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Adaptive Methods for Convection-Diffusion Equations

Master Thesis

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Abstract

This thesis present a numerical study of the a posteriori error estimators for steady-state convection-diffusion equations. The study involves the gradient indicator, which based on the gradient recovery, four residual based error estimators for different norm and also introduces two error estimators defined by the solution of the local Neumann problems. These estimators' quality are studied through two examples with respect to two aspects. One is the accuracy of the estimated solution with respect to the real solution on the mesh cells, which is implemented by computation of the efficiency index. Another is comparison of these estimators with respect to the quality of the adaptive grid refinement. The computation of the examples is performed using ParMooN [11] [28].

Keywords: Steady-state Convection-diffusion equation, SUPG method, A posteriori error estimators, ParMooN

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Contents

1 Introduction

This thesis presents adapative methods for convection-diffusion equations.
The convection–diffusion equation is a combination of the diffusion and convection (advection) equations, and describes the flow of heat, particles, or other physical quantities in situations where there is both diffusion and convection or advection.
The numerical methods for convection-diffusion problems go back to 1955, more than 60 years ago. In 1969, two significant papers [3] [15] from Russia analysed new numerical methods for convection-diffusion ordinary differential equations.In [3], Bakhvalov considered an upwinded difference scheme on a layer-adapted graded mesh. Such meshes are based on a logarithmic scale, which are very fine inside the boundary layer and coarse outside. In [15] was used a uniform mesh. Angermann [2] gives an example of an exponentially-fitted scheme that does a remarkable job of capturing an interior layer on a uniform mesh. In the next 20 years, the Il'in-type schemes were applied for some PDEs by researchers, see [24] for reference. This doesn't explicitly contain exponentials, but it is based on the idea of solving a local problem exactly [9], as is Il'in's method [23]. In 1990 the Russian mathematician Grisha Shishkin showed that in-stead one could use a simpler piecewise uniform mesh. This idea has been enthusiastically propagated throughout the 1990s by a group of Irish mathematicians: Miller, O'Riordan, Hegarty, Stynes and Farrell. See [8] , [24] and their bibliographies.
The finite element method obtained its real impetus in the 1960s and 1970s by the developments of J. H. Argyris (Johann Hadji Argyris (1913-2004)) with co-workers. A rigorous mathematical basis to the finite element method was provided in 1973 with the publication by Strang and Fix [25]. The method has since been generalized for the numerical modeling of physical systems in a wide variety of engineering disciplines, e.g., electromagnetism, heat transfer, and fluid dynamics [21], [4]. It is a good choice for analyzing problems over complicated domains (like cars and oil pipelines), when the domain changes (as during a solid state reaction with a moving boundary), when the desired precision varies over the entire domain, or when the solution lacks smoothness.
In the past years, several techniques to stabilize finite element methods have been proposed for steady-state convection-diffusion equations. And the streamline-upwind Petrov/Galerkin method (SUPG method) is one of the most popular one among them. It was introduced by Hughes and co-workers [6]. After that, the SUPG method has then been improved by addition of the variable multiscale method, which was proposed also by Hughes [13]. The developed scheme result in a reasonable numerical solution and the accuracy is quite good outside layers [10] [19].
Here we only consider the steady-state convection-diffusion equation and present the Galerkin method and SUPG method based on the derivative of the weak solution of the equation. After that, the error estimation is presented accordingly in Section 2. In Section 3, we present a posteriori error estimation based on the residual and introduce several common a posteriori error estimators including gradient indicator, Zienkiewicz-Zhu estimator, residual-based error estimators in different norm and two error estimators based on the solution of local Neumann problems. The main goal of the a posteriori error estimators is to provide ideally bounds and an estimate to the solution error in a specified norm. For

an effective error estimator, there are some characteristics that it should include [12].The study of the behaviour of the a posteriori error estimators with respect to the estimation of the global error will be studied through one simple problem, which shown in the example 1 of Section 4. And the second example studies the adaptive grid refinement. In Section 5, we summarized the results of the numerical studies.

2 Steady-State Convection-Diffusion Equation

In this section, we give the introduction of the class of problems and corresponding mathematical equations we are interested in, derive the strong and weak form of the equation and the Streamline-Upwind Petrov-Glalerkin(SUPG) method of the problem, and introduce the notation which will be used.
Generally, we cannot expect that a partial differential equation has a classical solution. Hence there are many conditions to ensure the existence of a classical solution of a partial differential equation, such as all parameters have to be sufficiently smooth, the domain has to satisfy certain regularity conditions when in higher dimensions and so on. All these conditions become barriers when we require a classical solution for a partial differential equation derived from realistic problems.

Definition: Convection-diffusion-reaction equation
A linear convection-diffusion(-reaction) equation with homogeneous Dirichlet boundary condition (further detail of the Dirichlet boundary condition refer to Example 4.2.2.1 below) defined on a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in 1,2,3$ and Lipschitz boundary $\partial\Omega$ will be written

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f \ in \ \Omega \\ u = 0 \ on \ \partial\Omega \end{aligned} \tag{1}$$

where

- $-\varepsilon\Delta u$ is the diffusion term, $\varepsilon \in \mathbb{R}, \ \ \varepsilon > 0$.
- $\mathbf{b}\cdot\nabla u$ is the convection term (advective term, or transport term), $\mathbf{b}(x) \not\equiv 0$.
- cu is the reaction term.

Here we may call ε, b and c as the diffusion, convection and reaction parameter, repectively. They reflect the weight of the different parts.
Obviously these equations have the solution, even though it might not satisfy the smoothness or regularity conditions. Therefore one needs an extension for the notion of the solution.

2.1 Weak Solution

Denote $(\cdot,\cdot)$ as the inner product of $L^2(\Omega)$. Consider problem (1) and multiple the equation (1) with an appropriate function $v(x)$ satisfying $v = 0$ on $\partial\Omega$, intergrate the resulting equation on Ω, and apply integration by parts (applied with Gaussian Theorem [22]), we have

$$\begin{aligned} &\int_\Omega (-\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu)(\mathbf{x})v(\mathbf{x})\,d\mathbf{x} \\ &= \int_{\partial\Omega} (-\varepsilon(\nabla u\cdot\mathbf{n})v(\mathbf{s}))\,d\mathbf{s} + \int_\Omega (\varepsilon\nabla u\cdot\nabla v + (\mathbf{b}\cdot\nabla u + cu)(\mathbf{x})\,d\mathbf{x} \\ &= \int_\Omega (\varepsilon\nabla u\cdot\nabla v + (\mathbf{b}\cdot\nabla u + cu)v(\mathbf{x})\,d\mathbf{x} \\ &= \int_\Omega (f(\mathbf{x})v(\mathbf{x}))\,d\mathbf{x}. \end{aligned}$$

Here, $\mathbf{n}$ is the outward pointing unit normal vecotr on $\partial\Omega$. The highest order derivative of $u(\mathbf{x})$ has been transferred to $v(\mathbf{x})$.

Definition: Variational or weak formulation
If $b, c \in L^\infty(\Omega)$ and $f \in H^{-1}(\Omega)$, the convection-diffusion-reaction equation (1) can be written in the variational or weak formulation as: Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$\varepsilon(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v) = (f, v), \tag{2}$$

a solution of (2) is called variational or weak solution, the space where the solution searched is called solution or ansatz space, and the functions $v(\mathbf{x})$ are called test functions and the space from which they come is the test space. For the definition above, u is the weak solution and the solution space and test space are both $H_0^1(\Omega)$.
Pay attention that for different boundary conditions, the weak formulations are also different, which will be shown in the following examples.
There are different kind of boundary conditions with respect to different properties, such as Dirichlet boundary conditions, Neumann boundary conditions and Robin boundary conditions and so on, see [26]. In this paper, we will mainly concentrate on Dirichlet boundary conditions and Neumann boundary conditions.

Example 4.2.2.1 Inhomogeneous Dirichlet boundary conditions
Consider inhomogeneous Dirichlet boundary conditions

$$u(\mathbf{x}) = g(\mathbf{x}) \; on \; \partial\Omega, \tag{3}$$

with $g \in H^{\frac{1}{2}}(\partial\Omega)$, which are included into the definition of the ansatz space

$$V_a = \{v \in H^1(\Omega) : v|_{\partial\Omega} = g\},$$

the test space is still $V = H_0^1(\Omega)$.
Then the weak formulation can be written asFind uV_a such that

$$\varepsilon(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v) = (f, v) \; \forall v \in V. \tag{4}$$

Under this kind of description, the test space is still $H_0^1(\Omega)$, but the solution space is V_a.

There also exists a different way to express the variational problem by using an extension $u_g \in H^1(\Omega)$ of $g(x)$ to Ω. This kind of weak formulation reads as: Find $u \in H^1(\Omega)$ such that $\tilde{u} = u - u_g \in V$ and

$$\varepsilon(\nabla \tilde{u}, \nabla v) + (\mathbf{b} \cdot \nabla \tilde{u} + c\tilde{u}, v) = (\tilde{f}, v) \; \forall v \in V. \tag{5}$$

In this formulation, the solution and test space are same, which is $H_0^1(\Omega)$. Hereby the term $(\tilde{f}, v)$ has the form

$$(\tilde{f}, v) = (f, v) + \varepsilon(\nabla u_g, \nabla v) + (\mathbf{b} \cdot \nabla \; u_g + cu_g, v). \tag{6}$$

Example 4.2.2.2 Neumann boundary conditions
Consider Neumann boundary conditions

$$\varepsilon(\nabla u \cdot \mathbf{n}) = g_N \ on \ \partial\Omega_N. \tag{7}$$

Let $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ with $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, and assume that $u(x) = 0$ for all $x \in \partial\Omega_D$ for simplicity.
Let $V_0 = \{v \in H^1(\Omega) : v|_{\partial}\Omega_D = 0\}$, then the variational formulation reads as: Find $u \in V_0$ such that

$$\varepsilon(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v) = (f, v) + \int_{\partial\Omega_N} g_N(\mathbf{x})v(\mathbf{s})\, d\mathbf{s} \ \forall v \in V_0, \tag{8}$$

a bilinear form in mathematics on a vector space V is a bilinear map $V \times V \to K$ where K is the field of scalars. In other words, a bilinear form is a function $B : V \times V \to K$ that is linear in each argument separately. Therefore we have following properties.

Property2.2.1 Let $(V, \|\cdot\|_V)$ be a Banach space. A map $\mathbf{s} : V \times V \to \mathbb{R}$ is called

- Bilinear, if $a(\cdot, \cdot)$ is linear in both arguments,
- Symmetric, if $a(u, v) = a(v, u)$ for all $u, v \in V$,
- Positive, if $a(v, v) \geq 0$ for all $v \in V$,
- Strictly positive or coercive or V-elliptic or positive definite if there is a $m > 0$ such that $a(v, v) \geq m\|\|_V^2$ for all $v \in V$,
- Bounded if there is a $M > 0$ such that

$$|a(u, v)| \geq M\|u\|_V\|v\|_V \tag{9}$$

for all $u, v \in V$.

Lemma For the bilinear form of the weak formulation of problem (2)
First of all, for the weak formulation of problem (2) we study in this section, it is

$$a(u, v) := \int_\Omega (\varepsilon\nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) + \mathbf{b}(\mathbf{x}) \cdot \nabla u(\mathbf{x})v(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x})v(\mathbf{x}))\, d\mathbf{x}, \tag{10}$$

a bilinear form in the space $V = H_0^1(\Omega)$. Then the conditions of different properties can be written in the form below:

- Symmetric, if $b(x) = 0$ for all $x \in \Omega$,
- Coercive, if $-\frac{1}{2} \cdot \mathbf{b}(\mathbf{x}) + c(\mathbf{x}) \geq 0$,
- Bounded, and the corresponding $M = \varepsilon + C_{PF}\|\mathbf{b}\|_{L^\infty(\Omega)} + C_{PF}^2\|\mathbf{c}\|_{L^\infty(\Omega)}$ such that

$$|a(u, v)| \geq M\|u\|_V\|v\|_V \tag{11}$$

for all $u, v \in V$.

Proof: If $b(x) = 0$ for all $x \in \Omega$, then we have

$$a(u,v) := \int_\Omega (\varepsilon \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x}))\, d\mathbf{x}, \tag{12}$$

and exchange of the position of u and v won't change the form of $a(u,v)$. It is symmetric. Let $\mathbf{b} \in C^1(\bar{\Omega})$ and $c \in C(\bar{\Omega})$. By integration by parts and the product rule, one obtains

$$\int_\Omega (\mathbf{b}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) u(\mathbf{x}))\, d\mathbf{x} = -\frac{1}{2} \int_\Omega (\nabla \cdot \mathbf{b}(\mathbf{x}) v(\mathbf{x}) u(\mathbf{x}))\, d\mathbf{x}. \tag{13}$$

Insert it into $a(u,v)$ and let $u(x) = v(x)$ yields

$$a(v,v) = \int_\Omega (\varepsilon (\nabla v(\mathbf{x}))^2 + (-\frac{1}{2} \nabla \cdot \mathbf{b}(\mathbf{x}) + c(\mathbf{x})) v(\mathbf{x})^2\, d\mathbf{x}. \tag{14}$$

Hence if $-\frac{1}{2}\nabla \cdot \mathbf{b}(\mathbf{x}) + c(\mathbf{x}) \geq 0$, then

$$a(u,v) \geq \int_\Omega (\varepsilon (\nabla v(\mathbf{x}))^2\, d\mathbf{x} = \varepsilon \|\nabla v(\mathbf{x})\|^2_{L^\infty(\Omega)} = \varepsilon \|v(\mathbf{x})\|^2_V, \tag{15}$$

which means $a(u,v)$ is coercive.
By the Cauchy-Schwarz inequality, Hoelder's inequality, and the Poincare-Friedrichs inequality, we have

$$\begin{aligned}
|a(u,v)| \leq \varepsilon \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} &+ \|\mathbf{b}\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\
&+ \|\mathbf{c}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}, \\
\leq \varepsilon \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} &+ C_{PF} \|\mathbf{b}\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\
&+ C_{PF}^2 \|\mathbf{c}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\
= (\varepsilon + C_{PF} \|\mathbf{b}\| &+ C_{PF}^2 \|\mathbf{c}\|) \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\
= (\varepsilon + C_{PF} \|\mathbf{b}\| &+ C_{PF}^2 \|\mathbf{c}\|) \nabla u\|_V \|\nabla u\|_V,
\end{aligned}$$

where we assume that $b, c \in L^\infty(\Omega)$.
According to the Lax-Milgram theorem [20], when the map $a(\cdot,\cdot)$ is a bounded and coercive bilinear form on the Hilbert space V, then for each bounded linear functional $f \in V^{'}$, there is exactly one $u \in V$ satisfying (cf. e.g. Ciarlet [7])

$$a(u,v) = f(v)\ \forall v \in T. \tag{13}$$

2.2 Finite Element Formulations

2.2.1 The Galerkin Method

Definition:The Galerkin Method
Repacing the space V in weak formulation (2) with a finite-dimensional space

V^h, we have the standard finite element method, i.e., Galerkin Method: Find $u^h \in V^h$, such that for all $v^h \in V^h$

$$\varepsilon(\nabla u^h, \nabla u^h) + (\mathbf{b}\nabla u^h + c\nabla u^h, \nabla v^h) = (f, \nabla v^h), \tag{14}$$

where V^h is a finite-dimensional space.
The method is called conforming if $V^h \subset V$.

Cea's Lemma Let $V^h \subset V$ and $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bounded and coercive bilinear form on the Hilbert space V, then there is a unique solution of the problem to find $u^h \in V^h$ such that

$$a(u^h, v^h) = f(v^h) \ \forall v^h \in V^h \tag{15}$$

and the error between the original and the Galerkin solution admits the estimate

$$\|u - u^h\|_V \leq \frac{M}{m} \inf_{v^h \in V^h} \|u - v^h\|_V. \tag{16}$$

That is to say, the Galerkin solution u^h is "the best" approximation of the original solution u inV^h, up to the constant $\frac{M}{m}$, where m is a positive constant satisfying $a(v,v) \geq m\|v\|_{V^2}$ for all $v \in V$, and M is also a positive constant such that $|a(u,v)| \leq M\|v\|_V\|u\|_V$ for all $u, v \in V$.
The error estimate can be simply proved with the boundedness and ellipticity of the bilinear form and Galerkin orthogonality.
Proof: Since the subspace of a Hilbert space is also a Hilbert space and the properties of the bilinear form carry over from V to V^h, and based on the Thorem of Lax-Milgram, we can directly derive the existence of a unique solution of the discrete problem.
Copmuting the difference of (3) and (5), yield the error estimator

$$a(u - u^h, v^h) = 0 \ \forall v^h \in V^h, \tag{17}$$

with

$$m\|v\|^2_{V^2} \leq a(v,v) \tag{18}$$

and

$$|a(u,v)| \leq M\|v\|_V\|u\|_V. \tag{19}$$

It follows for all $v^h \in V^h$ that

$$\|u - u^h\|^2_V \leq \frac{1}{m} a(u - u^h, u - v^h) \leq \frac{M}{m}\|u - u^h\|_V\|u - v^h\|_V. \tag{20}$$

This equality is equivalent to the statement of the theorem.

Definition:Petrov-Galerkin Method.
Petrov-Galerkin method is a finite element method whose ansatz space and test space are different.Let A^h and $\mathcal{T}^h$ be the ansatz and test spaces respectively, with$dim(A^h) = dim(\mathcal{T}^h)$. Then the Petrov-Galerkin method reads as follows: Find $u^h \in A^h$ such that

$$a(u^h, v^h) = f(v^h) \ \forall v^h \in \mathcal{T}^h. \tag{21}$$

2.2.2 SUPG Method

In this section, we consider a coercive problem (2), which means $-\frac{1}{2}\nabla \cdot \mathbf{b}(\mathbf{x}) + c(\mathbf{x}) \geq 0$ is satisfiey.

Definition:Residual-based stabilization
Methods that consist of a penalization of the residual if the so-called strong residual has large value. Given a linear partial differential equation whose Galerkin finite element discretization is same as (6) in its strong form

$$A_{str} u_{str} = f,\ f \in L^2(\Omega). \tag{22}$$

Modify A_{str} as a well-defined for finite element functions with a linear operator

$$A_{str} : V^h \to L^2(\Omega). \tag{23}$$

The residual can be defined by

$$r^h(u^h) = A_{str} u_{str} - f \in L^2(\Omega). \tag{24}$$

Instead of finding the solution of the equations, we look for the minimizer of the residual, which can be present as

$$\arg\min_{v^h \in V^h} \|r^h(u^h)\|^2_{L^2(\Omega)} = \arg\min_{v^h \in V^h} (r^h(u^h), r^h(u^h)). \tag{25}$$

From the optimization problem's necessary condition is vanishing the Gateaux derivative, we can have a generalization consisting in the minimization problem

$$\arg\min_{v^h \in V^h} \|\delta^{1/2} r^h(u^h)\|^2_{L^2(\Omega)} = \arg\min_{v^h \in V^h} (\delta^h r^h(u^h), r^h(u^h)) \tag{26}$$

with the positive weighting function $\delta(x)$. The minimization problem above can be derivated by using the linearity of A^h_{str} and the bilinearity of the inner product in $L^2(\Omega)$

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \frac{(r^h(u^h + \varepsilon v^h), r^h(u^h + \varepsilon v^h)) - (r^h u^h, r^h u^h))}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{(r^h u^h + \varepsilon A^h_{str} v^h, r^h u^h + \varepsilon A^h_{str} v^h) - (r^h u^h, r^h u^h))}{\varepsilon} \\ &= 2(r^h u^h, {}^h_{str} v^h)\ \forall v^h \in V^h. \end{aligned} \tag{27}$$

For the solution of certain problem that possesses structures (such as layers for convection-diffusion equations) that are important but are not resolved by the used grid, the Garlerkin method failed. When it comes to sharp layers, which is generally coarser than the mesh width, the solution is not accurate enough. Hence we consider combine residual-based stabilizations with the Galerkin discretization (possessing not sufficient diffusion) and the minimization of the residual (is over-diffusive),

$$a^h(u^h, v^h) + (\delta r^h(u^h), A^h_{str} v^h) = (f, v^h)\ \forall v^h \in V^h. \tag{28}$$

The goal of numerical analysis consists in determining the weighting function δ optimally in an asymptotic sense.

Definition:Consistent finite element method
Let $u(\mathbf{x})$ be a sufficiently smooth solution of: Find $u \in V$ such that

$$a(u, v) = f(v) \ \forall v \in V, \tag{29}$$

where $a(\cdot, \cdot)$ is an appropriate bilinear form and $f(\cdot)$an appropriate functional. A finite element method related to this problem: Find $u^h \in V^h$ such that

$$a^h(u^h, v^h) = f(v^h) \ \forall v^h \in V^h \tag{30}$$

is called consistent, if

$$a^h(u, v^h) = f(v^h) \ \forall v^h \in V^h. \tag{31}$$

Hence the consistency of a finite element method means that a sufficiently smooth solution satisfies also the discrete equation, which is quite different from the consistency of finite differential operator.
The upwind finite element discretization remove the unneeded oscillations but the accuracy attained is usually poor since too much introduced numerical diffusion, just like in the finite difference method. Besides, they are not consistent either, which lead to the accuracy limited to first order. Even worse condition is that the discrete solution is less accurate than the one produced by Galerkin method (cf. e.g. Brooks and Hughes [6]for a discussion on shortcomings of upwind methods).

Definition:Streamline-Upwind Petrov-Galerkin FEM, SUPG method
The Streamline-Upwind Petrov-Diffusion(SUPG) FEM or Streamline-Diffusion FEM(SDFEM) has the form: Find $u^h \in V^h$, such that

$$a^h(u^h, v^h) = f(v^h) \ \forall v^h \in V^h \tag{32}$$

with

$$a^h(v, w) := a(v, w) \tag{33}$$

$$+ \sum_{K \in \mathcal{T}^h} \int_K \delta_K (-\varepsilon \Delta v(\mathbf{x}) + \mathbf{b}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) + c(\mathbf{x}) v(\mathbf{x}))(\mathbf{b}(\mathbf{x}) \cdot \nabla w(\mathbf{x})) \, d\mathbf{x} \tag{34}$$

$$f^h(w) := (f, w) + \sum_{K \in \mathcal{T}^h} \int_K \delta_K f(\mathbf{x})(\mathbf{b}(\mathbf{x}) \cdot \nabla w(\mathbf{x})) \, d\mathbf{x}. \tag{35}$$

Here, δ_K are user-chosen weights, which are called stabilization parameters or SUPG parameters.

Remarks

- The SUPG method is consistent.
  Proof: It can be proved by inserting a sufficient smooth solution of (1) in the SUPG formulation, which results in a vanishing of the stabilization term and remains

$$a^h(u^h, v^h) = f(v^h) \ \forall v^h \in V^h, V^h \subset V. \tag{36}$$

- The SUPG method introduces artificial diffusion only in the streamline direction $\mathbf{b}(\mathbf{x}) \cdot \nabla w(\mathbf{x})$.

- The SUPG parameter δ_K will be chosen as a constant function in practice. Finite element error estimation's goal is proposing a good asymptotic choice of it.

Lemma:Galerkin orthogonality
A consistent finite element method has the property of the Galerkin orthogonality if

$$a^h(u - u^h, v^h) = 0 \ \forall v^h \in V^h. \tag{37}$$

The error is "orthogonal" to the finite element space.
Proof: Through subtracting (1) and (33-35), we can directly deduce it.
Since SUPG method is a consistant finite element method, we know that SUPG method also satisfies Galerkin orthogonality.

Example: SUPG method for P_1 finite elements in one dimension
Consider $\Omega = (0, 1)$ and $V^h = P_1$ on an equidistant grid with $h_i = h, i = 1, \ldots, N$. Assume all coefficients and the SUPG parameter are constant and $c = 0$, then the left-hand of the SUPG method reduces to $\varepsilon((u^h)', (v^h)') + b((u^h)', v^h) + \varepsilon b^2((u^h)', (v^h)')$, and the right-hand side is hf_i.
Overall, the SUPG method is

$$-\varepsilon(1 + \delta\frac{b^2}{\varepsilon})D^+D^-u_i + bD^0u_i = f_i. \tag{38}$$

Definition: SUPG norm
Let for almost all $x \in \Omega$ hold

$$-\frac{1}{2}\nabla \cdot \mathbf{b}(\mathbf{x}) + c(\mathbf{x}) \geq \omega > 0, \tag{39}$$

in V^h,the SUPG norm is defined by

$$|||v^h|||_{SUPG} := (\varepsilon|v^h|^2_{H^1(\Omega)} + \omega\|v^h\|^2_{L^2(\Omega)} + \sum\|\delta_K^{\frac{1}{2}}\mathbf{b} \cdot \nabla v^h\|^2_{L^2(\Omega)})^{1/2}. \tag{40}$$

Theorem: Coercivity of the SUPG bilinear form
Assume that $b \in W^{1,\infty}(\Omega), c \in L^\infty(\Omega), -\frac{1}{2}\nabla \cdot \mathbf{bx} + c(\mathbf{x}) \geq \omega > 0$, and let

$$0 < \delta_K \leq \frac{1}{2}min\left\{\frac{h_K^2}{\varepsilon C_{inv}^2}, \frac{\omega}{\|c\|^2_{L_{\inf(K)}}}\right\}, \tag{41}$$

where C_{inv} is a constant that depends only on $k, l, p, q, \hat{K}, P(\hat{K})$ with $0 \leq k \leq 1$ such that

$$\|D^l v^h\|_{L^q(K)} < C_{inv}h_K^{(k-l)-d(p^{-1}-q^{-1})}\|D^k v^h\|_{L^p(K)} \ \forall v^h \in V^h. \tag{42}$$

Then the SUPG bilinear form is coercive with respect to the SUPG norm, i.e., it is

$$a^h(v^h, v^h) \geq \frac{1}{2}|||v^h|||^2_{SUPG} \ \forall v^h \in V^h. \tag{43}$$

Proof: Integration by parts gives,

$$a^h(u^h, v^h) \geq \tfrac{1}{2}|||v^h|||^2_{SUPG} \; \forall v^h \in V^h. \tag{44}$$

With the definition of ω, one obtains

$$a^h(u^h, v^h) = \varepsilon|v^h|_1^2 + \int_\Omega (c(\mathbf{x}) - \tfrac{\nabla\cdot\mathbf{b}(\mathbf{x})}{2} + c)(v^h)^2(\mathbf{x})d\mathbf{x} + \sum_{K\in\mathcal{T}^h} \|\delta_K^{\frac{1}{2}}(\mathbf{b}\cdot\nabla v^h)\|^2_{L^2(K)} \tag{45}$$

$$+ \sum_{K\in\mathcal{T}^h} \int_K \delta_K(-\varepsilon\Delta v^h(\mathbf{x}) + c(\mathbf{x})v^h(\mathbf{x}))(\mathbf{b}(\mathbf{x})\cdot\nabla v^h(\mathbf{x}))\, d\mathbf{x} \tag{46}$$

$$\geq |||v^h|||_{SUPG} - |\sum_{K\in\mathcal{T}^h} \int_K \delta_K(-\varepsilon\Delta v^h(\mathbf{x}) + c(\mathbf{x})v^h(\mathbf{x}))(\mathbf{b}(\mathbf{x})\cdot\nabla v^h(\mathbf{x}))\, d\mathbf{x}|. \tag{47}$$

Using (5) and Young's inequality [5], it is for each $K \in \mathcal{T}^h$

$$|\sum_{K\in\mathcal{T}^h} \int_K \delta_K(-\varepsilon\Delta v^h(\mathbf{x}) + c(\mathbf{x})v^h(\mathbf{x}))(\mathbf{b}(\mathbf{x})\cdot\nabla v^h(\mathbf{x}))\, d\mathbf{x}| \tag{48}$$

$$\leq \int_K (\delta_K^{\frac{1}{2}}\varepsilon|\Delta v^h|)(\delta_K^{\frac{1}{2}}|\mathbf{b}(\mathbf{x})\cdot\nabla v^h(\mathbf{x})|)\, d\mathbf{x}$$

$$+ \int_K (\delta_K^{\frac{1}{2}}|c(\mathbf{x})||v^h(\mathbf{x})|)(\delta_K^{\frac{1}{2}}|\mathbf{b}(\mathbf{x})\cdot\nabla v^h(\mathbf{x})|)\, d\mathbf{x} \tag{49}$$

$$\leq \tfrac{\varepsilon}{2}\|\cdot\nabla v^h\|^2_{L^2(K)} + \tfrac{1}{4}\|\delta_K^{\frac{1}{2}}(\mathbf{b}\cdot\nabla v^h\|^2_{L^2(K)} + \tfrac{\omega}{2}\|v^h\|^2_{L^2(K)} + \tfrac{1}{4}\|\delta_K^{\frac{1}{2}}(\mathbf{b}\cdot\nabla v^h\|^2_{L^2(K)} \tag{50}$$

$$= \tfrac{1}{2}|||v^h|||_{SUPG,K}. \tag{51}$$

Insert it to the obove equation, the proof is finished.

Corollary:Coercivity of the SUPG bilinear form for P_1 finite elements
Based on the theorem above, for piecewise linear finite elements, the SUPG bilinear form (4) is coercive with respect to the SUPG norm if

$$0 < \delta_K \leq \tfrac{\omega}{\|c\|^2_{L^\infty(K)}}. \tag{52}$$

Proof: Based on theorem of SUPG bilinear form's coercivity, considering that for piecewise linear finite elements $\Delta v^h(x)|_K = 0$ for all $K \in \mathcal{T}^h$. Thus the corresponding terms do not appear.

If the assumptions of the corollary above valid, then the SUPG FEM has a unique solution.

Corollary:Stability of the SUPG method.
From the theorem of coercivity of the SUPG bilinear norm and the definition of stability, we can deduce that an appropriate norm of the solution can be estimated with the data of the problem. For SUPG method, the norm is the SUPG norm shown above. The estimation is shown as below:

$$|||u^h|||^2_{SUPG} \leq 2a^h(u^h, u^h) = 2f^h(u^h)$$

$$= 2(f, u^h) + 2 \sum_{K \in \mathcal{T}^h} \int_K \delta_K (\mathbf{b}(\mathbf{x}) \cdot \nabla w(\mathbf{x}))\, d\mathbf{x} \tag{53}$$

$$\overset{CS}{\leq} \frac{2}{\sqrt{\omega}} \|f\|_{L^2(\Omega)} \sqrt{\omega} \|u^h\|_{L^2(\Omega)} + 2 \sum_{K \in \mathcal{T}^h} \|\delta_K^{1/2} f\|_{L^2(K)} \|\delta_K^{1/2} (\mathbf{b} \cdot \nabla u^h)\|_{L^2(K)} \tag{54}$$

$$\overset{Young}{\leq} C\|f\|_{L^2(\Omega)} + \frac{1}{2}(\omega \|u^h\|^2_{L^2(\Omega)} + \sum_{K \in \mathcal{T}^h} \|\|\delta_K^{1/2} (\mathbf{b} \cdot \nabla u^h)\|^2_{L^2(K)} \tag{55}$$

$$\leq C\|f\|_{L^2(\Omega)} + \frac{1}{2} |||u^h|||^2_{SUPG}. \tag{56}$$

where constant C depends on ω and the upper bound of δ_K.
The SUPG method is also stable with respect to the norm $\|\cdot\|_\varepsilon$, because all $v^h \in V^h$ satisfy

$$|||u^h|||_{SUPG} \geq min\,\{1, \omega\}\, \|v^h\|_\varepsilon. \tag{57}$$

And the Garlerkin Finite Element Method is also stable with respect to this norm. But it is not stable with respect to stability of the SUPG norm. Which means the stability of SUPG method is stronger than the Garlerkin finite element method.

2.3 Error Estimate

Theorem: Convergence of the SUPG method.
Let $u \in H^(k+1)(\Omega)$, $k \geq 1$, $b \in W^{1,\infty}(\Omega)$, $c \in L^\infty(\Omega)$, let the assumptions of the coercivity theorem be satisfied, and consider the SUPG method for P_k finite elements. Using the following SUPG parameter

$$\delta_K = \begin{cases} C_0 \frac{h_K^2}{\varepsilon}, & \text{for } h_K < \varepsilon \text{ (covection-dominated)} \\ C_0 h_K, & \text{for } \varepsilon \leq h_K \text{ (diffusion-dominated)} \end{cases} \tag{58}$$

where the constant $C_0 > 0$ is sufficiently small such that (4) is satisfied for $k \geq 2$ or (5) for $k = 1$, respectively. Then the solution of the SUPG method satisfies the error estimate shown below

$$|||u - u^h|||_{SUPG} \leq C(\varepsilon^{\frac{1}{2}} h^k + h^{k+\frac{1}{2}})|u|_{H^{K+1}(\Omega)}, \tag{59}$$

where the constant C is independent of ε and h.
Proof: Let $u_I^h \in V^h$ be the Lagrangian interpolant of $u(x)$. One obtains with the triangle inequality

$$|||u - u^h|||_{SUPG} \leq |||u - u_I^h|||_{SUPG} + |||u_I^h - u|||_{SUPG}. \tag{60}$$

The first term on the right-hand side is the interpolant error. Note that for both regimes it is

$$\delta_K \leq C_0 h_K \leq Ch. \tag{61}$$

Using this after having applied the interpolation error estimate to each term of the SUPG norm individually gives

$$|||u - u^h|||_{SUPG} \leq (C\varepsilon h^{2k} |u|^2_{H^{k+1}(\Omega)} + C^{2(k+1)} |u|^2_{H^{k+1}(\Omega)} \tag{62}$$

$$+C\sum_{K\in\mathcal{T}^h}\delta_K\|\mathbf{b}\|_{L^\infty(K)}h_K^{2k}|u|^2_{H^{k+1}(\Omega)})^{1/2} \tag{63}$$

$$\leq C(\varepsilon h^{2k}+h^{2(k+1)}+h^{2k+1})^{1/2}|u|_{H^{k+1}(\Omega)} \tag{64}$$

$$\leq C(\varepsilon^{1/2}h^k+h^{k+1/2})|u|_{H^{k+1}(\Omega)}. \tag{65}$$

Based on the coercivity and the Galerkin orthogonality, the second term on the right-hand side yields

$$\frac{1}{2}|\|u_I^h-u^h\||^2_{SUPG}\leq a^h(u_I^h-u^h,u_I^h-u^h)=a^h(u_I^h-u,u_I^h-u). \tag{66}$$

Therefore, the triangle inequality is applied to $a^h(u_I^h-u^h,u_I^h-u^h)$ and each term is estimated independentendly. Apply local interpolation error estimate and let $w^h=u_I^h-u^h$, for the diffusive term, reative term we obtain

$$|\varepsilon(\nabla(u_I^h-u^h),\nabla w^h)|\leq C\varepsilon^{\frac{1}{2}}h^k|u|_{H^{K+1}(\Omega)}|\|w^h\||_{SUPG}. \tag{67}$$

$$|(c(u_I^h-u^h),w^h)|\leq Ch^{k+1}|u|_{H^{K+1}(\Omega)}|\|w^h\||_{SUPG}. \tag{68}$$

For the rest terms, we consider which come from the SUPG stabilization. And for both regimes it is

$$\varepsilon\delta_K\leq C_0h_K^2, \tag{69}$$

one gets

$$|\sum_{K\in\mathcal{T}^h}(-\varepsilon\Delta(u_I^h-u),\delta_K\mathbf{b}\cdot\nabla w^h)_K| \tag{70}$$

$$\overset{CS}{\leq}\sum_{K\in\mathcal{T}^h}\varepsilon^{1/2}\|\Delta(u_I^h-u)\|_{L^2(K)}\|\delta_K^{1/2}(\mathbf{b}\cdot\nabla w^h)\|_{L^2(K)} \tag{71}$$

$$\overset{CS}{\leq}C_0^{1/2}\varepsilon^{1/2}h(\sum_{K\in\mathcal{T}^h}\|\Delta(u_I^h-u)\|^2_{L^2(K)})^{1/2}\times(\sum_{K\in\mathcal{T}^h}\|\delta_K^{1/2}(\mathbf{b}\cdot\nabla w^h)\|_{L^2(K)})^{1/2} \tag{72}$$

$$\leq C\varepsilon^{1/2}h^k|u|_{H^{K+1}(\Omega)}|\|w^h\||_{SUPG}. \tag{73}$$

and

$$|\sum_{K\in\mathcal{T}^h}(\mathbf{b}\cdot\nabla(u_I^h-u)+c(u_I^h-u),\delta_K\mathbf{b}\cdot\nabla w^h)_K| \tag{74}$$

$$\overset{CS}{\leq}\sum_{K\in\mathcal{T}^h}\|\mathbf{b}\|_{L^\infty(\Omega)}\|\Delta(u_I^h-u)\|_{L^2(K)}\delta_K^{1/2}\|\delta_K^{1/2}(\mathbf{b}\cdot\nabla w^h)\|_{L^2(K)} \tag{75}$$

$$\overset{CS}{\leq}Ch^{1/2}[(\sum_{K\in\mathcal{T}^h}\|\nabla(u_I^h-u)\|^2_{L^2(K)})^{1/2}+(\sum_{K\in\mathcal{T}^h}^{h}{}_I-u\|^2_{L^2(K)})^{1/2}] \tag{76}$$

$$\times(\sum_{K\in\mathcal{T}^h}\|\delta_K^{1/2}(\mathbf{b}\cdot\nabla w^h)\|^2_{L^2(K)})^{1/2} \tag{77}$$

$$\leq C(h^{k+\frac{1}{2}}+h^{k+\frac{3}{2}})|u|_{H^{K+1}(\Omega)}|\|w^h\||_{SUPG}. \tag{78}$$

Next, apply first integration by parts to obtain an optimal estimate for the convective term

$$(\mathbf{b}\cdot\nabla(u_I^h-u),w^h)=(\nabla(u_I^h-u),\mathbf{b}w^h)=-(u_I^h-u,\nabla\cdot(\mathbf{b}w^h)) \tag{79}$$

$$=-(u_I^h-u,(\nabla\cdot\mathbf{b})w^h)-(u_I^h-u\mathbf{b}\cdot\nabla w^h). \tag{80}$$

In the right-hand side, both terms are estimated seperately. Using same tools to the other estimates we have

$$|(u_I^h-u,\nabla\cdot(\mathbf{b}w^h)|\leq Ch^{k+1}|u|_{H^{K+1}(\Omega)}|||w^h|||_{SUPG}. \tag{81}$$

In the estimates of the others, we have to distinguish if in the mesh cell K it is $\varepsilon\leq h_K$. We get

$$|(u_I^h-u,\nabla\cdot(\mathbf{b}w^h)| \tag{82}$$

$$\overset{CS}{\leq}\sum\varepsilon\leq h_K\delta_K^{-1/2}\|u_I^h-u\|_{L^2(K)}\|\delta_K^{1/2}(\mathbf{b}\cdot\nabla w^h)\|_{L^2(K)} \tag{83}$$

$$+\sum\varepsilon>h_K\|\mathbf{b}\|_{L^\infty(\Omega)}\|u_I^h-u\|_{L^2(K)}\|\nabla w^h\|_{L^2(K)} \tag{84}$$

$$\overset{C_0h_K=\delta_K,\varepsilon>h_K}{\leq} C(\sum_{\varepsilon\leq h_K}C_0^{-1/2}h_K^{-1/2}h_K^{k+1}|u|_{H^{K+1}(K)\|\nabla\mathbf{b}\cdot\nabla w^h\|_{L^2(K)}} \tag{85}$$

$$+\sum_{\varepsilon>h_K}h_K^{k+1/2}|u|_{H^{K+1}(K)}\varepsilon^{1/2}\|\nabla w^h\|_{L^2(K)} \tag{86}$$

$$\leq Ch^{k+1}|u|_{H^{K+1}(\Omega)}|||w^h|||_{SUPG}. \tag{87}$$

Overall, the statement of the theorem is proved.

Remarks

Therefore we can deduce that in the convection-dominated regime $\varepsilon\ll h$, the order of convergence in the SUPG norm is $k+1/2$ and in the diffusion-dominated case it is k. For obtaining an estimate with a constant C, which is independent of ε, the term $(\sum K\in\mathcal{T}^h\|\mathbf{b}\cdot\nabla w^h\|^2_{L^2(\Omega)})^{1/2}$ is part of the norm.

When taking the polynomial degree k of the finite element into account, the stabilization parameter is proposed

$$\delta_K=\begin{cases}C_0\frac{h_K}{\|\mathbf{b}\|_{L^\infty(K)}} & \text{for } Pe_K\geq 1,\\ C_0\frac{H_k^2}{\varepsilon} & \text{else}\end{cases} \tag{88}$$

with

$$Pe_K=\frac{\|\mathbf{b}\|_{L^\infty(K)}h_K}{2p\varepsilon}. \tag{89}$$

For linear and bilinear finite elements in practice, one takes the parameter

$$\delta_K=\frac{h_K}{2\|\mathbf{b}\|_{L^\infty(K)}}(coth(Pe_K)-\frac{1}{Pe_K}), \tag{90}$$

$$Pe_K=\frac{\|\mathbf{b}\|_{L^\infty(K)}h_K}{2\varepsilon}. \tag{91}$$

Where Pe_K is the local Peclet number.

3 A Posterori Error Estimators and Conditions

3.1 Residual A Posteriori Error Estimation

Remark:Goal
There are mainly two goals for a posteriori error estimators.
Firstly, they should provide computable estimates of the error between a computed solution u_h and the unknown solution u of the continuous problem (1). The errors are measured usually in norm of Sobolev spaces defined on Ω, hence we have the form of an upper bound of the estimates of global error

$$\|u - u^h\|_\Omega \leq C\eta, \tag{92}$$

where η can be computed with the information gained from the numerical solution process and C is a positive constant that is generally independent of h (mesh width) and u (the solution).
Secondly, controlling an adaptive mesh refinement is another task. When the local error is larger in certain subregions, then these subregions can be refined and hence the error can be reduced at least significantly. Thus, a lower estimate of local error has the form

$$\eta_K \leq \|u - u^h\|_{\omega(K)}, \tag{93}$$

where $\omega(K)$ denotes a small neighborhood of a mesh cell K and η_K is the local error estimator.
Consider the stationary convection-diffusion-reaction equations we concentrating in this paper of the form

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f \ in \Omega, u = 0 \ on \ \Omega, \\ u = 0 \ on \ \Gamma_D, \\ \varepsilon\frac{\mathrm{d}u}{\mathrm{d}n} = g_N \ on \ \Gamma_N, \end{aligned} \tag{94}$$

where Ω is a polygonal domain in R^d, $d \geq 2$, with Lipschitz boundary Γ satisfying $\Gamma = \Gamma_D + \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Γ_D is the Dirichlet part of the boundary which has a positive (d-1)-dimensional Lebesgue measure and $\partial\Omega^-$, where Ω^- is the inflow boundary of Ω. And there will also be some assumptions: $0 < \varepsilon$, $b \in W^{1,\infty}(\Omega) c \in L^\infty(\Omega)$, $f^2(\Omega)$ and the equations above will be scaled such that $0 < \varepsilon \ll 1$, $\|\mathbf{b}\|_{L^\infty(\Omega)} = \|c\|_{L^\infty(\Omega)} = \mathcal{O}(1)$. And in order to satisfy the condition to use SUPG method, we assume here that the following condition is fulfilled:

$$c(x) - \frac{1}{2}div(b(x)) = \mu(x) \geq \mu_0 > 0 \ \forall x \in \Omega. \tag{95}$$

Then, it will be deduced that the equation (94) has a unique weak solution $u \in H^1_D(\Omega) = v \in H^1(\Omega) : v|_{\Gamma_D} = 0$ that satisfies

$$\varepsilon(\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u, v) + (cu, v) = (f, v) + (g, v)_{\Gamma_N} \ \forall v \in H^1_D(\Omega). \tag{96}$$

The corresponding SUPG finite element method reads as follows: Find $u_h \in V^h \subset V$ such that

$$a_{SUPG}(u_h, v_h) = (f, v_h) + (g, v_h)_{\Gamma_N} + \sum_{K\in\mathcal{T}^h} \delta_K(f, \mathbf{b}\cdot\nabla v_h)_K \ \forall v_h \in H^h_V. \tag{97}$$

Hypothesis: Assumed that several norms of the interpolation error $u - I_h u$ can be bounded by norms of the error $u - u_h$:

$$\sum_{K\in\mathcal{T}^h} \delta_K^{-1}\|u - I^h u\|^2_{L^2(K)} \le 2|||u - u^h|||^2_{SUPG}, \tag{98}$$

$$\sum_{K\in\mathcal{T}^h} \delta_K\|\mathbf{b}\cdot\nabla(u - I^h u)\|^2_{L^2(K)} \le 2|||u - u^h|||^2_{SUPG}, \tag{99}$$

$$\sum_{E\in\mathcal{E}^h} \delta_K\mathbf{b}_{L^\infty(K)}\|u - I^h u\|^2_{L^2(K)} \le 2|||u - u^h|||^2_{SUPG}, \tag{100}$$

where $\mathcal{E}^h$ is the set of all faces of the triangulation.

Theorem: Residual a posteriori error estimate
Let $e = u - u^h$, $u \in H^2(\Omega)$ being the solution of (15) and u^h its SUPG approximation computed bu solving (16). Chose the SUPG parameters satisfying $\delta_K \le \frac{h_K^2}{8C_{inv}^2\varepsilon}$, $K \le \frac{\omega}{2\|c\|^2_{L^2(K)}C^2_{inv}}$, which means the SUPG bilinear form with the error as argument is coercive, and under the hypothesis from the remark above, one gets the following global upper bound

$$|||u-u^h|||_{SUPG} \le \eta_1^2+\eta_2^2+\eta_3^2+\sum_{K\in\mathcal{T}^h} 16\delta_K h_K^{-2}\varepsilon^2 C^2_{inv}\|\nabla(u-I^h u)\|^2_{L^2(K)} \tag{101}$$

$$+\sum_{K\in\mathcal{T}^h} 8\delta_K\varepsilon^2\|\Delta(u - I^h u)\|^2_{L^2(K)}, \tag{102}$$

where

$$\eta_1^2 = \sum_{K\in\mathcal{T}^h} min\left\{\frac{C}{\omega}, C\frac{h_K^2}{\omega}, 24\delta_K\right\}\|\ R_K(u^h)\|^2_{L^2(K)}, \tag{103}$$

$$\eta_2^2 = \sum_{K\in\mathcal{T}^h} 24\delta_K\|\ R_K(u^h)\|^2_{L^2(K)}, \tag{104}$$

$$\eta_3^2 = \sum_{E\in\mathcal{E}^h} min\left\{\frac{24}{\|\mathbf{b}\|_{L^\infty(E)}}, C\frac{h_E}{\omega}, \frac{c}{\varepsilon^{1/2}\omega 1/2}\right\}\|\ R_E(u^h)\|^2_{L^2(E)}, \tag{105}$$

where the mesh cell and the face residuals are defined by

$$R_K(u^h) := f - \varepsilon\nabla u^h - b\cdot u^h - cu^h|_K, \tag{106}$$

$$R_E(u^h) = \begin{cases} -\varepsilon[|\partial_{\mathbf{n}_E}u^h|]_E & \text{if } E\in\mathcal{E}^h_\Omega,\\ g - \varepsilon\partial_{\mathbf{n}_E}u^h & \text{if } E\in\mathcal{E}^h_N,\\ 0 & \text{if } E\in\mathcal{E}^h_D,\end{cases} \tag{107}$$

with

$$\mathcal{E}^h_\Omega - set\ of\ all\ interior\ faces, \tag{108}$$

$$\mathcal{E}^h_N - set\ of\ all\ faces\ on\ the\ Neumann\ boundary, \tag{109}$$

$$\mathcal{E}^h_D - set\ of\ all\ faces\ on\ the\ Dirichlet\ boundary. \tag{110}$$

Proof: Following the proof of coercivity of the SUPG bilinear form, and based on the conditions and suitable chosen of the SUPG parameters above, we can similarly deduce that

$$a_{SUPG}(e,e) \geq \frac{1}{2}|||e|||_{SUPG}^2 - \sum_{K\in\mathcal{T}^h} 4\delta_K h_K^{-2}\varepsilon^2 C_{inv}^2 \|\nabla(u - I^h u)\|_{L^2(K)}^2 \tag{111}$$

$$- \sum_{K\in\mathcal{T}^h} 2\delta_K \varepsilon^2 \|\Delta(u - I^h u)\|_{L^2(K)}^2. \tag{112}$$

Using the Galerkin orthogonality (37), the weak form of the equation (96), and integration by parts gives for all $v \in H_D^1(\Omega)$

$$\begin{aligned} a_{SUPG}(u - u^h, v) = &\sum_{K\in\mathcal{T}^h} (R_K(u^h), v - I^h v)_K \\ &+ \sum_{K\in\mathcal{T}^h} \delta_K (R_K(u^h), \mathbf{b}\cdot\nabla(v - I^h v))_K \\ &+ \sum_{E\in\mathcal{E}^h} (R_E(u^h), v - I^h v)_E. \end{aligned}$$

Setting $v = u - u^h$, observing that $v - I^h v = (u - u^h) - I^h(u - u^h) = u - I^h u$ and $I^h u^h = u^h$, and together with (17) we can deduce that

$$\begin{aligned} \frac{1}{2}|||u - u^h||| \leq &\sum_{K\in\mathcal{T}^h} (R_K(u^h), u - I^h u)_K + \sum_{K\in\mathcal{T}^h} \delta_K (R_K(u^h), \mathbf{b}\cdot\nabla(u - I^h u))_K \\ &+ \sum_{E\in\mathcal{E}^h} (R_E(u^h), u - I^h u)_E + \sum_{K\in\mathcal{T}^h} 4\delta_K h_K^{-2}\varepsilon^2 C_{inv}^2 \|\nabla(u - I^h u)\|_{L^2(K)}^2. \end{aligned}$$

For illustration, only the estimate for the first term of above is presented. In the first step of the estimate, the Cauchy-Schwarz inequality, stability of the Lagrangian interpolant in $L^2(\Omega)$ and Young's inequality lead to

$$\begin{aligned} \sum_{K\in\mathcal{T}^h} (R_K(u^h), u - I^h u)_K &\leq \sum_{K\in\mathcal{T}^h} C\|R_K(u^h)\|_{L^2(K)} \|u - u^h\|_{L^2(K)} \\ &\leq \sum_{K\in\mathcal{T}^h} \frac{c}{\omega}\|R_K(u^h)\|_{L^2(K)} + \frac{1}{12}\|\mu^{1/2} u - u^h\|_{L^2(K)}^2. \end{aligned}$$

One can apply interpolation estimate and obtain

$$\begin{aligned} \sum_{K\in\mathcal{T}^h} (R_K(u^h), u - I^h u)_K &\leq \sum_{K\in\mathcal{T}^h} \|R_K(u^h)\|_{L^2(K)} h_K \|\nabla(u - u^h)\|_{L^2(K)} \\ &\leq \sum_{K\in\mathcal{T}^h} \frac{Ch_K^2}{\omega}\|R_K(u^h)\|_{L^2(K)}^2 + \frac{\varepsilon}{12}\|_{L^2(K)} h_K \|\nabla(u - u^h)\|_{L^2(K)}^2. \end{aligned}$$

Eventually, the term with streamline derivative can be used for the bound. Young's ineuqality yields

$$\sum_{K\in\mathcal{T}^h} (R_K(u^h), u - I^h u)_K \leq 6\sum_{K\in\mathcal{T}^h} \|R_K(u^h)\|_{L^2(K)}^2 + \frac{1}{24}\delta_K^{-1}\|u - I^h u\|_{L^2(K)}^2. \tag{113}$$

Collecting the three estimates (98)-(100) in the hypothesis above leads to

$$\sum_{K\in\mathcal{T}^h}(R_K(u^h), u - I^h u)_K$$
$$\leq \sum_{K\in\mathcal{T}^h} min\left\{\frac{C}{\omega}, C\frac{h_K^2}{\omega}, 24\delta_K\right\} \| \, \|R_K(u^h)\|^2_{L^2(K)} + \frac{1}{12}|||u - I^h u|||^2_{SUPG}.$$

For the reminding proof, it is refer to [18].

3.2 A Posteriori Error Estimators

In the numerical solution of convection-diffusion equations, an A posteriori error estimator is quite useful as it can not only estimate the global error but it can obtain information for adaptive mesh-refinement techniques. Some of the a posteriori error estimators studied in this paper will be presented as below Table 1 ([16])

Type	Name	Norm
Gradient indicator	$\eta_{gradind}$	
Residual-based	η_{res-H^1}	H^1-semi norm
Residual-based	η_{res-L^2}	L^2-norm
Residual-based	$\eta_{res-eng}$	Energy norm + dual norm
Residual-based	$\eta_{res-supg}$	SUPG norm

Table 1

Figure 1: Table 1.

According to the Remark of Goal above, we can deduce that the comparison in these a posterior error estimators contains two parts. First, the accuracy of the approximated global error. Second, the adaptively refined grids used for computation of solutions with sharp layer. And we can compare the adaptive meshes generated by these error estimators by comparing the accuracy of the solutions computed on these meshes.

Definition: The gradient indicator $\eta_{gradind}$
To control the adaptive grid refinement in software packages, the gradient indicator is useful since it is simple. The definition of it on the mesh cell K is written as below

$$\eta_{gradind,K} := \|\nabla u^h\|_{L^2(K)}. \tag{114}$$

As it can be deduced from the definition that if the L^2-norm of ∇u^h is large, the indicator will be large. Then the corresponding mesh cells will be refined to reduce the indicator. So this indicator is easy to use and independent of the problems' class. However, with this indicator's use, it is not possible to estimate the global error.Therefore we can not analysis its quality about the accuracy of the estimated error with respect to the computed one through computing the efficient index.

Definition: The Zienkiewicz-Zhu estimator η_{ZZ-H^1}

Different from the gradient indicator, the Zienkiewicz-Zhu estimator's aim is to estimate not $\|\nabla u\|_{L^2}$ but $\|u - \nabla u^h\|_(L^2(K))$. In order to achieve that, a higher-order recovery Ru of ∇u will be constructed with only u^h, i.e.

$$\|\nabla u - Ru\|_{L^2(K)} \leq c\|\nabla u - \nabla u^h\|_{L^2(K)},\ c < 1. \tag{115}$$

Therefore, we have

$$\frac{1}{c+1}\|Ru - \nabla u\|_{L^2(K)} \leq \|\nabla u - \nabla u^h\|_{L^2(K)} \leq \frac{1}{1-c}\|Ru - \nabla u^h\|_{L^2(K)}. \tag{116}$$

Then the Zienkiewicz-Zhu estimator on the mesh cell K can be written as

$$\eta_{ZZ-H^1,K} := \|Ru - \nabla u\|_{L^2(K)}, \tag{117}$$

which is an estimator for the H^1-semi norm. This estimator is also independent of the problems' class as the gradient indicator.

Generally, we can construct Ru by defining the function at a point to be the average is the average of the gradient of u^h in a neighborhood of that point. Assume that the recovery of ∇u is totally defined by its values in triangles' vertices. Hence Ru can be identified by a piecewise linear and continuous function in each component. Ru can be defined as

$$Ru(N) = \sum_{K \in U_N} \frac{|K|}{|U_N|}\nabla u^h|_K, \tag{118}$$

where N is a node of the mesh, U_N is the union of all mesh cells containing node N, $|K|$ is the area of K and $|U_N|$ the area of U_N. It shows that $Ru(N)$ is the weighted average of the gradients of u^h of all mesh cells in U_N with node N.

For convection-diffusion problems, residual-based a residual-based error estimators has the general form

$$\begin{aligned}\eta_{*,K} &:= \alpha_K\|f^h + \varepsilon\Delta u^h - b^h \cdot \nabla u^h - c^h u^h\|_{0,K}^2 \\ &+ \sum_{E \subset \partial K, E \not\subset \partial\Omega_N} \frac{\beta_E}{2}\|\left[|\varepsilon\nabla u^h \cdot n_E|\right]_E\|_{0,K}^2 \\ &+ \sum_{E \subset \partial K, E \subset \partial\Omega_N} \beta_E\|\varepsilon\nabla u^h \cdot n_E - g_N^h\|_{0,K}^2.\end{aligned}$$

To ensure the restriction to each mesh cell K of $f^h + \varepsilon\Delta u^h - b^h \cdot \nabla u^h - c^h u^h$ and the restriction of g_{Nh} to each edge $E \subset \partial\Omega_N$ are polynomials of some fixed degree k, f^h, b^h, c^h, and g_{Nh} are approximations of f, b, c, and g_N, respectively. Verfürch present the traditional approach to derive the residual-based error estimators detailed in [27]. We take care only how the weights depend on the local mesh width and gets the error estimators as shown below.

Definition: A residual-based error estimator in the H^1-semi norm η_{res-H^1}
The a sposteriori error estimator η_{res-H^1} for the H^1-semi norm is obtained by picking $\alpha_K = h_k^2$ and $\beta_E = h_E$

$$\|\nabla(u-u^h)\|_0^2 \leq C(\sum_{K \in \mathcal{T}^h} h_k^2\|R_K(u^h)\|_{0,K}^2 + \sum_{E \in \mathcal{E}^h} h_E\|R_E(u^h)\|_{0,E}^2) + h.o.t. \tag{119}$$

Due to constants C in the estimates depend on the Peclet number, η_{res-H^1} is not robust, cf. the study in [16]. Only in the diffusion dominates case, it becomes robust. And the higher order terms represent only data approximation errors instead of another term of the forms (101)-(102).

Definition: A residual-based error estimator in the L^2-norm η_{res-L^2}
Picking $\alpha_K = h_K^4$ and $\beta_E = h_E^2$, we obtain the residual-based a posteriori error estimator η_{res-L^2}).

$$\|(u-u^h)\|_0^2 \leq C(\sum_{K\in\mathcal{T}^h} h_k^4\|R_K(u^h)\|_{0,K}^2 + \sum_{E\in\mathcal{E}^h} h_E^2\|R_E(u^h)\|_{0,E}^2) + h.o.t. \quad (120)$$

Same as the residual-based error estimator in the H1 norm. It is also only robust in the diffusion dominates case.

Definition: A residual-based error estimator in the energy norm plus a dual norm $\eta_{res-eng}$
A non-robust residual-based error estimator in the energy norm was derived in [**?**]. And later it was refined to a robust error estimator through plus a dual norm to the energy norm in [27], which is showing bellowing:

$$\begin{aligned}
&\|(u-u^h)\|_{en}^2 + \sup_{v\in H_D^1(\Omega)} \frac{\langle \mathbf{b}\cdot\nabla(u-u^h), v\rangle}{\|v\|_{en}^2} \\
&\leq C(\sum_{K\in\mathcal{T}^h} min\left\{\frac{1}{\mu_0}, \frac{h_k^2}{\varepsilon}\right\}\|R_K(U^h)\|_{0,K}^2 \\
&+ \sum_{E\in\mathcal{E}^h} min\left\{\frac{h_E}{\varepsilon}, \frac{1}{\varepsilon^{1/2}\mu_0^{1/2}}\right\}\|R_E(u^h)\|_{0,E}^2) \\
&+h.o.t..
\end{aligned}$$

It is robust under the condition of small mesh Peclet numbers; while it may be not robust when the constants in the estimates depend on the coefficients of the problem in the case of large mesh Peclet numbers.Note that the weights in front of $\|R_E(u^h)\|_{0,E}^2$ appear also in (103)-(105).

Definition: A residual-based error estimator in the SUPG norm $\eta_{res-supg}$
In the case of the diffusion-dominated regime, one obtains

$$\begin{aligned}
&\|(u-u^h)\|_{SUPG}^2 \leq C(\sum_{K\in\mathcal{T}^h} \frac{h_K^2}{\varepsilon}\|R_K(u^h)\|_{0,K}^2 + \sum_{E\in\mathcal{E}^h} \frac{h_E}{\varepsilon}\|R_E(u^h)\|_{0,E}^2) \\
&+h.o.t..
\end{aligned}$$

Both h_K and h_E are effective and the weights in η_{res-H^1} divided by and are recovered. In the case of the diffusion-dominated regime i.e., $Pe_K \gg 1$, the minimum achieves at these values of the weights, one gets a residual-based a posteriori error estimator for the SUPG norm $\eta_{res-supg}$

$$\begin{aligned}
&\|(u-u^h)\|_{SUPG}^2 \leq C(\sum_{K\in\mathcal{T}^h} \frac{h_K}{\|\mathbf{b}\|_{\infty,K}}\|R_K(u^h)\|_{0,K}^2 + \sum_{E\in\mathcal{E}^h} \frac{1}{\|\mathbf{b}\|_{\infty,E}}\|R_E(u^h)\|_{0,E}^2) \\
&+h.o.t..
\end{aligned}$$

Which also means, the dividing by ε residual-based error estimator in the H^1-semi norm η_{res-H^1}'s weights are recovered. The norm of $\|\cdot\|_{SUPG}$ and $\|\cdot\|_{eng}$ are equivalent also in this case. Therefore, the error bounds for the energy norm can be applied replacing the SUPG norm.
The approach for error estimation with the solution of local Neumann problems defined on a single mesh cell K is called element residual method.
We consider a mesh cell K with edges E_i, $i = 1, 2, 3$. And E_i's barycentric coordinates can be written as λ_{E_i}. Define a space V_K on K as

$$V_K = span\{B_K, B_{E_1}, B_{E_2}, B_{E_3}\}, \tag{121}$$

where

$$B_{E_i} = 4\lambda_{E_{(i+1)modulo3}}\lambda_{E_{(i+2)modulo3}}, i = 1, 2, 3 \tag{122}$$

$$B_K = 27\lambda_{E_1}\lambda_{E_2}\lambda_{E_3}. \tag{123}$$

B_{E_i} represents the edge bubble functions defined in K and B_K represents the element bubble function defined in K. Then an approximation of the solution of the global error residual problem is computed in the space $\sum_{K\in\mathcal{T}^h} V_K$.
Definition: An error estimator based on the solution of local Neumann problems, Galerkin discretization, $\eta_{NeumGa-H^1}$
In the space $\sum_{K\in\mathcal{T}^h} V_K$, the functions are discontinuous, therefore the solution of the global equation can be separated into solutions of Neumann problems defined in a single mesh cell K: Find $u_K \in V_K$ such that $\forall v_K \in V_K$

$$\begin{aligned}
&a(u_K, v_K) + b(u_K, v_K) + c(u_K, v_K) \\
&= (f + \varepsilon\Delta u^h - \mathbf{b}\cdot\nabla u^h, v^h)_K - \frac{1}{2}\sum_{E, E\subset\partial K, E\not\subset\partial\Omega}([|\varepsilon\nabla u^h\cdot\mathbf{n}_E|]_E, v_E)_E \\
&+ \sum_{E, E\subset\partial K, E\subset\partial\Omega}(g - \varepsilon\nabla u^h\cdot\mathbf{n}_E)_E.
\end{aligned}$$

Therefore we can derive the corresponding error estimator

$$\eta_{NeumGa-H^1,K} : = |u_K|_{1,K}. \tag{124}$$

Note that the solution u_K might have oscillations because Garlerkin discretization is not stable.

Definition: An error estimator based on the solution of local Neumann problems, SDFEM discretization, $\eta_{NeumSD-H^1}$
Adding a stabilization in the definition of the local Neumann problems will solve the non-stable discretization above:
Find $u_K \in V_K$ such that $\forall v_K \in V_K$

$$\begin{aligned}
&a(u_K, v_K) + b(u_K, v_K) + c(u_K, v_K) + \delta_K(\mathbf{b}\cdot\nabla u_K + cu_K, \mathbf{b}\cdot\nabla v_K)_K \\
&= (f + \varepsilon\Delta u^h - \mathbf{b}\cdot\nabla u^h, v^h + \delta_K\mathbf{b}\cdot\nabla v_K)_K \\
&-\frac{1}{2}\sum_{E, E\subset\partial K, E\not\subset\partial\Omega}([|\varepsilon\nabla u^h\cdot n_E|]_E, v_E)_E + \sum_{E, E\subset\partial K, E\subset\partial\Omega}(g - \varepsilon\nabla u^h\cdot n_E)_E.
\end{aligned}$$

Therefore we can derive the corresponding error estimator

$$\eta_{NeumSD-H^1,K} : = |u_K|_{1,K}. \tag{125}$$

Algorithm

For the numerical solution of the problem, the program flow shown in Figure 2 is applied.The program is applied by using ParMooN. First of all, a level on which the adaptive grid refinement should start ($refinement\_max\_n\_adaptive\_steps$) must be chosen appropriate, which in general can be found through numerical tests. The uniform grid refinement will be applied up to $refinement\_max\_n\_adaptive\_steps$. And the main and most important feature of the solution of the problem should be recognized well on the $refinement\_max\_n\_adaptive\_steps$. Secondly, the error estimator type also should be chosen. Then the computation of the error estimator will compute a number $\eta_{*,K}$ for each mesh cell K to make the decision of if the mesh cell K should be refined or coarsened. In the past experiences [16], the coarsening of cells is not important for the stationary problems and $refinement\_max\_n\_adaptive\_steps$. A sufficient increase of the number of degrees of freedom after an adaptive refinement step is important to obtain an efficient adaptive algorithm for the stationary problems' solutions.

To compare different a posteriori error estimators fairly, the same number of degrees of freedom should be possessed approximately by different adaptive meshes. Hence the computation will stop after the first mesh on which the sum of degrees of freedom and Dirichlet nodes exceeded 100 000.

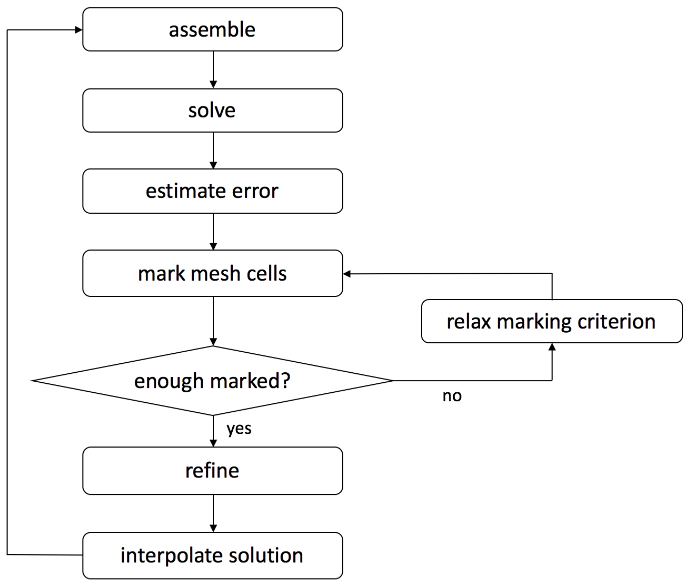


Figure 2: Algorithm for adaptive mesh refinement

4 Numerical Study

In this section, we present the results of two numerical tests to compare the quantity of a posteriori error estimators introduced in Section 3. And the numerical analysis of the behavior of these a posteriori error estimators with respect to the estimated global error η starts from two aspects. One is the efficiency of the error estimators, which means the ability of the a posteriori error estimators estimate the error in a certain norm, i.e., the accuracy of the estimated solution with respect to the real solution on the mesh cells. The measurement is usually made by computation of the efficiency index I_{eff}

$$I_{eff} := \frac{\eta}{\|u - u^h\|} \tag{126}$$

where η is the estimated error and the $\|u - u^h\|$ is the computed error in the corresponding norm $\|\cdot\|$ the estimator is designed for.
For the results from using different estimators in this section, the estimated global error η is

$$\eta = (\eta_1^2 + \eta_2^2 + \eta_3^2)^{1/2}, \tag{127}$$

where η_1^2, η_2^2, η_3^2 are given in (116)-(118). And the constant C is chosen to be 1, since one can gets same weights as in the error estimator on the energy norm plus a dual norm.
On the other hand, a posteriori error estimators should also be compared with respect to the quality of the adaptive grid refinement. In other words, one should consider if the error estimator control an adaptive grid refinement well. In the examples of this section, the SUPG stabilization parameter η_K is chosen same as in [17], where a comprehensive discussion about the possible choices of the SUPG stabilization parameter has been done.

$$\eta_K = \frac{\tilde{h_K}}{2r|\mathbf{b}|}\xi(\tilde{Pe_K}) \tag{128}$$

with

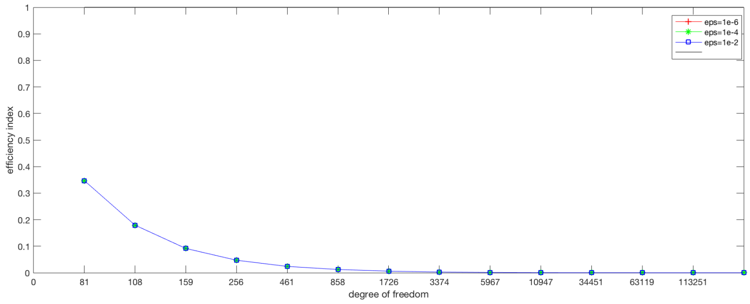
$$\xi(\alpha) = coth\ \alpha - \frac{1}{\alpha},\ \tilde{Pe}_K = \frac{|\mathbf{b}|\tilde{h_K}}{2r\varepsilon}, \tag{129}$$

where $|\mathbf{b}|$ is the Euclidean norm of the convection vector $\mathbf{b}$. And δ_K is the cell diameter in the direction of $\mathbf{b}$. All simulations were performed with the code ParMooN [11] [28].
The results are obtained through the adaptive mesh refinement in Algorithm 1, which was introduced in Section 3.

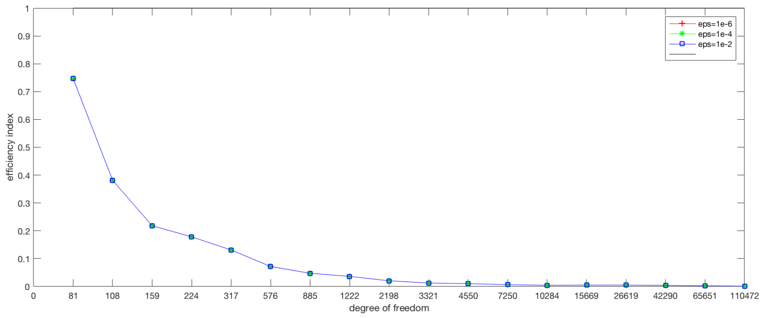
Example 4.1 (A known two dimensional solution with a boundary layer) This example was proposed in [1] and we solve (103)-(105) for different values of ε, $\mathbf{b} = (2,1)^T$, $c = 1$, $u_D = 0$, and $\Omega = (0,1)^2$. The right-hand side f and the boundary conditions are chosen such that the exact solution is given by

$$u(x,y) = y(1-y)(x - \frac{e^{-\frac{(1-x)}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}}). \tag{130}$$

The derivatives of the solution $u(x,y)$ depend on ε. This example is typical for solutions of convection-diffusion equations. The initial mesh for this case is the square Ω divided into two triangles by the straight line $y = x$. we will study the results obtained for the a posteriori error estimators we studied above. And both the accuracy of the results and the adaptively refined grids are analysed in this example. Notice that the computation process will stop after having computed the solution on the first grid with more than 10^5 degree of freedoms.

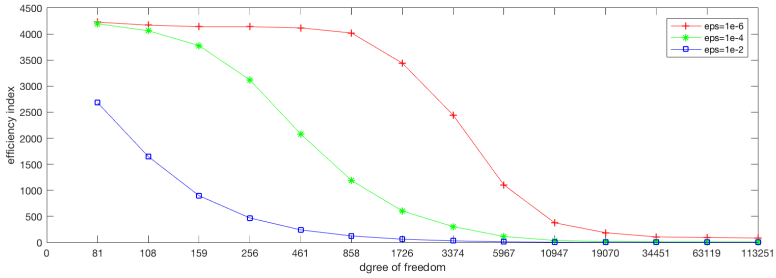


(a) Efficiency index,η_{res-H^1}

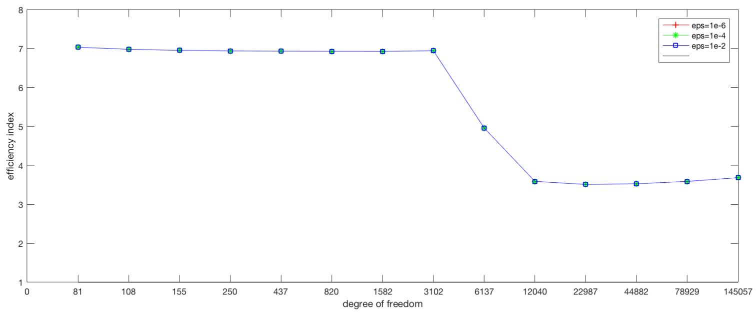


(b) Efficiency index,η_{res-L^2}

Figure 3: Efficiency indices, η_{res-H^1}(left) and η_{res-L^2} (right), Example 4.1.



(a) Efficiency index, $\eta_{res-eng}$

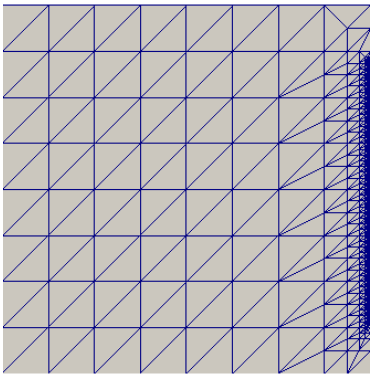


(b) Efficiency index, $\eta_{res-supg}$

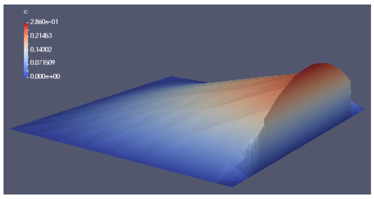
Figure 4: Efficiency indices, $\eta_{res-eng}$ (left) and $\eta_{res-supg}$ (right), Example 4.1.

From Figure 3, we can notice that the efficiency indices of η_{res-H^1} and η_{L^2} are both smaller than 1, which shows that these two estimators both underestimate the error in this example. Whereas from estimator $\eta_{res-eng}$ gives reliable error estimates based on the provided grids with the boundary layers is sufficiently fine for relatively large value of ε (under the condition of $\varepsilon = 10^{-2}$), as one can observe from Figure 4 (left). And strongly overestimated the error when ε gets smaller. The reason is that the constants in this estimate depend on the problem's parameters. And due to small value of ε, this estimator is not robust in the case of large mesh Peclet numbers, which is described in the definition of $\eta_{res-eng}$ in Section 3. Therefore the smaller ε, the stronger overestimation the estimator will perform, which can also be found in the Figure 4 (left). The estimator $\eta_{res-supg}$'s efficiency indices for different diffusion parameters ε are all in the interval $[3, 4]$. For the three a posteriori error estimators η_{res-H1},η_{res-L2} and $\eta_{res-supg}$, the efficiency indices are same with respect to different ε. The reason might be that this example is quite simple and the appropriate property cannot be presented well. As we described earlier in Section 3, neither η_{H^1} nor η_{L^2} is robust, and therefore the efficiency index should vary following the change of value of ε accordingly. Notice that for the gradient indicator, the estimated global value is unknown, and the efficiency index is incalculable.
The final adaptively refined grids of different values of ε (the value differs from 10^{-2}, 10^{-4} to 10^{-6}) of the a posteriori error estimators are presented in Figure 5-9. It can be seen that the grids generated by $\eta_{gradind}$ looks quite differently from the others. It does not refine all the boundary very well but only the steepest region of the layer. And one can also observe that error estimator η_{res-L^2} tends

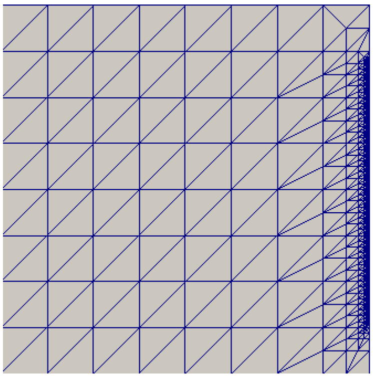
to refine large mesh cells earlier than η_{res-H^1} and $\eta_{res-eng}$ by comparing the left side of Figure 6-8. Which can also be derived from the weights α_K and β_E in the error estimator form in η_{res-H^1}, η_{res-L^2} and $\eta_{res-eng}$. We can also find a broad refinement for the estimator η_{res-L^2}. It is caused by its weights and is good for refinement [18]. As for the estimator $\eta_{res-supg}$, the refined grids is similar to η_{res-H^1} and η_{res-H^1}, which is also depicted the boundary well.
This example shows that there still some error estimators have problems to generate appropriate adaptively refined grids even for the solution of a quite simple problem, which possesses only one layer.



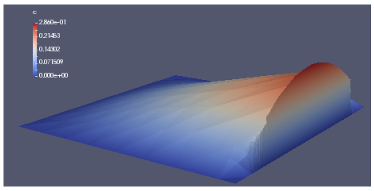
(a) Mesh with $\eta_{gradind}$, $\varepsilon = 10^{-2}$



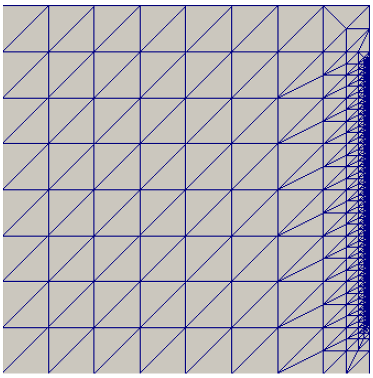
(b) Solution of $\eta_{gradind}$, $\varepsilon = 10^{-2}$



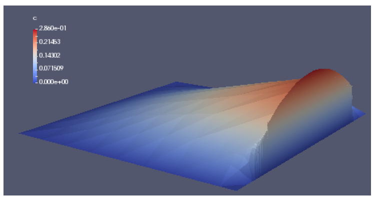
(c) Mesh with $\eta_{gradind}$, $\varepsilon = 10^{-4}$



(d) Solution of $\eta_{gradind}$, $\varepsilon = 10^{-4}$

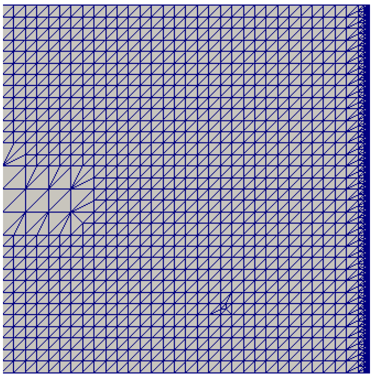


(e) Mesh with $\eta_{gradind}$, $\varepsilon = 10^{-6}$

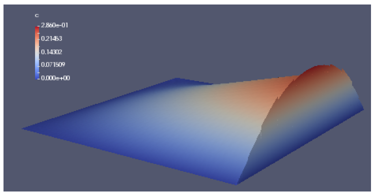


(f) Solution of $\eta_{gradind}$, $\varepsilon = 10^{-6}$

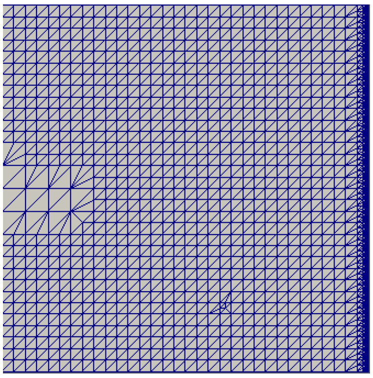
Figure 5: Mesh (left) and solution (right) with $\eta_{gradind}$ for different values of ε, Example 4.1.



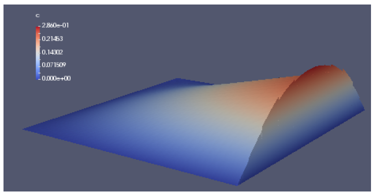
(a) Mesh with η_{res-H^1}, $\varepsilon = 10^{-2}$



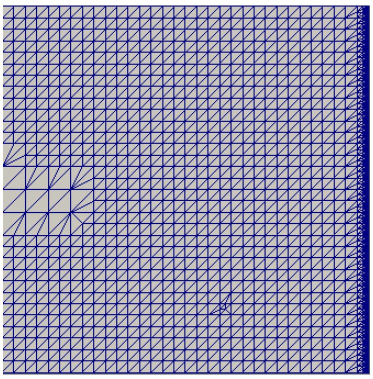
(b) Solution of η_{res-H^1}, $\varepsilon = 10^{-2}$



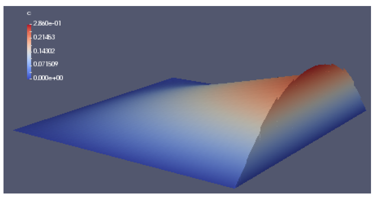
(c) Mesh with η_{res-H^1}, $\varepsilon = 10^{-4}$



(d) Solution of η_{res-H^1}, $\varepsilon = 10^{-4}$

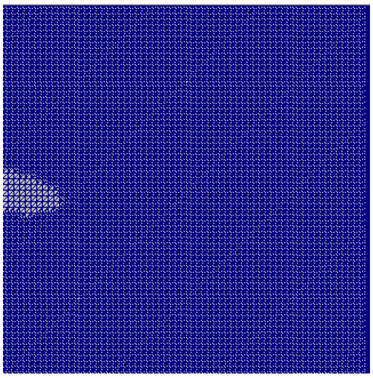


(e) Mesh with η_{res-H^1}, $\varepsilon = 10^{-6}$

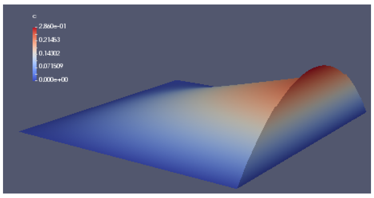


(f) Solution of η_{H^1}, $\varepsilon = 10^{-6}$

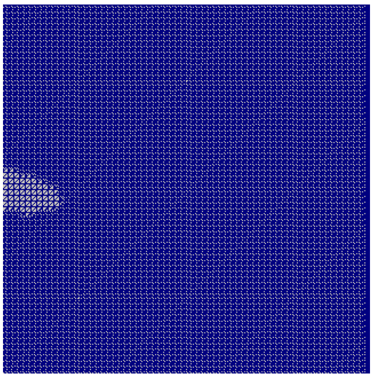
Figure 6: Mesh (left) and solution (right) with η_{res-H^1} for different values of ε, Example 4.1.



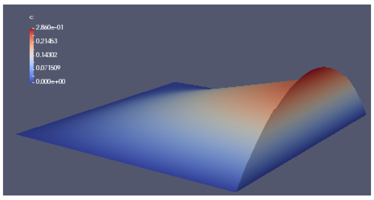
(a) Mesh with η_{res-L^2}, $\varepsilon = 10^{-2}$



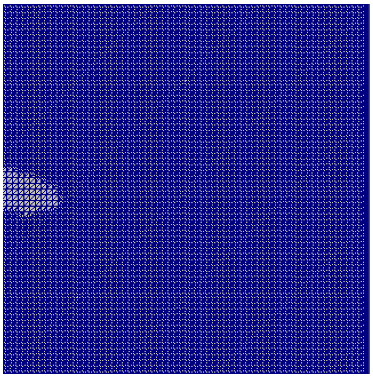
(b) Solution of η_{res-L^2}, $\varepsilon = 10^{-2}$



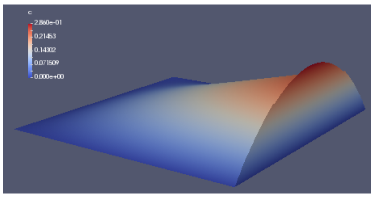
(c) Mesh with η_{res-L^2}, $\varepsilon = 10^{-4}$



(d) Solution of η_{res-L^2}, $\varepsilon = 10^{-4}$

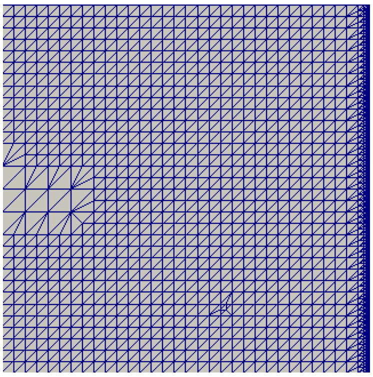


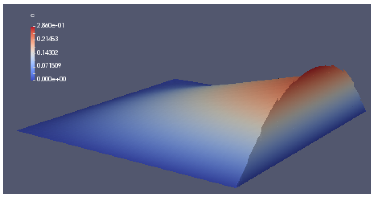
(e) Mesh with η_{res-L^2}, $\varepsilon = 10^{-6}$



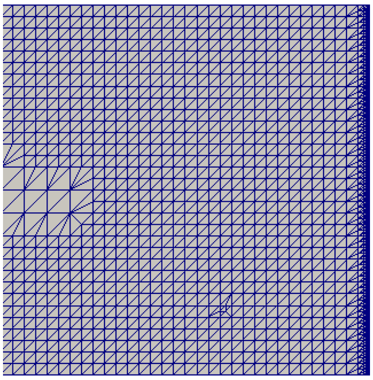
(f) Solution of η_{res-L^2}, $\varepsilon = 10^{-6}$

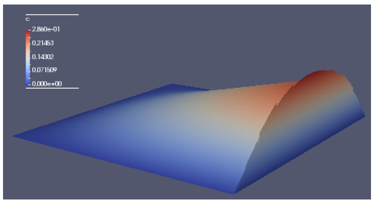
Figure 7: Mesh (left) and solution (right) with η_{res-L^2} for different value of ε, Example 4.1.



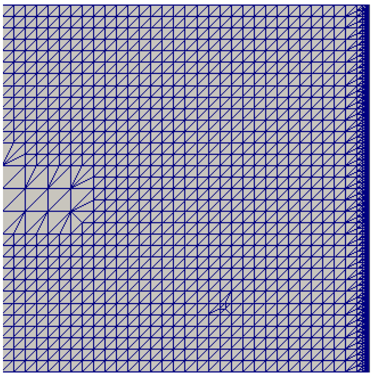


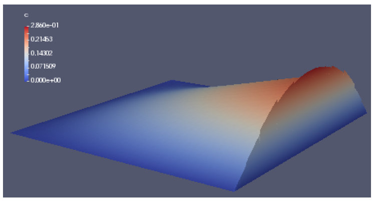
(a) Mesh with $\eta_{res-eng}$, $\varepsilon = 10^{-2}$ (b) Solution of $\eta_{res-eng}$, $\varepsilon = 10^{-2}$





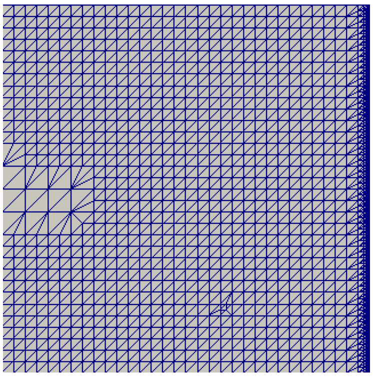
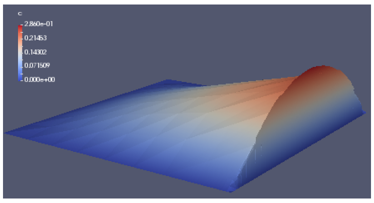
(c) Mesh with $\eta_{res-eng}$, $\varepsilon = 10^{-4}$ (d) Solution of $\eta_{res-eng}$, $\varepsilon = 10^{-4}$



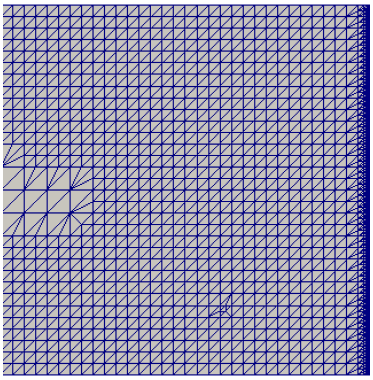
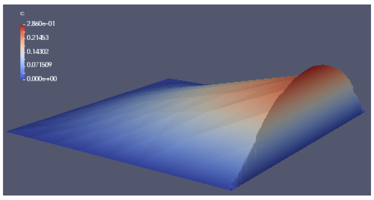


(e) Mesh with η_{eng}, $\varepsilon = 10^{-6}$ (f) Solution of $\eta_{res-eng}$, $\varepsilon = 10^{-6}$

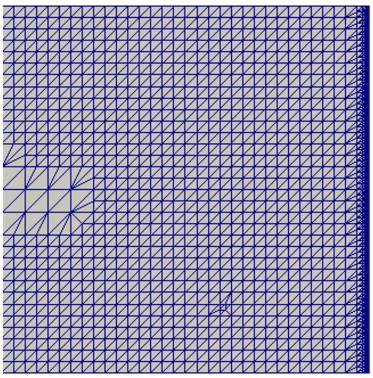
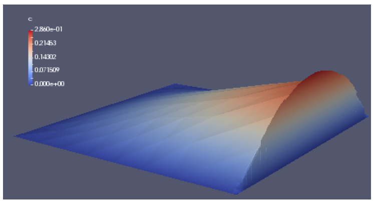
Figure 8: Mesh (left) and solution (right) with $\eta_{res-eng}$ for different value of ε, Example 4.1.

(a) Mesh with $\eta_{res-supg}$, $\varepsilon = 10^{-2}$ (b) Solution of $\eta_{res-supg}$, $\varepsilon = 10^{-2}$

(c) Mesh with $\eta_{res-supg}$, $\varepsilon = 10^{-4}$ (d) Solution of $\eta_{res-supg}$, $\varepsilon = 10^{-4}$

(e) Mesh with $\eta_{res-supg}$, $\varepsilon = 10^{-6}$ (f) Solution of $\eta_{res-supg}$, $\varepsilon = 10^{-6}$

Figure 9: Mesh (left) and solution (right) with $\eta_{res-supg}$ for different value of ε, Example 4.1.

Example 4.2 (Unknown solution with inner layer and exponential boundary layer)
The convection-diffusion equation (103)-(105) is considered in $\Omega = (0,1)^2$, with the data $\varepsilon = 10^{-6}$, $\mathbf{b} = (\cos(-\frac{\pi}{3}), \sin(-\frac{\pi}{3}))^T$, $c = 0$, $\Omega = (0,1)^2$ and

$$\mathrm{u}_D(x,y) = \begin{cases} 0, & \text{for } x = 1 \text{ or } y \le 0.7 \\ 1, & \text{else.} \end{cases}$$

This example was defined and used in [14]. The exponential layers are developed on the boundary $x = 1$ and the right part of the boundary $y = 0$. The computation were implemented on the regular Grid 1, see Figure 10. Since we do not know the analytical solution of this problem, the efficient index cannot be computed. Hence we study the adaptive grids generated by the investigated error estimators and the graphical representations of the computed solutions here, which are presented in Figure 11-15. We use the sharpness of the inner layer and the boundary layer as the criteria for the comparison of the quality between different a posteriori error estimators.

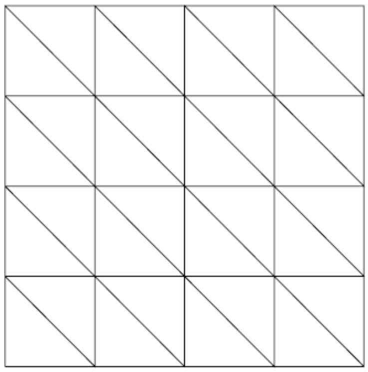
1.png 

Figure 10: Grid 1, Example 4.2.

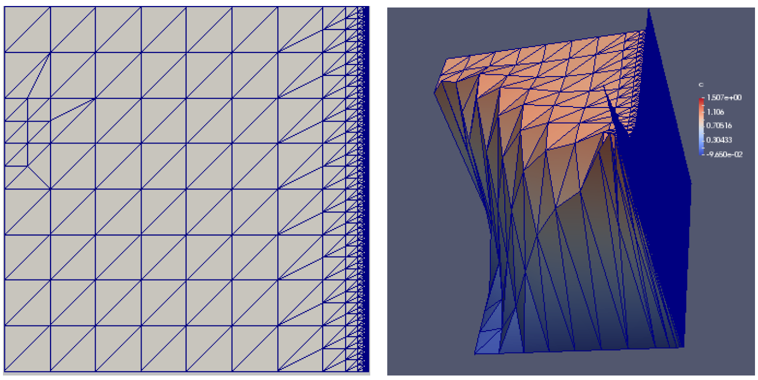


Figure 11: Mesh (left) and Solution (right) obtained with $\varepsilon = 10^{-6}$ for $\eta_{gradind}$, Example 4.2.

From the left side of Figure 11, the adaptive refined grids generated by the gradient indicator $\eta_{gradind}$ failed to depict the exponential layer on the right part of the boundary $y = 0$ and the inner layer. Which failure occurred similar in the example 4.1. The grids did not refine all the layers but only the steepest region of the layers in this probelm.

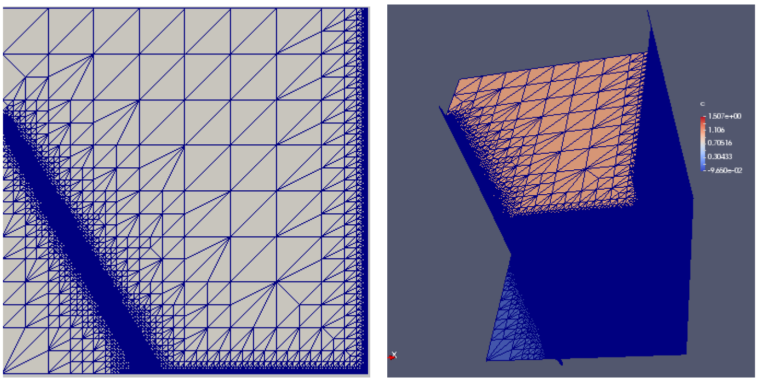


Figure 12: Mesh (left) and Solution (right) obtained with $\varepsilon = 10^{-6}$ for η_{res-H^1}, Example 4.2.

Compare the final refined grids of η_{res-H^1},η_{res-L^2},$\eta_{res-eng}$ and $\eta_{res-supg}$, it can be seen that for $\varepsilon = 10^{-6}$, η_{res-H^1} and $\eta_{res-eng}$ produce meshes which are well refined within all layers. The layers on the final adaptively refined grids is sharp, Which can be observe from Figure 12 and Figure 14. From the left side of Figure 15, for error estimator $\eta_{res-supg}$, the generated adaptive grid only refine the layers on the boundary well, but it does not refine the inner layer sufficiently, which can be derived from width of the inner layer presented on the left side of Figure 15 is wider than Figure 12 and Figure 14. The corresponding discrete solutions are unsatisfactory. And compare Figure 12 and Figure 14, we can observe that $\eta_{res-supg}$ refined the grids even worse.

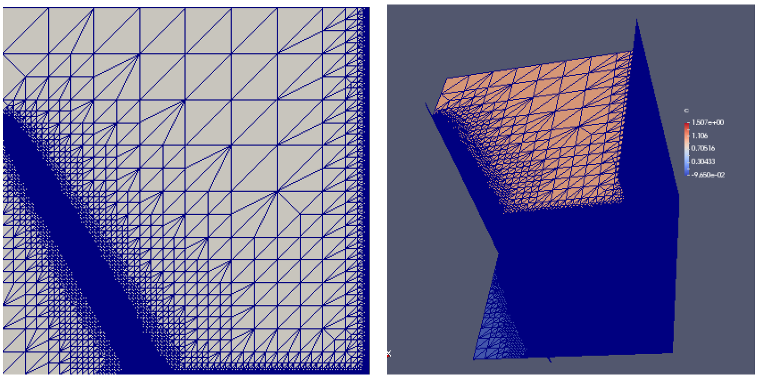


Figure 13: Mesh (left) and Solution (right) obtained with $\varepsilon = 10^{-6}$ for η_{res-L^2}, Example 4.2.

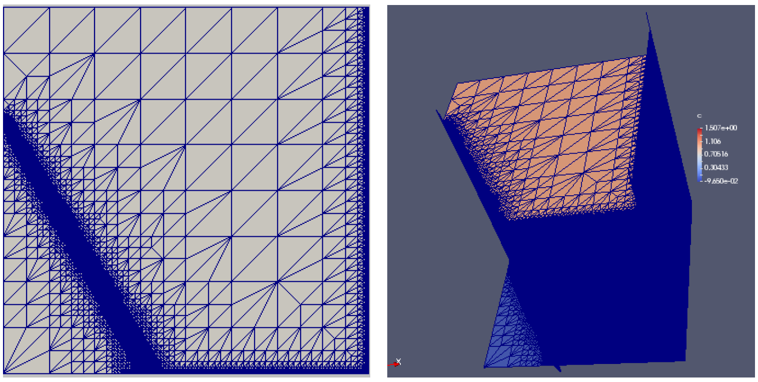


Figure 14: Mesh (left) and Solution (right) obtained with $\varepsilon = 10^{-6}$ for $\eta_{res-eng}$, Example 4.2.

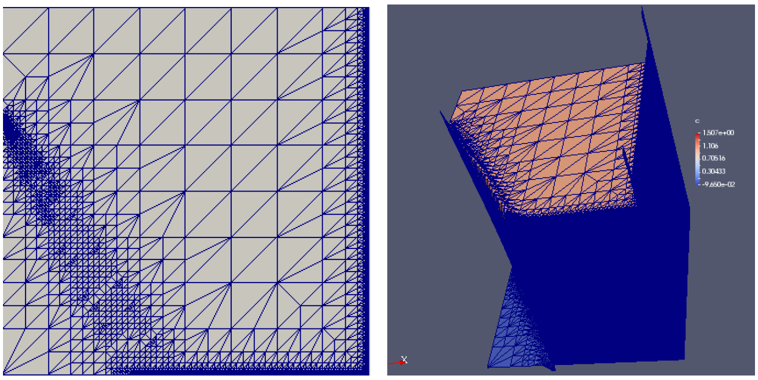


Figure 15: Mesh (left) and Solution (right) obtained with $\varepsilon = 10^{-6}$ for $\eta_{res-supg}$, Example 4.2.

This example shows again that gradient indicator $\eta_{gradind}$ has problems to generate appropriate adaptively refined grids even in the relatively simple situation.

5 Summary

Overall, for the behaviour of the a posteriori error estimator with respect to the generated adaptively refined grids to solve the two examples in Section 4, the gradient indicator $\eta_{gradind}$ worked unsatisfactorily in both of the two examples in Section 4. Wheareeas the other four residual-based error estimators worked acceptable. The estimator η_{res-L^2} worked good in the both examples in Section 4. While in Example 4.1, η_{res-H^1} and $\eta_{res-eng}$ generated the better adaptively refined grids. It means for the solution of different steady-state convection-diffusion problems, there is no one determined optimal a posteriori error estimator to apply.

For study of the behaviour of the a posteriori error estimator with respect to the estimation of the global error, the estimator $\eta_{res-eng}$ estimated the global error most accurate in the case of small Peclet Number. The residual-based estimators η_{res-H^1} and η_{res-L^2} underestimated the error in the Example 4.1. While $\eta_{res-eng}$ slightly overestimated the error. As for the estimator η_{SUPG}, it gave good efficiency indices independent of the Peclet number.

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Affidavit

I, Yurong Ding, hereby declare that this master thesis in question was written single-handed and no further as the denounced resources and sources were employed.

Beijing, 26 December 2019

Yurong Ding