|_1$ for the special case of a constant viscosity ν in finite element spaces Q_2/P_1^{disc}(left) and for the Scott-Vogelius finite element space (right).

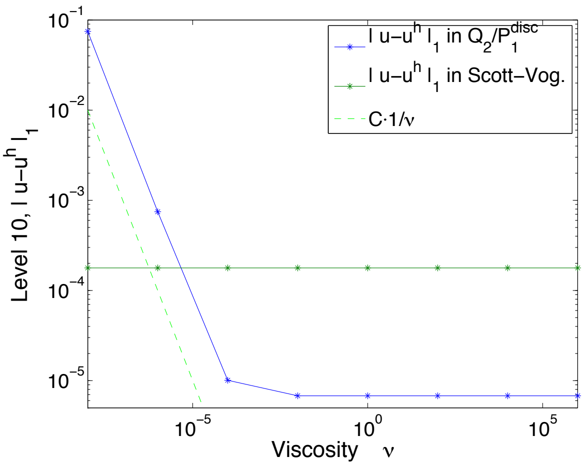
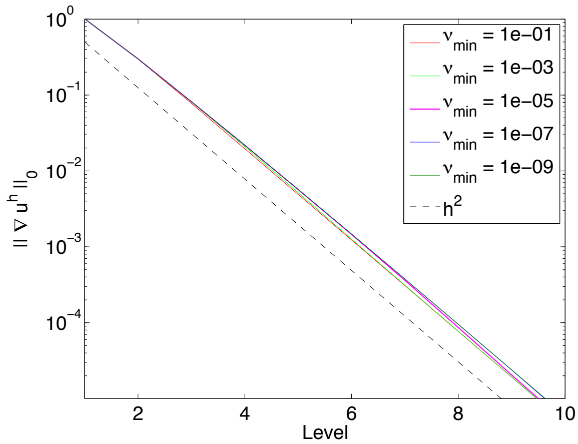


Figure 13: Error $\left|\mathbf{u}-\mathbf{u}^h\right|_1$ for level 10 and constant ν. The blue line shows how the error depends on ν in the spaces Q_2/P_1^{disc}. For comparison the light green line with slope ν^{-1} has been plotted.
The dark green line shows the ν-independence of the error for the Scott-Vogelius space.

5.6. Order of Convergence: $\|\nabla\cdot\mathbf{u}^h\|_0$

As expected, the $L^2(\Omega)$-norm of the divergence of the velocity for Q_2/Q_1, behaves like $\left|\mathbf{u}-\mathbf{u}^h\right|_1$. For the problem with linear viscosity $\nu_1(\mathbf{x})$, Figure 14 shows the optimal order of convergence 2 and a slight increase of the error when $\nu_{\min}$ is decreased.



i	$\nu_{\min}$	Level 10 $\text{err}_i = \|\nabla\cdot\mathbf{u}^h\|_0$	growth $\text{err}_{i+1}/\text{err}_i$
1	10^{-1}	$4.8060\cdot 10^{-6}$	
2	10^{-3}	$4.8071\cdot 10^{-6}$	1.0002
3	10^{-5}	$4.9244\cdot 10^{-6}$	1.0244
4	10^{-7}	$5.8542\cdot 10^{-6}$	1.1888
5	10^{-9}	$6.0173\cdot 10^{-6}$	1.0279

Figure 14: Order of convergence for the divergence of the finite element solution in the $L_2(\Omega)$-norm $\|\nabla\cdot\mathbf{u}^h\|_0$ in finite element spaces Q_2/P_1^{disc} (left) for different values of $\nu_{\min}$ in the linear viscosity function $\nu_1(x,y)$ defined in (5.6). The table reveals a small growth in the errors for decreasing $\nu_{\min}$.

5.7. Order of Convergence: $\|p - p^h\|_0$

Looking at estimate (4.61), one expects the finite error estimate of the pressure field to grow when $\nu_{\max}$ is increased. In Figure 15 the finite element errors of the pressure field for a Stokes problem with linear viscosity $\nu_1(\mathbf{x})$ are shown. For $\nu_{\max} = 1$ one obtains the same $\nu_{\min}$-independent errors (left) whereas different values of $\nu_{\max}$ for $\nu_{\min} = 0.1$ (right) show that the error grows by the same factor as $\nu_{\max}$.
The optimal error of convergence 2 can be observed only for $\nu_{\max} = 1$ (left) while for large values of $\nu_{\max}$ a higher order of error reduction, 3, occurs again.

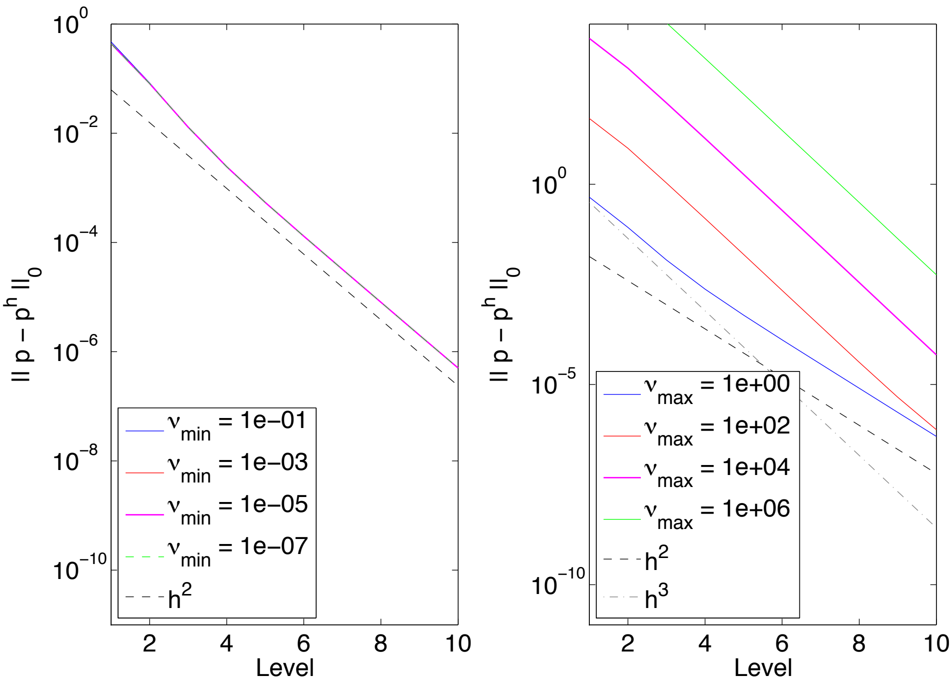


Figure 15: Order of convergence for the finite element error of the pressure field in the $L_2(\Omega)$-norm $\|p - p^h\|_0$ in finite element spaces Q_2/P_1^{disc} for varying $\nu_{\min}$ (left) and varying $\nu_{\max}$ (right) in the linear viscosity function $\nu_1(x, y)$ defined in (5.6).

In Figure 16 the growth of the error $\|p - p^h\|_0$ for different values of the parameters $\nu_{\max}$ and $\nu_{\min}$ of the quadratic viscosity function $\nu_2(x)$ is shown. As expected the error grows only for large values of $\nu_{\max}$. Furthermore, in the plot one cannot see a $\nu_{\min}$-dependence of the error which could be expected from (4.61), albeit not as strong as the dependence on $\nu_{\max}$.

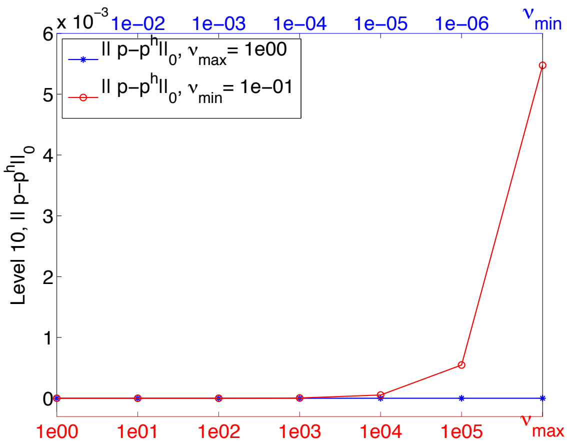


Figure 16: Error $\|p-p^h\|_0$ for level 10 in Q_2/P_1^{disc} and varying values of $\nu_{\min}$ and $\nu_{\max}$ in the quadratic viscosity function $\nu_2(x,y)$ defined in (5.7). The blue line shows how the error depends on $\nu_{\min}$ when $\nu_{\max}=1$ and the red one indicates the dependence on $\nu_{\max}$ for fixed $\nu_{\min}=0.1$.

In Figures 17 and 18 similar behavior of the finite element error of the pressure field is illustrated for the problem with constant viscosity $\nu(\mathbf{x})=\nu$.
The plots show that the error grows linearly in ν for $\nu>1$ for Q_2/P_1^{disc} (Figure 17) as well as for the Scott-Vogelius element (Figure 18), as stated in in estimates (4.40) and (4.61) for $\nu_{\min}=\nu_{\max}=\nu$.
This linear growth in ν is illustrated also in the plot on the right in Figure 17 for Q_2/P_1^{disc}.

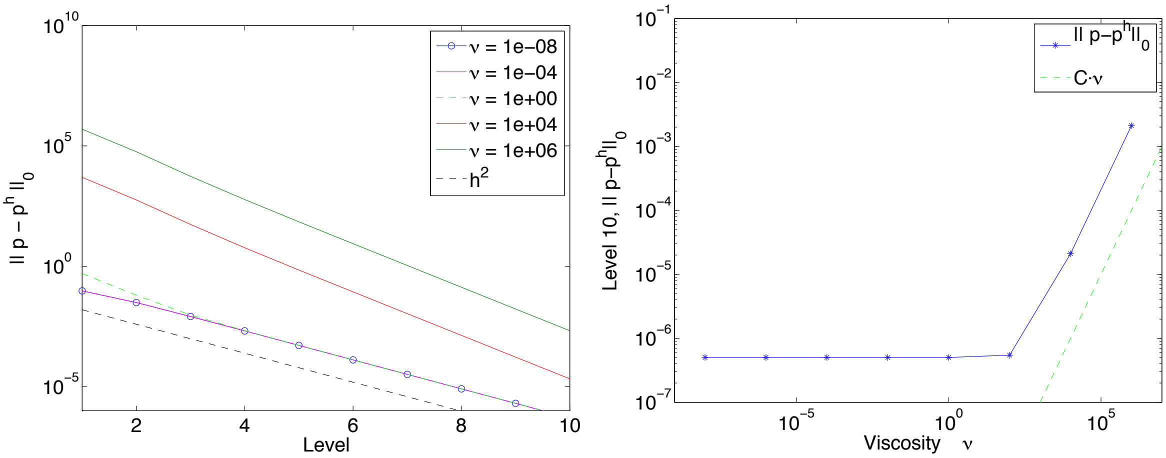
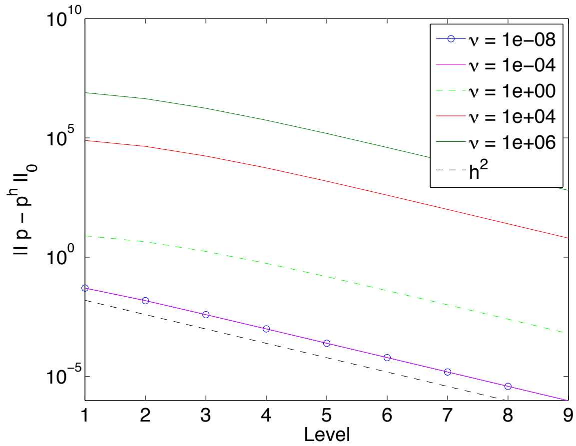


Figure 17: Order of convergence for the finite element error of the pressure field in the $L^2(\Omega)$-norm $\|p-p^h\|_0$ for the special case of a constant viscosity ν in finite element spaces Q_2/P_1^{disc} (left) and growth of the error for level 10 with respect to the constant viscosity ν (left). The green line shows that the error grows linearly in ν for $\nu>1$.

For $\nu>1$, the left plot in Figure 17 indicates a higher order of error reduction than the proved order of convergence 2 for Q_2/P_1^{disc} whereas the plot as well as the table in Figure 18 show the optimal order of convergence for the Scott-Vogelius element. Again, we refer to Section 6.1 for a discussion of this phenomenon.



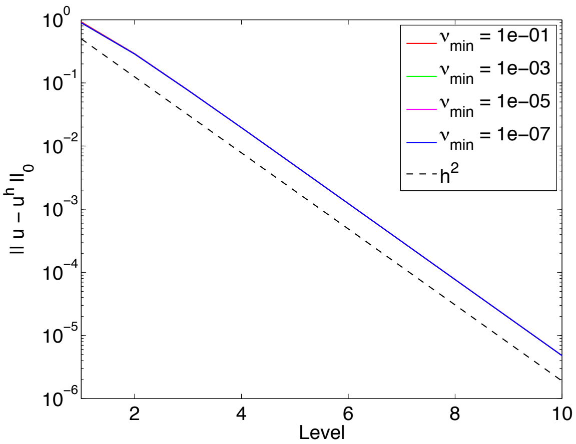
ν	$\text{err}_{\text{Level}} = \|p - p^h\|_0$ Level 8	Level 9	order p $\frac{\ln(\text{err}_9/\text{err}_8)}{\ln(2)}$
10^{-8}	$3.86 \cdot 10^{-6}$	$9.65 \cdot 10^{-7}$	1.99998
10^{-4}	$3.87 \cdot 10^{-6}$	$9.67 \cdot 10^{-7}$	1.99995
10^{0}	$2.52 \cdot 10^{-3}$	$6.30 \cdot 10^{-4}$	1.99890
10^{4}	$2.52 \cdot 10^{1}$	$6.30 \cdot 10^{0}$	1.99890
10^{6}	$2.52 \cdot 10^{3}$	$6.30 \cdot 10^{2}$	1.99890

Figure 18: Order of convergence for the finite element error of the pressure field in the $L^2(\Omega)$-norm $\|p - p^h\|_0$ for the special case of a constant viscosity ν for the Scott-Vogelius finite element space (right).

5.8. Order of Convergence: $\|\mathbf{u} - \mathbf{u}^h\|_0$

For $\nu(\mathbf{x}) = \nu_2(\mathbf{x})$, Figure 19 confirms the order of convergence[9] 3 for $\|\mathbf{u} - \mathbf{u}^h\|_0$ that was stated in estimate (4.65) for Q_2/P_1^{disc}.
The variation of the values of $\nu_{\min}$ and $\nu_{\max}$ has little influence on the size of the errors. The dependence of the estimate (4.65) on the viscosity therefore seems not to be sharp.
In Figure 20, one can see how the error $\|\mathbf{u} - \mathbf{u}^h\|_0$ changes with different values of $\nu_{\min}$ (blue) and $\nu_{\max}$ (red). Note that the influence of $\nu_{\min}$ on the error is stronger than that of $\nu_{\max}$. However, the influence of both parameters is, as mentioned already, much smaller than expected.
A detailed discussion can be found in Section 6.1.

[9] It is $k = 2$.



i	$\nu_{\min}$	Level 10 $\mathrm{err}_i = \|\mathbf{u} - \mathbf{u}^h\|_0$	growth $\mathrm{err}_{i+1}/\mathrm{err}_i$
1	10^{-1}	$1.0245 \cdot 10^{-9}$	
2	10^{-3}	$1.0989 \cdot 10^{-9}$	1.0727
3	10^{-5}	$1.4843 \cdot 10^{-9}$	1.3507
4	10^{-7}	$1.4986 \cdot 10^{-9}$	1.0096

Figure 19: Order of convergence for the finite element error of the velocity field in the $L_2(\Omega)$-norm $\|\mathbf{u}-\mathbf{u}^h\|_0$ in finite element spaces Q_2/P_1^{disc} for varying $\nu_{\min}$ in the quadratic viscosity function $\nu_2(x,y)$ defined in (5.7) (left). The table reveals a small growth in the errors for decreasing $\nu_{\min}$.

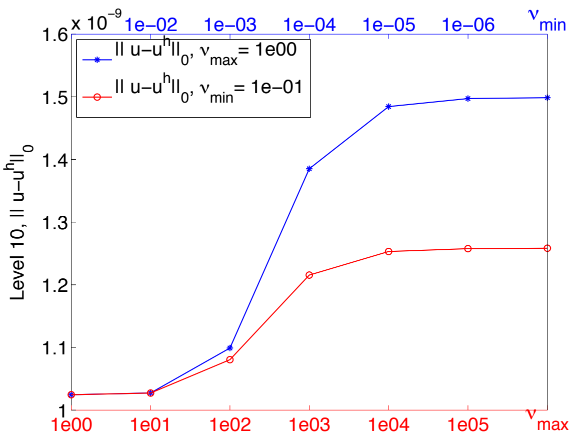


Figure 20: Error $\|\mathbf{u}-\mathbf{u}^h\|_0$ for level 10 in Q_2/P_1^{disc} and varying values of $\nu_{\min}$ and $\nu_{\max}$ in the quadratic viscosity function $\nu_2(x,y)$ defined in (5.7) . The blue line shows how the error depends on $\nu_{\min}$ while the red one indicates the dependence on $\nu_{\max}$.

In contrast, for the **exp** viscosity function $\nu_5(x,y)$, defined in (5.9), the influence of the values chosen for $\nu_{\min}$ and $\nu_{\max}$ seems to be of importance on coarse grids, i.e., for the levels 1 to 5.
On fine grids, one cannot see the influence of varying $\nu_{\max}$ any longer, whereas different $\nu_{\min}$ lead to different errors also for levels 6 and 7.
However for higher levels, one could also expect the errors to converge like those of $\nu_{\min} = 10^{-1}$ and $\nu_{\min} = 10^{-3}$ or those of different values for $\nu_{\max}$.
For the Scott-Vogelius space similar behavior can be found. The corresponding plots are omitted for the sake of brevity.

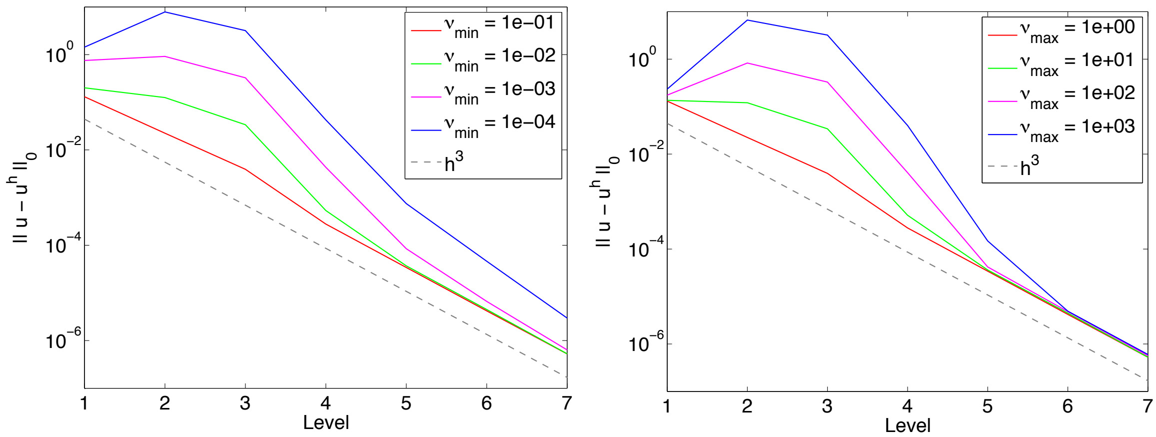


Figure 21: Order of convergence for the finite element error of the velocity field in the $L_2(\Omega)$-norm $\|\mathbf{u}-\mathbf{u}^h\|_0$ in finite element spaces Q_2/P_1^{disc} for varying $\nu_{\min}$ (left) and $\nu_{\max}$ (right) in the **exp** viscosity function $\nu_5(x,y)$ defined in (5.9).

For the special case of a constant viscosity function $\nu(\mathbf{x}) = \nu$ estimate (4.65) becomes (4.68),

$$\|\mathbf{u}-\mathbf{u}^h\|_0 \leq Ch^{k+1}\left(\|\mathbf{u}\|_{H^{k+1}} + \frac{1}{\nu}\|p\|_{H^k(\Omega)}\right) \times \left(\|\boldsymbol{\phi}_{\hat{\mathbf{f}}}\|_{H^2(\Omega)} + \|\xi_{\hat{\mathbf{f}},1}\|_{H^1(\Omega)}\right),$$

as was discussed in Remark 4.34. In Figure 22, one sees how the error grows for small values of ν and scales linearly with ν^{-1}.
As expected, varying $\nu > 1$ has no effect on the error since this leads only to $\nu^{-1}\|p\|_{H^k(\Omega)}$ being dominated by the other terms in estimate (4.68).
For the Scott-Vogelius element, estimate (4.66) simplifies to (4.69),

$$\|\mathbf{u}-\mathbf{u}^h\|_0 \leq Ch^{k+1}\|\mathbf{u}\|_{H^{k+1}} \times \|\boldsymbol{\phi}_{\hat{\mathbf{f}}}\|_{H^2(\Omega)}. \tag{5.10}$$

According to the estimate one can observe the ν-independence of the error $\|\mathbf{u}-\mathbf{u}^h\|_0$ in Figure 22 (right).
This effect is illustrated also by the plot in Figure 13 for $\left|\mathbf{u}-\mathbf{u}^h\right|_1$. It confirms the findings in the corresponding estimates (4.59) and (4.63), where we set $\nu_{\min} = \nu_{\max} = \nu$. However, it stays unclear why the ν-dependence for the Taylor-Hood element can be observed only in combination with a higher order of error reduction (left plot in Figure 22). (See Section 6.1 for a discussion.)

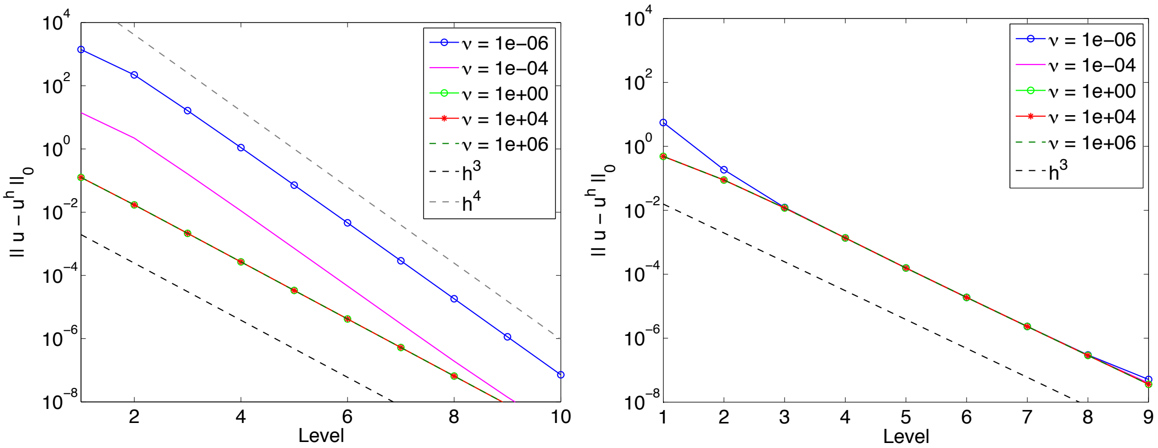


Figure 22: Order of convergence for the finite element error of the pressure field in the $L^2(\Omega)$-norm $\|\mathbf{u}-\mathbf{u}^h\|_0$ for the special case of a constant viscosity ν in finite element spaces $Q_2 P_1^{\text{disc}}$ (left) and for the Scott-Vogelius finite element space (right).

5.9. Scale Invariance of the $H_0^1(\Omega)$-seminorm of the Finite Element Error of the Velocity for the Scott-Vogelius Element

For the Scott-Vogelius space, the error $\left|\mathbf{u}-\mathbf{u}^h\right|_1$ depends only on the ratio $\frac{\nu_{\max}}{\nu_{\min}}$ rather than on the specific choices for $\nu_{\min}$ and $\nu_{\max}$ which can be seen in estimate (4.31) that becomes

$$\left|\mathbf{u}-\mathbf{u}^h\right|_1 \leq 2\, C_K \sqrt{\frac{\nu_{\max}}{\nu_{\min}}} \left(1 + C C_K \sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\right) \inf_{\mathbf{v}^h \in V^h} \left|\mathbf{u}-\mathbf{v}^h\right|_1, \tag{5.11}$$

and in estimate (4.63).

In order to demonstrate this effect, the errors $\left|\mathbf{u}-\mathbf{u}^h\right|_1$ have been plotted for different combinations of $\nu_{\min}$ and $\nu_{\max}$ in a checkerboard plot (left plot in Figure 23), where the errors are the same on fields with the same color. As expected, all fields with the same ratio of $\nu_{\max}/\nu_{\min}$ have the same color, e.g., $10^0/10^{-3}$ and $10^2/10^{-1}$, where both combinations yield $\left|\mathbf{u}-\mathbf{u}^h\right|_1 = 4.455 \cdot 10^{-5}$.

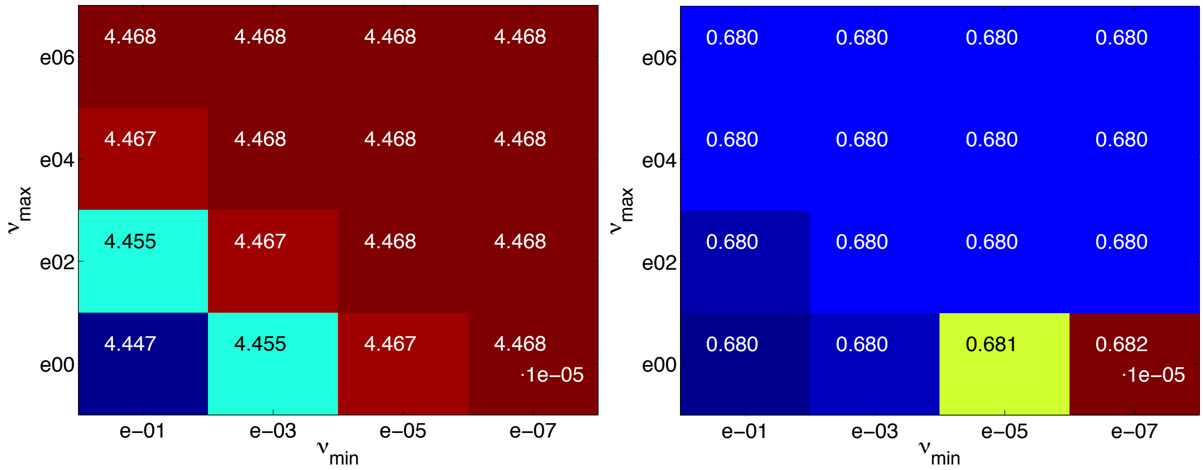


Figure 23: Error $\left|\mathbf{u}-\mathbf{u}^h\right|_1$ for level 10 and varying values for $\nu_{\min}$ and $\nu_{\max}$. Each field shows the error for the quadratic viscosity function $\nu_2(x,y)$ parametrized by $\nu_{\min}$ and $\nu_{\max}$.
Left: Scott-Vogelius finite element space.
Right: Q_2/P_1^{disc}, the influence of $\nu_{\min}^{-1}$ dominates that of the factor $\nu_{\max}/\nu_{\min}$.

For comparison, if one uses the Taylor-Hood pair of finite element spaces Q_2/Q_1 an additional dependence on $\nu_{\min}$ is expected since (4.23) gives

$$\begin{aligned}\left|\mathbf{u}-\mathbf{u}^h\right|_1 \leq & 2\,C_K\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\left(1+CC_K\sqrt{\frac{\nu_{\max}}{\nu_{\min}}}\right)\inf_{\mathbf{v}^h\in V^h}\left|\mathbf{u}-\mathbf{v}^h\right|_1 \\ & +\frac{C_K^2}{\nu_{\min}}\inf_{q^h\in Q^h}\|p-q^h\|_0. \end{aligned} \tag{5.12}$$

This influence of $\nu_{\min}$ is illustrated in Figure 24 by the blue line, whereas the negligible growth of the error due to an increase in $\nu_{\max}$ is depicted by the red line.

This effect can also be found in the right checkerboard plot in Figure 23, where one can see the error increase with smaller values of $\nu_{\min}$ for a fixed $\nu_{\max}=1$.

One notices another rather unexpected effect when looking carefully at the right plot in Figure 23, namely that the errors decrease for increasing values of $\nu_{\max}$, e.g.,for $\nu_{\min}=10^{-7}$. This contradicts the theory developed so far. In particular, from estimate (5.12) one would expect increasing errors for increasing $\nu_{\max}$.
The observed differences are of order 10^{-8}, i.e., very small and could be due to round-off errors that appear when the solver encounters an ill-conditioned problem which is the case for viscosities with extreme maximal and minimal values, like $\nu_{\min}=10^{-7}$.

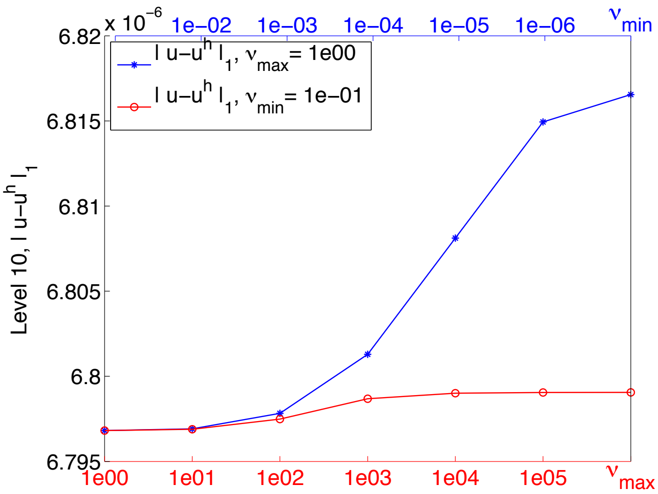


Figure 24: Error $\left|\mathbf{u}-\mathbf{u}^h\right|_1$ for level 10 and varying values for $\nu_{\min}$ and $\nu_{\max}$ in the quadratic viscosity function $\nu_2(x, y)$ defined in (5.7). The blue line shows the error for decreasing $\nu_{\min}$ while $\nu_{\max}$ is set to one and the red line shows the error for increasing $\nu_{\max}$ while $\nu_{\min} = 0.1$.

However, what is more plausible, is that the observed effect is due to the fact that for large values of $\nu_{\max}$ the function is stretched such that the critical small value for $\nu_{\min}$ is taken only at the boundary whereas away from the boundary the function grows very fast. In most of the domain, the viscosity is larger than one, i.e., the harming effect expected for large values for $\nu_{\max}$ is compensated by rather mild values larger than $\nu_{\min}$ in most of the domain, (see Figure 25,) such that one observes a decreasing error for increasing $\nu_{\max}$.

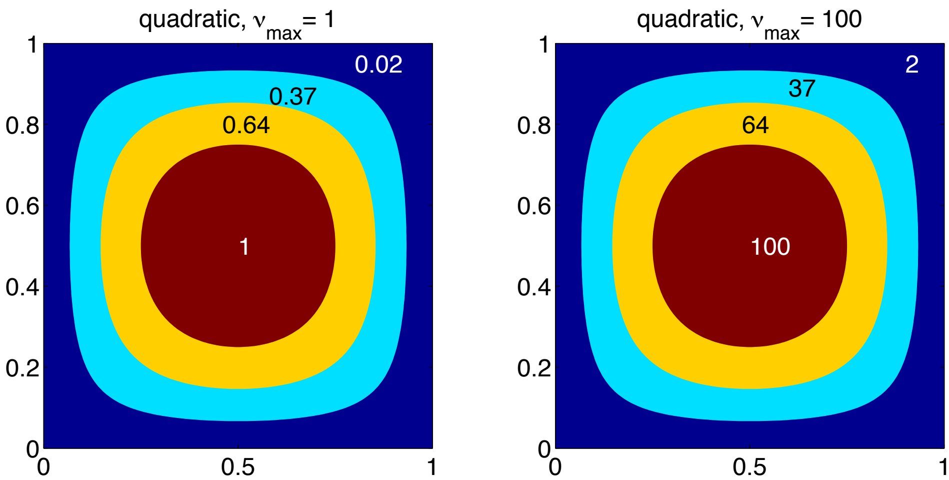


Figure 25: Effect of changing $\nu_{\max}$ for the values in the whole domain. For $\nu_{\max} = 100$ the viscosity takes values $\nu(x, y) \gg \nu_{\min} = 10^{-4}$ in most of the domain than for $\nu_{\max} = 1$.

A natural choice for a viscosity function that avoids this effect is an exponentially shaped one like $\nu_5(\mathbf{x})$ defined in (5.9). The representative checkerboard plots can be found in Figure 26. Here one can clearly see the scale invariance stated in (5.11) for the Scott-Vogelius element on the left and the dominance of

$$\nu_{\min}^{-1} C \inf_{q^h \in Q^h} \|p - q^h\|_0$$

for Q_2/P_1^{disc} finite element spaces on the right.

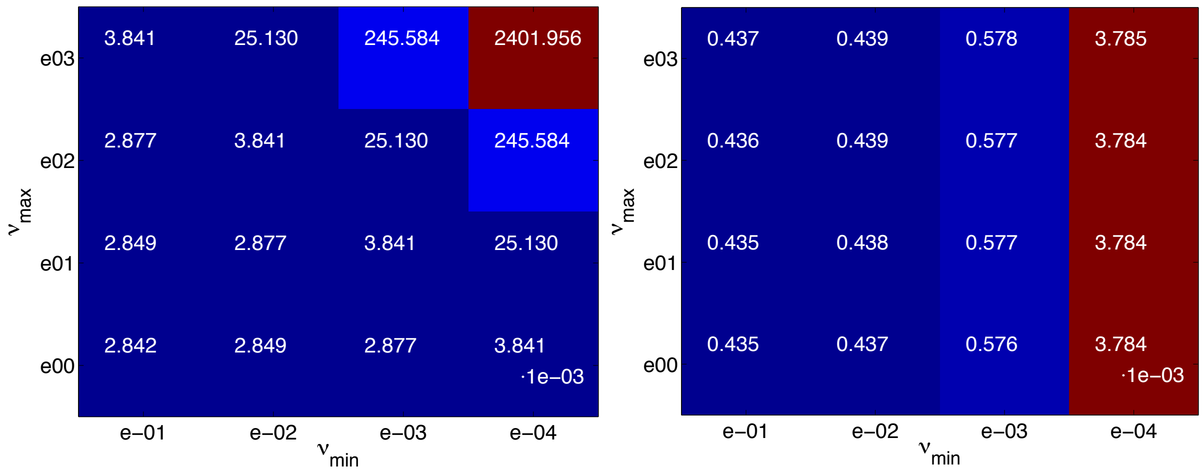


Figure 26: Error $\left|\mathbf{u} - \mathbf{u}^h\right|_1$ for level 7 and varying values for $\nu_{\min}$ and $\nu_{\max}$. Each field shows the error (multiplied by 10^3) for the exponential viscosity function $\nu_5(x, y)$ defined in (5.9) parametrized by $\nu_{\min}$ and $\nu_{\max}$.
Left: Scott-Vogelius finite element space.
Right: Q_2/P_1^{disc}, the influence of $\nu_{\min}^{-1}$ dominates that of the factor $\nu_{\max}/\nu_{\min}$.

Looking again at estimate (5.12) for the pair of finite element spaces Q_2/P_1^{disc}, one notices that scale invariance could be achieved also for this pair of finite element spaces if the latter term vanishes, i.e., if the prescribed solution for the pressure p lies in the ansatz space P_1^{disc} such that the best approximation error is zero. Clearly, for the function $p(\mathbf{x}) = 0$, $\forall \mathbf{x} \in \Omega$ this is the case. Figure 27 confirms the scale invariance for that choice for p.

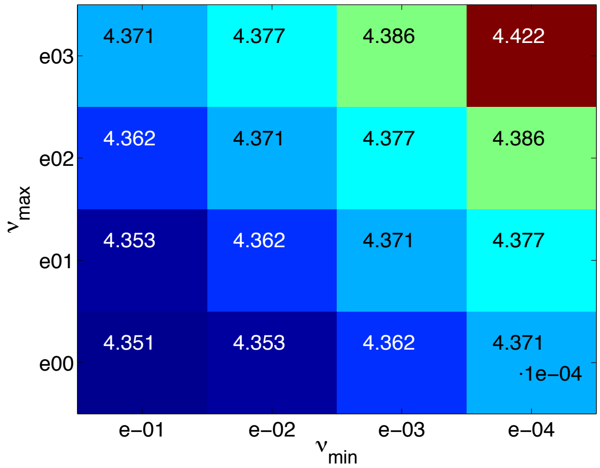


Figure 27: Error $\left|\mathbf{u} - \mathbf{u}^h\right|_1$ for level 7 and Q_2/P_1^{disc} and varying values for $\nu_{\min}$ and $\nu_{\max}$ when the pressure solution is zero. Each field shows the error (multiplied by 10^4) for the exponential viscosity function $\nu_5(x, y)$ defined in (5.9) parametrized by $\nu_{\min}$ and $\nu_{\max}$.

6. Conclusion and Outlook

6.1. Open Questions

Remark 6.1. Superconvergence?
As noted several times before, in many of the examples a higher order of error reduction on coarse grids than the proved order of convergence in the estimates (4.58)-(4.66) is observed, e.g., for the $H_0^1(\Omega)$-seminorm in Figure 11 for the Scott-Vogelius space and in Figure 12 for a constant viscosity for Q_2/P_1^{disc}. As this behavior appears for both types of finite element spaces, one might conclude that it is not a problem of the chosen element.
It rather seems like the higher order of error reduction occurs whenever the (expected) ν-dependence can be observed[10]. Comparing Figures 11 and 10 one finds the $\nu_{\max}$-dependence in Q_2/P_1^{disc} (Fig. 10, top left) only until the optimal order of convergence is reached in level 5.
The same phenomenon appears for the convergence of the finite element error of the pressure $\|p - p^h\|_0$, illustrated in Figure 15. Again a higher order of error reduction is observed together with linear growth of the error in ν.
In Figures 17 and 18, the higher order of convergence occurs only in Q_2/P_1^{disc} whereas for the Scott-Vogelius element the error decays according to the optimal order of convergence 2. The theory developed in this work cannot explain this behavior. One could guess that the errors in Q_2/P_1^{disc} converge all to a ν-independent one and decay then at optimal order like observed for the velocity in Figure 10.
Obviously, there is a considerable need for future research on that phenomenon.

Remark 6.2. ν-independence of $\|\mathbf{u} - \mathbf{u}^h\|_0$.
In Figure 19, one can hardly make out the influence of the different values of $\nu_{\min}$ and $\nu_{\max}$ on the error $\|\mathbf{u} - \mathbf{u}^h\|_0$ that one might expect from estimate (4.65). This corroborates the belief that the dependence on the viscosity predicted in this estimate is not sharp.
The fact that the influence of $\nu_{\min}$ on the error is stronger than that of $\nu_{\max}$ suggests that the terms $\nu_{\max}/\zeta_1(\nu)$ and $\zeta_2(\nu)/\zeta_1(\nu)$ are of order 1, as shown for the constant case in Remark 4.34.
Understanding what $\zeta_1(\nu)$ and $\zeta_2(\nu)$ look like for variable ν can be an interesting and fundamentally important goal of future analysis and simulations.
A question that suggests itself is whether the estimates are not sharp with respect to their predictions on the ν-dependence or whether the problem lies in the simplicity of the considered example which

- has a polynomial solution,
- is defined for a simple domain and
- and has been solved only on structured uniform meshes.

Remark 6.3. Iterative Solvers.
Some of the above mentioned open questions might be answered if it was possible to

[10]Only exception is Figure 18

solve the finite element problems on finer grids, i.e., for levels > 10.
As already mentioned, computing even ten levels was hardly possible for some of the viscosity functions presented in Section 5.1.
The used direct solver cannot go further than until Level 7. For finer grids one needs different direct solvers or iterative solvers which work fine for some of the viscosity functions (e.g.,the linear and the quadratic one, $\nu_1(\mathbf{x})$ and $\nu_2(\mathbf{x})$) but did not converge at all or only after countless iterations for the rougher viscosities, i.e., those with steep gradients (e.g., **atan(steep)**, $\nu_4(\mathbf{x})$ and **exp**, $\nu_5(\mathbf{x})$.
It is beyond question that there is a lot of room for improvement in simulating non-constant viscosities as well. Furthermore new ideas and approaches like using different finite element spaces, testing different viscosity functions, prescribing different solutions, should be pursued.

6.2. Conclusion

In spite of the aforementioned unanswered questions, we can come to the pleasant conclusion that this work presents a theory for the incompressible Stokes equations with non-constant viscosity that is able to explain the behavior of a finite element method for a Stokes problem with non-constant viscosity. None of the described observations contradicts this theory.
Furthermore, this theory is consistent with the theory for the case of a constant viscosity in the corresponding spaces as was discussed for example in Remarks 4.17 and 4.18.
Many expectations could be confirmed like for example the dependence of the stability estimates like described in Remark 4.6 and on the bigger part of the finite error estimates on the viscosity ν.
Rather positively surprising was the fact that the finite element error analysis is seemingly independent of the shape of the viscosity function since the gradient or higher derivatives of ν do not appear in the error estimates. (This, however could change for more complicated examples.)
Another pleasant discovery was the scale invariance of the error in some cases described in Section 5.9, i.e., the independence of the error estimates only on the ration $\nu_{\max}/\nu_{\min}$ and not of the concrete choices for those values.

Summing up, one can conclude that the problem of simulating and numerically solving the Navier-Stokes equations, albeit studied for many years already, still offers numerous interesting questions that ask to be answered.
Also in the future, researchers will work on those questions - presumably without knowing whether a proof for the existence of an analytical solution to the Navier-Stokes equations even exists.

A. Functional Analysis

Theorem A.1. Korn's inequality.
Let $\mathbb{D}(\mathbf{v})$ denote the deformation tensor of $\mathbf{v}$. For $p \in (1, \infty)$ there is a constant $C_K > 0$ such that

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)}^p \leq C_K \left(\|\mathbf{v}\|_{L^p(\Omega)}^p + \|\mathbb{D}(\mathbf{v})\|_{L^p(\Omega)}^p \right), \forall \mathbf{v} \in W^{1,p}(\Omega). \tag{A.1}$$

Let $|\cdot|$ denote a seminorm on $L^p(\Omega)$. Then it is

$$\|\mathbf{v}\|_{L^p(\Omega)} \leq C_K \left(|\mathbf{v}| + \|\mathbb{D}(\mathbf{v})\|_{L^p(\Omega)}^p \right), \forall \mathbf{v} \in W^{1,p}(\Omega). \tag{A.2}$$

For a detailed discussion of Korn's inequalities we refer to [7].

B. Finite Element Theory

Let $\hat{K} \subset \mathbb{R}^d, \quad d \in \{2, 3\}$, be the reference mesh cell, a compact polyhedron and K be an arbitrary mesh cell with diameter h_K.
The polynomial space of dimension N is denoted by $\hat{P}(\hat{K})$ and is assumed to be unisolvent with respect to the continuous linear functionals

$$\hat{\phi}_1, \ldots, \hat{\phi}_N : C^s(\hat{K}) \to \mathbb{R}.$$

Definition B.1. The polynomial space P_k.
Let $x = (x_1, \ldots, x_d)$, $k \in N \cup \{0\}$, and denote by $\alpha = (\alpha_1, \ldots, \alpha_d)$ a multi-index. Then, the polynomial space P_k is given by

$$P_k = \operatorname{span} \left\{ \prod_{i=1}^d x_i^{\alpha_i} = \mathbf{x}^{\boldsymbol{\alpha}} : \alpha_i \in \mathbb{N} \cup \{0\} \quad \text{for } i = 1 \ldots d, \quad \sum_{i=1}^d \alpha_i \leq k \right\}.$$

Definition B.2. The polynomial space Q_k.
Let $x = (x_1, \ldots, x_d)$, $k \in N \cup \{0\}$, and $\alpha = (\alpha_1, \ldots, \alpha_d)$. Then, the polynomial space Q_k is given by

$$Q_k = \operatorname{span} \left\{ \prod_{i=1}^d x_i^{\alpha_i} = \mathbf{x}^{\boldsymbol{\alpha}} : 0 \leq \alpha_i \leq k \quad \text{for } i = 1 \ldots d \right\}.$$

Theorem B.3. Interpolation error estimate on a reference mesh cell.
Let $P_m(\hat{K}) \subset \hat{P}(\hat{K})$ and $p \in [1, \infty)$ with $(m + 1 - s)p > d$ and let the interpolant of $\hat{v} \in C^s(\hat{K})$ be denoted by $I_{\hat{K}}\hat{v} \in \hat{P}(\hat{K})$.
Then, it is for all $\hat{v}$ from the Sobolev space $W^{m+1,p}(\hat{K})$

$$\|\hat{v} - I_{\hat{K}}\hat{v}\|_{W^{m+1,p}(\hat{K})} \leq C \|D^{m+1}\hat{v}\|_{L^p(\hat{K})}, \tag{B.1}$$

where the constant C is independent of $\hat{v}(\hat{\mathbf{x}})$.

Theorem B.4. Local interpolation estimate.
For an affine family of finite elements with reference cell $\hat{K}$, functionals $\hat{\phi}_i$ and a space of polynomials $\hat{P}(\hat{K})$ let all assumptions of Theorem B.3 be fulfilled and let $I_K v \in P(K)$ denote the interpolant of v.
Then, for all $v \in W^{m+1,p}(K)$ with $p \in [1, \infty)$ it is

$$\|D^k(v - I_K v)\|_{L^p(K)} \le C h_K^{m+1-k} \|D^{m+1} v\|_{L^p(K)}, \quad k \le m+1, \tag{B.2}$$

where the constant C is independent of v.

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Eigenständigkeitserklärung

Hiermit erkläre ich, dass ich die vorstehende Masterarbeit mit dem Titel "Finite Element Methods for the Incompressible Stokes Equations with Non-Constant Viscosity" selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel erstellt habe.
Die Stellen, die anderen Werken dem Wortlaut oder dem Sinn nach entnommen wurden, habe ich in jedem einzelnen Fall durch die Angabe der Quelle als Entlehnung kenntlich gemacht.

Berlin, den 02. September 2014